Minimax Theory and its Applications

Volume 11 (2026), No. 2s, 80 to 97

Harnessing Fractional Calculus: Modeling The Stabilizing Effect Of Marigold On Onion Borer-Natural Enemy Dynamics

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Abstract
This study thoroughly investigates an ecological pest control strategy centred on integrating the marigold plant as a companion crop. The core goal is to suppress onion borer by enhancing the effectiveness of their natural enemies, a form of biocontrol. To precisely capture the complex dynamics of this system, an advanced fractional-order model was employed. This model incorporates the Holling type-I functional response to illustrate the predatory interaction between the onion borer and its natural enemies while also accounting for the specific biocontrol influence provided by the marigold. The analysis went deep, first establishing the foundational rigor of the model by proving the existence and uniqueness of solutions and confirming their biological plausibility through the non-negativity condition. The investigation then moved to dynamic stability to examine both the local and global stability at equilibrium points. Crucially, all the analytical findings derived from this rigorous theoretical analysis were conclusively validated by comprehensive numerical simulations.

Keywords: Fractional order model, Non-negativity, Existence and uniqueness of solutions, Dynamical Analysis.

Introduction

As a prominent member of the Amaryllidaceae family, the onion holds significant global importance as a commercial vegetable and staple food. Lamani et al.[1] emphasizes its culinary versatility and high market demand, which underscores its economic value. Given that India is the second-largest producer of vegetables globally, Jat et al.[2] and Gupta et al.[3] highlight the country's substantial contribution to onion cultivation, noting that the crop is extensively integrated into various regional cuisines as both a primary ingredient and a condiment. The historical significance of the onion is equally profound; Gupta et al.[4] observes that its pharmacological and medicinal qualities have ensured its cultivation for over 5000 years, with ancient medical texts like the Charaka Samhita and Ayurveda documenting its use for various ailments. Modern scientific studies conducted by Velasco González de Peredo et al.[5] and Wu et al.[6] further substantiate these traditional claims. Their research demonstrates that the rich antioxidant content of onions, particularly flavanols, contributes to the prevention of oxidative stress-related diseases while providing anticancer, antiplatelet, and anti-asthmatic benefits. Despite its global significance, production is frequently hampered by insect pests. Research indicates that pests such as thrips can inflict devastating damage by rasping plant tissues and sucking sap. This leads to stunted growth and distorted bulbs, which can result in yield reductions of up to 90% if infestations occur early in the crop cycle.

Marigolds are increasingly recognized for their multifaceted utility beyond ornamental purposes. Sharma G et al.[7] recognizes their significant role in sustainable agriculture as a biological pest deterrent, noting that this annual flower is widely cultivated for its inherent hardiness and adaptability. Investigations into the marigold's capacity to naturally control pests have focused particularly on the onion borer, which poses a substantial threat to the viability of onion crops. Silveira LC et al. [8] explains that marigolds emit volatile compounds, most notably α-terthienyl, which effectively repel harmful nematodes and insects. Furthermore, the researcher notes that these plants act as a trap crop while providing nectar and shelter to bolster populations of beneficial natural enemies.

The study of these biological interactions requires a sophisticated mathematical approach to assess the intricate dynamics of pest suppression. Kartal S et al. [12], Misra AK et al. [26], and Jha N et al. [27] utilize fractional models to capture non-local and memory effects, allowing for a more accurate representation of the long-term impact of marigold intercropping on pest dynamics compared to traditional integer-order differential equations. This methodology accounts for the historical dependence of pest populations on environmental factors and the cumulative impact of intercropping strategies. By fostering a healthy ecosystem through organic practices and diligent field sanitation, farmers can significantly enhance the resilience of onion crops against harmful infestations. This innovative approach provides a robust framework for predicting population changes and optimizing marigold integration strategies in modern agricultural practices.

One of the conventional techniques of researching mathematical ecology is the application of mathematical structures such as ODE for expressing interactions between species, especially prey-predator dynamics. Recently, fractional-order differential equations have been an excellent tool for describing inherited traits and the memory of many biological aspects, and due to their possessing similar relationships to fractals, researchers are interested in using them to create mathematical models Li W, Chen Y et al. [11]. These models can provide deeper insights into the complex behaviours of ecosystems by capturing the nuances of species interactions over time. As a result, employing fractional-order equations may lead to more accurate predictions and better management strategies in agricultural practices and conservation efforts. These are the main advantages of a mathematical model based on fractional-order derivatives, as opposed to models based on classical ordinary-order derivatives, which overlook these unique characteristics. Karmaker S et al. [28] and Kundu P et al.[29] demonstrate that fractional-order differentiation is more realistic than ordinary differentiation, as it provides deeper insights into the complex behaviours of ecosystems by accurately capturing the nuances of species interactions over time. With the publication of various popular texts on fractional-order differential equations , researchers and practitioners are increasingly adopting these advanced methodologies. Kilbas AA [16], Hariprasad S et al. [18], and Singh J et al. [25] assert that this shift toward fractional-order methodologies not only enhances the understanding of complex systems but also fosters innovative approaches to tackling ecological challenges.



Onion leaves showing blight Onion bulb infested by onion maggot



Onion bulbs showing progressive stages of basal rot and decay

Fractional-order differential equations have several standard definitions, such as those by Riemann-Liouville, Grünwald-Letnikov, and Caputo. The Caputo definition is frequently chosen in mathematical ecology because it allows for the use of initial conditions in a manner familiar from standard integer-order differential equations. As the initial conditions in the Aćimović M [9], Podlubny I [13], Li HL et al. [17], and El-Saka HA et al.[24] observe that while the Caputo definition is laid out analogously to classical differentiation and remains the most frequently used form in mathematical ecology, there is still scant theoretical backing for mathematical models exploring dynamics that involve fractional-order derivatives. This paper introduces a novel fractional-order model to describe the dynamics between an onion borer and its natural enemy, within the model parameters. The fractional order α quantitatively represents the system's ecological memory, where lower values of α correspond to a stronger and more persistent damping effect induced by the marigold.Vargas-De-León C [19], Naim M et al.[20], and Li Y et al. [23] utilized the Lyapunov method to demonstrate the ability to apply fractional calculus for evaluating the stability of multiple stationary points within complex systems. These approaches have opened new avenues for understanding complex ecological interactions and their stability over time. As researchers

continue to explore the implications of fractional-order modelling, we can expect to gain deeper insights into the dynamics of ecosystems and the effects of various perturbations on their equilibrium states.

Companion planting of a flowering trap crop alongside a field crop susceptible to pest infestation is a recognized biological pest management strategy: the companion crop draws in natural enemies of the pest through nectar, shelter, or volatile cues, reducing dependence on chemical pesticides while opening a secondary revenue stream from the companion crop itself. Following this intercropping principle, we pair onion (*Allium cepa*) with a surrounding or interspersed marigold (*Tagetes* spp.) planting. The general modelling approachpairing a primary crop with a companion flowering plant to link biological pest control with a secondary flower-sale revenue stream — follows the methodological lineage established by Kundu and Mallick (2023)[29] for guava and tuberose, though the crop pairing, spatial arrangement, and underlying parameters here are independently formulated for the onion–marigold system. We adopt this companion-planting arrangement for the physical layout but reinterpret its management practices through the lens of ecological memory, motivating the fractional-order reformulation developed in the next section.

Proposed Cultivation Layout for Onion and Marigold Intercropping

The cultivation layout partitions a fixed field area into equal beds, with onion grown as the primary row crop and marigold planted as a border or interspersed row at a set spacing, close enough that natural enemies reliably move between the marigold and adjacent onion rows. Three categories of management decision drive the pest–predator dynamics, and each carries a temporal or cumulative dimension that a classical (integer-order, memoryless) model cannot capture:

- **Sanitation history:**The removal of infested foliage, decaying bulbs, and crop debris is not a single action but an ongoing practice; breeding sites for the onion pest accumulate or clear depending on the history of sanitation effort. This process is naturally modelled using a fractional-order sanitation term.

- **Pesticide application:**An organic pesticide degrades over time and is reapplied repeatedly throughout the growing season. Consequently, its suppressive effect on the pest population at time t depends on its cumulative application history rather than the instantaneous spray alone—a hereditary property that the Caputo derivative is specifically designed to capture.

- **Flower harvesting and natural-enemy carrying capacity.** Removing marigold flowers for sale reduces the nectar resource available to natural enemies; because natural-enemy populations respond to resource availability with a lag (established populations don't vanish the instant flowers are cut), harvesting effects are better modelled as memory-dependent than as an instantaneous proportional loss.

Formulation of Mathematical Model

$$\frac{do}{dt} = r\left(1 - \frac{o}{k}\right)o - vom + \Upsilon_1 o - \beta o$$

$$\frac{dm}{dt} = -sm + wom - \gamma_2 m \quad (1)$$

The analysis assumes the onion borer functions as the prey, and natural enemies are considered the predator that aids in controlling the pest population within the onion crop. Also,o and m in the above system represents the density of prey and predator population at the time t respectively. Here all the parameters used in the model (1) $r, u, k, \Upsilon_1, \beta, s, w$ and Υ_2 are taken as positive. In the proposed model, r symbolizes intrinsic growth rate of the prey, the natural death rate of the predator or the mortality rate of the predator is referred by s, k represents the logistic growth rate of onion borer, v refers to decline rate of the onion borer due to natural enemies, Υ_1 refers to the sanitation, β Suppression rate via organic pesticide, γ_2 natural enemies loss due to harvesting,here $w = \Omega\times\delta$, Ω represents the rate at which the natural enemies population increases as a result of consuming the onion borers and the δ represents the area of the onion plant,calculated as $\delta = a^2 - 4\left(\frac{a}{2} - 2b\right)^2$,$a$ is the area undercultivation of both onion and marigold plantand b is the width of the marigold plant. The above equation represents the Holling type – I functional response of the predator.

Fractional order form of our proposed model (1) as follows

$${}^{c}D^{\alpha_1} o(t) = \hat{r}\left(1 - \frac{o}{k}\right)o - \hat{v}om + \hat{\gamma}_1 o - \hat{\beta}o, o(t_0) = o_0 > 0$$

$^{c}D^{\alpha_2}m(t) = \hat{s}m + \hat{w}om - \hat{\gamma}_2 m, m(t_0) = m_0 > 0$ (2)

Here, for i=1,2 , $\alpha_i \epsilon$ (0,1), $^{c}D^{\alpha_i}$ are the fractional order derivatives in the sense of Caputo [13][19].

To ensure dimensional consistency within the model (2), the orders of the fractional differentiations are assigned distinct values, α_1 for the onion borer and α_2 for the natural enemies, reflecting the general ecological principle that different species possess varying degrees of memory. Considering that the left-hand sides of the first and second equations have dimensions of $time^{-\alpha_1}$ and $time^{-\alpha_2}$respectively, while the right-hand sides have a dimension of $time^{-1}$, the correct system dimensions must be adjusted to maintain homogeneity. The below equation shows the correct system of dimensions,

$^{c}D^{\alpha_1}o(t) = \hat{r}^{\alpha_1}\left(1 - \frac{o}{k}\right)o - \hat{v}^{\alpha_1}om + \tilde{\gamma}_1{}^{\alpha_1}o - \hat{\beta}^{\alpha_1}o,\ o(t_0) = o_0 > 0$

$^{c}D^{\alpha_2}o(t) = \hat{s}^{\alpha_2}m + \hat{w}^{\alpha_2}om - \hat{\gamma}_2{}^{\alpha_2}m, m(t_0) = m_0 > 0$(3)

substituting with simpler variables in the above system (3) as $r = \hat{r}^{\alpha_1}, u = \hat{u}^{\alpha_1}, v = \hat{v}^{\alpha_1}, \hat{\gamma}_1 = \hat{\gamma}_1{}^{\alpha_1}, \hat{\beta} = \hat{\beta}^{\alpha_1}, \hat{s} = \hat{s}^{\alpha_1}, \hat{w} = \hat{w}^{\alpha_2}, \hat{\gamma}_2 = \hat{\gamma}_2{}^{\alpha_2}$. Now the system (3)reduces as

$^{c}D^{\alpha_1}o(t) = ro\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 o - \beta o,\ o(t_0) = o_0 > 0$

$^{c}D^{\alpha_2}o(t) = -sm + wom - \gamma_2 m\ , m(t_0) = m_0 > 0$(4)

A system (4) where the fractional derivatives have different orders ($\alpha_1 \neq \alpha_2$) is called as an incommensurate fractional-order system. Suppose the system whose fractional derivatives have the same order (α_1=α_2=α, where α ϵ (0,1)) is known as a commensurate fractional-order system. Now the system (4) becomes,

$^{c}D^{\alpha}o(t) = r\left(1 - \frac{o}{k}\right)o - vom + \Upsilon_1 o - \beta o$

$^{c}D^{\alpha}m(t) = -sm + wom - \gamma_2 m$(5)

This outlines the remaining steps of the analysis. Important lemmas and theorems that aid in the system analysis during some preliminary work will be recalled. The text then determines a number of equilibria of (5) in dynamical behaviour of fractional order system. These lemmas and theorems are subsequently used to prove the boundedness and non-negativity of the solutions for the model (5). This work also discusses the examination of the local and global stability of a number of equilibria. To validate the analytical results, a few numerical simulations are conducted. Finally, a succinct discussion and conclusion are provided.

Some preliminaries

Definition 1 [14] The Caputo differential operator for α>0 can be expressed as

$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t f^m(s)(t-s)^{m-\alpha-1}\,ds$ where $\Gamma(.)$ is the gamma function. Whereas$0 < \alpha < 1$ is simply

$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(s)(t-s)^{-\alpha}\,ds$

Lemma 1[15]Let $\alpha \in (0,1]$ and consider a function $f(t)$ such that its Caputo fractional derivative $^{c}D^{\alpha}f(t)$ exists and is continuous on $[a,b]$, i.e., $^{c}D^{\alpha}f(t) \in C[a,b]$.Then, the following holds for all $t \in [a,b]$:

1. If $^{c}D^{\alpha}f(t) \geq 0$, then $f(t)$ is monotonically increasing.
2. If $^{c}D^{\alpha}f(t) \leq 0$, then $f(t)$ is monotonically decreasing.

Lemma 2[16]Let $\alpha \in (0,1)$ and $a, b \in \mathbb{R}$ with $a > 0$. Suppose a function $o: [t_0, \infty) \to \mathbb{R}$ is continuous and satisfies the fractional differential inequality:

$^{c}D^{\alpha}o(t) + a\,o(t) \leq b, o(t_0) = o_0$,where $^{c}D^{\alpha}$ denotes the Caputo fractional derivative.

Then, the following inequality holds for all $t \geq t_0$:

$o(t) \leq (o_0 - \frac{b}{a})E_\alpha[-a(t-t_0)^\alpha] + \frac{b}{a}$,where E_α is the one-parameter Mittag-Leffler function

Lemma 3 [13]Consider the following linear fractional-order system of order $\alpha \in (0,1]$: ${}^cD^\alpha o(t) = Po(t), o(t_0) = o_0 > 0$,where $o \in \mathbb{R}^n$and $P \in \mathbb{R}^{n\times n}$. The stability of the system is characterized as follows: The system is asymptotically stable if and only if $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ is satisfied for every eigenvalue λ of the matrix P.
The system is stable if and only if the following two conditions are met,

For every eigenvalue λ of P,$|\arg(\lambda)| \geq \alpha\frac{\pi}{2}$.Any eigenvalue that satisfies the critical condition $|\arg(\lambda)| = \alpha\frac{\pi}{2}$ has a geometric multiplicity of one.

Lemma 4 [14]

Consider a nonlinear fractional-order system of order $\alpha \in (0,1]$:

${}^cD^\alpha o(t) = Po, o(t_0) = o_0 > 0$,where $o\epsilon\mathbb{R}^n, P\,\epsilon\mathbb{R}^{n\times n}$.The local stability of a stationary point is determined by the linearization of the system around this point. A system is locally asymptotically stable if and only if, for every eigenvalue λ_i $(i = 1,2,\dots,n)$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at a stationary point, the following condition holds:$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$. Conversely, it is unstable if all the eigenvalues $\lambda_i(i = 1,2 \dots \dots . n)$ satisfy $|arg\,\lambda_i| < \alpha\frac{\Pi}{2}$

Dynamical behaviour of fractional order framework

When $\alpha_1 = \alpha_2 = \alpha\,\epsilon\,(0,1]$ in the incommensurate system (4),then the system (5) becomes commensurate fractional-order system

Non-negativity and uniform boundedness

The results for the population of onion borers and their natural enemies were proven to non-negative.

The non-negative and bounded solution of the most important and useful solutions in the context of mathematical ecology. Assume that $\aleph^+$={(o,m) ϵ $\aleph$: o,m ϵ $\mathbb{R}^+$} where $\mathbb{R}^+ = \{o\epsilon\mathbb{R} : o \geq 0\}$.

Theorem 1 All the solutions of the onion pest and natural enemies initiates in $\mathbb{R}^{+2}$ are non-negative and uniformly bounded.

Proof (Non-negativity)Let $\hat{o}(t_0) = (o(t_0), m(t_0)) \in \aleph^+$ be a non-negative initial condition for the system (5). The proof demonstrates that any solution $o(t)$ remains non-negative for all future time. Suppose, for the sake of contradiction, that the solution $o(t)$ is not non-negative. If a function $o(t)$ satisfies $o(t) \geq 0$ for $t < T$ but $o(t) < 0$ for $t > T$, then T^* is defined as the smallest such T where this change happens, so that:

$$o(t) \geq 0 \text{ for } t < T^*, o(T^*) = 0, o(t) < 0 \text{ for } t > T^*$$

From the first equation of system (5), the Caputo derivative evaluated at this point is:

$$^cD^\alpha o(t)\,|_{t=T^*} = 0.$$

By **Lemma 1**, the condition ${}^cD^\alpha o(t) = 0$ at $t = T^*$ implies that $o(T^*) = 0$. This result contradicts the initial assumption that $o(T^*) < 0$.A similar argument is used for other variables such as $m(t)$, ensuring all system components remain non-negative throughout their evolution, for all $t \in (0,T]$.

Uniform Boundedness: To establish the uniform boundedness of the solutions, we introduce the following auxiliary function: $N(t) = o(t) + \frac{v}{w}m(t)$which will be used to demonstrate the boundedness of the system's solutions.

${}^cD^\alpha$ N(t) = ${}^cD^\alpha$ o(t) $+\frac{v}{w}{}^cD^\alpha$ m(t)

$${}^cD^\alpha\, N(t) = ro\left(1 - \frac{o}{k}\right) + \Upsilon_1 o - \beta o - \frac{vsm}{w} - \frac{v\gamma_2 m}{w}$$

$$\eta(t)N(t) = ao(t) + \frac{o(t)}{w}$$

${}^{c}D^{\alpha}\, N(t) + a(t) = ro\left(1 - \frac{o}{k}\right) + vom + \Upsilon_1 o - \beta o - \frac{smv}{w} - \frac{\gamma_2 mv}{w} + ao + a\frac{v}{w}$

$$\leq -\frac{r}{k}o^2 + o(r + \Upsilon_1 - \beta + a) + \frac{mv}{w}(a - \zeta)$$

$$= -\frac{r}{k}o^2 + o(\eta + a), o(t) > 0$$

${}^{c}D^{\alpha}\, N(t) + \eta(t) \leq \frac{k(\eta+a)^2}{4r}$

By using the lemma(2) we have,

$$N(t) \leq \left(N(t_0) - \frac{k(\eta + e)^2}{4r}\right) E_{\alpha}[-x(t - t_0)^{\alpha}] + \frac{k(\eta + a)^2}{4r} \longrightarrow \frac{k(\eta + a)^2}{4r}\ as\ t \longrightarrow \infty$$

Therefore, all solutions of the system that originate in the region ρ^{+}remain confined within the set δ={(o,m) ϵ ρ^{+}: o+$\frac{m}{w} \leq \frac{k(\eta+a)^2}{4r} + \epsilon, \epsilon > 0$}

Existence and uniqueness of the solution

Lemma 5[23]

Let us consider the following fractional-order differential equation of order $\alpha \in (0,1]$:

$${}^{c}D^{\alpha}o(t) = f(t, o), o(t_0) = o_0 > 0 (6)$$

where $f: \aleph \times [t_0, \infty) \to \mathbb{R}^n$and $\aleph \subset \mathbb{R}^n$.
If the function $f(t, o)$satisfies the Lipschitz condition with respect to o on $\aleph \times [t_0, \infty)$, then a unique solution of equation (6) exists on this domain.

To establish the existence and uniqueness of the system (5), Lemma (5) is utilized. The uniqueness holds in the region $\aleph \times [t_0, a)$, where

$$\aleph = \{(o, m) \in \mathbb{R}^2 : \max(|\ o\ |, |\ m\ |) \leq b\},$$

and $a$and $b$are finite positive real constants.

Let$\mathbb{V} = (o, m)$ and $\widetilde{\mathbb{V}}$=(o_1, m_1) be two points in ℵ. Consider the mapping ψ: ℵ → $\mathbb{R}^2$ defined by $\psi(V) = (\psi_1(\mathbb{V}),$ $\psi_2(\widetilde{\mathbb{V}}))$ were

$\psi_1(\mathbb{V}) = ro\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 - \beta o$

$\psi_2(\widetilde{\mathbb{V}}) = -sm + wom - \gamma_2 m$

For any $\mathbb{V}$,$\widetilde{\mathbb{V}} \epsilon\ \mathbb{N}$be arbitrary it follows that

$$\left\|\psi(\mathbb{V}) - \psi(\widetilde{\mathbb{V}})\right\| = \left|\psi_1(\mathbb{V}) - \psi_1(\widetilde{\mathbb{V}})\right| + \left|\psi_2(\mathbb{V}) - \psi_2(\widetilde{\mathbb{V}})\right|$$

$$= |ro\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 o - \beta o - ro_1 + uo_1^2 + vo_1 m_1 - \gamma_1 o_1 + \beta o_1| \quad + |-sm + wom - \gamma_2 m + sm_1 + vwo_1 m_1 - \gamma_2 m_1|$$

$$\leq \left|(r - \frac{r}{k}(o - o_1) - vm + \Upsilon_1 - \alpha)(o - o_1)\right| + |-vo_1(m - m_1)(o - o_1) + wm(o - o_1)| + |s + \Upsilon_2|(m - m_1)$$

$$\leq |(r + \frac{2br}{k} + vb + \Upsilon_1 - \beta + wb)(o - o_1)| + |(s + \gamma_2)(m - m_1)|$$

$=[\eta + (w + \frac{2r}{k} + v)b]\,|(o - o_1)| + b[v + w] + \zeta\,|(m - m_1)|$

$\leq L\|\mathbb{V} - \widetilde{\mathbb{V}}\|$

Where L=max$\{\eta + \left(w + \frac{2r}{k} + v\right)b, \zeta + b(v + w)\}$.Thus $\psi(V)$ satisfies Lipschitz condition, initial condition $\mathbb{V}_0 = (o_0, m_0)$ and hence there exist a unique solution $\mathbb{V}(t)$.

Analysis of equilibrium stability in a fractional-order framework

Local stability of equilibrium points

Local stability analysis is a process that involves linearizing a system around its equilibrium points. The Jacobian matrix, of the system (5) at any point (o, m) as follows.

$$J(E) = \begin{pmatrix} r - \frac{2r}{k}o - vm + \Upsilon_1 - \beta & -vo \\ vwm & -s + wo - \Upsilon_2 \end{pmatrix}$$

Theorem 2

The trivial equilibrium points $E_0(0,0)$ of the system (5) is locally asymptotically stable if $r + \Upsilon_1 < \beta$

Proof

For $E_0 = (0,0)$ the Jacobian matrix is given by

$$J(E_0) = \begin{pmatrix} r + \Upsilon_1 - \beta & 0 \\ 0 & -s - \gamma_2 \end{pmatrix}$$

Hence the eigen values at E_0 are $\lambda_1 = r + \Upsilon_1 - \beta$,$\lambda_2 = -m - \gamma_2$,were $\eta = r + \Upsilon_1 - \beta$ and $\zeta = -s - \gamma_2$.

Its clear that $|\arg(\lambda_1)| = \pi > \alpha\frac{\pi}{2}$ if $r + \Upsilon_1 - \beta < 0$ i.e $r + \Upsilon_1 < \beta$ otherwise we get $|\arg(\lambda_1)| = 0 < \alpha\frac{\pi}{2}$ and also $|\arg(\lambda_2)| = \pi$, which leads to $|\arg(\lambda_2)| > \alpha\frac{\pi}{2}$ for $0 < \alpha < 1$. Therefore E_0 is locally asymptotically stable.

Theorem 3

The natural enemies -free equilibrium point $E_1 = (\frac{\eta k}{r}, 0)$ where $\eta = r + \gamma_1 - \beta$ of the system (5) is locally asymptotically stable if $\frac{r}{k}(\zeta) < w\eta$ otherwise it is unstable

Proof

For natural enemies freethe Jacobian Matrix E_1as follows

$$J(E_1) = \begin{pmatrix} r - \frac{2r}{k}o^* - vm + \Upsilon_1 - \beta & -vo^* \\ wm & -s + wo^* - \Upsilon_2 \end{pmatrix}$$

$$J(E_1) = \begin{pmatrix} -\eta & \frac{-v\eta k}{r} \\ 0 & -\zeta + \frac{w\eta k}{r} \end{pmatrix}$$

Hence the eigenvalues are $\lambda_1 = -\eta$ and the eigen values $\lambda_2 = \frac{w\eta k}{r} - \zeta$. Hence all the eigen values will be negative if $\frac{w\eta k}{r} - \zeta < 0$ that is $w\eta < \frac{r}{k}(\zeta)$ otherwise it is unstable.

Now let us analyse the coexisting equilibrium point $E^*(o^*, m^*) = \left(\frac{\zeta}{wv}, \frac{kw\eta - r(\zeta)}{vwk}\right)$ of the system (5) is locally asymptotically stable, interior equilibrium point. J, the Jacobian matrix of the model(5) at $E^*(o^*, m^*)$ is presented by

J(E^*) = $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where

$$A_{11} = -\frac{r\zeta}{kw} A_{12} = \frac{-v\zeta}{w}$$

$$A_{21} = \frac{\eta kw - r\zeta}{vk} A_{22} = 0$$

Therefore, the characteristic equation of the matrix J(E^*) is $\lambda^2 - A_{11}\lambda + A_{12}A_{21} = 0$.Let λ_1 and λ_2 be eigen values. Then $\lambda_1+\lambda_2 = A_{11}$ and $\lambda_1\lambda_2= A_{12}A_{21}$.

when $A_{11} < 0$,the eigen values of following matrixes

$$\lambda_1 = \frac{1}{2}(A_{11} + \sqrt{A_{11}^2 - 4A_{12}A_{21}}\ )$$

$$\lambda_2 = \frac{1}{2}(A_{11} + \sqrt{A_{11}^2 - 4A_{12}A_{21}}\ )$$

The values of the stability parameters,λ_1 and λ_2 are determined by the system's coefficientsA_{11}, A_{12} and A_{21}.

Case (i) If the conditionA_{11}< 0 and $A_{11}^2 \geq 4A_{12}A_{21}$,it results in both eigen values λ_1 andλ_2 being negative. This negative result satisfies the necessary stability condition |arg(λ_1)|=$\pi > \alpha\frac{\pi}{2}$ and |arg(λ_2)|=$\pi > \alpha\frac{\pi}{2}$. Therefore, based on Lemma 3.3 and 3.4 the equilibrium point $E^*(o^*, m^*)$ can be concluded to be asymptotically stable.

Case(ii) If the condition A_{11}>0 and $A_{11}^2 < 4A_{12}A_{21}$, it results in both eigen values$\lambda_1 and\ \lambda_2$ are pair of complex conjugate roots.

$\lambda_{1,2}$=$\frac{1}{2}(A_{11} \pm i\sqrt{A_{11}^2 - 4A_{12}A_{21}}\ )$ where $i = \sqrt{-1}$

Thus |arg($\lambda_{1,2}$)| = $tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right|$. Therefore $E^*(O^*, M^*)$ is asymptotically stable, if $tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right|$>$\alpha\frac{\pi}{2}$ in the interval 0<$\alpha < \frac{2}{\pi}\ tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right|$. Hence the interior equilibrium E* is asymptotically stable in this interval.

Case(iii) If the condition A_{11}< 0 and $A_{11}^2 < 4A_{12} - A_{21}$,then |arg($\lambda_{1,2}$)| = $\pi - tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right|$. The resulting stability condition obtained |arg($\lambda_{1,2}$)| > $\alpha\frac{\pi}{2}$ .Then $E^*(o^*, m^*)$ is asymptotically stable if $\pi - tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right| > \alpha\frac{\pi}{2}$ in the interval $0 < \alpha < 2 - \frac{2}{\pi}\ tan^{-1}\left|\frac{\sqrt{A_{11}^2-4A_{12}A_{21}}}{A_{11}}\right|$,where the interior equilibrium E* is asymptotically stable in this interval.

Case(iv) If the condition A_{11} = 0 and $A_{11}^2 < 4A_{12}A_{21}\ holds$,then |arg($\lambda_{1,2}$)| = $\frac{\Pi}{2}$>$\frac{\alpha\pi}{2}$.Therefore the resulting stability condition obtained is |arg($\lambda_{1,2}$)| > $\alpha\frac{\Pi}{2}$. Hence, the $E^*(o^*, m^*)$ is asymptotically stable.

Case(v) If the condition$A_{11} \geq 0$ and $A_{11}^2 \geq 4A_{12}A_{21}$holds,then one of the eigen value λ_1> 0 or λ_2 = 0.Therefore,the resulting stability the condition obtain |arg($\lambda_{1,2}$)| = 0 < $\frac{\alpha\Pi}{2}$ for i=1 or 2,where α ϵ (0,1).Hence by the lemma 3.3 and 3.4 the interior equilibrium $E^*(o^*, m^*)$ is unstable.

Global stability

The analysis introduces the necessary lemmas for establishing the criterion for global asymptotic stability at both the natural enemies-free E_1 and the interior equilibrium point E^*

Lemma[6][19]

Let $\xi: \mathbb{R} \to \mathbb{R}^+$be a fractional continuously differentiable function of order $\alpha \in (0,1)$. Then, for any time instant $t > t_0$ and $\rho \epsilon \mathbb{R}^+$:

$${}^{c}D^{\alpha}(\xi(t) - \rho - \rho\, log\, \frac{\xi(t)}{\rho}) \leq (1 - \frac{\rho}{\xi(t)})\, {}^{c}D^{\alpha} o(t)$$

The following two theorems are stated and proven to address the global asymptotic stability of the natural enemies-free and its interior equilibrium points.

Theorem-4

The natural enemies -free equilibrium point E_1 is globally asymptotically stable if $2r - \Upsilon_1 + \beta < 0$ and $k < \frac{\zeta}{w}$

$W(o,m) = o - k(1+\log\frac{o}{k}) + \frac{vm}{w}$

By lemma 6,

$${}^{c}D^{\alpha}W(o,m) \leq (1 - \frac{k}{o})\, {}^{c}D^{\alpha} o(t) + \frac{v}{w}\, {}^{c}D^{\alpha} m(t)$$

$$= \left(1 - \frac{k}{o}\right)\left(ro\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 o - \beta o\right) + \left(\frac{v}{w}\right)(-sm + wom - \Upsilon_2 m)$$

$$= \left(\frac{o-k}{o}\right)\left(r\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 - \beta\right) + \left(\frac{v}{w}\right)(-sm + wom - \gamma_2 m)$$

$$= -\frac{r}{k}(o-k)^2 + (\Upsilon_1 - \beta)(o-k) + vm(o-k) - (s + \Upsilon_2)\frac{m}{w} + vmo$$

$$= -\frac{r}{k}(o-k)^2 + (\Upsilon_1 - \beta)(o-k) - vmo + vmk - \frac{(s + \Upsilon_2)vm}{w} + vmo$$

$$= -\frac{r}{k}(o-k)^2 + (\Upsilon_1 - \beta)(o-k) + vmk - \frac{(s + \Upsilon_2)vm}{w}$$

$${}^{c}D^{\alpha}W(o,m) \leq -\frac{ro^2}{k} + k(-r - \Upsilon_1 + \beta) + o(2r + \Upsilon_1 - \beta) + \left(kv - \frac{\zeta v}{w}\right)m$$

$${}^{c}D^{\alpha}W(o,m) \leq -\frac{ro^2}{k} - k\eta + o(2r + \Upsilon_1 - \beta) + \left(k - \frac{\zeta}{w}\right)vm$$

$${}^{c}D^{\alpha}W(o,m) \leq 0$$

Since $\eta > 0$ for the existence of E_1. ${}^{c}D^{\alpha}W(o,m) \leq 0$ if $2r + \Upsilon_1 < \beta$ and $k < \frac{\zeta}{w}$.Hence E_1 is globally asymptotically stable.

Theroem-5

The coexisting equilibrium point $E^*(o^*, m^*)$ is globally asymptotically stable if $\frac{r}{k} > 0$

$W(o,m) = o - o^*(1+\log\frac{O}{O^*}) + \frac{v}{w}(m - m^*)(1 + \log\frac{m}{m^*})$

$${}^{c}D^{\alpha}W(o,m) \leq \left(\frac{o-o^*}{o}\right){}^{c}D^{\alpha} o(t) + \frac{v}{w}\left(\frac{m-m^*}{m}\right){}^{c}D^{\alpha} m(t)$$

$$= \left(\frac{o-o^*}{o}\right)\left(ro\left(1 - \frac{o}{k}\right) - vom + \Upsilon_1 o - \beta o\right) + \frac{v}{w}\left(\frac{m-m^*}{m}\right)(sm + wom - \gamma_2 m)$$

$$= (o - o^*)\left(r\left(1 - \frac{o}{k}\right) - vm + \Upsilon_1 - \alpha\right) + \frac{v}{w}(m - m^*)\,(s + wo - \gamma_2)$$

$$= (o - o^*)(-\frac{r}{k}(o - o^*) - v(m - m^*)) + \frac{v}{w}w(m - m^*)(o - o^*)$$

$$= -\frac{r}{k}(o - o^*)^2 - v(m - m^*)(o - o^*) + v(m - m^*)(o - o^*)$$

${}^cD^\alpha W(o,m) \leq -\frac{r}{k}(o - o^*)^2$

Therefore ${}^cD^\alpha W(o,m) \leq 0$, if $\frac{r}{k} > 0$. Hence E* is globally asymptotically stable.

Numerical Stimulation and Stability Verification:

Diethelm K. [21] and Diethelm K. et al. [22] establish that the analysis of multi-term fractional equations can be effectively verified through numerical techniques. By adopting this framework, the present study verifies the analytical results of the developed model using the modified predictor-corrector (MPC) method algorithm, which is implemented in the MATLAB environment for this purpose. This numerical simulation demonstrates that the repellent effect of companion plants, such as marigolds, effectively supports the population of natural enemies which serve as a biological control for the onion borer.The behaviour of the onion borer and its natural enemies are analysed using various initial values for the 80–120 days interval. While the complete onion growth cycle encompasses 150 days, this specific timeframe is essential for the intensive bulb development phase. The analysis aims to evaluate the effectiveness of biological control in suppressing the onion borer population during this high-risk developmental stage.

Parameter estimation

In the absence of field-collected time-series data for the onion borer and its natural enemies, model parameters presented in Table [1] were estimated through discussions with local agricultural practitioners cultivating onion together with marigold as a intercrop, together with insights drawn from published literature on analogous pest–natural enemy systems [8],[25],[29] andgeneral biological reasoning regarding growth, predation, and mortality rates. These assumed values were chosen to ensure a biologically feasible interior equilibrium and were used to numerically simulate the model over a 150-day horizon, allowing qualitative analysis of system stability — particularly during the critical 80–120-day bulb development period. This approach provides a theoretical foundation for future IPM predictive modelling until field data becomes available.

Table 1: Parameter Values and their description

Symbol	Parameter Description	Value(day^{-1})
r	Growth rate of onion borer	0.07
k	Rate of onion borer logistic growth	70000
v	Declining rate of onion borer in presence of natural enemies	0.000046
w	Improving rate of natural enemy in absence of onion borer	0.0006
Υ_1	Onion borer increasing rate because of the improper sanitation	0.025
Υ_2	Decreasing rate of natural enemy due to flower harvesting	0.03
β	Decreasing rate of borer due to natural pesticides	0.05
s	Falling rate of natural enemy in the absence of onion borer	0.05

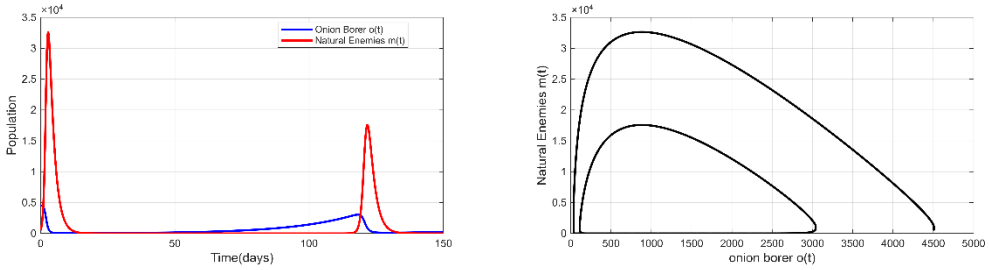


Figure 1: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initialvalues$o_0 = 4500, m_0 = 500$ with α=1 around the interior equilibrium E*

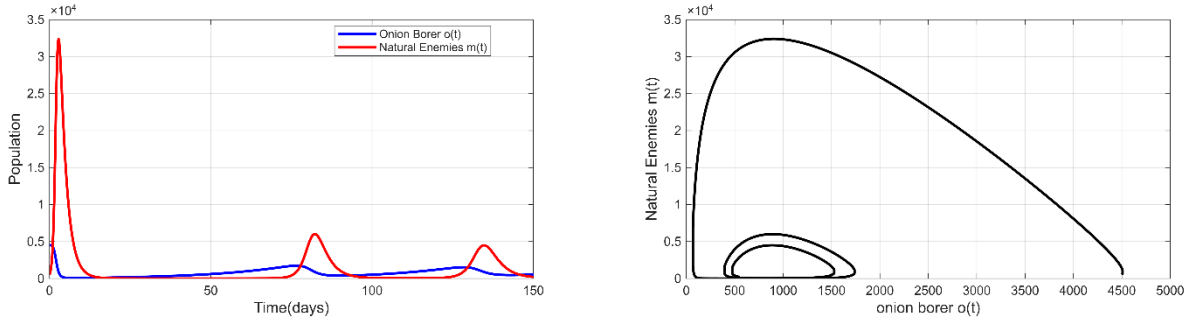


Figure 2: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 4500$,$m_0 = 500$ with α=0.99 around the interior equilibrium E*

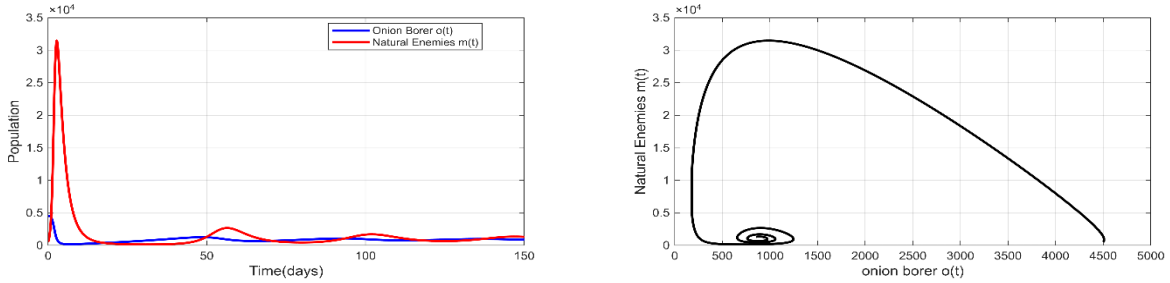


Figure 3: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 4500$,$m_0 = 500$ with α=0.95 around the interior equilibrium E*

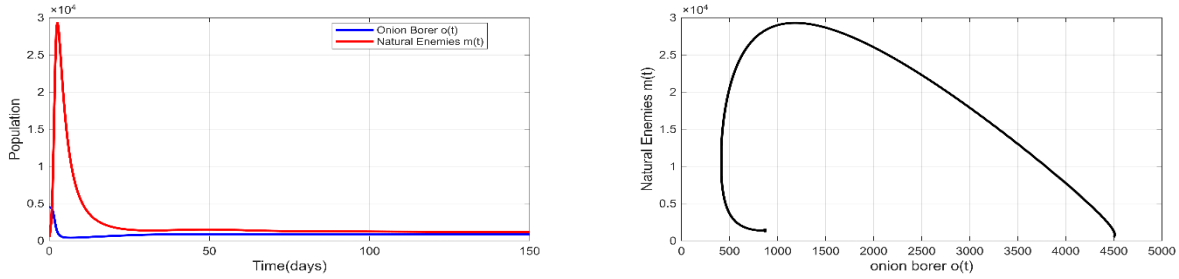


Figure 4: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 4500$,$m_0 = 500$ with α=0.90 around the interior equilibrium E*

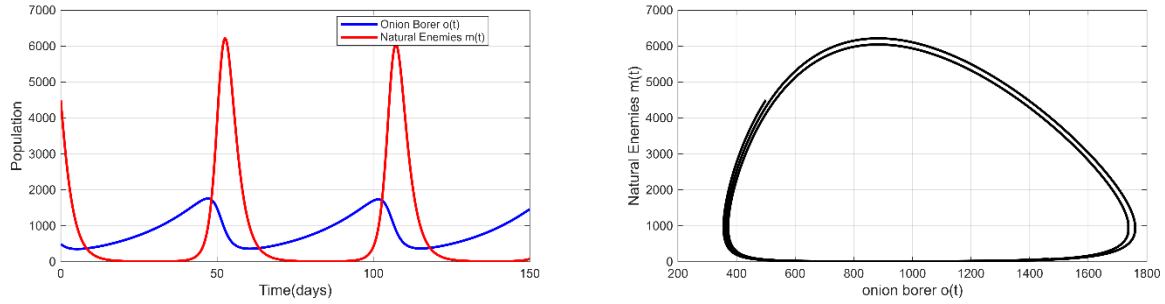


Figure 5: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 500, m_0 = 4500$ with α=1 around the interior equilibrium E*

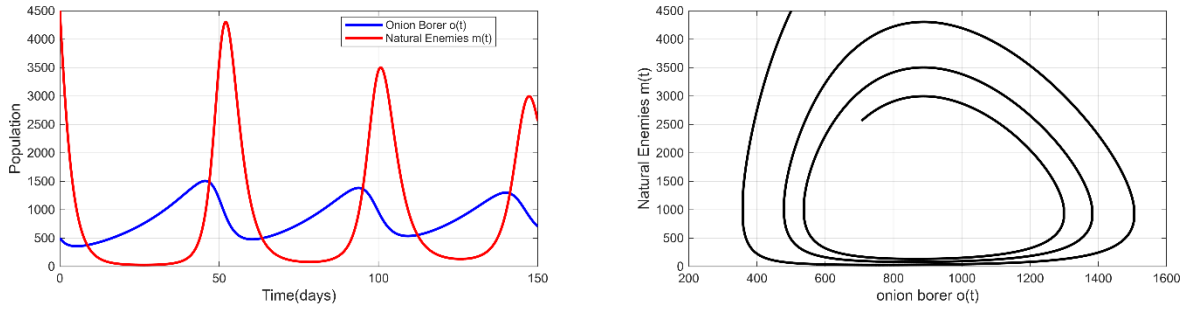


Figure 6: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 500, m_0 = 4500$ with α=0.99 around the interior equilibriumE*

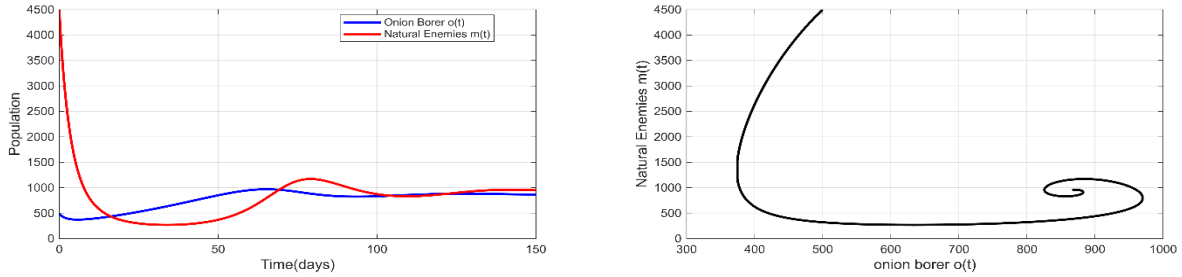


Figure 7: The dynamical behaviour of the onion borer and the natural enemies demonstrates the trajectory and phase portrait is examined for the initial values $o_0 = 500, m_0 = 4500$ with α=0.95 around the interior equilibrium E*

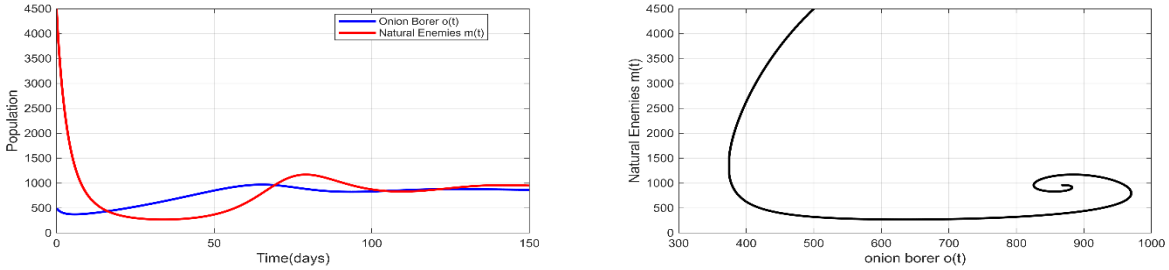


Figure 8: The dynamical behaviour of the onion borer and the natural enemies demonstrate the trajectory and phase portraitis examined for the initial values $o_0 = 500, m_0 = 4500$ with α=0.90 around the interior equilibrium E*

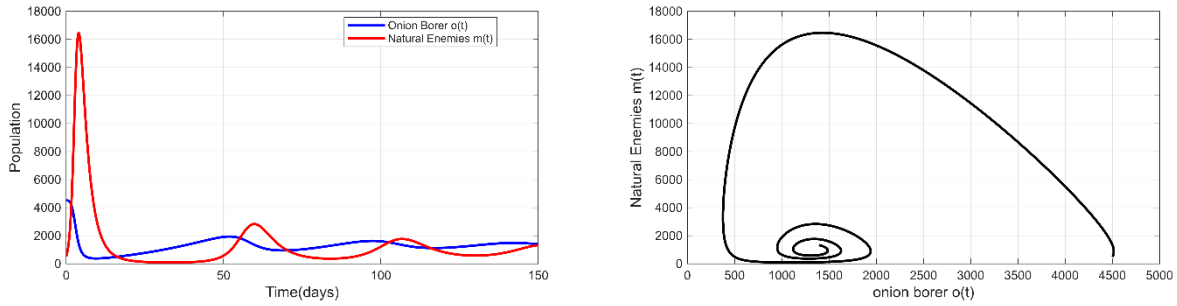


Figure 9: The population dynamical behaviour of the onion borer and its natural enemiesare analysed under a decrease in w=0.0004.The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order $\alpha = 0.95$

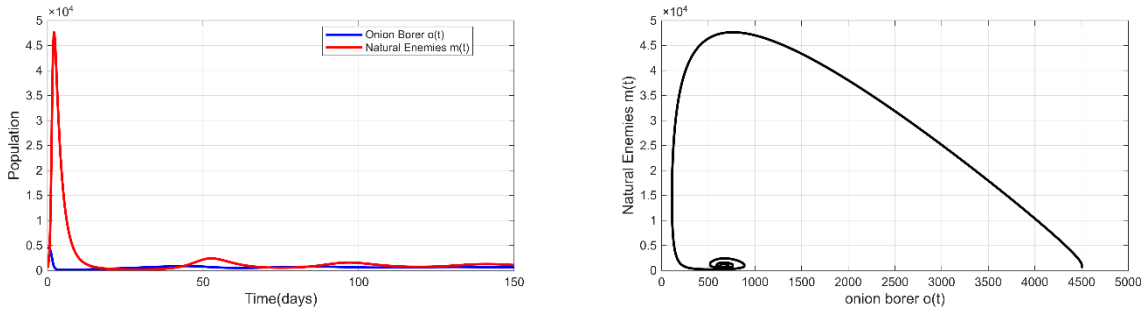


Figure 10: The population dynamical behaviour of the onion borer and its natural enemies are analysed under an increase in w=0.0008.The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order $\alpha = 0.95$.

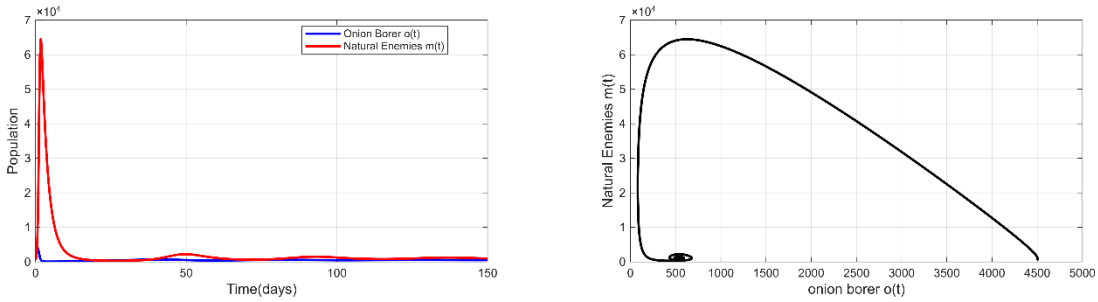


Figure 11: The population dynamical behaviour of the onion borer and its natural enemies are analysed under an increase in w=0.001.The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order $\alpha = 0.95$

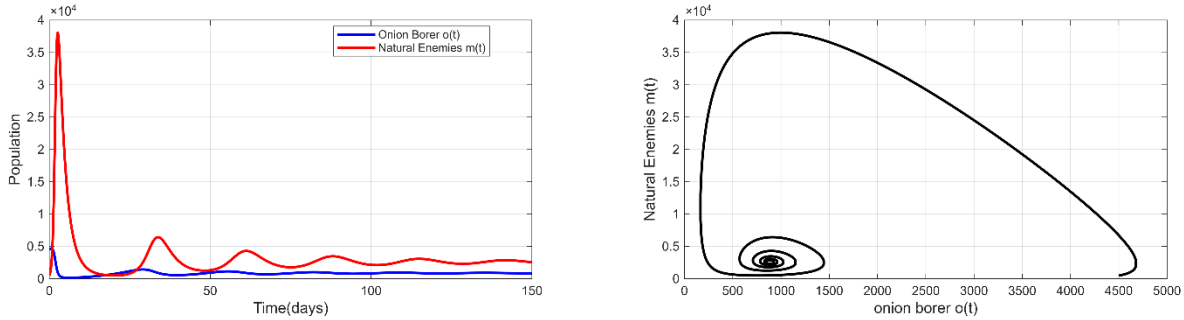


Figure 12: The Population dynamics of the onion borer and its natural enemies are analysed under a decrease in $\beta = 0.00001$. The system is analyzed near the interior equilibrium E^* using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order $\alpha = 0.95$.

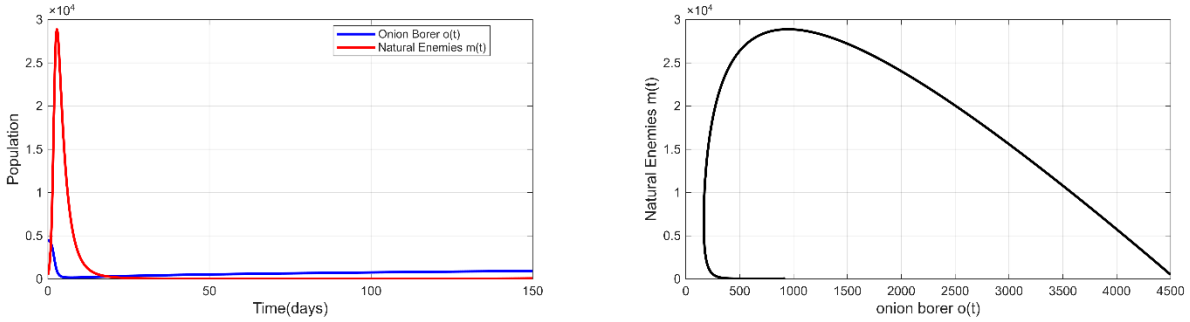


Figure 13: The population dynamics of the onion borer and its natural enemiesare analysed under an increase in β=0.09 The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order$\alpha = 0.95$.

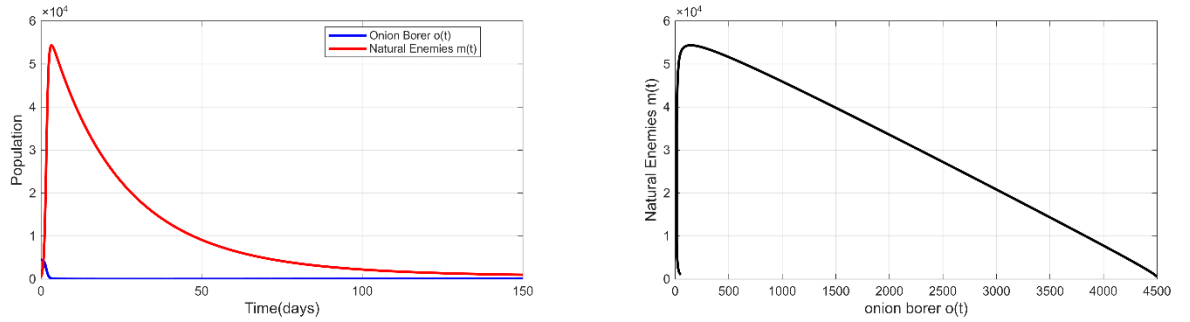


Figure 14: The population dynamics of the onion borer and its natural enemies are analysed under a decrease in Υ_1 =0.00001.The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order $\alpha = 0.95$

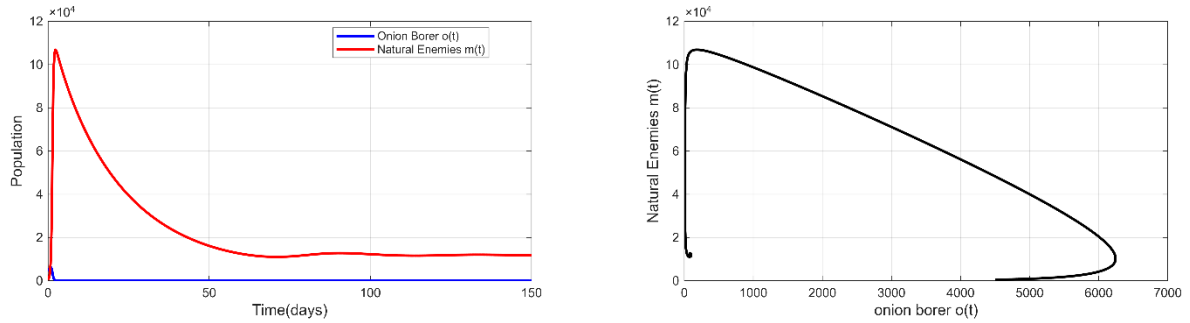


Figure 15: The population dynamics of the onion borer and its natural enemies are analysed under an increase in Υ_1 =0.5 The system is examined near the interior equilibrium E^*, using the initialvalues $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order$\alpha = 0.95$.

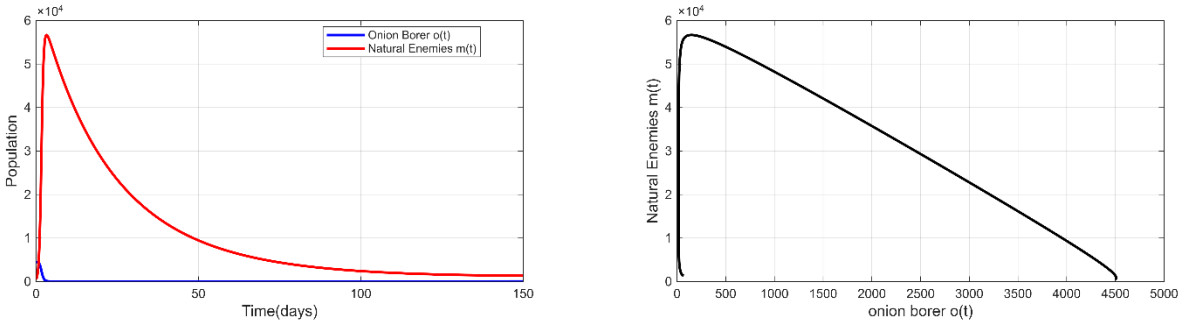


Figure-16: The population dynamics of the onion borer and its naturalenemies are analysed under a decrease in Υ_2 =0.00003. The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order$\alpha = 0.95$

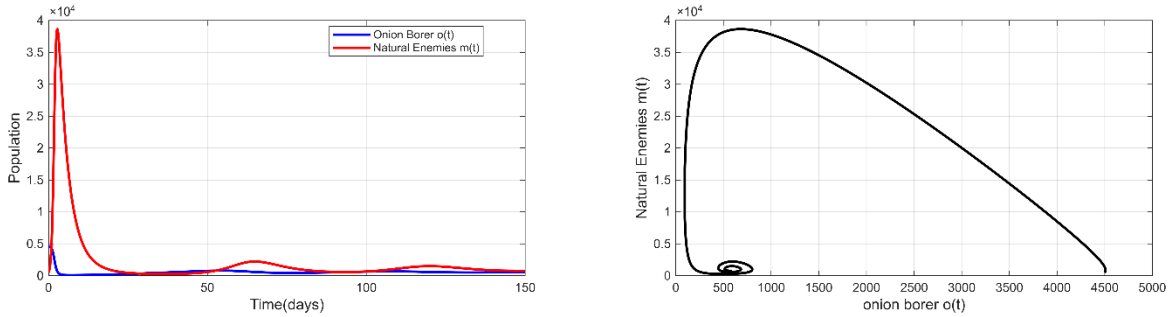


Figure 17: The population dynamics of the onion borer and its natural enemies are analysedunder an increase in Υ_2 =0.3 The system is examined near the interior equilibrium E^*, using the initial values $(o_0, m_0) = (4500,500)$ over the time interval $t \in [0,150]$ at a fractional order$\alpha = 0.95$.

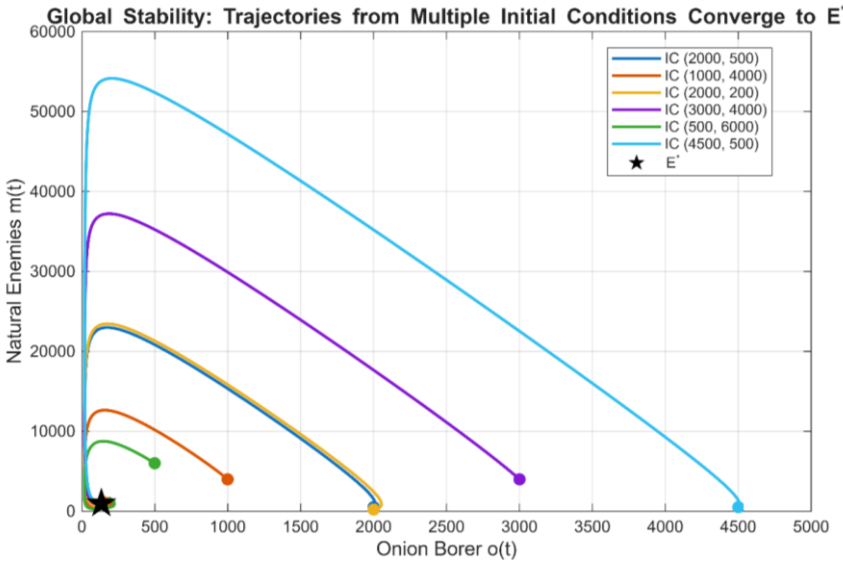


Figure 18: Globally asymptotically stability is observed for different initial values, as the trajectories are converging towards the interior equilibrium E*

Observed results and inferences

The Caputo fractional derivative models the control of onion pests by natural enemies, which are attracted by planting marigolds near the onion crop. The time dimension discrepancy in the original model system (3) was successfully resolved. Subsequently, the system was reformulated into its commensurate configuration (5), in which all state variables share a common fractional order. The analysis confirmed that the stability of various equilibria points remains independent of the temporal dimension.The research rigorously derived the existence and uniqueness criteria for the diverse equilibrium states of the commensurate fractional-order system (5).Furthermore, the paper explored the sufficient conditions for both the non-negativity and boundedness of the model's solutions, ensuring mathematical and physical plausibility. Additionally, Lyapunov functions were employed to establish the global stability of both the predator-free equilibrium point (E_1) and the coexisting equilibrium point (E*).The study investigated the impact of fractional differentiation, which introduces a memory effect into the model. This influence is explored by analysing the system's behaviour under a constant fractional order (α), applied uniformly across all state variables. For these numerical simulations, the predictor-corrector method and the Adams-Moulton algorithm were applied using MATLAB software. Figures 1–8 show that the system (5) converges to its equilibrium point for various point of initial values and α values. As α decreases, the system loses its instability and stabilizes, with the onion borer population falls under control within the80–120 day which is corresponding to the bulb development stage. The phase portrait and trajectory diagrams (4,8) demonstrate that reductions in the α value successfully control the onion borer over the 80–120-day interval, which is corresponding to the onion bulb development phase. Figures 12 and 13 illustrate that lower β values, corresponding to reduced organic pesticide application, maintain a stable, biologically feasible equilibrium while preserving natural enemy suppression of the pest population**.** Conversely, increased rates of pesticide application gradually disrupt this equilibrium and may ultimately lead to the elimination of the natural

enemy population. This finding supports a fundamental IPM principle: even organic pesticides must be used sparingly to maintain, rather than replace, natural pest-suppression processes. Figures 9,10 and 11 indicate that increasing w – reflecting a larger area planted with marigolds – boosts the natural enemy population and improves control of the onion borer. Figures 16 and 17 demonstrate that maintaining low marigold harvesting rates (γ_2) retains natural enemy populations within the field. Prolonged flower persistence preserves the floral resource base required for these beneficial organisms. Conversely, higher harvesting rates accelerate resource depletion, thereby reducing natural enemy survival and reproduction, which ultimately diminishes biological control efficacy. Figures 14 and 15 demonstrate that improved sanitation also supports natural enemy populations. Since γ_1 represents the sanitation rate, increasing it directly suppresses the onion borer through cultural and mechanical removal without the harmful side effects associated with pesticide use—making sanitation a safer, more sustainable IPM tool. However, because natural enemies depend on the pest population as a food source, excessively high sanitation levels could limit their growth by reducing prey availability. A balanced approach is therefore advisable.

These results demonstrate that combining reduced organic pesticide application with enhanced sanitation and minimal floral harvesting optimizes onion borer suppression during the critical bulb development phase. The fractional-order model (Figures 1 and 5) provides clear evidence of a critically stable onion borer–natural enemy system, achieved through the biological damping effect of marigold intercropping. Simulations at the integer order ($\alpha = 1$) confirm the analytically predicted critical stability: the time-series results show persistent, undamped oscillations in both populations, while the phase portrait reveals a closed orbit — illustrating the endless boom-and-bust cycle characteristic of a failed control strategy in which natural enemies alone cannot achieve lasting suppression. When the fractional order is reduced, the system undergoes a decisive shift to asymptotic stability. The time-series plots show damped oscillations, with the amplitude of both populations decaying exponentially until they converge to equilibrium. The corresponding phase portrait shows trajectories spiralling inward toward equilibrium point E*, indicating that the borer population is maintained at a low, economically sustainable level.

Conclusion

In conclusion, the Caputo fractional-order model definitively demonstrates that marigold intercropping acts as a systemic stabilizer to achieve predictable and sustainable pest control. The research confirms that the fractional differentiation, by introducing a vital memory effect, fundamentally changes the system dynamics. Specifically, the core finding is that a reduction in the fractional order value causes the undesirable, persistent pest-enemy oscillations, a characteristic of failed control at the integer order ($\alpha=1$), to transition into a rapid convergence toward a low-pest equilibrium state (E*). This study reveals that the lower organic pesticide rates (β) protect natural enemies and maintain system equilibrium. Consequently, organic pesticides must supplement natural enemy suppression rather than replace it.Increasing the marigold-planted area (w) strengthens natural enemy populations and improves borer control, while low harvesting rates (γ_2) sustain the floral resources these enemies depend on. Higher sanitation rates (Υ_1) directly suppress the pest without pesticide-related harm, though excessive sanitation can limit natural enemy growth by reducing prey availability. Hence, these results show that optimal suppression arises from balancing reduced pesticide use, expanded marigold coverage, minimal flower harvesting, and moderate sanitation, rather than maximizing any single intervention. Therefore, this biological damping effect keeps the pest at a level that is economicallymanageable, shifting companion plants from passive components into strong, reliable regulators of agricultural stability. These findings highlight the importance of incorporating biological controls into pest management plans to enhance crop resilience and advance sustainable farming practices.

Reference

1. Lamani K, Deshpande VK, BIRADAR PN, Shashidhar TR. Effect of modified atmospheric packaging on seed longevity of onion (Allium cepa L.) cv. Arka Kalyan. Int. J. Curr. Microbiol. App. Sci. 2020;9(3):198-209.

2. Jat PK, Khandelwal SK, Maharia JS. Performance of Nutrients and Plant Growth Regulators on Growth and Quality attributes of Onion (Allium cepa L.). Int. J. Curr. Microbiol. App. Sci. 2020;9(10):2977-84.

3. Gupta V, Namgyal D, Kumar A, Namgyal D, Angchuk S, Safal R. Assessment of integrated pest management against onion maggot in Trans Himalaya Leh. International Journal of Current Microbiology and Applied Sciences. 2019;8(10):2180-3.

4. Gupta AJ, Kaldate S, Volaguthala S, Mahajan V. Onion nutritional and nutraceutical composition and therapeutic potential of its phytochemicals assessed through preclinical and clinical studies. Journal of Functional Foods. 2025 Jun 1;129:106889.

5. Velasco González de Peredo A, Vázquez Espinosa M, Espada Bellido E, Ferreiro González M, Carrera Fernández CA, Fernández Barbero G, Palma Lovillo M. Development of Optimized Ultrasound-Assisted Extraction Methods for the Recovery of Total Phenolic Compounds and Anthocyanins from Onion Bulbs.

6. Wu S, Ning F, Wu X, Wang W. Proteomic characterization of differential abundant proteins accumulated between lower and upper epidermises of fleshy scales in onion (Allium cepa L.) bulbs. PLoS One. 2016 Dec 30;11(12):e0168959.

7. Sharma G, Rajhansa KC, Sharma P, Singh A, Sharma A, Sahu MK, Sharma R, Pandey AK. Marigold (Tagetes spp.): a diverse crop with multipurpose value for health and environment: a review. Agricultural Reviews. 2024;45(4):618-26.

8. Silveira LC, Berti Filho E, Pierre LS, Peres FS, Louzada JN. Marigold (Tagetes erecta L.) as an attractive crop to natural enemies in onion fields. Scientia agricola. 2009;66:780-7.

9. Aćimović M, Navarro Rocha J, Ibraliu A, Červenski J, Sikora V, Winter S, Lončar B, Pezo L, Salamon I. Intercropping medicinal and aromatic plants with other crops: Insights from a review of sustainable farming practices. Agronomy. 2025;15(12):2692

10. Rafikov M, Angelelli T. Optimization of biological pest control of sugarcane borer. In2009 IEEE Control Applications,(CCA) & Intelligent Control,(ISIC) 2009 Jul 8 (pp. 1254-1258). IEEE.

11. Li W, Chen Y, Huang L, Wang J. Global dynamics of a filippov predator-prey model with two thresholds for integrated pest management. Chaos, Solitons & Fractals. 2022 Apr 1;157:111881.

12. Kartal S. Caputo and conformable fractional order guava model for biological pest control: discretization, stability and bifurcation. Journal of Computational and Nonlinear Dynamics. 2023 Dec 1;18(12):121002.

13. Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. elsevier; 1998 Oct 27.

14. Petráš I. Fractional-order nonlinear systems: modeling, analysis and simulation. Springer Science & Business Media; 2011 May 30.

15. Odibat ZM, Shawagfeh NT. Generalized Taylor's formula. Applied Mathematics and computation. 2007 Mar 1;186(1):286-93.

16. Kilbas AA. Theory and applications of fractional differential equations. North-Holland Mathematics Studies. 2006;204.

17. Li HL, Zhang L, Hu C, Jiang YL, Teng Z. Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing. 2017 Jun;54(1):435-49.

18. Hariprasad S, Phani Kumar N, Shiva Reddy K, Acharyulu KV, Srinivas MA, Nisar KS. Fractional order eco-epidemiological model for the dynamics of a prey-predator system. Modeling Earth Systems and Environment. 2025 Jun;11(3):186.

19. Vargas-De-León C. Volterra-type Lyapunov functions for fractional-order epidemic systems. Communications in Nonlinear Science and Numerical Simulation. 2015 Jul 1;24(1-3):75-85.

20. Naim M, Zeb A, Mohsen AA, Sabbar Y, Yıldız M. Local and global stability of a fractional viral infection model with two routes of propagation, cure rate and non-lytic humoral immunity. Mathematical Modelling and Numerical Simulation with Applications. 2024 Dec 12;4(5-Special Issue: ICAME'24):94-115.

21. Diethelm K. Efficient solution of multi-term fractional differential equations using P (EC) m E methods. Computing. 2003 Nov;71(4):305-19.

22. Diethelm K, Ford NJ, Freed AD. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics. 2002 Jul;29(1):3-22.

23. Li Y, Chen Y, Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability. Computers & Mathematics with Applications. 2010 Mar 1;59(5):1810-21.

24. El-Saka HA, Lee S, Jang B. Dynamic analysis of fractional-order predator–prey biological economic system with Holling type II functional response. Nonlinear Dynamics. 2019 Apr 1;96(1):407-16.

25. Singh J, Ganbari B, Kumar D, Baleanu D. Analysis of fractional model of guava for biological pest control with memory effect. Journal of Advanced Research. 2021 Sep 1;32:99-108.

26. Misra AK, Yadav A, Bonyah E. A fractional model for insect management in agricultural fields utilizing biological control: AK Misra et al. International Journal of Dynamics and Control. 2025 Jan;13(1):4.

27. Jha N, Yadav A, Pandey R, Misra AK. Analyzing crop production: Unraveling the impact of pests and pesticides through a fractional model. Nonlinear Analysis: Modelling and Control. 2024 Jul 1;29(5):858-77.

28. Karmaker S, Ruhi FY, Mallick UK. Mathematical Modelling of Engineering Problems. Journal homepage: http://iieta. org/Journals/MMEP. 2018 Dec;5(4):427-40.

29. Kundu P, Mallick UK. Modeling the impact of natural pesticides on optimize guava nutrients in an integrated cultivation. Modeling Earth Systems and Environment. 2024 Apr;10(2):2329-48.

30. R. Prashanth. (2026). Entropy-Guided Detection of Chaos Transitions in Electrochemical Transport Systems. *Applied Nonlinearity in Science and Technology*, 1-6.