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Mean Time between Failure on Sequential Order Statistics from (k, s) System

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Abstract
This paper focuses on estimating the reliability characteristics of differently structured independent sequential (k, s) system. In this system, each component failure parametrically alters subsequent components' failure rates based on prior failures. The system functions only if at least k out of s components are operational. By considering the different models of sequential (k, s) system, the study derives system failure time and Mean Time Between Failure(MTBF). Component failure times are modelled using exponential and gamma distributions with location and scale parameters. The current investigation reveals that system performance can be expressed as a function of a distribution's location and scale parameters that best describes the system's failure pattern. A linear association between MTBF and these parameters is observed for the sequential system models analyzed. An illustrative example shows that increasing the location parameter with a fixed scale parameter or increasing the scale parameter with a fixed location parameter, leads to a rise in MTBF. It is also found that the lines are parallel and shift upward, indicating a positive trend in MTBF with both the cases. The results obtained here are consistent with the previously published work. This linearity highlights how tuning these parameters directly improves system reliability.

Keywords: Failure time, Mean time between failure, Exponential and Gamma distributions, Sequential (k, s) system, Scale parameter and Location parameter.

1. Introduction

Product reliability has become increasingly important in today's competitive market, where consumers demand high quality and long-lasting performance. To attract customers, manufacturers often provide product life guarantees through well-structured warranties. Developing such cost-effective warranties requires an understanding of the product's failure-time distribution, which is obtained through life testing and reliability studies conducted before the product is released. Real-world examples of k-out-of-s systems include an aircraft with four engines that can continue flying as long as at least two are operational, or a satellite that functions effectively if at least four of its ten batteries work. Data from life testing are also useful for determining dosage levels, expiry dates and warranty durations. In reliability analysis, order statistics play a crucial role in modelling lifetimes. When s identical items are placed on a life test, the weakest fails first, followed by the next, and so on, resulting in ordered lifetime observations. These observations provide an ideal framework for statistical study. In general, a system remains functional if at least k out of s components work.

A k-out-of-s system consists of s components that begin operating simultaneously. The system functions as long as at least k components remain operative and fails when $s - k + 1$ or more components stop working. Let the component lifetimes be denoted by the random variables $X_1, X_2, \ldots, X_s$. When these lifetimes are statistically identical and independently distributed, the system failure time corresponds to the $(s - k + 1)$-th order statistics among them, usually represented by $X_{(s-k+1)}$ or $X_{s-k+1:s}$. In such a conventional system, it is assumed that the failure of one component does not affect the performance or lifetime of the remaining components. However, this assumption rarely holds true in practical situations. The failure of one component generally imposes an additional load or stress on the surviving components, which may result in reduced efficiency, higher failure rates, or both.

To address this limitation, a more flexible and realistic model has been developed. In this model, the lifetime distribution of the active components are assumed to share the increased stress leading to a modification in their lifetime distribution. This enhanced modelling approach is known as the sequential k-out-of-s system. The order statistics associated with this system are defined as SOS. It is assumed that the failure of each component causes a

corresponding change in the failure rates of the active components. A sequential (k, s) system is a (k, s) system in which a component failure changes the life length distribution of the remaining components. The life length of a sequential $(s - r + 1, s)$ system is modelled by r-th SOS denoted by $X^*_{(r)}$, $1 \leq r \leq s$.

MTBF is a fundamental reliability engineering metric that quantifies the average operational time between consecutive failures of component under specified conditions. In practical contexts, it estimates the average duration a repairable system operates between successive failures. It helps guiding engineers in maintenance scheduling, lifecycle cost analysis, and design optimization. Enhancements in sensor networks and AI-powered analytics have further improved MTBF estimation, enabling predictive maintenance and real-time condition monitoring in mission-critical domains such as industry, aerospace, and infrastructure.

Numerous studies have contributed to the modelling and analysis of reliability systems. Aji and Uchendu [1] highlighted that monitoring MTBF, MTTR and SLA is essential for operational efficiency, noting significant improvements required in the Western region. Alavian et al. [2] introduced an (α, β)-precision method to determine minimum sample sizes and observation times for accurate MTBF and MTTR estimation. Amari [3] refined MTBF approximations under periodic maintenance, showing exponential models to estimate MTBF for IFRA systems. Work by Basu and El-Mawaziny [4] compared maximum likelihood and minimum variance unbiased estimators for k-out-of-m systems under exponential lifetimes. Braglia et al. [5] applied multivariate analysis for MTBF-based reliability assessment and prediction. Butnicu and Bonteanu [6] compared MIL-HDBK-217F and SN 29500 standards in buck converter failure rate predictions, emphasizing design reliability in power electronics. Cao and Liang [7] utilized fault tree analysis with MTBF to identify weaknesses in NNBI beam source systems, strengthening tokamak device reliability. Cheshmehzangi [8] examined urban infrastructure resilience, stressing reliance on high MTBF to withstand disruptions. Kullstam [17] developed explicit MTBF and MTTR formulas for m-out-of-n redundant systems, demonstrating simplicity comparable to availability calculations. Utami [20] evaluated PT XYZP's IOT-driven hospitality system, achieving 98.85% availability, increasing MTBF from 608 to 744 hours, and eliminating MTTR, reflecting enhanced efficiency and user satisfaction.

Despite these advancements, MTBF modelling for sequential (k, s) systems considering location and scale parameters of component lifetime distributions remains underexplored. This work presents a novel insights by including location and scale parameters into MTBF modelling for sequential (k, s) systems with the gamma distribution family. This article aims to compute the system failure time and mean time between failure for various models of sequential (k, s) system, under the assumption that the failure time distribution belongs to the exponential or gamma location-scale family. The structure of the article is as follows: Section 2 presents the cumulative distribution, marginal and joint density functions for different system. Section 3 focuses on the system failure time for the proposed models. In Section 4, the MTBF is evaluated based on system failure times. Section 5 offers numerical illustration to support the analysis, and Section 6 provides conclusion about the result.

2. Sequential Order Statistics

Sequential order statistics play a central role in modelling reliability structures where system failure depends on the ordered occurrences of component breakdowns. Kamps [16] established that if the $F_1, \dots, F_n$ are continuous distribution functions with their corresponding densities $f_1, f_2, \dots, f_n$, then the joint density function for the first r SOS $X^*_{(1)}, X^*_{(2)}, \dots, X^*_{(r)}$, where $(1 \leq r \leq n)$ is given by

$$f^*(x_1, x_2\ , \dots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left[\left\{ \frac{1 - F_i(x_i)}{1 - F_i(x_{i-1})} \right\}^{n-i} \frac{f_i(x_i)}{1 - F_i(x_{i-1})} \right] \qquad (1)$$

$$-\infty = x_0 < x_1 < \cdots < x_n < \infty.$$

In this section, we focus on the cumulative distribution function, marginal and joint density functions of the component failure times for various sequential (k, s) systems.

2.1 Notations

- s – Total number of components
- k – Minimum components required for operation
- X_i – Lifetime of individual components
- $X^*_{(r)}$ – r-th ordered failure time
- PDF – Probability density function

- F_i – Cumulative distribution function of X_i
- f_i – Marginal density function of X_i
- $f^*(x_1, x_2, \dots, x_s)$ – Joint density function of $(X^*_{(1)}, X^*_{(2)}, \dots, X^*_{(s)})$
- $f^*_r(x_r)$ – System failure time function, $1 \le r \le s$
- α – Location parameter
- ξ – Scale parameter
- c – Shape parameter
- SOS – Sequential order Statistics
- (k, s) system – At least k out of s components
- MTBF – Mean time between Failure

The lifetime of each component is assumed to follow Gamma distributions with location parameter α, scale parameter ξ and shape parameter c. Its probability density function is

$$f(x; \alpha, \xi, c) = \frac{1}{\xi^c \Gamma(c)} (x-\alpha)^{c-1} e^{-\frac{1}{\xi}(x-\alpha)}, \quad x > \alpha; \ \xi, \alpha, c > 0 \tag{2}$$

where $\Gamma(c)$ is the Gamma function. For $c = 1$, this reduces to the exponential distribution with mean ξ. To simplify model presentation, Table 1 summarizes components with Gamma $(\alpha, \xi, 2)$ and Exponential (α, ξ) in each sequential (k, s) system.

Table 1: Distributional Assumption of Components for sequential (k, s) system

Sequential (k, s) system	Gamma $(\alpha, \xi, 2)$ Components	Exponential (α, ξ) Components
(1,3)	f_1, f_2	f_3
(2,3)	f_1	f_2
(1,4)	f_1, f_2	f_3, f_4
(2,4)	f_1	f_2, f_3

2.2 Model 1: Sequential $(1, 3)$ System

The system operates as long as at least one component works and fails only when all three components fail. The system's lifetime is the time of the last failure $X^*_{(3)}$. Suppose the marginal density functions are

$$f_1(x) = f_2(x) = \frac{1}{\xi^2}(x-\alpha) e^{-\frac{1}{\xi}(x-\alpha)},$$

$$f_3(x) = \frac{1}{\xi} e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0, \ x > \alpha, \ \alpha \in \mathbb{R}.$$

Their cumulative distribution functions are

$$F_1(x) = F_2(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)} \left(1 + \frac{x-\alpha}{\xi}\right),$$

$$F_3(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0, \ x > \alpha, \ \alpha \in \mathbb{R}.$$

The joint density of the sequential order statistics $(X^*_{(1)}, X^*_{(2)}, X^*_{(3)})$ is

$$f^*(x_1, x_2, x_3) = \frac{6}{\xi^5}(x_1 - \alpha)(x_2 - \alpha) \left(1 + \frac{x_2 - \alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_1 + x_2 + x_3 - 3\alpha)}, \tag{3}$$

$\alpha < x_1 < x_2 < x_3 < \infty$, $\alpha \in \mathbb{R}$, $\xi > 0$.

The joint probability density function of the three ordered failure times $(X^*_{(1)}, X^*_{(2)}, X^*_{(3)})$ quantifies the likelihoods for all possible sequences and intervals of component failures.

2.3 Model 2: Sequential $(2, 3)$ System

In a (2,3) system, the system works if at least two components are operational, it fails upon the second

component's failure $X^*_{(2)}$. Suppose the marginal density functions are

$$f_1(x) = \frac{1}{\xi^2}(x-\alpha)e^{-\frac{1}{\xi}(x-\alpha)},$$

$$f_2(x) = \frac{1}{\xi}e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

The cumulative distribution functions are

$$F_1(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}\left(1 + \frac{x-\alpha}{\xi}\right),$$

$$F_2(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

Thus the joint probability density function of $(X^*_{(1)}, X^*_{(2)})$ is

$$f^*(x_1, x_2) = \frac{6}{\xi^3}(x_1-\alpha)\left(1 + \frac{x_1-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_1+2x_2-3\alpha)}, \tag{4}$$

$\alpha < x_1 < x_2 < \infty,\ \alpha \in \mathbb{R},\ \xi > 0.$

The joint probability density function of the first two ordered failure times details the probability structure of the system's survival until the second failure.

2.4 Model 3: Sequential $(1, 4)$ System

The system functions when at least one of four components is operational, failing only after the last component fails $X^*_{(4)}$. Suppose the marginal density functions are

$$f_1(x) = f_2(x) = \frac{1}{\xi^2}(x-\alpha)e^{-\frac{1}{\xi}(x-\alpha)},$$

$$f_3(x) = f_4(x) = \frac{1}{\xi}e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

Their cumulative distribution functions are

$$F_1(x) = F_2(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}\left(1 + \frac{x-\alpha}{\xi}\right),$$

$$F_3(x) = F_4(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}; \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

Thus the joint probability density function of $(X^*_{(1)}, X^*_{(2)}, X^*_{(3)}, X^*_{(4)})$ is

$$f^*(x_1, x_2, x_3, x_4) = \frac{24}{\xi^6}(x_1-\alpha)(x_2-\alpha)\left(1 + \frac{x_2-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_1+x_2+x_3+x_4-4\alpha)} \tag{5}$$

$\alpha < x_1 < x_2 < x_3 < x_4 < \infty,\ \alpha \in \mathbb{R},\ \xi > 0.$

The joint probability density function of the four ordered failure times captures all possible failure sequences.

2.5 Model 4: Sequential $(2, 4)$ System

For the $(2,4)$ system, at least two components must work for a system operation. The system fails with the third component failure $X^*_{(3)}$. Suppose the marginal density function are

$$f_1(x) = \frac{1}{\xi^2}(x-\alpha)e^{-\frac{1}{\xi}(x-\alpha)},$$

$$f_2(x) = f_3(x) = \frac{1}{\xi}e^{-\frac{1}{\xi}(x-\alpha)}, \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

Their cumulative distribution functions are

$$F_1(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}\left(1 + \frac{x-\alpha}{\xi}\right),$$

$$F_2(x) = F_3(x) = 1 - e^{-\frac{1}{\xi}(x-\alpha)}, \quad \xi > 0,\ x > \alpha,\ \alpha \in \mathbb{R}.$$

Hence the joint probability density function of $(X^*_{(1)}, X^*_{(2)}, X^*_{(3)})$ is

$$f^*(x_1, x_2, x_3) = \frac{24}{\xi^4}(x_1 - \alpha)\left(1 + \frac{x_1-\alpha}{\xi}\right)^3 e^{-\frac{1}{\xi}(x_1+x_2+2x_3-4\alpha)}, \tag{6}$$

$\alpha < x_1 < x_2 < x_3 < \infty,\ \alpha \in \mathbb{R},\ \xi > 0.$

The joint pdf of the first three ordered failure times $(X^*_{(1)}, X^*_{(2)}, X^*_{(3)})$ determines the system's survivability, as the system is degraded with each failure and fails after the third.

3.System Failure Time

This section derives the probability density function of system failure time for various sequential (k, s) systems, utilizing the joint density function in equations (3), (4), (5) and (6). The system failure time corresponds to a specific order statistic in the sequence of component failure times and its PDF is obtained by marginalizing the joint density over the other order statistics. Table 2 provides the occurrence of failure time and order statistics.

Table 2: Failure Times and Order Statistics for Sequential Models

System model	Failure time	Corresponding order statistics
(1,3)	Time of last failure	$X^*_{(3)}$
(2,3)	Time of second failure	$X^*_{(2)}$
(1,4)	Time of last failure	$X^*_{(4)}$
(2,4)	Time of third failure	$X^*_{(3)}$

3.1 Failure time of $(1, 3)$ system

For the $(1,3)$ system, failure occurs when the third component fails, denoted by the order statistic $X^*_{(3)}$. The PDF of the system failure time is derived by integrating the joint density of all three ordered failure times over the first two, effectively marginalizing these out to focus on $X^*_{(3)}$. The limits of integration are set according to the order constraints.

After detailed integration (omitted for brevity), the failure time PDF can be expressed as:

$$f^*_3(x_3) = \sum_{i=1}^{8} w_i f_i(x_3) \tag{7}$$

where the weights are $w_1 = \frac{2}{9}$, $w_2 = \frac{7}{9}$, $w_3 = \frac{3}{2}$, $w_4 = \frac{9}{4}$, $w_5 = -\frac{3}{2}$, $w_6 = -\frac{9}{2}$, $w_7 = -9$, $w_8 = \frac{45}{4}$, and the component functions are appropriate gamma and exponential densities.

3.2 Failure time of $(2, 3)$ system

In the $(2,3)$ system, the system fails at the second ordered failure time, $X^*_{(2)}$.The PDF of the system failure time is derived by integrating the joint density of two ordered failure times over the first, effectively marginalizing these out to focus on $X^*_{(2)}$.

After integration, we obtain: (The detailed derivation of theeqaution is provided in Appendix A)

$$f^*_2(x_2) = \sum_{i=1}^{5} w_i f_i(x_2) \tag{8}$$

where $w_1 = -\frac{4}{9}$, $w_2 = -\frac{20}{9}$, $w_3 = -\frac{22}{3}$, $w_4 = -22$, $w_5 = 33$.

3.3 Failure time of $(1,4)$ system

For the $(1,4)$ system, failure occurs when all four components have failed, i.e., at $X^*_{(4)}$, the last ordered failure time. The PDF is derived by integrating the joint density over the first three ordered times, marginalizing them out to obtain the distribution of $X^*_{(4)}$.

The resulting PDF is:

$$f_4^*(x_4) = \sum_{i=1}^{11} w_i f_i(x_4) \qquad (9)$$

with weights $w_1 = -\frac{3}{32}$, $w_2 = -\frac{19}{32}$, $w_3 = -\frac{101}{48}$, $w_4 = -\frac{409}{72}$, $w_5 = -\frac{1439}{108}$, $w_6 = \frac{305}{27}$, $w_7 = -\frac{195}{2}$, $w_8 = \frac{8}{9}$, $w_9 = \frac{52}{9}$, $w_{10} = \frac{70}{3}$, $w_{11} = 79$.

3.4 Failure time of $(2,4)$ system

In the $(2,4)$ system, failure is defined by the third ordered failure time $X^*_{(3)}$. The corresponding PDF is derived through marginalization of the joint density over $X^*_{(1)}, X^*_{(2)}$ integrating within order-respecting limits to isolate the distribution of $X^*_{(3)}$.

We obtain:

$$f_3^*(x_3) = \sum_{i=1}^{7} w_i f_i(x_3) (10)$$

where $w_1 = \frac{9}{32}$, $w_2 = \frac{81}{32}$, $w_3 = \frac{225}{16}$, $w_4 = \frac{519}{8}$, $w_5 = \frac{1107}{4}$, $w_6 = -392$, $w_7 = \frac{69}{2}$.

4 . Mean Time Between Failure

This section derives the mean time between failure for sequential (k,s) systems using the system failure time distributions obtained in the previous section. Mathematically, MTBF is computed by taking the expected value of the system lifetime random variable, which is often expressed as the integral of the product of system failure time x_r and its probability density function:

$$\text{MTBF} = \int_{\alpha}^{\infty} x_r f^*(x_r)\, dx_r \qquad (11)$$

where $f^*(x_r)$ is the probability density function of the system failure time $X^*_{(r)}$, $1 \le r \le s$, and α is the location parameter representing the minimum possible failure time. For sequential (k,s) systems, $X^*_{(r)} (1 \le r \le s)$ corresponds to a sequential order statistic, such as the r-th failure time in the ordered sequence of failures. The exact MTBF expression varies among systems due to differences in how component failure distributions combine and interact.

4.1 MTBF of the sequential $(1,3)$ system

Consider the $(1,3)$ system, which fails when the last of three components fails. The system failure $X^*_{(3)}$ has pdf $f_3^*(x_3)$ derived by integrating over the joint failure distributions.

Through detailed integration and using the substitution $y = x_3 - \alpha$, we obtain:

$$\text{MTBF} = \alpha + \frac{305}{108}\xi \qquad (12)$$

4.2 MTBF of sequential $(2,3)$ system

Following similar integration procedures for the $(2,3)$ system:

$$\text{MTBF} = \alpha + \frac{79}{54}\xi \qquad (13)$$

4.3 MTBF of sequential $(1,4)$ system

For the $(1,4)$ system:

$$\text{MTBF} = \alpha + \frac{10153}{3456}\xi \tag{14}$$

4.4 MTBF of $(2,4)$ system

For the $(2,4)$ system:

$$\text{MTBF} = \alpha + \frac{629}{384}\xi \tag{15}$$

In the considered sequential (k,s) system model with Gamma-distributed component lifetimes, the Mean Time Between Failures is obtained as a linear combination of both the location parameter α and the scale parameter ξ, for various sequential (k,s) systems. Here, α can be interpreted as a location parameter related to the baseline life or failure start time, while ξ serves as a scale parameter influencing the spread or variability of the failure time distribution. The coefficient of ξ indicates how it proportionally affects the MTBF value. This formulation shows that MTBF increases linearly with both α and ξ, reflecting their combined impact on the system's expected lifetime.

5. Numerical Illustration

This section provides comprehensive numerical illustrations of Mean Time Between Failures for sequential (k,s) systems under exponential and gamma distributions. The analysis explores how varying the location parameter α and the scale parameter ξ influences MTBF through systematic computational experiments.

5.1 Algorithm for MTBF Computation

The following algorithm outlines the systematic procedure for computing MTBF for sequential (k,s) systems:

MTBF Computation for Sequential (k,s) Systems

Input:

- System configuration (k,s)
- Location parameter α
- Scale parameter ξ
- Shape parameter c (for Gamma distribution)

Output: MTBF value for the specified system

Step 1: Define Component Distributions each component $i = 1,2,\dots,s$ .Specify $f_i(x)$ based on Table 1 Gamma$(\alpha,\xi,2)$ $f_i(x) = \frac{1}{\xi^2}(x-\alpha)e^{-\frac{1}{\xi}(x-\alpha)}$ Exponential(α,ξ) $f_i(x) = \frac{1}{\xi}e^{-\frac{1}{\xi}(x-\alpha)}$

Step 2: Compute Cumulative Distribution Functions each component i.Calculate $F_i(x)$ corresponding to $f_i(x)$

Step 3: Derive Joint Density Function Using equation (1), compute: $f^*(x_1,x_2,\dots,x_r)$ for $r = s-k+1$ where r represents the critical failure order

Step 4: Marginalize to System Failure Time Integrate joint density over non-critical order statistics: $f_r^*(x_r) = \int \cdots \int f^*(x_1,\dots,x_r)\,dx_1 \cdots dx_{r-1}$. Resulting in weighted sum: $f_r^*(x_r) = \sum w_i f_i(x_r)$

Step 5: Compute MTBF Calculate expected value: MTBF $= \int_\alpha^\infty x_r f_r^*(x_r)\,dx_r$ Using substitution $y = x_r - \alpha$: MTBF $= \alpha + (\text{coefficient}) \cdot \xi$ where coefficient depends on system type: $(1,3)$ system: $\frac{305}{108}$ $(2,3)$ system: $\frac{79}{54}$ $(1,4)$ system: $\frac{10153}{3456}$ $(2,4)$ system: $\frac{629}{384}$

Step 6: MTBF value

Computational Complexity: $O(s \cdot m)$ where s is the number of components and m is the number of

integration steps required for marginalization.

5.2 Extended Numerical Results

Table 3 presents comprehensive MTBF values for different combinations of α and ξ across all four system configurations, now extended to include $\alpha = 3.60$.

Table 3: MTBF of sequential (k, s) system for different locationα and scale parameters ξ

Parameters		**MTBF**			
α	ξ	(1,3)	(2,3)	(1,4)	(2,4)
0.45	0.40	1.57963	1.03519	1.62512	1.10521
0.45	0.80	2.70926	1.62037	2.80023	1.76042
0.45	1.20	3.83889	2.20556	3.97535	2.41563
0.45	1.60	4.96852	2.79074	5.15046	3.07083
0.45	2.00	6.09814	3.37593	6.32558	3.72604
1.50	0.40	2.62963	2.08519	2.67512	2.15521
1.50	0.80	3.75926	2.67037	3.85023	2.81042
1.50	1.20	4.88889	3.25556	5.02535	3.46563
1.50	1.60	6.01852	3.84074	6.20046	4.12083
1.50	2.00	7.14814	4.42593	7.37558	4.77604
2.55	0.40	3.67963	3.13519	3.72512	3.20521
2.55	0.80	4.80926	3.72037	4.90023	3.86042
2.55	1.20	5.93889	4.30556	6.07535	4.51563
2.55	1.60	7.06852	4.89074	7.25046	5.17083
2.55	2.00	8.19815	5.47592	8.42557	5.82604
3.60	0.40	4.72963	4.18519	4.77512	4.25521
3.60	0.80	5.85926	4.77037	5.95023	4.91042
3.60	1.20	6.98889	5.35556	7.12535	5.56563
3.60	1.60	8.11852	5.94074	8.30046	6.22083
3.60	2.00	9.24815	6.52592	9.47557	6.87604
0.20	0.45	1.47083	0.85833	1.52201	0.93711
0.60	0.45	1.87083	1.25833	1.92201	1.33711
1.00	0.45	2.27083	1.65833	2.32201	1.73711
1.40	0.45	2.67083	2.05833	2.72201	2.13711
1.80	0.45	3.07083	2.45833	3.12201	2.53711
0.20	1.50	4.43611	2.39444	4.60668	2.65703
0.60	1.50	4.83611	2.79444	5.00668	3.05703
1.00	1.50	5.23611	3.19444	5.40668	3.45703
1.40	1.50	5.63611	3.59444	5.80668	3.85703
1.80	1.50	6.03611	3.99444	6.20668	4.25703
0.20	2.55	7.40139	3.93056	7.69136	4.37695
0.60	2.55	7.80139	4.33056	8.09136	4.77695
1.00	2.55	8.20139	4.73056	8.49136	5.17695
1.40	2.55	8.60139	5.13056	8.89136	5.57695
1.80	2.55	9.00139	5.53056	9.29136	5.97695
0.20	3.60	10.36667	5.46667	10.77604	6.09688
0.60	3.60	10.76667	5.86667	11.17604	6.49688
1.00	3.60	11.16667	6.26667	11.57604	6.89688
1.40	3.60	11.56667	6.66667	11.97604	7.29688
1.80	3.60	11.96667	7.06667	12.37604	7.69688

5.3 Graphical Analysis: Effect of Scale Parameter ξ

Figures 1–4 illustrate the relationship between MTBF and the scale parameter ξ for fixed values of location parameter α.

Figure 1: MTBF vs ξ for (1,3) system

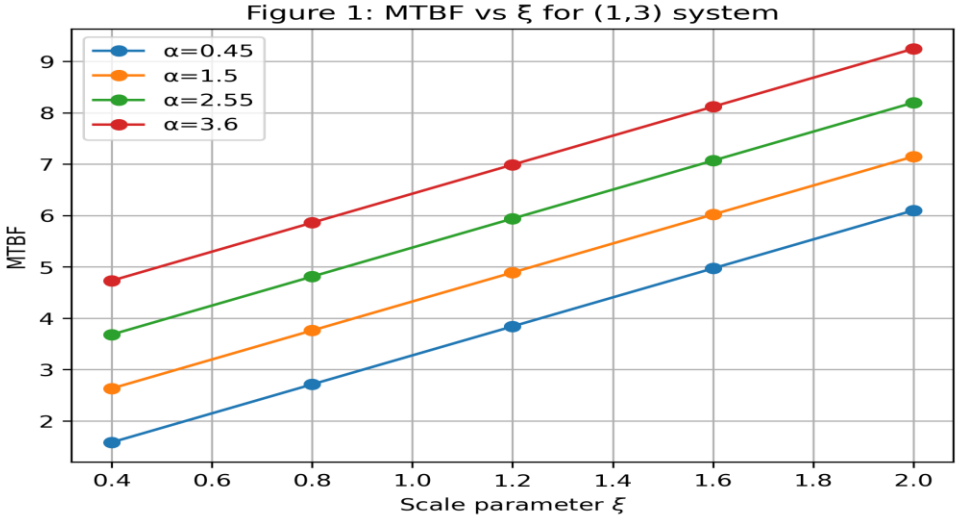


Figure 2: MTBF vs ξ for (2,3) system

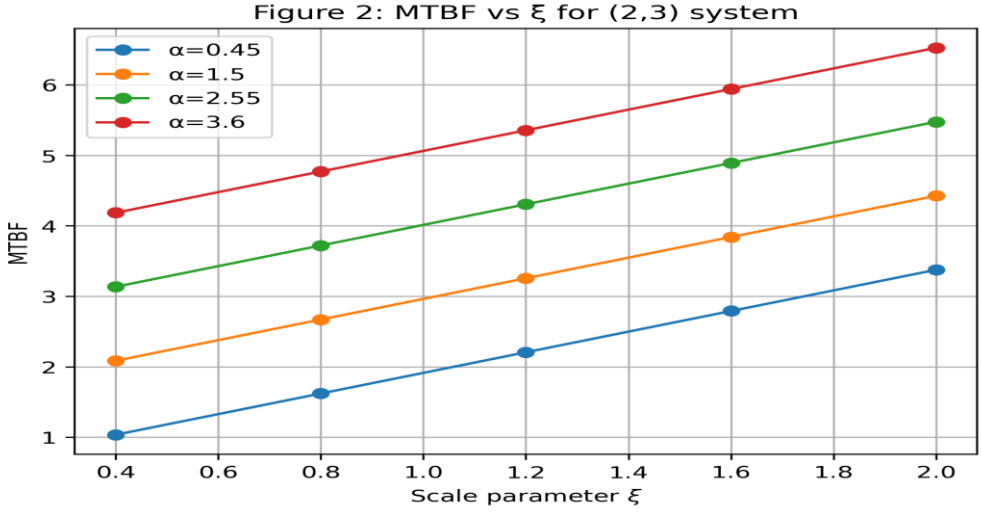
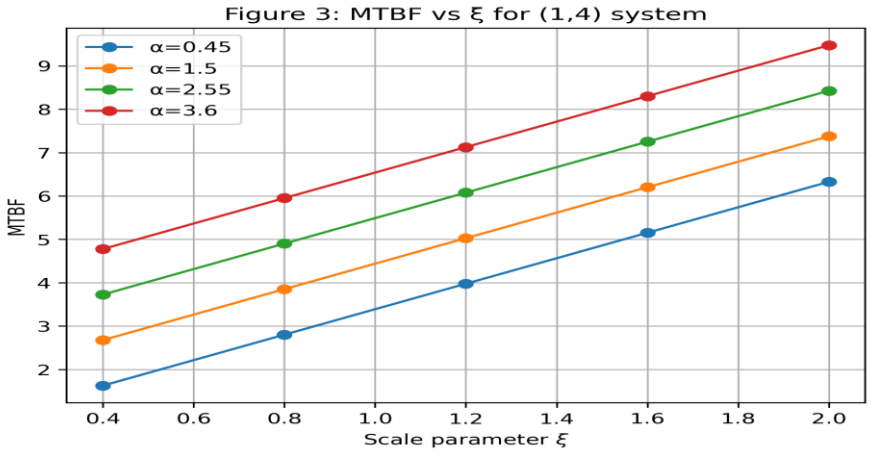
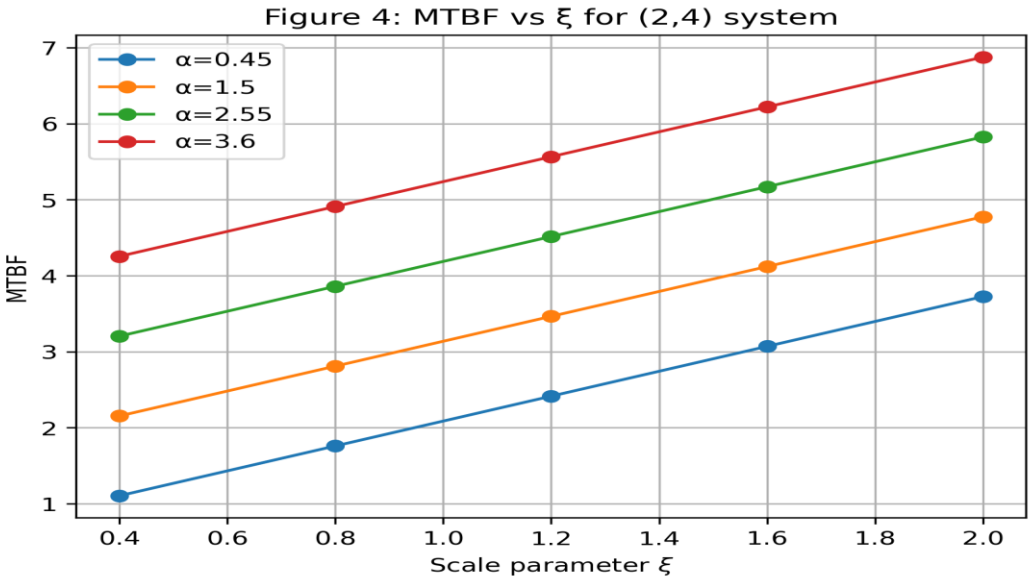


Figure 3: MTBF vs ξ for (1,4) system





5.3.1 Analysis of Figures 1–4

From Figures 1–4, several key observations emerge:

1. Linear Positive Relationship: MTBF increases linearly with ξ for all fixed values of α across all system configurations, confirming the theoretical results from equations (12)–(15).

2. Parallel Trend Lines: The plots exhibit perfectly parallel lines for different α values, indicating that the rate of MTBF increase with respect to ξ is independent of the location parameter. This parallelism confirms the additive structure $\text{MTBF} = \alpha + c \cdot \xi$.

3.Upward Shift with α: Higher values of α shift the entire MTBF curve upward by exactly $(\alpha_2 - \alpha_1)$ units, demonstrating the direct additive contribution of the location parameter.

4. System Redundancy Effect: The $(1,4)$ system consistently shows the highest MTBF values, followed by $(1,3)$, $(2,4)$, and $(2,3)$ systems, reflecting the degree of redundancy built into each configuration.

5.Slope Analysis: The slope of MTBF versus ξ equals:

- $(1,3)$ system: $\frac{305}{108} \approx 2.824$
- $(2,3)$ system: $\frac{79}{54} \approx 1.463$
- $(1,4)$ system: $\frac{10153}{3456} \approx 2.938$
- $(2,4)$ system: $\frac{629}{384} \approx 1.638$

5.4 Graphical Analysis: Effect of Location Parameter α

Figures 5–8 illustrate the relationship between MTBF and the location parameter α for fixed values of scale parameter ξ.

Figure 5: MTBF vs α for (1,3) system

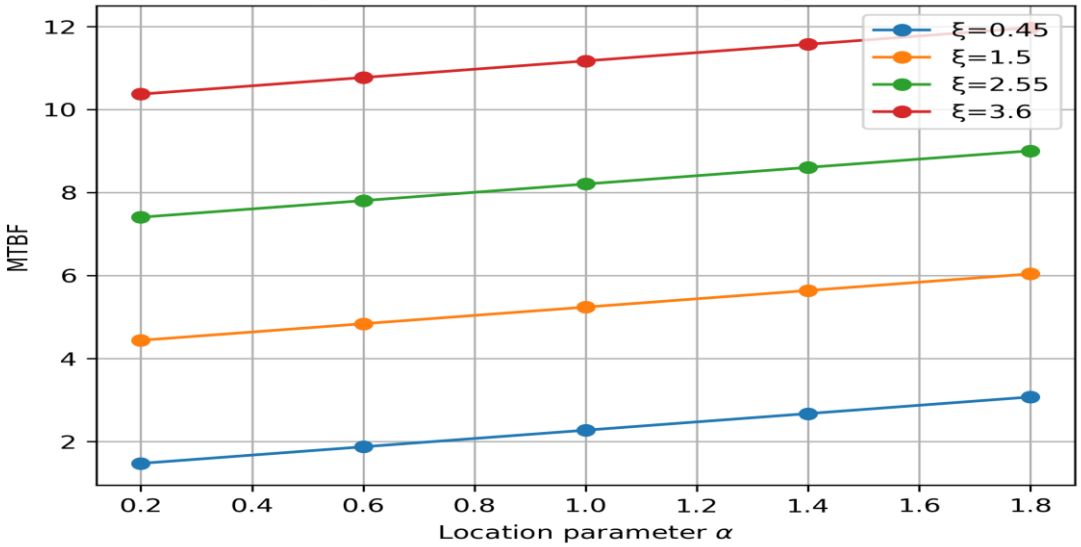


Figure 6: MTBF versus α for (2,3) system

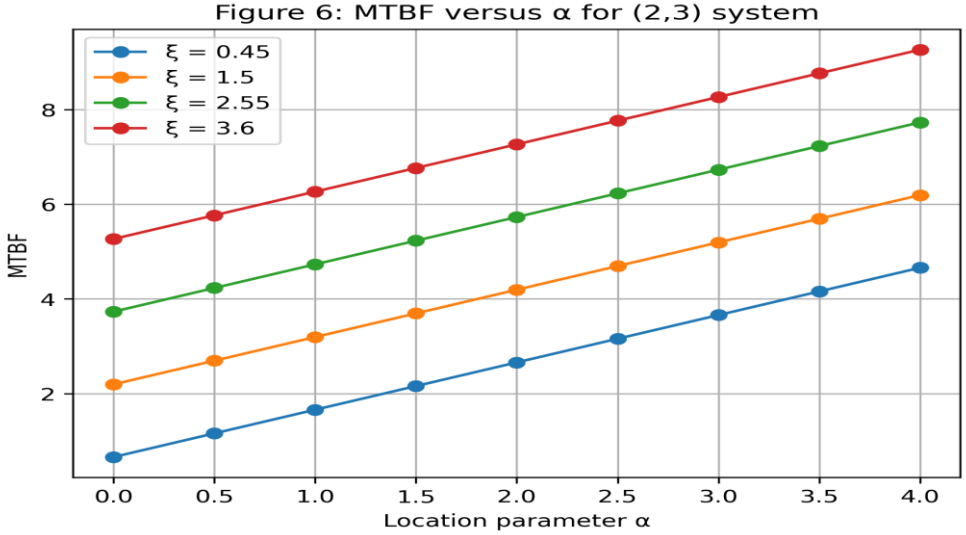
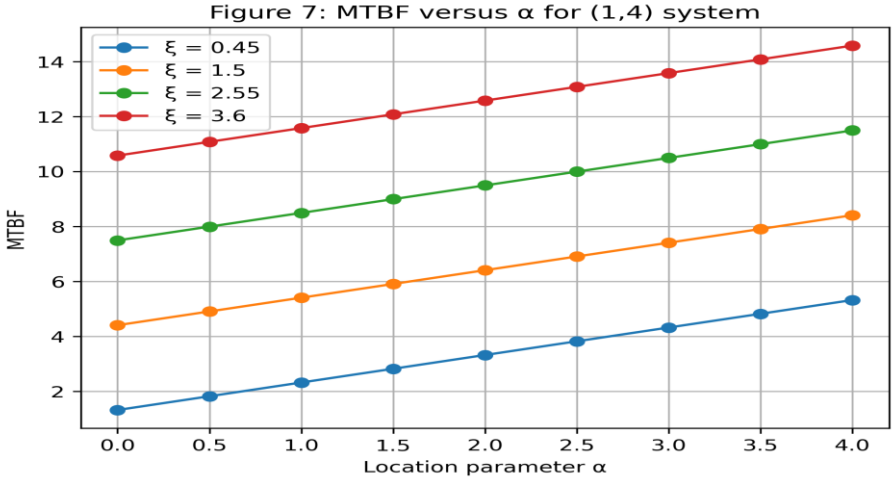
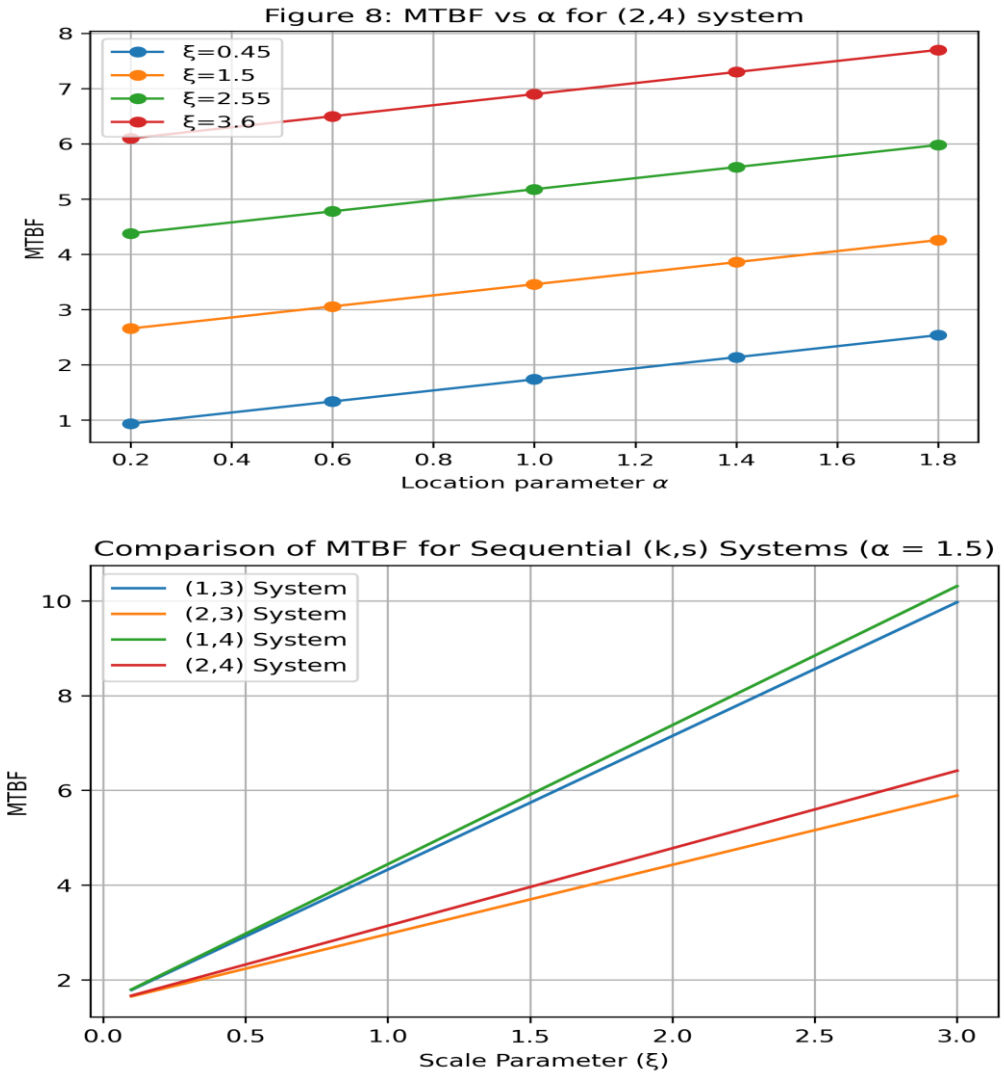


Figure 7: MTBF versus α for (1,4) system





Comparison of MTBF versus scale parameter ξ for different sequential (k,s) systems with fixed location parameter α = 1.5 as shown in figure 9. The (1,4) system exhibits the highest MTBF growth, while the (2,3) system shows the lowest reliability improvement. The differing slopes illustrate the impact of system configuration on reliability performance.

5.4.1 Analysis of Figures 5–8

These figures reveal complementary insights to Figures 1–4:

1. **Unit Slope Linear Relationship:** MTBF increases linearly with α with a slope of exactly 1.0 for all fixed values of ξ, consistent with the theoretical form $\text{MTBF} = \alpha + c \cdot \xi$.

2. **Parallel Lines with Different Intercepts:** For different ξ values, the lines remain parallel with intercepts determined by the product of ξ and the system-specific coefficient.

3. **Intercept Spacing:** The vertical distance between consecutive lines (for different ξ values) equals $(\xi_2 - \xi_1) \times$ coefficient, where the coefficient is system-dependent.

4. **Direct Translation:** Increasing α by $\Delta\alpha$ translates the entire MTBF by exactly $\Delta\alpha$ units, regardless of the ξ value or system configuration.

5. **Parameter Dominance:** For larger α values, the location parameter becomes the dominant contributor to MTBF, while for smaller α, the scale parameter contribution becomes relatively more significant.

5.5 Detailed Numerical Examples

Example 1: Consider a $(1,3)$ sequential system where:

- Initial configuration: $\alpha_1 = 0.45,\ \xi_1 = 0.40$
- Modified configuration: $\alpha_2 = 2.55,\ \xi_2 = 1.60$

Calculations:

$$\text{InitialMTBF} = 0.45 + \frac{305}{108} \times 0.40 = 1.57963 \text{ hours}$$

$$\text{ModifiedMTBF} = 2.55 + \frac{305}{108} \times 1.60 = 7.06852 \text{ hours}$$

$$\text{Improvement} = 7.06852 - 1.57963 = 5.48889 \text{ hours } (347.5\% \text{increase})$$

Component Analysis

- Contribution from α increase: $2.55 - 0.45 = 2.10$ (38.3%)
- Contribution from ξ increase: $\frac{305}{108} \times (1.60 - 0.40) = 3.38889$ (61.7%)

Example 2: Comparison across system configurations for $\alpha = 1.50,\ \xi = 1.20$:

- $(1,3)$ system: MTBF = 4.88889 hours
- $(2,3)$ system: MTBF = 3.25556 hours (33.4% lower)
- $(1,4)$ system: MTBF = 5.02535 hours (2.8% higher than $(1,3)$)
- $(2,4)$ system: MTBF = 3.46563 hours (29.1% lower than $(1,3)$)

This demonstrates that requiring more operational components (higher k) reduces MTBF, while adding more total components (higher s) with the same k increases MTBF.

5.6 Sensitivity Analysis

The sensitivity of MTBF to parameter changes can be quantified as:

Sensitivity to ξ:

$$\frac{\partial(\text{MTBF})}{\partial \xi} = \text{coefficient}(\text{system} - \text{dependent constant}) \qquad (16)$$

Sensitivity to α:

$$\frac{\partial(\text{MTBF})}{\partial \alpha} = 1(\text{constant across all systems}) \qquad (17)$$

This analysis reveals that the location parameter has a uniform additive effect, while the scale parameter's impact varies significantly with system configuration. The $(1,4)$ system shows the highest sensitivity to ξ changes (coefficient ≈ 2.938), making it most responsive to improvements in component scale parameters.

5.7 Practical Implications

1. **Design Optimization:** Engineers can strategically adjust α and ξ to achieve target MTBF values using the linear relationships established.
2. **Maintenance Planning:** The predictable linear relationship enables accurate scheduling of preventive maintenance intervals.
3. **Cost-Benefit Analysis:** The quantified sensitivities allow evaluation of the return on investment for improving component quality (affecting ξ) versus baseline reliability (affecting α).
4. **System Selection:** The comparative analysis across (k, s) configurations provides guidance for choosing

appropriate redundancy levels based on reliability requirements and resource constraints.

5.8 Summary of Key Findings

The numerical illustrations confirm that:

- MTBF exhibits perfect linearity with both α and ξ
- All trend lines are parallel within parameter families
- System configuration strongly influences the ξ coefficient
- The $(1,4)$ system provides maximum reliability for given parameters
- Parameter optimization can achieve substantial MTBF improvements
- The additive model $\text{MTBF} = \alpha + c \cdot \xi$ provides excellent predictive capability.

6.CONCLUSION

The results of this study demonstrate that the reliability of differently structured independent sequential (k, s) systems can be effectively modelled and quantified through the careful analysis of component failure patterns and their parametric adjustments. By focusing on sequential (k, s) systems where at least k out of s components remain operational and considering gamma distributions for component failure times, this investigation shows that the Mean Time Between Failures can be described as functions of the location and scale parameters. A key finding is the linear relationship observed between MTBF and the distribution parameters for the analyzed models. Illustrative examples showing that varying either parameter while holding the other constant results in a predictable upward trend. This manifests visually as parallel, upward-shifting lines, confirming that optimized parameter selection consistently improves system reliability as measured by MTBF.

The present analysis, however, has certain limitations. It is restricted to Gamma distribution with location and scale parameter for $(1,3)$, $(2,3)$, $(1,4)$ and $(2,4)$ systems. Future research can be extended to more general lifetime distributions, such as Weibull or generalized gamma with three parameters. A promising direction is applying the model to real-world systems such as sensor networks, manufacturing systems, or large-scale data centers, where interdependent failures are common. These extensions would further enhance the practical utility and predictive power of sequential (k, s) system modelling.

The developed model not only contributes to theoretical advancements in reliability analysis but also offers tangible value for engineering practice. By appropriately adjusting the location and scale parameters within practical operating boundaries, engineers and system designers can enhance system uptime, defer failure occurrences, and schedule maintenance more strategically. Consequently, the proposed model serves as an effective decision-support tool for optimizing mission-critical systems where reliability and accurate failure prediction are crucial to overall performance.

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A DetailedMathematicalDerivations

This appendix presents the complete mathematical derivations for the system failure time probability density functions for all sequential (*k,s*) systems analyzed in this study.

A.1 Derivation of Failure Time for Sequential (1,3)System

For the (1,3) system, failure occurs when the third component fails, denoted by the order

statisticX^{*} $^{(3)}$. The PDF of the system failure time is derived by integrating the joint density of all three ordered failure times over the first two:

$$f_3^*(x_3) = \int_{\alpha}^{x_3}\int_{x_1}^{x_3} f^*(x_1,x_2,x_3)\,dx_2dx_1, \alpha < x_1 < x_2 < x_3 < \infty$$

$$= \int_{\alpha}^{x_3}\int_{x_1}^{x_3} 6\frac{1}{\xi^5}(x_1-\alpha)(x_2-\alpha)\left[1+\frac{x_2-\alpha}{\xi}\right] e^{-\frac{1}{\xi}(x_1+x_2+x_3-3\alpha)}\,dx_2dx_1$$

$$= \frac{6}{\xi^3}e^{-\frac{1}{\xi}(x_3-\alpha)}\int_{\alpha}^{x_3}\int_{x_1}^{x_3}\left(\frac{x_1-\alpha}{\xi}\right)\left(\frac{x_2-\alpha}{\xi}\right)\left[1+\frac{x_2-\alpha}{\xi}\right]e^{-\frac{1}{\xi}(x_1+x_2-2\alpha)}dx_2dx_1$$

$$= \frac{6}{\xi^3}e^{-\frac{1}{\xi}(x_3-\alpha)}\left\{\int_{\alpha}^{x_3}\left(\frac{x_1-\alpha}{\xi}\right)\left[\int_{x_1}^{x_3}\left(\frac{x_2-\alpha}{\xi}\right)\left[1+\frac{x_2-\alpha}{\xi}\right]e^{-\frac{1}{\xi}(x_2-\alpha)}dx_2\right]e^{-\frac{1}{\xi}(x_1-\alpha)}dx_1\right\} \quad (18)$$

Consider the inner integral

$$\int_{x_1}^{x_3}\left(\frac{x_2-\alpha}{\xi}\right)\left[1+\frac{x_2-\alpha}{\xi}\right]e^{-\frac{1}{\xi}(x_2-\alpha)}dx_2 \quad (19)$$

$$Put\ u = \frac{x_2-\alpha}{\xi}, then\ \xi du = dx_2$$

The limit varies from $\frac{x_1-\alpha}{\xi}$ to $\frac{x_3-\alpha}{\xi}$

$$\int_{x_1}^{x_3}\left(\frac{x_2-\alpha}{\xi}\right)\left[1+\frac{x_2-\alpha}{\xi}\right]e^{-\frac{1}{\xi}(x_2-\alpha)}dx_2 = \xi\int_{\frac{x_1-\alpha}{\xi}}^{\frac{x_3-\alpha}{\xi}}(u+u^2)e^{-u}du$$

$$= \xi\left\{\left[\left(\frac{x_1-\alpha}{\xi}\right)^2+3\left(\frac{x_1-\alpha}{\xi}\right)+3\right]e^{-\frac{1}{\xi}(x_1-\alpha)} - \left[\left(\frac{x_3-\alpha}{\xi}\right)^2+3\left(\frac{x_3-\alpha}{\xi}\right)+3\right]e^{-\frac{1}{\xi}(x_3-\alpha)}\right\} (20)$$

substituting (20) in (18),we get

$$f_3^*(x_3) = \frac{6}{\xi^2}e^{-\frac{1}{\xi}(x_3-\alpha)}\left\{\int_{\alpha}^{x_3}\left(\frac{x_1-\alpha}{\xi}\right)\left\{\left[\left(\frac{x_1-\alpha}{\xi}\right)^2+3\left(\frac{x_1-\alpha}{\xi}\right)+3\right]e^{-\frac{1}{\xi}(x_1-\alpha)}dx_1 - \left[\left(\frac{x_3-\alpha}{\xi}\right)^2+3\left(\frac{x_3-\alpha}{\xi}\right)+3\right]e^{-\frac{1}{\xi}(x_3-\alpha)}\right\}e^{-\frac{1}{\xi}(x_1-\alpha)}dx_1\right\} (21)$$

$$f_3^*(x_3) = \frac{6}{\xi^2} e^{-\frac{1}{\xi}(x_3-\alpha)} \left\{ \int_\alpha^{x_3} \left(\frac{x_1-\alpha}{\xi}\right) \left[\left(\frac{x_1-\alpha}{\xi}\right)^2 + 3\left(\frac{x_1-\alpha}{\xi}\right) + 3 \right] e^{-\frac{2}{\xi}(x_1-\alpha)} dx_1 \right.$$
$$\left. - \left[\left(\frac{x_3-\alpha}{\xi}\right)^2 + 3\left(\frac{x_3-\alpha}{\xi}\right) + 3 \right] e^{-\frac{1}{\xi}(x_3-\alpha)} \int_\alpha^{x_3} \left(\frac{x_1-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_1-\alpha)} dx_1 \right\} (22)$$

$$Let\ v = \frac{x_1-\alpha}{\xi}, then\ \xi dv = dx_1$$

The limit varies from 0 to $\frac{x_3-\alpha}{\xi}$

$$f_3^*(x_3) = \frac{6}{\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} \left\{ \int_\alpha^{\frac{x_3-\alpha}{\xi}} v(v^2+3v+3)e^{-2v} dv \right.$$
$$\left. - \left[\left(\frac{x_3-\alpha}{\xi}\right)^2 + 3\left(\frac{x_3-\alpha}{\xi}\right) + 3 \right] e^{-\frac{1}{\xi}(x_3-\alpha)} \int_\alpha^{\frac{x_3-\alpha}{\xi}} ve^{-v} dv \right\} (23)$$

Using integration by parts and standard integral formulas:

$$f_3^*(x_3) = \frac{6}{\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} \left\{ \left[-\frac{1}{2}\left(\frac{x_3-\alpha}{\xi}\right)^3 e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{9}{4}\left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{15}{4}\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{2}{\xi}(x_3-\alpha)} \right.\right.$$
$$\left. - \frac{15}{8} e^{-\frac{2}{\xi}(x_3-\alpha)} + \frac{15}{8} \right]$$
$$- \left[\left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_3-\alpha)} + 3\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} + 3e^{-\frac{1}{\xi}(x_3-\alpha)} \right] \left[-\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} \right.$$
$$\left.\left. - e^{-\frac{1}{\xi}(x_3-\alpha)} + 1 \right] \right\} (24)$$

Expanding and simplifying the second term:

$$\left[\left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_3-\alpha)} + 3\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} + 3e^{-\frac{1}{\xi}(x_3-\alpha)} \right] \left[-\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} - e^{-\frac{1}{\xi}(x_3-\alpha)} + 1 \right]$$

$$= -\left(\frac{x_3-\alpha}{\xi}\right)^3 e^{-\frac{2}{\xi}(x_3-\alpha)} - 4\left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} - 6\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{2}{\xi}(x_3-\alpha)} - 3e^{-\frac{2}{\xi}(x_3-\alpha)} + \left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_3-\alpha)} +$$
$$3\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} + 3e^{-\frac{1}{\xi}(x_3-\alpha)} (25)$$

Combining all terms:

$$f_3^*(x_3) = \frac{6}{\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} \left\{ \left[\left(1-\frac{1}{2}\right)\left(\frac{x_3-\alpha}{\xi}\right)^3 e^{-\frac{2}{\xi}(x_3-\alpha)} + \left(4-\frac{9}{4}\right)\left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} \right.\right.$$
$$+ \left(6-\frac{15}{4}\right)\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{2}{\xi}(x_3-\alpha)} + \left(3-\frac{15}{8}\right) e^{-\frac{2}{\xi}(x_3-\alpha)} - \left(\frac{x_3-\alpha}{\xi}\right)^2 e^{-\frac{1}{\xi}(x_3-\alpha)}$$
$$\left.\left. - 3\left(\frac{x_3-\alpha}{\xi}\right) e^{-\frac{1}{\xi}(x_3-\alpha)} - 3e^{-\frac{1}{\xi}(x_3-\alpha)} + \frac{15}{8} \right] \right\} (26)$$

Simplifying the coefficients:

$$f_3^*(x_3) = \frac{6}{\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} \left[\frac{1}{2} \left(\frac{x_3 - \alpha}{\xi} \right)^3 e^{-\frac{2}{\xi}(x_3-\alpha)} + \frac{7}{4} \left(\frac{x_3 - \alpha}{\xi} \right)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} + \frac{9}{4} \left(\frac{x_3 - \alpha}{\xi} \right) e^{-\frac{2}{\xi}(x_3-\alpha)} \right.$$
$$\left. + \frac{9}{8} e^{-\frac{2}{\xi}(x_3-\alpha)} - \left(\frac{x_3 - \alpha}{\xi} \right)^2 e^{-\frac{1}{\xi}(x_3-\alpha)} - 3 \left(\frac{x_3 - \alpha}{\xi} \right) e^{-\frac{1}{\xi}(x_3-\alpha)} - 3e^{-\frac{1}{\xi}(x_3-\alpha)} + \frac{15}{8} \right] (27)$$

Distributing the factor $\frac{6}{\xi} e^{-\frac{1}{\xi}(x_3-\alpha)}$:

$$f_3^*(x_3) = \frac{3}{\xi} \left(\frac{x_3 - \alpha}{\xi} \right)^3 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{21}{2\xi} \left(\frac{x_3 - \alpha}{\xi} \right)^2 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{2\xi} \left(\frac{x_3 - \alpha}{\xi} \right) e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{4\xi} e^{-\frac{3}{\xi}(x_3-\alpha)}$$
$$- \frac{6}{\xi} \left(\frac{x_3 - \alpha}{\xi} \right)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi} \left(\frac{x_3 - \alpha}{\xi} \right) e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi} e^{-\frac{2}{\xi}(x_3-\alpha)} + \frac{45}{4\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} (28)$$

Converting to standard form:

$$f_3^*(x_3) = \frac{3}{\xi^4} (x_3 - \alpha)^3 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{21}{2\xi^3} (x_3 - \alpha)^2 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{2\xi^2} (x_3 - \alpha) e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{4\xi} e^{-\frac{3}{\xi}(x_3-\alpha)}$$
$$- \frac{6}{\xi^3} (x_3 - \alpha)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi^2} (x_3 - \alpha) e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi} e^{-\frac{2}{\xi}(x_3-\alpha)} + \frac{45}{4\xi} e^{-\frac{1}{\xi}(x_3-\alpha)} (29)$$

This can be expressed as a weighted sum:

$$f_3^*(x_3) = \sum_{i=1}^{8} w_i f_i(x_3) (30)$$

$$w_1 = \frac{2}{9}, w_2 = \frac{7}{9}, w_3 = \frac{3}{2}, w_4 = \frac{9}{4}, w_5 = -\frac{3}{2}, w_6 = -\frac{9}{2}, w_7 = -9, w_8 = \frac{45}{4}$$

and

$$f_1(x_3) = \left(\frac{3}{\xi} \right)^4 \frac{1}{\Gamma 4} (x_3 - \alpha)^3 e^{-\frac{3}{\xi}(x_3-\alpha)} ,$$

$$f_2(x_3) = \left(\frac{3}{\xi} \right)^3 \frac{1}{\Gamma 3} (x_3 - \alpha)^2 e^{-\frac{3}{\xi}(x_3-\alpha)}$$

$$f_3(x_3) = \left(\frac{3}{\xi} \right)^2 \frac{1}{\Gamma 2} (x_3 - \alpha) e^{-\frac{3}{\xi}(x_3-\alpha)}$$

$$f_4(x_3) = \frac{3}{\xi} e^{-\frac{3}{\xi}(x_3-\alpha)}$$

$$f_5(x_3) = \left(\frac{2}{\xi} \right)^3 \frac{1}{\Gamma 3} (x_3 - \alpha)^2 e^{-\frac{2}{\xi}(x_3-\alpha)}$$

$$f_6(x_3) = \left(\frac{2}{\xi} \right)^2 (x_3 - \alpha) e^{-\frac{2}{\xi}(x_3-\alpha)}$$

$$f_7(x_3) = \left(\frac{2}{\xi} \right) e^{-\frac{2}{\xi}(x_3-\alpha)}$$

$$f_8(x_3) = \left(\frac{1}{\xi} \right) e^{-\frac{1}{\xi}(x_3-\alpha)}$$

A.2 DerivationofFailureTimeforSequential(2,3)System

Inthe(2,3)system,thesystemfailsatthesecondorderedfailuretime,

$X^*_{(2)}$. The PDF is system failure time is derived by integrating the joint density over the first ordered failure time:

$$f_2^*(x_2) = \int_\alpha^{x_2} f(x_1, x_2)\, dx_1; \alpha < x_1 < x_2 < \infty, \alpha > 0 \qquad (31)$$

Substituting the joint density from equation (4):

$$f_2^*(x_2) = 6\frac{1}{\xi^2}\int_\alpha^{x_2} \left(\frac{x_1 - \alpha}{\xi}\right)\left[1 + \frac{x_1 - \alpha}{\xi}\right]^2 e^{-\frac{1}{\xi}(x_1 + 2x_2 - 3\alpha)}\, dx_1 (32)$$

Through detailed integration (expanding the squared term and integrating term by term), we obtain:

$$f_2^*(x_2) = -\frac{6}{\xi^4}(x_2 - \alpha)^3 e^{-\frac{3}{\xi}(x_2-\alpha)} - \frac{30}{\xi^3}(x_2 - \alpha)^2 e^{-\frac{3}{\xi}(x_2-\alpha)} - \frac{66}{\xi^2}(x_2 - \alpha) e^{-\frac{3}{\xi}(x_2-\alpha)} - \frac{66}{\xi} e^{-\frac{3}{\xi}(x_2-\alpha)} + \frac{66}{\xi} e^{-\frac{2}{\xi}(x_2-\alpha)} (33) \qquad (6)$$

This can be expressed as:

$$f_2^*(x_2) = \sum_{i=1}^{5} w_i\, f_i(x_2) (34)$$

where:

$$w_1 = -\frac{4}{9}\ , w_2 = -\frac{20}{9}\ , w_3 = -\frac{22}{3}\ , w_4 = -22, w_5 = 33$$

$$f_1(x_2) = \left(\frac{3}{\xi}\right)^4 \frac{1}{\lceil 4}(x_2 - \alpha)^3 e^{-\frac{3}{\xi}(x_2-\alpha)}$$

$$f_2(x_2) = \left(\frac{3}{\xi}\right)^3 \frac{1}{\lceil 3}(x_2 - \alpha)^2 e^{-\frac{3}{\xi}(x_2-\alpha)}$$

$$f_3(x_2) = \left(\frac{3}{\xi}\right)^2 \frac{1}{\lceil 2}(x_2 - \alpha) e^{-\frac{3}{\xi}(x_2-\alpha)}$$

$$f_4(x_2) = \frac{3}{\xi} e^{-\frac{3}{\xi}(x_2-\alpha)}$$

$$f_5(x_2) = \frac{2}{\xi} e^{-\frac{2}{\xi}(x_2-\alpha)}$$

A.3 Derivation of Failure Time for Sequential(1,4)System

For the(1,4) system, failure occurs atX^*. The PDF is derived by marginalizing over the (4)

first three order statistics:

$$f_4^*(x_4) = \int_\alpha^{x_4}\int_{x_1}^{x_4}\int_{x_2}^{x_4} f^*(x_1, x_2, x_3, x_4)\, dx_3 dx_2 dx_1 \qquad (35)$$

$$\alpha < x_1 < x_2 < x_3 < x_4 < \infty$$

After extensive integration, the result is:

$$f_4^*(x_4) = -\frac{4}{\xi^5}(x_4-\alpha)^4 e^{-\frac{4}{\xi}(x_4-\alpha)} - \frac{76}{3\xi^4}(x_4-\alpha)^3 e^{-\frac{4}{\xi}(x_4-\alpha)} - \frac{202}{3\xi^3}(x_4-\alpha)^2 e^{-\frac{4}{\xi}(x_4-\alpha)}$$
$$-\frac{818}{9\xi^2}(x_4-\alpha)e^{-\frac{4}{\xi}(x_4-\alpha)} - \frac{1439}{27\xi}e^{-\frac{4}{\xi}(x_4-\alpha)} + \frac{305}{27\xi}e^{-\frac{2}{\xi}(x_4-\alpha)} - \frac{195}{\xi}e^{-\frac{2}{\xi}(x_4-\alpha)}$$
$$+\frac{12}{\xi^4}(x_4-\alpha)^3 e^{-\frac{3}{\xi}(x_4-\alpha)} + \frac{78}{\xi^3}(x_4-\alpha)^2 e^{-\frac{3}{\xi}(x_4-\alpha)} + \frac{210}{\xi^2}(x_4-\alpha)e^{-\frac{3}{\xi}(x_4-\alpha)}$$
$$+\frac{237}{\xi}e^{-\frac{3}{\xi}(x_4-\alpha)}(36) \qquad (7)$$

Expressed as a weighted sum with 11 terms.

A.4 DerivationofFailureTimeforSequential(2,4)System

Forthe(2,4)system,failureoccursatX^*: (3)

$$f_3^*(x_3) = \int_\alpha^{x_3}\int_{x_1}^{x_3} f^*(x_1,x_2,x_3)\,dx_2dx_1, \alpha < x_1 < x_2 < x_3 < \infty \qquad (37)$$

After integration:

$$f_3^*(x_3) = \frac{12}{\xi^5}(x_3-\alpha)^4 e^{-\frac{4}{\xi}(x_3-\alpha)} + \frac{108}{\xi^4}(x_3-\alpha)^3 e^{-\frac{4}{\xi}(x_3-\alpha)} + \frac{450}{\xi^3}(x_3-\alpha)^2 e^{-\frac{4}{\xi}(x_3-\alpha)}$$
$$+\frac{1038}{\xi^2}(x_3-\alpha)e^{-\frac{4}{\xi}(x_3-\alpha)} + \frac{1107}{\xi}e^{-\frac{4}{\xi}(x_3-\alpha)} - \frac{1176}{\xi}e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{69}{\xi}e^{-\frac{2}{\xi}(x_3-\alpha)}(38)$$

Expressed as a weighted sum with 7 terms.

A.5 Derivation of MTBF for Sequential (1,3)System

TheMTBFiscomputedas:

$$\text{MTBF} = \int_\alpha^\infty x_3 f_3^*(x_3)\,\mathrm{d}x_3 \qquad (39)$$

Substituting the derived PDF:

$$\text{MTBF} = \int_\alpha^\infty x_3\left(\frac{3}{\xi^4}(x_3-\alpha)^3 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{21}{2\xi^3}(x_3-\alpha)^2 e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{2\xi^2}(x_3-\alpha)e^{-\frac{3}{\xi}(x_3-\alpha)} + \frac{27}{4\xi}e^{-\frac{3}{\xi}(x_3-\alpha)}\right.$$
$$-\frac{6}{\xi^3}(x_3-\alpha)^2 e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi^2}(x_3-\alpha)e^{-\frac{2}{\xi}(x_3-\alpha)} - \frac{18}{\xi}e^{-\frac{2}{\xi}(x_3-\alpha)}$$
$$\left.+\frac{45}{4\xi}e^{-\frac{1}{\xi}(x_3-\alpha)}\right)\mathrm{d}x_3(40)$$

Using the substitution $y = x_3 - \alpha$ which implies$x_3 = y + \alpha$ *and limit become 0 to* ∞.

$$\text{MTBF} = \int_0^\infty (y+\alpha) f_3^*(y+\alpha)\,\mathrm{d}y$$

$$= \int_0^\infty y f_3^*(y+\alpha)\,\mathrm{d}y + \int_0^\infty \alpha f_3^*(y+\alpha)\,\mathrm{d}y \quad (41)$$

For the integrals, we use the standard result:

$$\int_0^\infty y^n e^{-cy}\,\mathrm{d}y = \frac{n!}{c^{n+1}} \quad (42)$$

Computing each term systematically:

For terms with $e^{-\frac{3}{\xi}y}$

$$\frac{3}{\xi^4}\int_0^\infty y^4 e^{-\frac{3}{\xi}y} dy = \frac{3}{\xi^4}\frac{4!}{\left(\frac{3}{\xi}\right)^5} = \frac{8}{27}\xi \quad (43)$$

$$\frac{21}{2\xi^3}\int_0^\infty y^3 e^{-\frac{3}{\xi}y} dy = \frac{21}{2\xi^3}\frac{3!}{\left(\frac{3}{\xi}\right)^4} = \frac{7}{9}\xi \quad (44)$$

$$\frac{27}{2\xi^2}\int_0^\infty y^2 e^{-\frac{3}{\xi}y} dy = \frac{27}{2\xi^2}\frac{2!}{\left(\frac{3}{\xi}\right)^3} = \xi \quad (45)$$

$$\frac{27}{4\xi}\int_0^\infty y e^{-\frac{3}{\xi}y} dy = \frac{27}{4\xi}\frac{1!}{\left(\frac{3}{\xi}\right)^2} = \frac{3}{4}\xi \quad (46)$$

For terms involving the exponential factor $e^{-\frac{2}{\xi}y}$

$$-\frac{6}{\xi^3}\int_0^\infty y^3 e^{-\frac{2}{\xi}y} dy = -\frac{6}{\xi^3}\frac{3!}{\left(\frac{2}{\xi}\right)^4} = -\frac{9}{4}\xi \quad (47)$$

$$-\frac{18}{\xi^2}\int_0^\infty y^2 e^{-\frac{2}{\xi}y} dy = -\frac{18}{\xi^2}\frac{2!}{\left(\frac{2}{\xi}\right)^3} = -\frac{9}{2}\xi \quad (48)$$

$$-\frac{18}{\xi}\int_0^\infty y e^{-\frac{2}{\xi}y} dy = -\frac{18}{\xi}\frac{1!}{\left(\frac{2}{\xi}\right)^2} = -\frac{9}{2}\xi \quad (49)$$

For terms with $e^{-\frac{1}{\xi}y}$

$$\frac{45}{4\xi}\int_0^\infty y e^{-\frac{1}{\xi}y} dy = \frac{45}{4\xi}\frac{1!}{\left(\frac{1}{\xi}\right)^2} = \frac{45}{4}\xi \quad (50)$$

For the second integral (constant α term), similar computations yield contributions that sum to α.Combining all terms:

$$MTBF = \frac{2}{9}\left(\frac{4\xi}{3}+\alpha\right) + \frac{7}{9}(\xi+\alpha) + \frac{3}{2}\left(\frac{2\xi}{3}+\alpha\right) + \frac{9}{4}\left(\frac{\xi}{3}+\alpha\right) - \frac{3}{2}\left(\frac{3\xi}{2}+\alpha\right) - \frac{9}{2}(\xi+\alpha)$$

$$-9\left(\frac{\xi}{2}+\alpha\right) + \frac{45}{4}(\xi+\alpha) \quad (51)$$

Simplifying:

$$MTBF = \alpha + \frac{305}{108}\xi. \quad (52)$$