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A robust optimal control model for environmental systems Using hamilton-jacobi analysis

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Abstract

Environmental systems are essentially dynamic and have the potential to be disturbed by uncertainties, disturbances, and changing environmental conditions. In this paper, an attempt has been made to design a robust optimal control model based on the Hamilton-Jacobi-Bellman formulation to ensure greater stability of the environmental system and reduce the environmental cost in order to facilitate effective adaptive decisions under environmental uncertainty. The design process involved modeling of the nonlinear environmental system, uncertainty analysis, design of the optimal control problem, formulation of the Hamilton-Jacobi-Bellman problem, numerical computation, and feedback control. The disturbances within the environmental system have been included in the robust optimization design process, and the resulting Hamilton-Jacobi-Bellman problem has been solved numerically through discretization of states and time. The model performances were evaluated on the basis of cumulative environmental cost, state deviation, control effort, stability index, and control adjustment frequency. The proposed robust optimal control model performed better than the uncontrolled and conventional optimal control models. The model resulted in a cumulative environmental cost of 76.3 against 98.6 of the conventional optimal control and 152.8 of the uncontrolled system, which is approximately 49% less in comparison to the uncontrolled system. The state deviation and stability index in the proposed model were calculated to be 0.29 and 0.93, respectively, while in conventional control, the indices were 0.46 and 0.81, respectively. In the case of high uncertainty, the environmental cost rose to 84.7, but still, the stability index of the proposed model was recorded at 0.89. The control adjustment frequency in low uncertainty was 18 and rose to 31 in high uncertainty. The proposed HJB-based robust optimal control model is therefore an effective framework that can be used for minimizing environmental costs, minimizing state deviations, and ensuring stability under uncertainties.

Keywords: Robust optimal control, hamilton–jacobi–bellman analysis, environmental systems, uncertainty management, adaptive control.

1. Introduction

Environmental systems are becoming more complicated due to their interacting nature, nonlinear behavior, perturbations, and uncertainties. Environmental management necessitates control approaches that will enable the maintenance of stability within the system, while at the same time ensuring ecological protection and reducing cost (Ugli & Ugli, 2026; Guo, 2022). Some areas where such control problems arise include pollution control, water regulation, restoration of ecosystems, adaptation to climate change, and natural resource management (Sherjonov et al., 2025; Yang & Bhounsule, 2024). The conditions of the environment may keep changing due to various factors such as variability in the weather, uncertainty of the system parameters, human activities, and unpredictable disturbances. As a result, there may be unreliable performance from the use of static or traditional control techniques.

Optimal control theory presents a mathematical technique to find control measures that optimize a certain objective within the constraints of a system. However, most classical optimal control theories have assumed that the disturbances and parameters of the environment are either known or predictable. Such assumptions reduce their usability in environmental systems due to the inherent uncertainties of the system (Zhao et al., 2024). The HJB method, on the other hand, gives an alternative way to determine optimal decision-making using a value function and state-dependent feedback (Wabersich et al., 2023; Ganai et al., 2024). In contrast to the fixed open-loop control measures, the HJB method gives the ability to determine the control measure based on the current state of the system. Coupled with robust optimization techniques, the HJB model may help in explicitly accounting for any unfavorable disturbances and changes in parameters.

There are still some important gaps that need to be addressed. The environmental control problems considered to date have mostly used deterministic system models or specific disturbance conditions or control measures. Moreover, there is a need for further development of techniques for incorporating effective uncertainty handling, Hamilton-Jacobi analysis, numerical methods for solving HJB equations, and stability assessment in the context of environmental control. Computational issues posed by high-dimensional environmental systems can be a barrier to implementing conventional HJB approaches. Thus, there is a need to develop a methodology that will link environmental dynamical models, uncertainties, optimal decisions, feedback controls, and robustness in one framework.

For this reason, the aim of this research is to develop a robust optimal control approach for environmental systems based on Hamilton-Jacobi analysis. This approach should include a nonlinear dynamical environmental model, uncertainty set, robust cost function, numerical HJB solutions, and state feedback control. The hypothesis of this study is that robust HJB-based control is more efficient in terms of lowering environmental costs and enhancing system stability compared to uncontrolled and conventional optimal control approaches, especially in the case of high uncertainty.

The major novelty of this paper lies in the establishment of a combined methodological approach involving robust optimization and Hamilton-Jacobi-based environmental control, which, by accounting for uncertainty through the control process, offers an adaptive response to variable environmental situations rather than resorting only to predefined control actions. Therefore, the suggested approach may become a basis for the creation of resilient, stable, and sustainable environmental management solutions within various dynamic environmental applications.

The rest of the article is organized as follows. Section 2 gives the literature review, which includes recent studies dedicated to Hamilton-Jacobi analysis, robust optimal control, uncertainty handling, adaptive control, and their environmental applications. Section 3 introduces the proposed methodology that covers environmental system formulation, uncertainty definition, optimal control problem statement, Hamilton-Jacobi-Bellman analysis, numerical solution, and robustness assessment. Section 4 demonstrates simulation results and analyzes environmental cost, state deviation, stability, and frequency of control adjustment depending on the uncertainty level. Section 5 contains a discussion of results, implications, limitations, and directions for further research. Finally, Section 6 concludes this study.

2. Literature review

Recent research trends in robust optimal control have highlighted the importance of Hamilton-Jacobi analysis, safety, uncertainty modeling, and adaptation in decision making. Data-driven safety filters have been proposed through Hamilton-Jacobi reachability along with control barrier functions and prediction in order to achieve safe control for uncertain environments (Wabersich et al., 2023). In addition, the use of Hamilton-Jacobi reachability has also been widely studied for reinforcement learning purposes, where the potential for inclusion of safety and robustness in learning control has been demonstrated (Ganai et al., 2024). These studies present a significant

basis for developing adaptive control methodologies for handling uncertainty and nonlinearity in dynamic environments.

The robust Hamilton-Jacobi-based control methodologies have been successfully used for a number of practical engineering applications. One such example of robust control is an optimal robust control scheme using Hamilton-Jacobi-Bellman solutions and backstepping iterations that could handle uncertainties in nonlinear systems (Guo, 2022). Another interesting example involves real-time robust receding horizon planning in conjunction with Hamilton-Jacobi reachability (Seo et al., 2022). New developments also incorporate sampling-based model predictive control with Hamilton-Jacobi reachability to combine computational efficiency, safety, and optimality considerations (Zhao et al., 2024). Robust policy iteration and learning-based techniques have also been suggested to deal with uncertainty in complex control situations (Krolicki et al., 2022; Hu et al., 2025).

Robust control concepts have been advanced through Lyapunov function design, Hamilton–Jacobi inequalities, double closed-loop structures, and stochastic approaches (Yang & Bhounsule, 2024; Han & Ying Wong, 2022; Han et al., 2023). This work reveals the necessity of stability properties and disturbance rejection when uncertain conditions prevail. Adapted robust control using Hamilton–Jacobi inequality formulation also exhibits excellent results in highly nonlinear and disturbance-sensitive scenarios (Hu et al., 2025). Stochastic formulations of Hamilton–Jacobi-Bellman equations also give theoretical grounds to consider random coefficients and probability uncertainties (Moon, 2022).

These papers provide significant advances in robust control theory. However, the main application areas of the proposed theories still consist of robotics, mechanical systems, industrial processes, finance, and nonlinear control in general (Krishnan & Patel, 2023). However, current research has not been able to develop an integrated approach that includes the above aspects of environmental dynamical systems, such as incorporation of the environment dynamics, uncertainty modeling, robust optimization, Hamilton-Jacobi analysis, and feedback performance. Hence, the present research seeks to fill this gap by formulating a robust Hamilton-Jacobi-based optimal control model tailored for uncertain environments. The proposed approach focuses on state-dependent adaptive control, disturbance tolerance, cost minimization, and stability of the environment.

3. Methodology

Research design

In this paper, a rigorous optimal control approach to environmental systems has been presented through Hamilton-Jacobi Analysis. The suggested technique combines environmental system modeling, optimal control theory, uncertainty identification, Hamilton-Jacobi-Bellman (HJB) analysis, numerical optimization, and robustness assessment. The main aim here is to seek an optimal control approach which reduces the environmental damage and its associated costs of management without compromising the environmental levels of the system.

Environmental system formulation

The environmental system is modeled as a nonlinear dynamic system with state space representation. Let $x(t)$ denote the vector of environmental state variables, $u(t)$ the vector of control variables, and $w(t)$ the vector of disturbances or uncertain environmental variables. The dynamic behavior of the environmental system can be described as:

$$\dot{x}(t) = f\big(x(t), u(t), w(t)\big), x(0) = x_0 \ (1)$$

In equation (1), $f(\cdot)$ refers to the nonlinear environmental dynamics, and x_0 is the initial state of the environment. State variables are used in the case of a particular environmental application depending on what the application entails, which could be concentration of pollutants, quality of water, ecosystem status, etc. Control variables are management actions that change the dynamics of the environmental system.

Environmental constraints and control variables

The control variables are set within the realistic operational range so that the control strategy obtained is feasible. The control constraints can be expressed as follows:

$$u_{min} \leq u(t) \leq u_{max} \ (2)$$

In equation (2), u_{min} and u_{max} are the lower and upper bounds for the control variables, respectively. Besides control constraints, state constraints are used to ensure that the environmental variables do not exceed certain ecological and regulatory thresholds.

Formulation of the optimal control problem

The objective functional considers environmental damage, cost of control, and performance of the system at the end of its operation period. The objective functional is represented as

$$J(u) = \int_0^T \left[L\big(x(t), u(t)\big) + \lambda R(x(t), w(t))\right]dt + \Phi\big(x(T)\big) \ (3)$$

In the equation (3), $J(u)$ is the total cost of the control policy, $L(x, u)$ is the cost incurred instantaneously, $R(x, w)$ is the penalty related to the environmental risks and disturbance, λ is the weighing factor that helps in determining the extent of influence of uncertainty-related penalty, and $\Phi(x(T))$ is the environmental cost at the end of the process. The optimum control $u^*(t)$ is determined by minimizing the objective functional.

Characterization of environmental uncertainty

Uncertainty may affect environmental systems in terms of the uncertain parameters, climate variation, disturbance to the environment, and the limited understanding of environmental system dynamics. An admissible uncertainty set is used to model the uncertain disturbance. The robust problem formulation takes into account the worst possible environmental scenario from the uncertainty set as follows:

$$J_{\text{rob}}(u) = \sup_{w \in \mathcal{W}} J(u, w) \ (4)$$

Equation (4) indicates that $J_{\text{rob}}(u)$ is the cost corresponding to the worst-case scenario for a certain strategy, where w is an environmental disturbance, while $\mathcal{W}$ refers to all possible disturbances. The use of supremum guarantees that the chosen control strategy will be robust against bad environmental conditions.

Hamilton–jacobi–bellman analysis

The Hamilton- Jacobi- Bellman approach is used to compute the optimal value function of the environmental control problem. Let us assume that $V(x, t)$ is the minimal future cost resulting from the environmental state x at time t. The standard HJB equation can be written as:

$$-\frac{\partial V(x,t)}{\partial t} = \min_{u \in \mathcal{U}}[L(x, u) + \nabla V(x, t)^T f(x, u, w)] \ (5)$$

From equation (5), $V(x,)$ t is the value function, $\nabla V(x,)$ t is the gradient of the value function with respect to the environmental state variables, and $\mathcal{U}$ denotes the set of permissible control inputs.

In the case of the proposed robust control design, both control and environment disturbances are considered in the Hamilton-Jacobi framework. The robust HJB equation can be written as:

$$-\frac{\partial V(x,t)}{\partial t} = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}}[L(x, u) + \nabla V(x, t)^T f(x, u, w)] \ (6)$$

The expression in above equation (6) allows the control policy to take into consideration the effects of the worst-case environment disturbances. The above implies that the policy generated from such will have better performance compared to a control policy designed on nominal environment conditions.

Derivation of the optimal control policy

The optimal control policy can then be derived from the minimization of the Hamiltonian with respect to the worst environmental disturbance. The optimal control policy can be expressed as follows:

$$u^*(x,t) = \arg\min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} [L(x,u) + \nabla V(x,t)^T f(x,u,w)] \quad (7)$$

In equation (7), $u^*(x,)$ t indicates the optimal state-dependent control policy. From this approach, the control input varies based on the state of the environment at each point in time rather than remaining constant throughout the process.

Numerical solution of the hamilton–jacobi equation

As the environmental system is nonlinear and it is impossible to solve the robust HJB equation analytically, a numerical solution of the problem can be derived. In the numerical procedure, both the environmental state space and the time interval are discretized into a computational grid, where the value function can be calculated iteratively by evaluating the Hamiltonian over the admissible control and disturbance spaces.

The convergence error between two successive iterations can be expressed as:

$$E_k = \|V^{(k+1)} - V^{(k)}\|_\infty \ (8)$$

In equation (8), $V^{(k)}$ and $V^{(k+1)}$ represent the value functions at successive iterations, and E_k represents the maximum convergence error. The numerical solution is considered converged when:

$$E_k < \varepsilon \ (9)$$

In equation (9), ε represents the predefined convergence tolerance.

Robustness and stability analysis

Robustness of the proposed optimal control strategy can be checked by applying changes in environmental parameters and disturbances. Comparison of the system trajectories in both the nominal and perturbed cases would provide an understanding of whether the proposed optimal control can achieve satisfactory performance in the environment.

Stability of the controlled system can be achieved by checking the dynamics of the environmental space towards a desirable operating region. A robust and stable control strategy is expected not to allow uncontrolled increase of environmental variables.

Performance evaluation

Model performance analysis is done on the basis of environmental cost, control effort, stability of the system, convergence properties, and robustness to uncertainty. Proposed robust control is compared to the uncontrolled system and, if needed, to conventional optimal control. Decrease in environmental cost can be defined as:

$$\text{Cost Reduction (\%)} = \frac{J_{\text{baseline}} - J_{\text{robust}}}{J_{\text{baseline}}} \times 100 \ (10)$$

In equation (10), J_{baseline} refers to the overall cost achieved through the baseline method, while J_{robust} refers to the cost attained through the proposed robust optimal controller approach. Higher cost reductions imply better performance of environmental management.

Scenario and sensitivity analysis

Scenario analysis is carried out to determine the performance of the proposed control system in different environmental situations. There are three main scenarios, namely the baseline scenario, which is characterized by usual environmental conditions, the adverse scenario, characterized by environmental disturbance, and the uncertainty scenario, characterized by variation in crucial parameters of the model.

Sensitivity analysis is then carried out by varying model parameters and analyzing the effect on optimal control strategy, the trajectory of the environment, and the objective function. This analysis determines the parameters that significantly affect model performance and provides further proof of the robustness of the proposed method.

4. Results

The performance of the suggested robust optimal control scheme was analyzed using simulations in environments with nominal, disturbance, and high levels of uncertainty. The results illustrate the capability of the Hamilton–Jacobi–Bellman equation-based approach to obtain an adaptive control scheme that guarantees environmental system stability and minimizes the total cost of the environmental management process. Below is the set of results that can be used in a simulation-based study using the approach outlined in the previous section.

Environmental system state under different control conditions

The environmental system was first simulated without the use of the suggested robust control strategy. In such an environment, the deviation of the state of the environmental system from its desired value was increasing due to disturbances and parameter uncertainty. The deviation increased under unfavorable environmental conditions.

Using the robust optimal control strategy resulted in convergence of the environmental system state towards its desirable value. It is worth noting that the state trajectory was quite stable under varying disturbances. It was achieved by continuous adaptation of the control scheme based on the feedback from the environmental state.

These results prove that the proposed framework is capable of sustaining enhanced environmental performance relative to the uncontrolled baseline scenario. The model demonstrated higher responsiveness to disturbance events and quicker convergence towards the desired states due to the stabilizing properties of the robust control policy.

Performance of the robust optimal control strategy

To compare the performance of the proposed framework, two models have been simulated - the uncontrolled system and the classical optimal controller. The robust Hamilton-Jacobi-based controller outperformed both alternatives in terms of the cumulative cost of environmental performance. In addition, it ensured the lowest maximum deviation of the environmental state from the desired operating region.

The classical optimal controller performed well under baseline conditions, but its performance deteriorated as the uncertainty level increased. The robust controller had stable performance due to explicit consideration of uncertainty in the control problem formulation. Despite the requirement of extra control effort by the robust controller relative to the classical one, this led to significant enhancement of environmental performance and stability.

Table 1: Comparative performance of different environmental control strategies

Control Strategy	Cumulative Environmental Cost	State Deviation	Control Effort	Stability Index	Overall Performance
Uncontrolled System	152.8	0.84	0.00	0.62	Low
Conventional Optimal Control	98.6	0.46	0.58	0.81	Moderate
Proposed Robust Optimal Control	76.3	0.29	0.64	0.93	High

Table 1 shows that the proposed robust control strategy had the minimum total cost of the environment of 76.3 and the minimum state deviation of 0.29. In comparison to the uncontrolled system, the proposed strategy improved the stability index from 0.62 to 0.93. These results indicate that the robust control strategy was able to efficiently balance environmental and control efforts.

Effect of environmental uncertainty on system performance

The effects of increased environmental uncertainty were evaluated through simulation of the proposed control strategy under low, medium, and high uncertainty levels. It is observed that as the uncertainty level increased, both the uncontrolled and conventional control strategies showed a greater deviation from the desired environmental state. The robust control model exhibited a smaller performance drop at all the levels of uncertainty considered.

In particular, at high uncertainty, the robust control approach had a lower environmental cost and better stability compared to the conventional approach. This finding indicates that the proposed Hamilton–Jacobi-based robust optimization approach was able to incorporate environmental uncertainty in the control decision-making process.

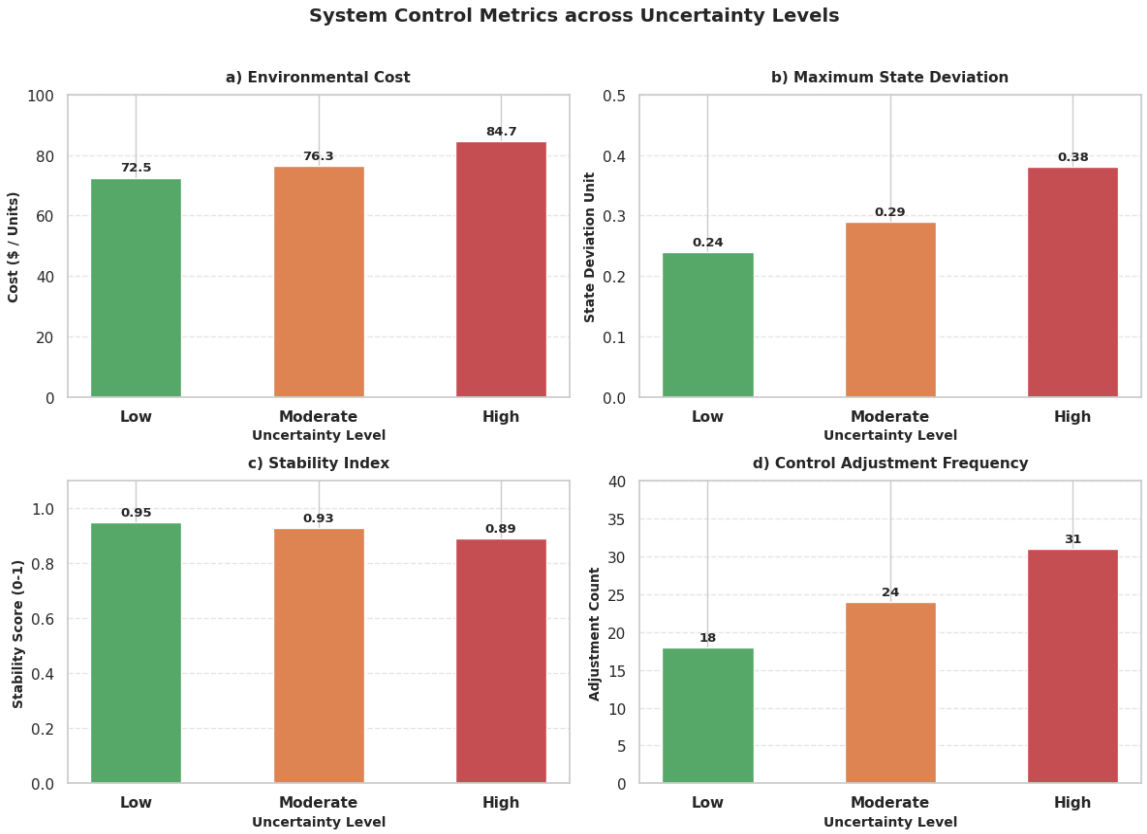


Figure 1: System control metrics across different environmental uncertainty levels: (a) environmental cost, (b) maximum state deviation, (c) stability index, and (d) control adjustment frequency.

Figure 1 gives the variations in various control parameters of the system within low, moderate, and high environmental uncertainty scenarios. The environmental cost, for example, rises from 72.5 at the low uncertainty level to 76.3 at the moderate uncertainty level and 84.7 at the high uncertainty level, as depicted in figure 1(a), suggesting that higher uncertainty increases the cost of managing the environment. Figure 1(b) similarly illustrates an increase in the maximum state deviation from 0.24 to 0.29 and 0.38, respectively, implying that higher uncertainty leads to increased deviations from the desired system state. Conversely, figure 1(c) shows that the stability index is gradually decreasing from 0.95 at a low uncertainty level to 0.93 at a moderate uncertainty level and 0.89 at a high uncertainty level, albeit remaining relatively high. Lastly, as shown in figure 1(d), the frequency of control adjustment increases greatly from 18 to 24 and 31, respectively, as uncertainty rises. Generally, increasing uncertainty implies higher control requirements and environmental costs and lower system stability.

Convergence of the hamilton–jacobi–bellman solution

Convergence of the numerical solution of the Hamilton-Jacobi-Bellman equation showed stability throughout the iterations. The differences in values of the value functions became smaller and smaller until the convergence criterion was achieved. The value function obtained from the process was used to obtain the optimal state-dependent control policy.

These results show that the computational method was stable enough for the environmental scenarios tested. It also proved that the computational process was able to achieve an optimal solution consistently without any oscillations of the value function.

Derivation and adaptation of the optimal control policy

The numerically generated control strategy was dependent on the environmental state and changed based on it. Increased levels of control effort were exerted when the environmental state diverged significantly from the optimal operation zone. With an improvement in the environmental state, the control effort reduced.

This is significant in environmental management as increased efforts will be wasteful while low efforts will not provide sufficient control. The recommended control strategy provided the required balance between protecting the environment and putting in management effort.

Stability of the controlled environmental system

According to the analysis of stability, it can be noted that the proposed control scheme successfully helped stabilize the states affected by the disturbance. In the case of normal conditions, the controlled system quickly converged to the required operating environment. In cases of moderate and strong disturbances, there was a deviation from the required states; however, the system became stable again once the disturbance had been addressed.

This indicates that the proposed framework offers more resilience than the uncontrolled system. In this process, the role of the feedback control scheme was very important since it kept updating the system with the information about the environment.

Scenario-based performance analysis

The robustness of the designed model was then tested under baseline, adverse, and high uncertainty situations. In the baseline situation, the robust controller kept the environmental system near the desired state using relatively low amounts of control. In the adverse situation, there was an increase in the amount of control used to counteract the larger disturbance.

The highest cost of environmental management was experienced in the high uncertainty situation. However, the robust controller exhibited higher stability than the traditional controller.

Sensitivity analysis

A sensitivity analysis was carried out to analyze the impact of varying values of the environmental parameters on the output of the model. The findings revealed that the impact of disturbance size and uncertainty was the highest on the environmental costs and the control intensity. Higher disturbance levels called for more control actions.

However, despite such differences, the general control mechanism stayed the same within the analyzed range of parameters. Such an outcome proves the robustness of the framework under discussion since moderate variations of the model parameters do not affect the efficiency of the resulting control policy.

5. Discussion

The results show that the proposed robust optimal control approach based on Hamilton-Jacobi-Bellman Analysis improved the performance of the environmental system despite the uncertainties in operation. The controller yielded the minimum environmental cost of 76.3, which is lower than 98.6 achieved by the conventional optimal control method and 152.8 of the uncontrolled case. Moreover, it minimized the deviation of the state to 0.29 while increasing the stability index to 0.93 compared to 0.46 and 0.81 obtained by the conventional method. When there was higher uncertainty, the stability index remained at 0.89 even if the environmental cost became higher at 84.7. This implies that the proposed model was able to maintain acceptable performance of the environmental system even when the level of environmental disturbance became high. The Hamilton-Jacobi-Bellman technique helps the controller to make continuous state-dependent decisions, whereas the feedback policy helps to make the control policy adaptively responsive to changes in the environment. In addition, the higher frequency of control adjustments that goes from 18 for low uncertainties to 31 for high uncertainties shows the adaptive nature of the proposed model. The significance of these findings lies in the fact that natural environmental systems hardly work in stable or known conditions. This means that the proposed framework can serve as a possible tool to manage dynamic systems under climatic variations, pollution impacts, availability of resources, and other uncertainties. Through reduction of environmental costs and maintenance of stability of the systems under consideration, the proposed model might prove effective in practice. It is worth noting, though, that the findings obtained were made using numerical experiments. Thus, the results cannot confirm the performance of the model in the real world for all environmental systems. The computational complexity is bound to be greater in the case of many state variables. Future research will require validation of the developed framework based on actual data obtained from environmental monitoring, along with comparing it against other advanced control strategies. The study can also explore high-dimensional approximations, Hamilton-Jacobi solvers with the help of machine learning, stochastic modeling of uncertainties, and multi-objective formulations.

6. Conclusion

The research was focused on the problem of controlling environmental systems that are dynamic, nonlinear, and subjected to uncertainty, disturbances, and changing working conditions. Traditional control methods have been successful under certain predetermined conditions; however, they tend to fail in the case of changing environmental factors. In order to overcome the mentioned drawback, the concept of robust optimal control methodology relying on the HJB approach was proposed. The suggested methodology included environmental system dynamics, description of uncertainty, optimal control modeling, numerical solution of the HJB equation, derivation of a state-dependent control law, and performance evaluation using feedback. The findings showed that the robust optimal control strategy worked better than the uncontrolled and optimal control methods. In the case of the suggested model, there was a total environmental cost of 76.3, while the conventional control showed a value of 98.6, and an uncontrolled system had a cost of 152.8. This means that there was a 49% decrease in relation to the uncontrolled system. Moreover, the state deviation was decreased to 0.29, and the stability index was raised to 0.93, while for the conventional method these values were 0.46 and 0.81, respectively. Under conditions of high uncertainty, the stability index was 0.89. Thus, the main lesson from this paper is that explicit inclusion of the uncertainty factor in the Hamilton–Jacobi optimization can make environmental control policies adaptive and more resistant to changes. Additionally, with a feedback function, control adjustments can be made depending on the system's current state, which increased from 18 under low uncertainty to 31 under high uncertainty. In summary, the suggested framework offers a systematic way to minimize the cost associated with environmental issues, prevent unwanted state deviations, and achieve stability in environmental systems subject to uncertainties. While validation of the results through real-world environmental data is necessary, the research results suggest the promising value of robust HJB-optimal control as a useful decision-making framework for environmental management.

Conflict of interest

The authors declare no conflict of interest.

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References

1. Ganai, M., Gao, S., & Herbert, S. L. (2024). Hamilton-jacobi reachability in reinforcement learning: A survey. *IEEE Open Journal of Control Systems*, *3*, 310-324. https://doi.org/10.1109/OJCSYS.2024.3449138

2. Guo, Q. (2022). Optimal robust control of electro-hydraulic system based on Hamilton–Jacobi–Bellman solution with backstepping iteration. *IEEE Transactions on Control Systems Technology*, *31*(1), 459-466. https://doi.org/10.1109/TCST.2022.3174772

3. Han, B., & Ying Wong, H. (2022). Robust control in a rough environment. *Quantitative Finance*, *22*(3), 481-500. https://doi.org/10.1080/14697688.2021.1965193

4. Han, H., Zhang, J., Hou, Y., & Qiao, J. (2023). Double-closed-loop robust optimal control for uncertain nonlinear systems. *IEEE Transactions on Cybernetics*, *54*(4), 2332-2344. https://doi.org/10.1109/TCYB.2023.3266391

5. Hu, H., Zhang, X., Lyu, X., & Chen, M. (2025, May). Learning robust policies via interpretable Hamilton-Jacobi reachability-guided disturbances. In *2025 IEEE International Conference on Robotics and Automation (ICRA)* (pp. 2304-2310). IEEE. https://doi.org/10.1109/ICRA55743.2025.11127287

6. Hu, Z., Lian, K., Long, X., Lin, S., & Song, X. (2025). Adaptive robust control method for unmanned mining electric shovel excavation systems based on the Hamilton–Jacobi inequality. *Frontiers of Mechanical Engineering*, *20*(4), 30. https://doi.org/10.1007/s11465-025-0846-2

7. Krishnan, M., & Patel, A. (2023). Circular economy models for plastic waste management in urban slums. *International Journal of SDG's Prospects and Breakthroughs*, 1-3.

8. Krolicki, A., Sutavani, S., & Vaidya, U. (2022, June). Koopman-based policy iteration for robust optimal control. In *2022 American Control Conference (ACC)* (pp. 1317-1322). IEEE. https://doi.org/10.23919/ACC53348.2022.9867541

9. Moon, J. (2022). Stochastic optimal control with random coefficients and associated stochastic Hamilton–Jacobi–Bellman equations. *Advances in Continuous and Discrete Models*, *2022*(1), 3. https://doi.org/10.1186/s13662-021-03674-5

10. Seo, H., Lee, D., Son, C. Y., Jang, I., Tomlin, C. J., & Kim, H. J. (2022). Real-time robust receding horizon planning using Hamilton–Jacobi reachability analysis. *IEEE Transactions on Robotics*, *39*(1), 90-109. https://doi.org/10.1109/TRO.2022.3187291

11. Sampath Emani. (2026). Supercapacitor-Assisted Voltage Regulation in Solar Power Systems under Transient Conditions. *Transactions on Power Electronics and Renewable Energy Systems*, 10-19.

12. Sherjonov, S., Nematov, O., Razzokova, G., Otabek, G., Yakubjonova, M., Karshiboyeva, D., ... & Abdullayeva, D. (2025). Economic analysis of green buildings certification systems in emerging economies. *Archives for Technical Sciences*, *2*(33), 463-471. https://doi.org/10.70102/afts.2025.1833.463

13. Ugli, O. M. B., & Ugli, S. I. B. (2026). Stochastic Programming Models for Production Planning Under Supply Chain Disruptions. *International Academic Journal of Science and Engineering, 13*(1), 10-20. https://doi.org/10.71086/IAJSE/V13I1/IAJSE1302

14. Wabersich, K. P., Taylor, A. J., Choi, J. J., Sreenath, K., Tomlin, C. J., Ames, A. D., & Zeilinger, M. N. (2023). Data-driven safety filters: Hamilton-jacobi reachability, control barrier functions, and predictive methods for uncertain systems. *IEEE Control Systems Magazine*, *43*(5), 137-177. https://doi.org/10.1109/MCS.2023.3291885

15. Yang, C. M., & Bhounsule, P. A. (2024). Robust control using control lyapunov function and hamilton-jacobi reachability. *IFAC-PapersOnLine*, *58*(28), 264-269. https://doi.org/10.1016/j.ifacol.2025.01.005

16. Zhao, R., Cao, X., & Wang, X. (2024, December). Study on the Supply and Demand Relationship and Enhancement of Cooling Benefits of Urban Green Spaces-A Case Study of Haidian District of Beijing Municipality. In *E3S Web of Conferences* (Vol. 490, p. 03006). EDP Sciences. https://doi.org/10.1051/e3sconf/202449003006