Evolution Hemivariational Inequalities for Non-stationary Navier-Stokes Equations: Existence of Periodic Solutions by an Equilibrium Problem Approach

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The main goal of this paper is to study the existence of solutions for non-stationary Navier-Stokes equations with a subdifferential boundary condition described by a superpotential function which is locally Lipschitz. The approach adopted in this paper is based on recent developments in the theory of equilibrium problems.

1. Introduction

In this paper, we study the existence of periodic solutions for non-stationary Navier-Stokes problems modeled by evolution hemivariational inequalities. The system of Navier-Stokes equations considered in this paper describes the motion of a viscous incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with smooth boundary Γ of class C^2. The non-stationary Navier-Stokes equations are as the following:

$$\frac{\partial u_i}{\partial t} - \chi \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial h}{\partial x_i} = f_i, \quad i = 1, \cdots, d \text{ in } Q, \qquad (1)$$

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$$\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{in} \ \ Q, \qquad u(0) = u(T) \quad \text{in} \ \ \Omega. \tag{2}$$

Here, $u(t,x) = \{u_i(t,x)\}_{i=1}^{d}$ is the flow velocity vector of the fluid, χ is the kinematic viscosity of the fluid ($\chi = \frac{1}{Re}$ where Re is the Reynolds number), $h(t,x) = p(t,x) + \frac{1}{2}|u(t,x)|^2$ is the total head of the flow with $p(t,x)$ is the pressure, $f(t,x) = \{f_i(t,x)\}_{i=1}^{d}$ is the volume density of external forces, $t \in [0,T]$ represents time, and $Q =]0,T[\times\Omega$. For a fixed $t \in [0,T]$, $u(t)$ means the function $u(t,\cdot)$ defined on Ω.

We suppose that on Γ the tangential components of the velocity vector are known and without loss of generality we put them equal to zero (the non-slip condition):

$$u_\tau(t,x) = u(t,x) - u_N(t,x)n = 0, \tag{3}$$

where $n = \{n_i\}_{i=1}^{d}$ is the unit outward normal on the boundary Γ and $u_N(t,x) = u(t,x) \cdot n = \sum u_i(t,x)n_i$ denotes the normal component of the vector $u(t,x)$.

The subdifferential boundary-value problem for the system (1)–(2) means to find a solution of this system that satisfies condition (3) and the subdifferential boundary condition

$$h(t,x) \in \partial^c j(t,x,u_N(t,x)), \quad \text{on} \ \ ]0,T[\times\Gamma, \tag{4}$$

where $j \colon]0,T[\times\Gamma \times \mathbb{R} \to \mathbb{R}$ is a given superpotential function which is locally Lipschitz with respect to the third variable and $\partial^c j$ is the Clarke's subdifferential of the function $j(t,x,\cdot)$. The presence of the argument x in the function j is understood as that the boundary conditions (4) can be of different character on different parts of the boundary Γ. The consideration of Navier-Stokes equations with subdifferential boundary condition allows to study a wide class of problems whose statement is of interest from the physical viewpoint. In particular, fluid flow control problems and the flow problems for semipermeable walls and membranes can be studied by this approach. More precisely, the problem under consideration describes a model in which the boundary orifices in a tube is regulated to reduce the pressure of the fluid on the boundary when the normal velocity attains a certain value. Note that if the function $j(t,x,\cdot)$ is convex, then the subdifferential boundary condition (4) becomes a particular case of the classical boundary condition. In this context the problem studied in this paper will be reduced to a problem studied by Chebotarev [15, 16] and its evolution counterpart by Konovalova [29]. It is worth noting that the approach considered in [15, 16] and [29] consists in formulating the problem as a variational inequality involving a maximal monotone operator. In the case where the superpotential function j does not satisfy any convexity condition, i.e. the function $j(t,x,\cdot)$ is not convex, the problem (1)–(4) can not be formulated as an evolution variational inequality but as an evolution hemivariational inequality. The theory of hemivariational inequalities has been introduced and studied since the 1980s by Panagiotopoulos [32, 33, 34] as a generalization of variational inequalities with the intention to model many problems coming from mechanics and economics when the energy

functionals are non-convex. For motivation and mathematical results on hemi-variational inequalities we refer to the book by Naniewicz-Panagiotopoulos [35] and the references therein.

The study of Navier-Stokes problems modeled by hemivariational inequalities has been initiated in the stationary and non-stationary cases by Migorski-Ochal [30, 31]. The study developed in [30] is essentially based on a surjectivity result for pseudomonotone operators, while in [31] it is based on a Galerkin method for a regularized problem.

In this paper, we study the existence of solutions for the problem (1)–(4) by a new approach based on the theory of equilibrium problems. In the recent past, equilibrium problems have been among the most interesting and intensively studied classes of problems. They include fundamental mathematical problems, for instance, optimization problems, Nash equilibrium, the fixed point problem, variational inequalities, minimax inequalities and complementarity problems, see [3, 12, 13, 19, 25, 26, 28] and for a recent survey on equilibria, see [4] and the references therein. It is well known that an hemivariational inequality is no longer a variational inequality due to the absence of the convexity of the superpotential j in its expression. The equilibrium problem formulation seems to be a very powerful tool to study hemivariational inequalities, in the sense that the techniques for solving variational inequalities could be used to study hemivariational inequalities through an equilibrium formulation. In fact, equilibrium problems lead to a common formulation of variational and hemivariational inequalities.

The method developed in this paper consists in presenting the evolution hemivariational inequality corresponding to the problem (1)–(4) as a mixed equilibrium problem. This approach is new and differs from most of the existing technics developed in literature, it is based on recent results obtained in [11] on the existence of solutions for mixed equilibrium problems described by the sum of a maximal monotone bifunction in the sense of Blum-Oettli [5] and a bifunction which is pseudomonotone in the topological sense, i.e. in the sense of Brézis. The concept of maximal monotone bifunctions is an extension of maximal monotone operators which was introduced by Blum-Oettli [5]. It was further studied and developed by Hadjisavvas-Khatibzadeh [23]. The notion of pseudomonotonicity for bifunctions in topological sense has been considered by Gwinner [21, 22] and is motivated by the concept of a topological pseudomonotone operator in the sense of Brézis [7].

Our study is divided in two steps. In a first step, we introduce an auxiliary problem which will be used afterwards to generate a sequence of approximated solutions of the evolution hemivariational inequality corresponding to the system (1)–(4). In a second step, we show that the sequence generated by the auxiliary problem converges weakly to a solution of the problem studied in this paper.

The paper is organized as follows. In Section 2, we give some concepts and preliminary results that are needed for our development. In Section 3, we present the statement and the weak formulation of the problem studied in this paper.

Section 4 is devoted to a study of the existence of periodic solutions for the problem (1)–(4).

2.   Preliminaries

Let X be a reflexive Banach space. We denote by X^* the *dual space* of X and by $\langle u^*, u \rangle$ the *dual pairing* between $u^* \in X^*$ and $u \in X$. The norms of X and X^* are denoted by $\| \cdot \|$. We use the standard notation $u_n \to u$ to denote the *strong convergence* of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ to u and $u_n \rightharpoonup u$ to denote the *weak convergence* of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ to u with respect to the weak topology $\sigma(X, X^*)$ of X. For a given set $A \subset K$, we use $co(A)$ to denote the *convex hull* of A. Let $J \colon X \to 2^{X^*}$ be the *duality mapping* defined by

$$J(x) = \left\{ x^* \in X^* : \langle x^*, x \rangle = \|x^*\|^2 \text{ and } \|x^*\| = \|x\| \right\}.$$

By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for any $x \in X$. We assume that the space X has been renormed so that X and its dual space are locally uniformly convex. Therefore, the duality mapping J is single-valued, continuous, see [38]. For a mapping $T \colon X \to X^*$, we shall denote by $\mathcal{D}(T)$ the *domain* of T.

Let K be a nonempty closed and convex subset of X and $\Phi \colon K \times K \to \mathbb{R}$ be a real-valued bifunction. By equilibrium problem, for short (EP), we mean the problem of finding $\bar{u} \in K$ such that

(EP) $\qquad\qquad\qquad\qquad \Phi(\bar{u}, v) \geq 0 \quad$ for all $v \in K.$ $\qquad\qquad\qquad$ (5)

Definition 2.1. A mapping $T \colon \mathcal{D}(T) \longrightarrow X^*$ is said to be

(i)   *monotone* if for any $u, v \in \mathcal{D}(T)$, the inequality $\langle Tu - Tv, u - v \rangle \geq 0$ holds;

(ii) *maximal monotone* if and only if T is monotone and $\langle u^* - Tv, u - v \rangle \geq 0$ for all $v \in \mathcal{D}(T)$ implies $u \in \mathcal{D}(T)$ and $u^* = Tu$.

Definition 2.2. A mapping $T \colon X \to X^*$ is said to be

(i)   *pseudomonotone* in the sense of Brézis if for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $u_n \rightharpoonup u$ in X and $\limsup\limits_{n\to+\infty}\langle Tu_n, u_n - u \rangle \leq 0$, we have

$$\liminf_{n\to+\infty}\langle Tu_n, u_n - v \rangle \geq \langle Tu, u - v \rangle, \quad \text{for all } v \in X;$$

(ii) *demicontinuous* if $u_n \to u$ in X implies $Tu_n \rightharpoonup Tu$ in X^*;

(iii) hemicontinuous (respectively, upper hemicontinuous) if for all $u, v, w \in X$, the functional $t \mapsto \langle T(u + tv), w \rangle$ is continuous (respectively, upper semicontinuous) on [0,1].

Definition 2.3. [17, 30] An operator $N \colon X \to X^*$ is called a *Navier-Stokes type operator* if $Nu = Au + B[u]$, where

(1) $A \colon X \to X^*$ is a linear, continuous, symmetric operator such that $\langle Au, u \rangle \geq \alpha\|u\|^2$ for all $u \in X$ with $\alpha > 0$;

(2) $B[u] := B(u, u)$ with $B \colon X \times X \to X^*$ is a bilinear continuous operator satisfying the conditions:

(2a) $\langle B(u,v), v \rangle = 0$ for $u, v \in X$,

(2b)   the mapping $B[\cdot] \colon X \to X^*$ is weakly continuous.

Lemma 2.4. *The Navier-Stokes type operator is pseudomonotone in the sense of Brézis.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset X$ such that $u_n \rightharpoonup u$ for $\sigma(X, X^*)$ and $\limsup \langle Nu_n, u_n - u \rangle \leq 0$. We need to verify that $\liminf_{n \to +\infty} \langle Nu_n, u_n - v \rangle \geq \langle Nu, u - v \rangle$, for all $v \in X$. First, we can easily show that A is pseudomonotone since it is continuous and symmetric. On the other hand, by condition (2a) of Definition 2.3, we have

$$\langle Nu_n, u_n - u \rangle = \langle Au_n, u_n - u \rangle - \langle B[u_n], u \rangle. \tag{6}$$

Since $B[\cdot]$ is weakly continuous, it follows from (6) that $\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0$. Hence, by taking the pseudomonotonicity of A into account we deduce that

$$\liminf_{n \to +\infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle \quad \text{for all } v \in X. \tag{7}$$

Now, for $v \in X$ we have

$$\langle Nu_n, u_n - v \rangle = \langle Au_n, u_n - v \rangle + \langle B[u_n], u_n - v \rangle. \tag{8}$$

Since $\langle B[u_n], u_n - v \rangle = \langle B[u_n], u_n \rangle - \langle B[u_n], v \rangle = -\langle B[u_n], v \rangle$, it follows

$$\langle Nu_n, u_n - v \rangle = \langle Au_n, u_n - v \rangle - \langle B[u_n], v \rangle. \tag{9}$$

Hence, from relation (7) and the condition (2b) of Definition 2.3, we deduce that

$$\liminf_{n \to +\infty} \langle Nu_n, u_n - v \rangle \geq \langle Au, u - v \rangle - \langle B[u], v \rangle.$$

From $\langle B[u], u \rangle = 0$ and the previous inequality we obtain

$$\liminf_{n \to +\infty} \langle Nu_n, u_n - v \rangle \geq \langle Nu, u - v \rangle.$$

This completes the proof. $\square$

Now we recall some concepts mentioned above for real-valued bifunctions considered in the literature in recent years. Most of these notions were inspired by similar monotonicity/continuity concepts defined for operators acting from a topological vector space to its dual space, see for instance [5, 22, 27].

Definition 2.5. Let K be a nonempty closed convex subset of X. A real-valued bifunction $\Phi \colon K \times K \longrightarrow \mathbb{R}$ is said to be

(i)   *monotone* if $\Phi(u, v) + \Phi(v, u) \leq 0$, for all $u, v \in K$;

(ii)   *pseudomonotone in the sense of Brézis*, for short *B-pseudomonotone*, if for any sequence $\{u_n\}_{n \in \mathbb{N}} \subset K$ such that $u_n \rightharpoonup u$ in X and $\liminf_{n \to +\infty} \Phi(u_n, u) \geq 0$, we have $\limsup_{n \to +\infty} \Phi(u_n, v) \leq \Phi(u, v)$ for all $v \in K$;

(iii)   *hemicontinuous* (respectively, *upper hemicontinuous*) if for all $u, v, w \in K$, the functional $t \in [0,1] \mapsto \Phi(tu + (1-t)v, w)$ is continuous (respectively, upper semicontinuous) on [0,1].

Remark 2.6. (a) It is easy to see that if $T : X \to X^*$ is pseusomonotone in the sense of Brézis, then the bifunction Φ defined by $\Phi(u, v) = \langle Tu, v - u \rangle$ is B-pseudomonotone.

(b) If the bifunction Φ is upper semicontinuous with respect to the first argument for the weak topology $\sigma(X, X^*)$, then it is B-pseudomonotone.

(c) If $\Phi, \Psi : K \times K \to \mathbb{R}$ are two real-valued B-pseudomonotone bifunctions such that $\Phi(u, u) \leq 0$ and $\Psi(u, u) \leq 0$ for all $u \in K$, then $\Phi + \Psi$ is also B-pseudomonotone, see [13].   □

Now we recall the concept of maximal monotonicity for bifunctions. This concept has been introduced by Blum-Oettli [5] with the objective to give an extension of the notion of maximal monotonicity (known for operators) to bifunctions. Another notion of maximal monotone bifunctions has been considered recently by Hadjisavvas-Khatibzadeh [23]. For a comparison between these two notions and some related properties, we refer to [23] and the references therein.

Definition 2.7. [5] Let K be a nonempty closed convex subset of X and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction with $\Phi(u, u) = 0$ for all $u \in K$. Φ is said to be *maximal monotone* if and only if for every $u \in K$ and for every convex function $\psi : K \to \mathbb{R}$ with $\psi(u) = 0$, we have

$$\Phi(v, u) \leq \psi(v) \ \text{ for all } v \in K \ \Rightarrow \ 0 \leq \Phi(u, v) + \psi(v) \ \text{ for all } v \in K. \qquad □$$

For comments, examples and characterizations of maximal monotone bifunctions, we refer to [10, 11].

The following recent result on the existence of solutions for mixed equilibrium problems will be used in the proof of the main results.

Theorem 2.8. [11] *Let X be a reflexive Banach space, K be a nonempty closed convex subset of X and $\Phi, \Psi : K \times K \to \mathbb{R}$ be real-valued bifunctions such that $\Phi(u, u) = \Psi(u, u) = 0$ for all $u \in K$. Let $\lambda > 0$ and $J : X \longrightarrow X^*$ be the duality mapping. Suppose that*

(i)   *Φ is monotone and maximal monotone;*

(ii)   *Ψ is B-pseudomonotone;*

(iii)   *Φ is weakly lower semicontinuous with respect to the second argument;*

(iv)   *For each finite subset A of K and each v in K fixed, the function $u \in K \mapsto \Psi(u, v)$ is upper semicontinuous on $co(A)$;*

(v)   *Φ and Ψ are convex with respect to the second argument;*

(vi)   *(Coercivity) there exist a nonempty weakly compact subset W of K such that for each $\lambda > 0$ (small enough) there exists a weakly compact convex subset B_λ of K such that, for each $u \in K \setminus W$ there exists $v \in B_\lambda$ satisfying $\Psi(u, v) + \lambda \langle Ju, v - u \rangle < \Phi(v, u)$.*

Then, there exists $\overline{u} \in K$ such that $\Phi(\overline{u}, v) + \Psi(\overline{u}, v) \geq 0$ for all $v \in K$. Furthermore, the solution set is weakly compact.

Remark 2.9. In the case when X is a reflexive Banach space endowed with the weak topology $\sigma(X, X^*)$, then the coercivity assumption (vi) in Theorem 2.8 is satisfied if we assume that there exists $v_0 \in K$ such that

$$\frac{\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle}{\|u - v_0\|} \to -\infty \quad \text{uniformly in} \quad \lambda > 0 \tag{10}$$

when $\|u - v_0\| \to +\infty$. Indeed, let $r_1 > 0$ and set $B(v_0, r_1) := \{u \in X : \|u - v_0\| \leq r_1\}$. The subset $B(v_0, r_1)$ is convex and $\sigma(X, X^*)$-compact in X. Since $\Phi(v_0, \cdot)$ is lower semicontinuous and convex, it follows that there exists $\gamma_0 \in \mathbb{R}$ such that $\Phi(v_0, z) > \gamma_0$ for all $z \in B(v_0, r_1)$. Let $u \in K \setminus B(v_0, r_1)$ and set $z := (r_1/\|u - v_0\|) \, u + (1 - r_1/\|u - v_0\|) \, v_0$. Since $\Phi(v_0, \cdot)$ is convex, $z \in B(v_0, r_1)$ and $\Phi(v_0, v_0) = 0$, it follows that

$$\Phi(v_0, u) \geq (\gamma_0/r_1)\|u - v_0\|, \quad \text{for all } u \in K \setminus B(v_0, r_1). \tag{11}$$

Hence, for all $u \in K \setminus B(v_0, r_1)$, we have

$$[\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle] - \Phi(v_0, u) \leq$$

$$\leq [\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle] - (\gamma_0/r_1)\|u - v_0\|$$

$$\leq \|u - v_0\| \left[\frac{\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle}{\|u - v_0\|} - (\gamma_0/r_1) \right]. \tag{12}$$

Since $[\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle]/\|u - v_0\| \to -\infty$ when $\|u - v_0\| \to +\infty$ uniformly in $\lambda > 0$, it follows that there exists $r_2 > 0$ independent from λ such that for $u \in K$ satisfying $\|u - v_0\| > r_2$ one has

$$\frac{\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle}{\|u - v_0\|} - (\gamma_0/r_1) < 0. \tag{13}$$

Let us set $C := \{u \in X : \|u - v_0\| \leq r\}$ where $r = \max\{r_1, r_2\}$. Note that C is independent from $\lambda > 0$. From relations (12) and (13) we deduce that

$$\Psi(u, v_0) + \lambda \langle Ju, v_0 - u \rangle < \Phi(v_0, u), \quad \text{for all } u \in K \setminus C.$$

Hence assumption (vi) in Theorem 2.8 is satisfied. $\qquad\square$

We recall the definitions of the directional derivative and the Clarke's subdifferential for a locally Lipschitz function.

Definition 2.10. (i) Let $f : X \to \mathbb{R}$ be a locally Lipschitz function defined on a Banach space X. The *generalized directional derivative* of f at $x \in X$ in the direction $v \in X$, denoted by $f^0(x; v)$, is defined by

$$f^0(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t} .$$

(ii) The *Clarke's subdifferential* of f at x , denoted by $\partial^c f(x)$, is a subset of the dual space X^* given by

$$\partial^c f(x) = \{\xi \in X^* : f^0(x; v) \geq \langle \xi, v \rangle, \text{ for all } v \in X\}.$$

3. Problem statement and weak formulation

In this section, we introduce the evolution Navier-Stokes equation describing a class of hydrodynamic problems with subdifferential boundary conditions. Afterward, we give the formulation of this problem as an evolution hemivariational inequality.

Let Ω be a bounded connected domain of $\mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary Γ of class C^2. We consider the following system of evolution Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} - \chi \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial h}{\partial x_i} = f_i, \quad i = 1, \dots, d \text{ in } Q, \qquad (14)$$

$$\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{in } Q, \qquad u(0) = u(T) \quad \text{in } \Omega. \qquad (15)$$

This system describes the flow of a viscous incompressible fluid in the domain Ω. Here, $u(t, x) = \{u_i(x, t)\}_{i=1}^{d}$ is the flow velocity vector of the fluid, χ is the kinematic viscosity of the fluid ($\chi = \frac{1}{Re}$ where Re is the Reynolds number), $h(t, x) = p(t, x) + \frac{1}{2}|u(t, x)|^2$ is the total head of the flow with $p(t, x)$ is the pressure, $f(t, x) = \{f_i(t, x)\}_{i=1}^{d}$ is the volume density of external forces, $t \in]0, T[$ represents time and $Q =]0, T[\times \Omega$. The first relation in (15) represents the equation for law of mass conservation and it states that the fluid is incompressible, it can be written in the divergence form as the following $\text{div } u = \nabla \cdot u = 0$. The vector $\{\sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^{d}$ can be written in the symbolic formulation $(u \cdot \nabla)u$. Hence the system (14)–(15) of the Navier-Stokes equations becomes

$$u' - \nu \triangle u + (u \cdot \nabla)u + \nabla h = f \quad \text{in } Q, \qquad (16)$$
$$\text{div } u = 0 \text{ in } Q, \quad u(0) = u(T) \quad \text{in } \Omega. \qquad (17)$$

In order to give a variational formulation of (16)–(17), we will use the approach developed by Chebotarev [15, 16] and Konovalova [29]. It consists in the use of the standard Lamb formulation and by means of the following identities of vector analysis that can be found in Girault-Raviart [20, Chapter I]

$$(u \cdot \nabla)u = \text{rot } u \times u + \frac{1}{2}\nabla(u \cdot u), \quad -\triangle u = \text{rot rot } u - \nabla \text{ div } u,$$

and the first relation in (17), which is the incompressible fluid condition, the system (16)–(17) can be written

$$u' + \nu \text{ rot rot } u + \text{rot } u \times u + \nabla h = f \quad \text{in } Q, \qquad (18)$$
$$\text{div } u = 0 \text{ in } Q, \quad u(0) = u(T) \quad \text{in } \Omega. \qquad (19)$$

We suppose that on Γ the tangential components of the velocity vector are known and without loss of generality we put them equal to zero (the non-slip condition):

$$u_\tau(t,x) = u(t,x) - u_N(t,x)n = 0, \tag{20}$$

where $n = \{n_i\}_{i=1}^d$ is the unit outward normal on the boundary Γ and $u_N(t,x) = u(t,x) \cdot n = \sum u_i(t,x)n_i$ denotes the normal component of the vector $u(t,x)$.

The subdifferential boundary-value problem for the system (18)–(19) means to find a solution of this system that satisfies condition (20) and the subdifferential boundary conditions

$$h(t,x) \in \partial^c j(t,x,u_N(t,x)), \quad \text{on }]0,T[\times\Gamma, \tag{21}$$

where $\partial^c j$ is the Clarke's subdifferential with respect to the third argument of a given superpotential function $j\colon]0,T[\times\Gamma \times \mathbb{R} \to \mathbb{R}$ such that $j(t,x,\cdot)$ is locally Lipschitz. The presence of the argument x in the functions j is understood as that the boundary condition (21) can be of different character on different parts of the boundary Γ.

In order to give the weak formulation of the problem (18)–(19), we consider the following set

$$W = \{w \in C^\infty(\Omega;\mathbb{R}^d) : \text{div}w = 0 \text{ in } \Omega, \ w_\tau = 0 \text{ on } \Gamma\}.$$

We denote by V and H the *closure* of W in the norms of $W_2^1(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. Here, $W_2^1(\Omega;\mathbb{R}^d))$ stands for the usual Sobolev space. Then, $V \subset H \simeq H^* \subset V^*$ with all embeddings being continuous and compact. The inner products in the spaces V and H are taken as follows:

$$(u,v)_V = \int_\Omega \text{rot}\,u \cdot \text{rot}\,v \ dx, \quad (u,v)_H = \int_\Omega u \cdot v \ dx.$$

We denote by $i\colon V \to H$ the *embedding injection*, by $\gamma\colon W_2^1(\Omega;\mathbb{R}^d) \to L^2(\Gamma;\mathbb{R}^d)$ the *trace operator*. For simplicity, we omit the notation of the embedding i and we write $\gamma(v) = \gamma(iv)$ for $v \in V$.

For the time interval [0,T] and the time derivative involved in the expression of the problem, which is considered here in the sense of vector-valued distributions, we consider the spaces

$$\mathcal{V} = L^2(0,T;V), \quad \mathcal{H} = L^2(0,T;H) \quad \text{and} \quad \mathcal{W} = \{u \in \mathcal{V} : u' \in \mathcal{V}^*\}.$$

The pairing between $\mathcal{V}$ and $\mathcal{V}^*$ is denoted by $\langle\langle\cdot,\cdot\rangle\rangle$.

The space $\mathcal{W}$ is a reflexive Banach space with the norm $\|u\|_{\mathcal{W}} := \|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}^*}$, and moreover it is continuously embedded in $\mathcal{C}(0,T;H)$, the space of all continuous functions on $[0,T]$ with values in H, see [38, Chapter 23]. Furthermore, since V is compactly embedded in H, we deduce that $\mathcal{W}$ is compactly embedded in $\mathcal{H}$. This follows from the theorem of Lions-Aubin (see [9, page 56]). Moreover, from a property of the trace operator, we have that $\mathcal{W}$ is compactly embedded in $L^2(0,T;L^2(\Gamma;\mathbb{R}^d))$.

We consider the following operators $A \colon V \to V^*$ and $B \colon V \times V \to V^*$ defined for $u, v, z \in V$ by

$$\langle Au, v \rangle = \nu \int_{\Omega} \operatorname{rot} u \cdot \operatorname{rot} v \, dx, \quad \text{and} \tag{22}$$

$$\langle B(u, v), z \rangle = \int_{\Omega} (\operatorname{rot} u \times v) \cdot z \, dx, \quad B[u] = B(u, u). \tag{23}$$

For $f(t) \in L^2(\Omega)$ and $u \in V$, $\langle F(t), u \rangle = \int_{\Omega} f(t, x) u(x) \, dx$.

Multiplying the equation of motion (17) by $(v - u(t))$ and applying Green's formula, we get for $v \in V$

$$\langle u'(t), v - u(t) \rangle + \langle Au(t) + B[u(t)], v - u(t) \rangle +$$

$$+ \int_{\Gamma} h(t, x)(v_N(x) - u_N(t, x)) \, d\sigma(x) = \langle F(t), v - u(t) \rangle. \tag{24}$$

From the boundary condition (21), and by using the definition of the Clarke's subdifferential, we obtain

$$\int_{\Gamma} h(t, x)(v_N(x) - u_N(t, x)) \, d\sigma(x) \leq$$

$$\leq \int_{\Gamma} j^0(t, x, u_N(t, x); v_N(x) - u_N(t, x)) \, d\sigma(x). \tag{25}$$

As a consequence of the relations (24) and (25) we derive the following weak formulation of the problem (18)–(19):

$$\begin{cases} \text{Find } u \in \mathcal{W} \text{ such that for all } v \in V \text{ and a.e. } t \in]0, T[\\ \langle u'(t), v - u(t) \rangle + \langle Au(t) + B[u(t)], v - u(t) \rangle + \\ + \int_{\Gamma} j^0(t, x, u_N(t, x); v_N(x) - u_N(t, x)) \, d\sigma(x) \geq \langle F(t), v - u(t) \rangle, \\ u(0) = u(T). \end{cases} \tag{26}$$

Remark 3.1. By following a similar argument to the one used in [17] and [31] we can show that in some sense the hemivariational inequality (26) leads to the relations (18)–(19). Indeed, consider $u \in \mathcal{W}$ and let us take $v = u \pm w \in \mathcal{W}$ in (26) for an arbitrary $w \in V \cap C_0^\infty(\Omega)$. Since w is arbitrary and $j^0(x, t, u_N(t, x); 0) = 0$, we obtain

$$\langle u'(t) + Au(t) + B[u(t)], w \rangle = \langle F(t), w \rangle \quad \text{for a.e. } t \in]0, T[. \tag{27}$$

By using Proposition 1.1 in [37] and relation (27), we obtain for some scalar function h defined on Q

$$u'(t) + Au(t) + B[u(t)] + \nabla h(t) = f(t) \quad \text{for a.e. } t \in]0, T[, \tag{28}$$

which implies equation (18). For $v \in V$, multiplying the equation (28) by $(v - u)$ and integrating by part over Ω, we obtain

$$
\begin{aligned}
\langle u'(t), v - u(t)\rangle + \nu \int_\Omega \operatorname{rot} u \cdot \operatorname{rot}(v - u) \ dx + \int_\Omega (\operatorname{rot} u \times u) \cdot (v - u) \ dx \\
+ \int_\Gamma h(t, x) \ (v_N(x) - u_N(t, x)) \ d\sigma(x) = \int_\Omega f(t, x)(v(x) - u(t, x)) \ dx.
\end{aligned}
\tag{29}
$$

Hence, from (26) and (29), we get for every $v \in V$

$$
\int_\Gamma [j^0(t, x, u_N(t, x); v_N(x) - u_N(t, x)) - h(t, x) \ (v_N(x) - u_N(t, x))] d\sigma(x) \geq 0. \tag{30}
$$

It follows, by arguing as in [34, Proposition 3.3.1], that

$$
j^0(t, x, u_N(t, x); v_N(x) - u_N(t, x)) \geq h(t, x) \ (v_N(x) - u_N(t, x))
$$

on Γ for a.e. $t \in]0, T[$, from which we get condition (21).

We close this section by the following lemma which will be useful in the sequel.

Lemma 3.2. *The operator $N \colon V \to V^*$ defined for $u \in V$ by $Nu = Au + B[u]$ is a Navier-Stokes type operator.*

Proof. We can easily verify that condition (1) of Definition 2.3 is satisfied. Now, we verify condition (2) of Definition 2.3. For this purpose we consider the trilinear form $b \colon [W_2^1(\Omega; \mathbb{R}^d)]^3 \to \mathbb{R}$ defined for $u, v, w \in W_2^1(\Omega; \mathbb{R}^d)$ by

$$
b(u, v, w) = \int_\Omega (\operatorname{rot} u \times v) \cdot w \ dx.
$$

From Lemmas 1.1, 1.3 and 1.5 in [37, Chapter II], we have that b is continuous, $b(u, v, w) = -b(u, w, v)$, $b(u, v, v) = 0$ for all $u, v, w \in W_2^1(\Omega; \mathbb{R}^3)$ and if $\{u_k\}$ is a sequence weakly converging to u in V, then $b(u_k, u_k, w) \to b(u, u, w)$ for all $w \in V$. As a consequence, we obtain that the operator B given by relation (23) satisfies condition (2) of Definition 2.3. $\qquad \square$

4. Main results

In this section, we study the existence of weak solutions of the problem (18)–(21) by means of the evolution hemivariational inequality (26). Our method is new and differs from most of the methods used in literature, it is based on recent results obtained in [11] on the existence of solutions for mixed equilibrium problems associated to a maximal monotone bifunction and a pseudomonotone bifunction in the sense of Brézis. More precisely, our approach works as follows: in a first step we introduce an auxiliary problem which will be used to generate a sequence of approximated solutions of problem (26). In a second step, we show that the obtained sequence converges weakly to a solution of the evolution hemivariational inequality (26).

In what follows, we suppose that the subdifferential of the superpotential j in the boundary condition (21) is obtained by the "filling in the jumps" procedure initiated by Rauch [36]. Furthermore, for simplicity we suppose that j does not depend on $(t, x) \in]0, T[\times \Gamma$.

4.1. The auxiliary problem

We denote by $j \colon \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function defined as follows: let $\beta \in L^\infty(\mathbb{R})$ and consider the functions

$$\overline{\beta}_\mu(r) = \operatorname*{ess\,sup}_{|s-r| \le \mu} \beta(s) \ \text{ and } \ \underline{\beta}_\mu(r) = \operatorname*{ess\,inf}_{|s-r| \le \mu} \beta(s)$$

They are increasing and decreasing functions of μ, respectively. Therefore the limits for $\mu \to 0^+$ exist. We denote them respectively by $\overline{\beta}(r)$ and $\underline{\beta}(r)$. Consider the multivalued function $\widehat{\beta}$ defined by:

$$\widehat{\beta}(r) = [\underline{\beta}(r), \overline{\beta}(r)].$$

The multifunction $\widehat{\beta}$ results from β, roughly speaking, by " filling in the jumps" in the generally discontinuous graphs $[r, \beta(r)]$. It has been shown by Chang [14] that if $\beta(r \pm 0)$ exists for every $r \in \mathbb{R}$, then a locally Lipschitz function $j \colon \mathbb{R} \to \mathbb{R}$ can be determined such that

$$\widehat{\beta}(r) = \partial^c j(r),$$

where j is obtained from β by simple integration, i.e. $j(r) = \int_0^r \beta(s) ds$. In this context, we suppose that the function j in the subdifferential boundary condition (21) is of the following form

$$j(r) = \int_0^r \beta(s) ds, \tag{31}$$

where $\beta \in L^\infty(\mathbb{R})$. A similar construction can be made when considering a function $\beta \colon]0, T[\times \Gamma \times \mathbb{R} \to \mathbb{R}$ such that β is measurable with respect to $(t, x) \in]0, T[\times \Gamma$ and $\beta(t, x, \cdot) \in L^\infty(\mathbb{R})$. To give a simple presentation of the approach developed in this section, we restrict the study to the case where the function j is taken as in relation (31). We suppose that β satisfies the following condition:

[H] $\beta \in L^\infty(\mathbb{R})$, $\beta(r \pm 0)$ exists for any $r \in \mathbb{R}$,   and there exists

$$r_0 \in \mathbb{R} \text{ such that } \operatorname*{ess\,sup}_{]-\infty, -r_0[} \beta(s) \le 0 \le \operatorname*{ess\,inf}_{]r_0, +\infty[} \beta(s). \tag{32}$$

Remark 4.1. Condition **[H]** means that the graph $[s, \beta(s)]$ ultimately increases.

Let $\varepsilon > 0$. We consider the mollification β_ε of β, i.e. for $p \in \mathcal{D}(]-1, 1[)$, the space of infinitely differentiable functions with compact support in $]-1, 1[$, such that $p \ge 0$ and $\int_{-\infty}^{+\infty} p(s) ds = 1$, let $\beta_\varepsilon = p_\varepsilon * \beta$ with $p_\varepsilon(s) = \frac{1}{\varepsilon} p(\frac{s}{\varepsilon})$, where $*$ denotes the convolution product. We consider the operator $B_\varepsilon \colon V \to V^*$ defined for $u, v \in V$ by

$$\langle B_\varepsilon u, v \rangle = \int_\Gamma \beta_\varepsilon(u_N(x))v_N(x)d\sigma(x).$$

For $\varepsilon > 0$, we consider the following auxiliary problem:

$$(\mathcal{P})_\varepsilon \begin{cases} \text{Find } u \in \mathcal{W} \text{ such that for all } v \in V, \text{ a.e. } t \in]0, T[\\ \langle u'(t), v - u(t) \rangle + \langle Nu(t), v - u(t) \rangle + \langle B_\varepsilon u(t), v - u(t) \rangle \geq \\ \geq \langle F(t), v - u(t) \rangle, \quad u(0) = u(T). \end{cases} \quad (33)$$

The generalized derivative $Lu = u'$ restricted to the subset

$$\mathcal{D}(L) = \{u \in \mathcal{V} : u' \in \mathcal{V}^* \text{ and } u(0) = u(T)\} = \{u \in \mathcal{W} : u(0) = u(T)\}$$

defines a linear operator $L \colon \mathcal{D}(L) \subset \mathcal{V} \to \mathcal{V}^*$ given by

$$\langle\langle Lu, v \rangle\rangle = \int_0^T \langle u'(t), v(t) \rangle \, dt \text{ for all } v \in \mathcal{V}.$$

Note that $\mathcal{D}(L)$ is a linear closed subspace of $\mathcal{W}$. $\mathcal{D}(L)$ equipped with the graph norm $\|u\|_L = \|u\|_\mathcal{V} + \|u'\|_{\mathcal{V}^*}$ is a reflexive Banach space (see [9]). Moreover, the operator $L \colon \mathcal{D}(L) \subset \mathcal{V} \to \mathcal{V}^*$ is densely defined, closed and maximal monotone, see [38, Proposition 32.10]. Now, we shall study the existence of solutions for the auxiliary problem by means of the following mixed equilibrium problem:

$$(\text{MEP})_\varepsilon \quad \begin{cases} \text{Find } u \in \mathcal{D}(L) \text{ such that} \\ \Phi(u, v) + \Psi_\varepsilon(u, v) \geq 0, \quad \text{for all } v \in \mathcal{D}(L), \end{cases} \quad (34)$$

where the bifunctions $\Phi, \Psi_\varepsilon \colon \mathcal{D}(L) \times \mathcal{D}(L) \to \mathbb{R}$ are defined by

$$\Phi(u, v) = \langle\langle Lu, v - u \rangle\rangle = \int_0^T \langle Lu(t), v - u(t) \rangle \, dt,$$

$$\Psi_\varepsilon(u, v) = \Theta(u, v) + H_\varepsilon(u, v) + G(u, v), \quad \text{with}$$

$$\Theta(u, v) = \langle\langle Nu, v - u \rangle\rangle, \ H_\varepsilon(u, v) = \langle\langle B_\varepsilon u, v - u \rangle\rangle \text{ and } G(u, v) = \langle\langle F, u - v \rangle\rangle.$$

When $\mathcal{D}(L)$ is equipped with the graph norm $\|u\|_L = \|u\|_\mathcal{V} + \|u'\|_{\mathcal{V}^*}$, the notion of pseudomonotonicity in the sense of Brézis for a bifunction $F \colon \mathcal{D}(L) \times \mathcal{D}(L) \to \mathbb{R}$ is traduced by the following: for any sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(L)$ satisfying $u_n \rightharpoonup u$ in $\mathcal{V}$, $Lu_n \rightharpoonup Lu$ in $\mathcal{V}^*$ and $\liminf_{n \to +\infty} F(u_n, u) \geq 0$, we have that $\limsup_{n \to +\infty} F(u_n, v) \leq F(u, v)$ for all $v \in \mathcal{D}(L)$.

We start by giving the following Hirano's type Lemma [24].

Lemma 4.2. *Let the operator A and B be given by* (22), (23) *and let* $N \colon V \to V^*$ *be the Navier-Stokes operator defined for* $u \in V$ *by* $Nu = Au + B[u]$. *Then, the bifunction* $\Theta \colon \mathcal{D}(L) \times \mathcal{D}(L) \to \mathbb{R}$ *defined for* $u, v \in \mathcal{D}(L)$ *by* $\Theta(u, v) = \langle\langle Nu, v - u \rangle\rangle$ *is B-pseudomonotone with respect to* $\mathcal{D}(L)$.

Proof. We know that the operator $N \colon V \to V^*$ is bounded since A is linear continuous and B is bilinear and continuous. Furthermore, we have that N is pseudomonotone in the sense of Brézis and coercive, i.e. there exists $\alpha > 0$ such that $\langle Nu, u \rangle \geq \alpha \|u\|^2$. For the rest of the proof, we follow a similar precedure to the one used in the proof of Lemma 3.4 in [11]. $\qquad\square$

In order to study the existence of solutions for the auxiliary problem $(\mathcal{P})_\varepsilon$, we need the following preliminary result.

Lemma 4.3. *Suppose $\beta \in L^\infty(\mathbb{R})$, then for every $r_1, r_2 \in \mathbb{R}$, we have*

$$|\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)| \leq \|\beta\|_{L^\infty(\mathbb{R})} \|p'_\varepsilon\|_{C([-\varepsilon,\varepsilon])} |r_1 - r_2| (|r_1 - r_2| + 2\varepsilon)$$

where p'_ε is the derivative of p_ε.

Proof. Suppose $r_1 < r_2$. Since $supp(p_\varepsilon) \subset [-\varepsilon, \varepsilon]$, we have

$$\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2) = \int_{r_1 - \varepsilon}^{r_2 + \varepsilon} (p_\varepsilon(r_1 - s) - p_\varepsilon(r_2 - s))\beta(s)ds.$$

Hence $\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)| \leq \|\beta\|_{L^\infty(\mathbb{R})} \int_{r_1 - \epsilon}^{r_2 + \epsilon} |p_\epsilon(r_1 - s) - p_\varepsilon(r_2 - s)|ds.$

By the mean value theorem, there exists $c \in]r_1 - \varepsilon, r_2 + \varepsilon[$ such that

$$p_\varepsilon(r_1 - s) - p_\varepsilon(r_2 - s) = (r_1 - r_2)p'_\varepsilon(c).$$

The case where $r_2 < r_1$ is similar. $\qquad\square$

Now, we show the following result on the existence of solutions for the mixed equilibrium problem $(MEP)_\varepsilon$.

Theorem 4.4. *Let the operators A and B be given by (22), (23) and let $N \colon V \to V^*$ be defined for $u \in V$ by $Nu = Au + B[u]$. Then, for each $\varepsilon > 0$ the mixed equilibrium problem $(MEP)_\varepsilon$ has at least one solution $u_\varepsilon \in \mathcal{D}(L)$. Furthermore, the set of approximate solutions $\{u_\varepsilon\}_{\varepsilon > 0}$ is bounded.*

Proof. We shall apply Theorem 2.8, where the problem $(MEP)_\varepsilon$ is considered on $\mathcal{D}(L)$. Here, $\mathcal{D}(L)$ is equipped with the graph norm $\|u\|_L = \|u\|_V + \|u'\|_{V^*}$. Therefore, we need to verify that all the assumptions of Theorem 2.8 are satisfied:

(1) The monotonicity of Φ follows from the one of L. According to [11, Lemma 2.3(b)] and [1, Proposition 3.8] the maximal monotonicity of Φ follows from the one of the operator L. Hence, assumption (i) of Theorem 2.8 is satisfied.

(2) We verify that the bifunction $\Psi_\varepsilon = \Theta + H_\varepsilon + G$ is B-pseudomonotone. According to Remark 2.6(c), we know that the sum of B-pseudomonotone bifunctions is B-pseudomonotone. Hence, we need to verify separately that the bifunctions Θ, H_ε and G are B-pseudomonotone. We know from Lemma 4.2 that the bifunction

Θ is B-pseudomontone, therefore we need only to verify that H_ε and G are B-pseudomonotone. To this aim, from Remark 2.6(b), it suffices to verify that the bifunctions H_ε and G are upper semicontinuous with respect to the first argument for the weak topology of $\mathcal{D}(L)$. Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(L)$ be such that $u_k \rightharpoonup u \in \mathcal{D}(L)$. Then $u_k \rightharpoonup u$ in $\mathcal{V}$ and $Lu_k \rightharpoonup Lu$ in $\mathcal{V}^*$. It follows that $u_k(t) \rightharpoonup u(t)$ in V. On the other hand, for $v \in \mathcal{D}(L)$ we have

$$H_\varepsilon(u_k, v) - H_\varepsilon(u, v) = \int_0^T \langle B_\varepsilon(u_k), v - u_k \rangle dt - \int_0^T \langle B_\varepsilon(u), v - u \rangle dt =$$

$$= \int_{\widetilde{Q}} [\beta_\varepsilon(u_{kN}(t,x)) - \beta_\varepsilon(u_N(t,x))][v_N(t,x) - u_N(t,x)] \, d\sigma(x)dt +$$

$$+ \int_{\widetilde{Q}} \beta_\varepsilon(u_{kN}(t,x))[u_{kN}(t,x) - u_N(t,x)] \, d\sigma(x)dt,$$

where $\widetilde{Q} = [0,T] \times \Gamma$. It follows,

$$|H_\varepsilon(u_k, v) - H_\varepsilon(u, v)| \leq \|\beta_\varepsilon(u_{kN}) - \beta_\varepsilon(u_N)\|_{L^2(\widetilde{Q})} \, \|v_N - u_N\|_{L^2(\widetilde{Q})} +$$

$$+ \|\beta_\varepsilon(u_{kN})\|_{L^2(\widetilde{Q})} \, \|u_{kN} - u_N\|_{L^2(\widetilde{Q})}. \tag{35}$$

Since $|\beta_\varepsilon(r)| \leq \|\beta\|_{L^\infty(\mathbb{R})}$ a.e. on $\mathbb{R}$, it follows that

$$\|\beta_\varepsilon(u_{kN}\|_{L^2(\widetilde{Q})} \leq (meas(\widetilde{Q}))^{\frac{1}{2}} \|\beta\|_{L^\infty(\mathbb{R})}.$$

On the other hand, since $\mathcal{W} \subset L^2(0,T; L^2(\Gamma))$ compactly, we have $\gamma u_k \to \gamma u$ in $L^2(0,T; L^2(\Gamma))$. Hence, $u_{kN} \to u_N$ in $L^2(0,T; L^2(\Gamma))$. Therefore, $u_{kN}(t,x) \to u_N(t,x)$ a.e $(t,x) \in \widetilde{Q}$. As $|\beta_\varepsilon(r)| \leq \|\beta\|_{L^\infty(\mathbb{R})}$ a.e. on $\mathbb{R}$, then by using Lemma 4.3 and the Lebesgue's theorem one has $\beta_\varepsilon(u_N^k) \to \beta_\varepsilon(u_N)$ in $L^2(\widetilde{Q})$. Hence, from relation (35) we obtain

$$\limsup_{k\to+\infty} H_\varepsilon(u_k, v) \leq H_\varepsilon(u, v).$$

Thus, the bifunction H_ε is B-pseudomonotone. We can easily verify that the bifunction G is upper semicontinuous with respect to the first argument, hence it is B-pseudomonotone. Consequently, the bifunction Ψ_ε is B-pseudomonotone on $\mathcal{D}(L)$ as a sum of B-pseudomonotone bifunctions.

(3) Conditions (iii) and (v) are easily obtained since for each fixed $u \in \mathcal{V}$, we have that $\Phi(u, \cdot)$ is linear and continuous and $\Psi_\varepsilon(u, \cdot)$ is linear.

(4) Condition (iv) is a direct consequence of the properties of the Navier-Stokes operator N (see Definition 2.3) and from the fact that H_ε is weakly upper semicontinuous with respect to the first argument on $\mathcal{D}(L)$.

(5) Now, we verify that the coercivity condition (vi) is satisfied. By taking account of Remark 2.9, we shall verify that there exists $v_0 \in \mathcal{V}$ such that

$$\frac{\Psi_\varepsilon(u, v_0) + \lambda\langle\langle Ju, v_0 - u \rangle\rangle}{\|u - v_0\|} \to -\infty \text{ uniformly in } \lambda > 0 \tag{36}$$

when $\|u - v_0\| \to +\infty$.

Indeed, let us take $v_0 = 0$. Then, we have

$$\Psi_\varepsilon(u, v_0) + \lambda\langle\langle Ju, v_0 - u\rangle\rangle = -\langle\langle Nu, u\rangle\rangle - \langle\langle B_\varepsilon u, u\rangle\rangle - \lambda\langle\langle Ju, u\rangle\rangle$$
$$= -\langle\langle Au, u\rangle\rangle - \langle\langle B[u], u\rangle\rangle - \langle\langle B_\varepsilon u, u\rangle\rangle - \lambda\|u\|_{\mathcal{V}}^2.$$

Since $\langle\langle Au, u\rangle\rangle \geq \alpha\|u\|_{\mathcal{V}}^2$ for some $\alpha > 0$ and $\langle\langle B[u], u\rangle\rangle = 0$, it follows that

$$\Psi_\varepsilon(u, v_0) + \lambda\langle\langle Ju, v_0 - u\rangle \leq -\alpha\|u\|_{\mathcal{V}}^2 - \langle\langle B_\varepsilon u, u\rangle\rangle - \lambda\|u\|_{\mathcal{V}}^2$$
$$\leq -\alpha\|u\|_{\mathcal{V}}^2 - \langle\langle B_\varepsilon u, u\rangle\rangle. \tag{37}$$

On the other hand we have

$$|\langle B_\varepsilon u(t), u(t)\rangle| = |\int_\Gamma \beta_\varepsilon(u_N(t, x)) u_N(t, x) d\sigma(x)|$$

$$\leq \int_\Gamma |\beta_\varepsilon(u_N(t, x))| \, |u_N(t, x)| d\sigma(x) \leq \|\beta_\varepsilon(u_N(t))\|_{L^2(\Gamma)} \|u_N(t)\|_{L^2(\Gamma)}$$

$$\leq (\text{meas}(\Gamma))^{\frac{1}{2}} \|\beta\|_{L^\infty(\mathbb{R})} c\|u(t)\|_V,$$

where c is a positive constant. Therefore, there exists a constant $c_1 > 0$ such that

$$|\langle\langle B_\varepsilon u, u\rangle\rangle| \leq c_1\|u\|_{\mathcal{V}}. \tag{38}$$

From the relations (37) and (38), we obtain uniformly in $\lambda > 0$

$$\frac{\Psi_\varepsilon(u, v_0) + \lambda\langle\langle Ju, v_0 - u\rangle\rangle}{\|u\|_{\mathcal{V}}} \leq -\alpha\|u\|_{\mathcal{V}} + c_1. \tag{39}$$

Therefore, there exists $v_0 \in \mathcal{D}(L)$ (here $v_0 = 0$) such that

$$\frac{\Psi_\varepsilon(u, v_0) + \lambda\langle\langle Ju, v_0 - u\rangle\rangle}{\|u - v_0\|_{\mathcal{V}}} \to -\infty \quad \text{uniformly in} \quad \lambda > 0 \tag{40}$$

when $\|u - v_0\|_{\mathcal{V}} \to +\infty$. Hence, all the assumptions of Theorem 2.8 are satisfied. It follows that for each $\varepsilon > 0$, the problem $(\text{MEP})_\varepsilon$ has a solution u_ε in $\mathcal{D}(L)$. In order to complete the proof, we need to verify that the set of approximate solutions $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded. To this aim, note that the right hand part of the inequality (39) does not depend on ε and the relation (40) is obtained uniformly in $\lambda > 0$ and $\varepsilon > 0$. Hence, in a similar way as in the proof of Remark 2.9, we can say that there exists a weakly compact C independent from λ and ε such that

$$\Psi_\varepsilon(u, v_0) + \lambda\langle Ju, v_0 - u\rangle < \Phi(v_0, u), \quad \text{for all } u \in K \setminus C,$$

where K is considered here as the hall set $\mathcal{D}(L)$. Furthermore, as in the proof of [11, Theorem 2.3], we can verify that the set of approximated solutions $\{u_\varepsilon\}_{\varepsilon>0}$ is contained in C, which completes the proof of the theorem. $\square$

From Theorem 4.4, we derive the following result on the existence of solutions for the auxiliary problem $(\mathcal{P})_\varepsilon$.

Theorem 4.5. *Under the assumptions of Theorem 4.4 and for each $\varepsilon > 0$, the auxiliary problem $(\mathcal{P})_\varepsilon$ has at least one solution.*

Proof. We have from Theorem 4.4 that for every $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{D}(L)$ such that

$$\langle\langle Lu_\varepsilon, v - u_\varepsilon\rangle\rangle + \langle\langle Nu_\varepsilon, v - u_\varepsilon\rangle\rangle + \langle\langle B_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle\rangle \geq \langle\langle F, v - u_\varepsilon\rangle\rangle$$

for all $v \in \mathcal{D}(L)$. Since $\mathcal{D}(L)$ is dense in $\mathcal{V}$, we can verify that

$$\langle\langle Lu_\varepsilon, v - u_\varepsilon\rangle\rangle + \langle\langle Nu_\varepsilon, v - u_\varepsilon\rangle\rangle + \langle\langle B_\varepsilon u_\varepsilon, v - u_\varepsilon\rangle\rangle \geq \langle\langle F, v - u_\varepsilon\rangle\rangle \tag{41}$$

for all $v \in \mathcal{V}$. Since $u_\varepsilon \in \mathcal{D}(L)$, it follows that $u_\varepsilon \in \mathcal{W}$ and $u_\varepsilon(0) = u_\varepsilon(T)$. From relation (41) we derive that for a.e. $t \in]0, T[$ and every $v \in V$

$$\langle Lu_\varepsilon(t), v - u_\varepsilon(t)\rangle + \langle Nu_\varepsilon(t), v - u_\varepsilon(t)\rangle + \langle B_\varepsilon u_\varepsilon(t), v - u_\varepsilon(t)\rangle \geq \langle F(t), v - u_\varepsilon(t)\rangle.$$

Hence, u_ε is a solution for the auxiliary problem $(\mathcal{P})_\varepsilon$. $\qquad\square$

4.2. An existence result

In this subsection, we study the existence of solutions for the evolution hemivariational inequality (26) by using the auxiliary problem $(\mathcal{P})_\varepsilon$ introduced and studied in the previous subsection.

We give first the following preliminary result that will be used in the sequel.

Lemma 4.6. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(L)$ such that $u_k \rightharpoonup u$ in $\mathcal{D}(L)$ and let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of real positive numbers converging to zero. Assume that the condition [**H**] is satisfied. Then, for each $v \in \mathcal{V}$ we have*

$$\limsup_{k\to\infty} \int_{\widetilde{Q}} \beta_{\varepsilon_k}(u_{kN}(t,x))(v_N(t,x) - u_{kN}(t,x)) \, d\sigma(x)dt \leq$$

$$\leq \int_{\widetilde{Q}} j^0(u_N(t,x); v_N(t,x) - u_N(t,x)) \, d\sigma(x)dt, \quad \text{where } \widetilde{Q} = [0,T] \times \Gamma.$$

Proof. For the proof of the lemma, we begin by showing the following result: Under hypothesis [**H**], there are constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\begin{aligned}
r\,\beta_\varepsilon(r) &\geq 0 & \text{for } |r| > \beta_1 \text{ and } \varepsilon \in]0,1[, \\
r\,\beta_\varepsilon(r) &\geq -\beta_1\beta_2 & \text{for } r \in \mathbb{R} \text{ and } \varepsilon \in]0,1[.
\end{aligned} \tag{42}$$

Indeed, we have

$$\beta_\varepsilon(r) = (p_\varepsilon * \beta)(r) = \int_{-\varepsilon}^{+\varepsilon} \beta(r-s)p_\varepsilon(s)ds \leq \operatorname*{esssup}_{|s|\leq\varepsilon} \beta(r-s) \tag{43}$$

and similarly

$$\underset{|s| \leq \varepsilon}{\mathrm{ess\,inf}}\ \beta(r-s) \leq \beta_\varepsilon(r). \tag{44}$$

In the above relations (43) and (44) let us set $x = r - s$ $(|x - r| \leq \varepsilon)$ and enlarge the bounds for $-\infty < x \leq \varepsilon + r$ and $r - \varepsilon \leq x < +\infty$, respectively. Then, for $r_1 \in \mathbb{R}$ arbitrary, the supremum and the infimum for $r \in]-\infty, -r_1[$ and $r \in]r_1, +\infty[$ respectively are formed and the bounds are enlarged by replacing $\varepsilon + r$ by $1 - r_1$ and $r - \varepsilon$ by $r_1 - 1$, here we consider $\varepsilon < 1$. Therefore, we obtain by means of assumption [**H**] that there exists $\tilde{r}_0 \in \mathbb{R}$ such that

$$\sup_{]-\infty, -\tilde{r}_0[} \beta_\varepsilon(r) \leq 0 \leq \inf_{]\tilde{r}_0, +\infty[} \beta_\varepsilon(r).$$

Hence, we can determine $\beta_1 > 0$ and $\beta_2 > 0$ such that $\beta_\varepsilon(r) \geq 0$ if $r > \beta_1$, $\beta_\varepsilon(r) \leq 0$ if $r < -\beta_1$ and $|\beta_\varepsilon(r)| \leq \beta_2$ if $|r| \leq \beta_1$ which entails relation (42).

Now, let $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(L)$ such that $u_k \rightharpoonup u$ in $\mathcal{D}(L)$ and let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a sequence of real positive numbers converging to zero. Hence, $u_k \rightharpoonup u$ in $\mathcal{V}$ and $Lu_k \rightharpoonup Lu$ in $\mathcal{V}^*$. From the properties of the evolution triple (see [38, Chapter 23]), we can write for any $k \in \mathbb{N}$, $u_k(t) = \int_0^t u_k'(s)ds$, where $u_k \colon [0, T] \to V$ is absolutely continuous. For each $z \in V \subset H$, we have

$$\langle u_k(t), z \rangle = \left\langle \int_0^t u_k'(s)ds, z \right\rangle = \int_0^t \langle u_k'(s), z \rangle\, ds.$$

Since $Lu_k \rightharpoonup Lu \in \mathcal{V}^*$, i.e., $u_k' \rightharpoonup u' \in \mathcal{V}^*$, we obtain

$$\lim_{k \to +\infty} \langle u_k(t), z \rangle = \lim_{k \to +\infty} \int_0^t \langle u_k'(s), z \rangle ds = \int_0^t \langle u'(s), z \rangle ds$$
$$= \left\langle \int_0^t u'(s)ds, z \right\rangle = \langle u(t), z \rangle,$$

hence, $u_k(t) \rightharpoonup u(t) \in V$ for all $t \in [0, T]$. Thanks to relation (42), we have

$$\beta_{\varepsilon_k}(u_{kN}(t, x))\left[v(t, x) - u_{kN}(t, x)\right] \leq \beta_1 \beta_2 + \|\beta\|_{L(\mathbb{R})^\infty}|v(t, x)| \text{ a.e. on } \widetilde{Q},$$

and by application of Fatou's Lemma we obtain

$$\limsup_{k \to \infty} \int_{\widetilde{Q}} \beta_{\varepsilon_k}(u_{kN}(t, x))\left[v(t, x) - u_{kN}(t, x)\right]\, d\sigma(x)dt \leq$$
$$\int_{\widetilde{Q}} \limsup_{k \to \infty} \beta_{\varepsilon_k} u_{kN}(t, x))\left[v(t, x) - u_{kN}(t, x)\right]\, d\sigma(x)dt. \tag{45}$$

In order to conclude, we need to show that for any $s \in \mathbb{R}$

$$\limsup_{k \to \infty} \beta_{\varepsilon_k}(u_{kN}(t, x))(s - u_{kN}(t, x)) \leq j^0(u_N(t, x); s - u_N(t, x)), \text{ a.e. on } \widetilde{Q}. \tag{46}$$

Since $u_k(t) \rightharpoonup u(t)$ in V for all $t \in]0, T[$ and the embedding $V \subset L^2(\Gamma)$ is compact, we have for a subsequence $\{u_{kN}(t)\}_{k \in \mathbb{N}}$, $u_{kN}(t, x) \to u_N(t, x)$ a.e. on Γ. For the proof of relation (46) it is sufficient to verify that for any $(t, x) \in \widetilde{Q}$ such that $u_{kN}(t, x) \to u_N(t, x)$ and for any convergent subsequence $\theta_m = \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x))(s - u_{k_m N}(t, x))$ the following inequality holds

$$\lim_{m \to +\infty} \theta_m \le j^0(u_N(t, x); s - u_N(t, x)). \tag{47}$$

We first show that $\liminf\limits_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x))$ and $\limsup\limits_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x))$ belong to $\partial^c j(u_N(t, x))$. As a consequence, we will have that for any $z \in \mathbb{R}$

$$\begin{aligned} \left[\limsup_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \right] z &\le j^0(u_N(t, x); z), \quad \text{and} \\ \left[\liminf_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \right] z &\le j^0(u_N(t, x); z). \end{aligned} \tag{48}$$

For the proof of the previous claim, let $(t, x) \in \widetilde{Q}$ with $u_{k_m}(t, x) \to u_N(t, x)$ and $\mu > 0$ be given. Since $\varepsilon_{k_m} \to 0$ and $u_{k_m N}(t, x) \to u_N(t, x)$, there exists $n_{\mu, t, x} > 0$ such that for all $m \ge n_{\mu, t, x}$ we have

$$|u_{k_m N}(t, x) - u_N(t, x)| \le \frac{\mu}{2} \quad \text{and} \quad 0 < \varepsilon_{k_m} \le \frac{\mu}{2}.$$

It follows,

$$\beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \le \operatorname*{ess\,sup}_{|r - u_{k_m N}(t, x)| \le \varepsilon_{k_m}} \beta(r) \le \operatorname*{ess\,sup}_{|r - u_N(t, x)| \le \mu} \beta(r) = \overline{\beta}_\mu(u_N(t, x)).$$

Therefore, $\limsup\limits_{k \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \le \overline{\beta}_\mu(u_N(t, x))$. Passing to the limit when $\mu \to 0+$, we derive $\limsup\limits_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \le \overline{\beta}(u_N(t, x))$.

By the same way, we obtain $\underline{\beta}(u_N(t, x)) \le \liminf\limits_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x))$.

Consequently,

$$\limsup_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \in [\underline{\beta}(u_N(t, x)), \overline{\beta}(u_N(t, x))] = \partial^c j(u_N(t, x)),$$

and

$$\liminf_{m \to \infty} \beta_{\varepsilon_{k_m}}(u_{k_m N}(t, x)) \in [\underline{\beta}(u_N(t, x)), \overline{\beta}(u_N(t, x))] = \partial^c j(u_N(t, x)).$$

Now, we are ready to prove relation (47). Let $s \in \mathbb{R}$, $(t, x) \in \widetilde{Q}$ such that $u_{k_m}(t, x) \to u_N(t, x)$ and θ_m as defined above.

(1) If $s = u_N(t, x)$, then $s - u_{k_m}(t, x) \to 0$. Since $|\beta_\varepsilon(r)| \le |\beta|_{L^\infty(\mathbb{R})}$ for a.e. $r \in \mathbb{R}$ and any $\varepsilon > 0$, it follows that $\theta_m \to 0$. Hence, relation (47) is satisfied.

(2) If $s > u_N(t,x)$, then by taking account of relation (48) we obtain

$$\lim_{m\to+\infty}\theta_m = \left(\lim_{m\to+\infty}\beta_{\varepsilon_{k_m}}\left(u_{k_m N}(t,x)\right)\right)(s-u_N(t,x)) \le$$

$$\le \left(\limsup_{m\to\infty}\beta_{\varepsilon_{k_m}}\left(u_{k_m N}(t,x)\right)\right)(s-u_N(t,x)) \le j^0(u_N(t,x);s-u_N(t,x)).$$

(3) The case $s < u_N(t,x)$ can be treated by a similar procedure to the one considered in the previous case, where instead of using the upper limit we use the lower limit. This completes the proof of the Lemma. $\square$

Now we give the following existence result of periodic solutions for the evolution hemivariational inequality (26) and from which we can derive the existence of weak solutions for the system of Navier-Stokes equations (18)–(19).

Theorem 4.7. *Let the operators A and B be given by (22),(23) and let $N: V \to V^*$ be the Navier-Stokes operator defined for $u \in V$ by $Nu = Au + B[u]$. Suppose that the assumption [H] is satisfied, then the evolution hemivariational inequality (26) has at least one solution.*

Proof. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\varepsilon_k \to 0$ when $k \to +\infty$. From Theorem 4.4 we know that for each $\varepsilon_k > 0$, there exists $u_k \in \mathcal{D}(L)$ such that u_k is a solution for the mixed equilibrium problem $(\text{MEP})_{\varepsilon_k}$. Moreover, the sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(L)$ is bounded, hence $u_k \rightharpoonup u \in \mathcal{D}(L)$. Since u_k is a solution of the mixed equilibrium problem $(\text{MEP})_{\varepsilon_k}$, we deduce that

$$\Phi(u_k,v)+\Theta(u_k,v)+\langle\langle B_{\varepsilon_k}u_k, v-u_k\rangle\rangle+\langle\langle F, u_k-v\rangle\rangle \ge 0 \quad \text{for all} \ \ v \in \mathcal{D}(L). \quad (49)$$

By taking $v = u$ in the previous relation and by using the monotonicity of Φ, we derive

$$\Theta(u_k,u) \ge \Phi(u,u_k) + \langle\langle B_{\varepsilon_k}u_k, u_k - u\rangle\rangle + \langle\langle F, u - u_k\rangle\rangle. \quad (50)$$

On the other hand, we have that

$$|\langle\langle B_{\varepsilon_k}u_k, u - u_k\rangle\rangle| = \left|\int_{\widetilde{Q}}\beta_{\varepsilon_k}(u_{kN}(t,x))[u_{kN}(t,x)-u_N(t,x)]d\sigma(x)dt\right|$$

$$\le \int_{\widetilde{Q}}|\beta_{\varepsilon_k}(u_{kN}(t,x))|\ |u_{kN}(t,x)-u_N(t,x)|d\sigma(x)dt$$

$$\le (\text{meas}(\widetilde{Q}))^{1/2}\ \|\beta\|_{L^\infty(\mathbb{R})}\ \|u_{kN}-u_N\|_{L^2(0,T;L^2(\Gamma))}.$$

Since $u_{kN} \to u_N$ in $L^2(0,T;L^2(\Gamma))$, it follows that

$$\lim_{k\to+\infty}|\langle\langle B_{\varepsilon_k}u_k, u - u_k\rangle\rangle| = 0.$$

From relation (50) and by taking account of the fact that $\Phi(u,\cdot)$ is weakly lower semicontinuous, we deduce that $\liminf_{k\to+\infty}\Theta(u_k,u) \ge 0$.

Therefore, since Θ is B-pseudomonotone with respect to $\mathcal{D}(L)$, we deduce that

$$\limsup_{k \to +\infty} \Theta(u_k, v) \leq \Phi(u, v) \quad \text{for all } v \in \mathcal{D}(L). \tag{51}$$

Now, by using the monotonicity of Φ, relation (49) becomes

$$\Theta(u_k, v) + \langle\langle B_{\varepsilon_k} u_k, v - u_k \rangle\rangle + \langle\langle F, u_k - v \rangle\rangle \geq \Phi(v, u_k) \quad \text{for all } v \in \mathcal{D}(L). \tag{52}$$

By considering the upper limit in the previous relation, we obtain, by taking account of relation (51) and Lemma 4.6, that for all $v \in \mathcal{D}(L)$ we have

$$\Theta(u, v) + \int_{\widetilde{Q}} j^0(u_N(t, x); v_N(t, x) - u_N(t, x)) \, d\sigma(x) dt + \langle\langle F, u - v \rangle\rangle \geq \Phi(v, u).$$

Since the bifunction Φ is maximal monotone, we deduce by using the properties of the Clarke's directional derivative (see [18, Proposition 2.1.1], that for all $v \in \mathcal{D}(L)$

$$\Phi(u, v) + \Theta(u, v) + \int_{\widetilde{Q}} j^0(u_N(t, x); v_N(t, x) - u_N(t, x)) \, d\sigma(x) dt + \langle\langle F, u - v \rangle\rangle \geq 0.$$

Hence, for all $v \in \mathcal{D}(L)$ we have

$$\langle\langle Lu, v - u \rangle\rangle + \langle\langle Nu, v - u \rangle\rangle +$$

$$+ \int_{\widetilde{Q}} j^0(u_N(t, x); v_N(t, x) - u_N(t, x)) \, d\sigma(x) dt \geq \langle\langle F, v - u \rangle\rangle. \tag{53}$$

Since $\mathcal{D}(L)$ is dense in $\mathcal{V}$, we can verify that the inequality (4.2) still true for any $v \in \mathcal{V}$. It follows that for a.e. $t \in]0, T[$ and all $v \in V$, we have

$$\langle u'(t), v - u(t) \rangle + \langle Nu(t), v - u(t) \rangle +$$

$$+ \int_{\Gamma} j^0(u_N(t, x); v_N(t, x) - u_N(t, x)) \, d\sigma(x) \geq \langle F(t), v - u(t) \rangle,$$

$u \in \mathcal{W}$ and $u(0) = u(T)$. Which completes the proof of the theorem. $\qquad \square$

Conclusion. In this paper, we study the existence of periodic solutions for non-stationary Navier-Stokes problems modeled by evolution hemivariational inequalities. Our approach is new, specially when dealing with periodic solutions, and differs from most of the existing techniques developed in the literature. It is based on recent results obtained in [11] on the existence of solutions for mixed equilibrium problems associated to a maximal monotone bifunction and a pseudomonotone bifunction in the sense of Brézis. This new approach is promising and can open new directions of investigations. Indeed, the equilibrium problem formulation seems to be a very powerful tool to study (evolution) hemivariational

inequalities, since the techniques and algorithms for solving variational inequalities could be used to study hemivariational inequalities through an equilibrium problem formulation.

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