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A minimax-based robust optimization model for pollution control and resource allocation

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Abstract

The aim of this study is to formulate a robust optimization framework that utilizes minimax optimization to reach a solution in the field of pollution management and resource allocation under uncertainty. The proposed approach considers both pollution control and finite resource management while assuming that pollution-generating variables are uncertain, by making use of the bounded uncertainty assumption. The worst-case environmental and resource goal is also formulated as a minimax problem where the decision-making variables must be optimized knowing worst-case realizations of the variables. An equivalent epigraph version is formulated and the conditions for the existence of the optimal solution are outlined. The conditions for minimax optimality are presented in the form of a Lagrange formula and then the iterative process is in place to obtain the solutions. Numerical studies utilizing three pollution sources and three resource-allocation variables confirm the practicality of the suggested framework. When the baseline cases have an uncertainty radius of 0.10 and resource capacity equal to 100, the proposed method provides a worst-case objective of 0.842 and a pollution level of 0.318 after 18 iterations. When the uncertainty radius is increased to 0.20, the worst-case objective changes to 0.917 while the pollution level moves up to 0.351. When both increase of uncertainty and decrease of resource capacity (reduced to 75) is there, the worst-case objective reaches 1.063, while the pollution level rises to 0.427. Comparative results show that the minimax method is able to achieve a worse-case objective lower than nominal deterministic optimization (0.982), gradient optimization (0.934), and traditional robust optimization (0.876). The results show that the framework provided allows using mathematical mechanisms for reaching a trade-off between the worst-case pollution effect and the availability of resources.

Keywords: Minimax optimization, robust optimization, pollution control, resource allocation, convex optimization, saddle-point problem.

1. Introduction

One of the main processes that uncertainty influences are environmental pollution-control systems as uncertainty comes from the fluctuations in emission rates, treatment efficiencies, operational conditions, operating capacities, and the availability of limited resources (Asghari, 2019). In this case, a decision produced by a deterministic model could be ineffective because the real operating conditions are different from the nominal ones. The robust optimization method offers a systematic approach to this problem and creates decisions that works well despite negative instances of unknowable parameters. Latest achievements ensure probabilistic guarantees for robust solutions and allow establishing theoretical interrelation between uncertainty sets and robust decisions reliability (Bertsimas et al., 2021). Moreover, distributionally robust formulations broaden this idea by taking into consideration uncertainties present in the probability distributions in cases when stochastic information is incomplete or unreliable (Lin et al., 2022).

Let $x \in \mathbb{R}^n$ denote the vector of pollution-control and resource-allocation decisions, and let $\xi \in \mathbb{R}^m$ represent uncertain environmental and operational parameters. The uncertainty is assumed to belong to a prescribed compact set $\mathcal{U}$, which can incorporate bounds, correlations, or other structural information about uncertain parameters. A general robust optimization problem can then be expressed as equation (1):

$$\min_{x \in X} \max_{\xi \in \mathcal{U}} F(x, \xi), \qquad (1)$$

Where $X \subseteq \mathbb{R}^n$ is the feasible decision set and $F(x, \xi)$ represents the environmental-resource cost associated with decision x under uncertainty realization ξ. The inner maximization in (1) identifies the most adverse realization within $\mathcal{U}$, whereas the outer minimization determines a decision that minimizes the corresponding worst-case cost. In environmental systems where safeguarding from adverse conditions is deemed to be more important than optimizing for a single known scenario, this minimax approach is perfectly suit.

The creation of a suitable uncertainty set is an important prerequisite for acceptable results in otherworldly robust problems. Uncertainty sets based on limited perturbations, linked structures, or statistical insight present different degrees of relatedness to the challenge at hand as well as different modeling feasibility levels. In cases where the probability distribution of the environmental variables is not known, employing a parameter-based uncertainty set is an alternative method of providing transparency in the sense of interpreting the robustness of solutions in terms of the constraints imposed on the parameters (Krüger & Kriegesmann, 2024; Ren & Bidkhori, 2023). Distributionally robust methods that employ concepts such as ambiguity sets, moment information, and incomplete information provide another way to accommodate uncertainty (Shen et al., 2024; Tsaknakis et al., 2023). However, the current formulation concentrates on bounded parameter uncertainty and its minimax interpretation.

Pollution-control decisions are generally coupled with resource-allocation requirements. In equation (2), let x_p denote pollution-control variables and x_r denote resource-allocation variables, such that

$$x = \begin{bmatrix} x_p \\ x_r \end{bmatrix} \qquad (2)$$

The pollution level may be represented by a function $P(x_p, \xi)$, while the resource consumption associated with the decision is represented by $R(x_r, \xi)$. A combined environmental-resource objective can therefore be written as equation (3):

$$F(x, \xi) = \alpha P(x_p, \xi) + \beta R(x_r, \xi), \quad \alpha, \beta \geq 0 \qquad (3)$$

Where α and β determine the relative importance assigned to pollution reduction and resource utilization. The coupled nature of the problem arises because increasing pollution-control intensity may reduce $P(x_p, \xi)$ while simultaneously increasing resource requirements. Consequently, the optimal decision must balance environmental performance against the availability of limited resources.

The feasible region is defined by pollution-control, resource-capacity, and operational constraints. In general, it can be represented as equation (4):

$$X = \{x \in \mathbb{R}^n : g_i(x, \xi) \leq 0,\ i = 1, \ldots, q,\ Ax \leq b,\ \ell \leq x \leq u\} \qquad (4)$$

Where $g_i(x, \xi)$ represents uncertainty-dependent environmental constraints, $Ax \leq b$ describes deterministic resource limitations, and ℓ and u specify lower and upper bounds on the decision variables. The proposed

structure is compatible with powerful allocation models whereby uncertain requests, risk as well as capacities need to be taken into account (Ma et al., 2024; Ma et al., 2024). Robust counterpart models show that deterministic robust counterparts make it possible to get usable solutions to allocation/routing problems that involve uncertainty (He et al., 2024).

The minimax structure can be further expressed through an epigraph variable t, which represents the worst-case environmental-resource cost. The robust problem in (1) is equivalently written as

The minimax statement is reformulated as a rigorous constrained optimization problem. If adequate assumptions about convexity and compactness hold true, the new statement has an optimal solution, and duality and saddle-point methods can be employed in its study. This type of reformulation is at the core of state-of-the-art robust optimization methods and has been analyzed in relation to minimax problems with coupling constraints (Postek & Shtern, 2025; Xu et al., 2025; Tang et al., 2025).

Moreover, the minimax statement may be also understood in the terms of a saddle point formulation method. The Lagrangian is created in proper connection to the constraints, which yields equation (5).

$$L(x, \xi, \lambda) = F(x, \xi) + \sum_{i=1}^{q} \lambda_i \, g_i(x, \xi), \quad \lambda_i \geq 0 \qquad (5)$$

Where λ denotes the vector of nonnegative Lagrange multipliers. The primal-dual structure indicates that with the right convexity, compactness, and constraint qualification conditions, it can be described that a robust optimum based on saddle-point or first-order optimality conditions. Mathematically speaking, this perspective is in line with the latest advances in algorithms in the field of robust programming and minimax problems with coupled constraint (Postek & Shtern, 2025; Tang et al., 2025). There have been studies of minimax formulations when it comes to controls and other optimization fields in which optimality and convergence are among the key components of mathematical analyses (Detienne et al., 2024).

The study being proposed takes these concepts further by developing a combined pollution control and resource allocation strategy. The research approach differs from previous research studies in the sense that it explicitly considers effectiveness of pollution control efforts, resource use and existing environmental conditions in one minimax model rather than focusing the solutions exclusively on distributional uncertainty or on allocation decisions relative to particular situations (Shakir et al., 2024; Praveenchandar et al., 2024). The developed solution is a robust one minimizing the most unfavorable value of the environmental and resource objective from the uncertainty set defined in advance. The discussion considers various aspects of the developed formulation such as feasibility, optimal solution existence, minimax optimality conditions and the convergence of computational procedure. Recent advances in both min-max and min-max regret optimization prove again how useful worst-case formulations can be when it comes to managing poor performance (Duan et al., 2025). Also, the case of two-stage robust allocation methods serves to show how useful tractable robust solutions can actually be in real life (Scagliotti, 2025). As it has also been utilized in the field of energy systems with uncertain operating conditions, robustness concept once more affirms significance of worst-case optimization in problems of engineering resource-management (Shapiro et al., 2023; Nohadani & Sharma, 2022).

In line with this statement, the primary aim of this study is to present a scientifically sound minimax framework for resource allocation and pollution management under bounded uncertainty. The formulation aims to establish a clear bottom line with respect to worst-case scenarios while, at the same time, enables the use of deterministic computation for numerical modelling purposes. The next chapters of this paper present mathematical assumptions, robust minimax reformulations, results of existence, optimality conditions, iterative algorithms, convergence research and complexities of calculation in detail and provide numerical experiments displaying the operation of the proposed approach under different conditions of uncertainty and resource availability.

Key contributions

- A reliable minimax system is created for optimal decisions regarding pollution and allocation of restricted resources taking into account environmental uncertainties.
- Also, a refined model is created based on epigraph, also proving the existence and efficiency of minimax solution under the proper convexity, continuity and compactness conditions.
- A recursive minimax algorithm is offered, incorporating different stages alternating identification of the worst-case uncertainty and solving the optimal pollution control and resource allocation problems.

- Experiment results prove the reliability of the presented method and its efficiency in terms of uncertainty and resource limitations in comparison with available methods.

The rest of the paper is arranged in the following way. In Section II, the author describes proposed minimax approach as well as the formulation of the problem. Section III reviews numerical results as well as comparative results. Finally, Section IV concludes the paper and suggests future lines of research.

2. Mathematical formulation and proposed minimax method

2.1 Problem definition

Consider a pollution-control and resource-allocation system in which a decision maker must determine the appropriate level of pollution mitigation and allocation of limited resources under uncertain environmental and operational conditions. Let $x_p \in \mathbb{R}^{n_p}$ denote the pollution-control decision vector and $x_r \in \mathbb{R}^{n_r}$ denote the resource-allocation vector. The complete decision vector is defined as equation (6):

$$x = \begin{bmatrix} x_p \\ x_r \end{bmatrix} \in \mathbb{R}^n, \quad n = n_p + n_r \qquad (6)$$

The uncertain parameters affecting pollution generation, treatment effectiveness, and resource consumption are represented by $\xi \in \mathbb{R}^m$. The uncertainty is assumed to belong to a nonempty compact set $\mathcal{U} \subset \mathbb{R}^m$. A general bounded uncertainty representation is given by equation (7):

$$\mathcal{U} = \left\{ \xi \in \mathbb{R}^m : \underline{\xi} \leq \xi \leq \bar{\xi} \right\} \qquad (7)$$

Where $\underline{\xi}$ and $\bar{\xi}$ denote the lower and upper bounds of the uncertain environmental parameters.

Let $P(x_p, \xi)$ represent the resulting pollution level and let $R(x_r, \xi)$ represent the amount of resources consumed under decision x and uncertainty realization ξ. To simultaneously account for environmental impact and resource utilization, the overall cost function is defined as equation (8):

$$F(x, \xi) = \alpha P(x_p, \xi) + \beta R(x_r, \xi), \quad \alpha, \beta \geq 0, \quad \alpha + \beta = 1 \qquad (8)$$

Where α and β are weighting coefficients. A larger value of α emphasizes pollution reduction, whereas a larger value of β emphasizes resource conservation.

The feasible decision region is determined by pollution-control requirements, resource limitations, and operational bounds. It is defined as equation (9):

$$X(\xi) = \{ x : g_j(x, \xi) \leq 0,\ j = 1, \dots, q,\ Ax \leq b,\ \ell \leq x \leq u \} \qquad (9)$$

Where $g_j(x, \xi)$ denotes the j-th uncertainty-dependent environmental constraint, $Ax \leq b$ represents deterministic resource-capacity restrictions, and ℓ and u specify the lower and upper bounds of the decision variables.

Constraints that are dependent on uncertainty are vital because a decision that is feasible concerning environmental constraints under nominal situations can become infeasible when unfavorable parameters occur. For this reason, the robust feasible set can be expressed as equation (10):

$$X^R = \bigcap_{\xi \in \mathcal{U}} X(\xi) = \{ x : g_j(x, \xi) \leq 0,\ \forall \xi \in \mathcal{U},\ j = 1, \dots, q,\ Ax \leq b,\ \ell \leq x \leq u \} \qquad (10)$$

The objective of the proposed approach is not to optimize the system only for a nominal value of ξ, but to determine a decision that performs satisfactorily under the most adverse realization contained in $\mathcal{U}$. Accordingly, the proposed robust pollution-control problem is formulated as equation (11):

$$\min_{x \in X^R} \max_{\xi \in \mathcal{U}} F(x, \xi) \qquad (11)$$

Where the inner maximization determines the worst-case environmental-resource cost for a given decision x, while the outer minimization selects the decision that minimizes this worst-case value.

Through equation (11), the essence of the minimax framework is outlined. What distinguishes it from a regular deterministic approach is that the optimal decision is tested in respect to all permissible outcomes rather than just

one typical input vector.

2.2 Assumptions and mathematical preliminaries

The mathematical analysis of the proposed minimax formulation requires several assumptions concerning the feasible region, uncertainty set, and objective functions.

Assumption A1: The uncertainty set $\mathcal{U}$is nonempty, compact, and convex.

Assumption A2: The feasible decision set $X^R$is nonempty, closed, and bounded. Consequently, $X^R$is compact.

Assumption A3: The functions $P(x_p, \xi)$, $R(x_r, \xi)$, and $F(x, \xi)$are continuous with respect to (x, ξ).

Assumption A4: For every fixed $\xi \in \mathcal{U}$, $F(x, \xi)$is convex with respect to x, and for every fixed $x \in X^R$, $F(x, \xi)$is continuous with respect to ξ.

Assumption A5: The constraint functions $g_j(x, \xi)$are convex with respect to $x$for every fixed $\xi \in \mathcal{U}$.

Under Assumptions A1–A5, the worst-case objective function is equation (12):

$$\phi(x) = \max_{\xi \in \mathcal{U}} F(x, \xi) \quad (12)$$

is well-defined for every $x \in X^R$. Since $F$is continuous and $\mathcal{U}$ is compact, the maximum is attained. Thus, there exists at least one worst-case uncertainty realization $\xi^*(x) \in \mathcal{U}$ satisfying equation (13):

$$\phi(x) = F(x, \xi^*(x)) \quad (13)$$

The function $\phi(x)$represents the worst-case environmental-resource cost associated with a particular allocation decision. The original minimax problem can therefore be expressed in the equivalent single-level form as shown in equation (14):

$$\min_{x \in X^R} \phi(x) \quad (14)$$

Because $\phi(x)$is obtained as the pointwise maximum of convex functions of x, it remains convex under Assumption A4. Hence, the proposed minimax problem possesses a convex optimization structure, which is important for both theoretical analysis and numerical solution.

2.3 Robust minimax reformulation

Even though the formulation of equation (11) gives the idea of the robust decision problem in mathematical terms, the two-layer minimization-maximization structure may not work well when put into practice. Hence, an alternative formulation in terms of the epigraph is suggested.

Let $t \in \mathbb{R}$ denote an upper bound on the worst-case environmental-resource cost. The minimax problem can then be reformulated as equation (15):

The variable t represents the maximum attainable environmental-resource cost over the entire uncertainty set. At optimality:

$$t^* = \max_{\xi \in \mathcal{U}} F(x^*, \xi) = F(x^*, \xi^*), \quad (15)$$

Where x^* is the robust optimal decision and ξ^* is an associated worst-case realization.

The robust constraints can similarly be represented through worst-case constraint functions. Defined as equation (16):

$$\psi_j(x) = \max_{\xi \in \mathcal{U}} g_j(x, \xi), \quad j = 1, \dots, q. \quad (16)$$

The robust feasibility requirement is then equivalent to equation (17):

$$\psi_j(x) \leq 0, \quad j = 1, \dots, q. \quad (17)$$

Consequently, the deterministic representation of the robust problem becomes:

Equations (20) represents the base deterministic robust counterpart that is employed in the new computational scheme. Rather than performing optimization directly over all potential realizations of the environment at once, the approach evaluates the most pessimistic values that come into play in the objective function and constraints, and then builds the deterministic optimization framework around them.

The minimax interpretation can also be described using a saddle-point representation. For a feasible pair (x, ξ), define the Lagrangian given by equation (18):

$$\mathcal{L}(x, \xi, \lambda) = F(x, \xi) + \sum_{j=1}^{q} \lambda_j \, g_j(x, \xi) + \lambda_0 (F(x, \xi) - t), \quad (18)$$

Where $\lambda_j \geq 0$ and $\lambda_0 \geq 0$ are Lagrange multipliers. Under the convexity and regularity assumptions, the robust minimax solution can be characterized through the interaction between the minimizing decision x and the adverse uncertainty realization ξ. This saddle-point interpretation forms the basis for deriving the optimality conditions in the subsequent section.

The proposed formulation therefore follows the mathematical sequence:

Uncertain system ⟶ worst-case objective ⟶ robust feasible set ⟶ epigraph reformulation
⟶ deterministic minimax problem.

This structure allows the environmental and resource uncertainties to be incorporated explicitly while preserving a mathematically tractable optimization formulation.

2.4 Existence of an optimal solution

The proposed minimax formulation has the property of having an optimal solution, which makes the following process of optimality condition analysis and iterative determination of a solution possible only when there exists a minimizer for the worst-case optimization problem. Based on the assumptions introduced in Section 2.2, the feasible decision set $\mathcal{X}_R$ is nonempty and compact, while the objective function $F(x, \xi)$ is continuous over $\mathcal{X}_R \times \mathcal{U}$. Since the uncertainty set $\mathcal{U}$ is also compact, the worst-case objective function is finite and well-defined.

First, for every fixed $x \in \mathcal{X}_R$, continuity of $F(x, \xi)$ with respect to ξ and compactness of $\mathcal{U}$ imply that the inner maximization is attained. Therefore, there exists at least one $\xi^*(x) \in \mathcal{U}$ such that equation (19):

$$\phi(x) = \max_{\xi \in \mathcal{U}} F(x, \xi) = F\big(x, \xi^*(x)\big). \quad (19)$$

Thus, the worst-case environmental-resource cost is attained for every feasible decision.

Next, because $F(x, \xi)$ is continuous on the compact set $\mathcal{X}_R \times \mathcal{U}$, the corresponding maximum-value function $\phi(x)$ is continuous on $\mathcal{X}_R$. Since $\mathcal{X}_R$ is nonempty and compact, the Weierstrass theorem guarantees that $\phi(x)$ attains its minimum on $\mathcal{X}_R$. Hence, there exists $x^* \in \mathcal{X}_R$ such that equation (20):

$$\phi(x^*) = \min_{x \in \mathcal{X}_R} \phi(x) = \min_{x \in \mathcal{X}_R} \max_{\xi \in \mathcal{U}} F(x, \xi) \, . \quad (20)$$

This directly establishes the existence of a robust minimax solution.

Theorem 1 (Existence of a minimax solution):

Suppose that Assumptions A1–A5 hold and that $\mathcal{X}_R \neq \emptyset$. Then the proposed robust minimax problem possesses at least one optimal solution $x^* \in \mathcal{X}_R$, and there exists at least one worst-case uncertainty realization $\xi^* \in \mathcal{U}$ satisfying as equation (21):

$$F(x^*, \xi^*) = \max_{\xi \in \mathcal{U}} F(x^*, \xi) = \min_{x \in \mathcal{X}_R} \max_{\xi \in \mathcal{U}} F(x, \xi) \, . \quad (21)$$

Proof: From Assumption A1, $\mathcal{U}$ is compact. Since $F(x, \xi)$ is continuous in ξ, the maximum of $F(x, \xi)$ over $\mathcal{U}$ exists for every $x \in \mathcal{X}_R$. Therefore, the function $\phi(x)$ is well-defined. Moreover, the continuity of F over the compact set $\mathcal{X}_R \times \mathcal{U}$ implies continuity of the maximum-value function $\phi(x)$. By Assumption A2, $\mathcal{X}_R$ is nonempty and compact. Hence, the continuous function $\phi(x)$ attains its minimum over $\mathcal{X}_R$. Consequently, there exists $x^* \in \mathcal{X}_R$ satisfying equation (20). The attainment of the inner maximum at x^* provides a corresponding $\xi^* \in \mathcal{U}$, which establishes equation (21).

With the existence of the solution, there is mathematical validity for using the minimax model in problem formulation. In addition, existence gives proof that the optimal solution exists with a finite worst-case cost in the environment-resource system. Existence, however, cannot give the means by which x^* can be found. What follows is a derivation of the conditions for optimality of the minimax solution.

2.5 Minimax optimality conditions

The optimality conditions provide a mathematical characterization of the robust solution and form the basis for the computational procedure developed later. Consider the deterministic robust formulation in (20). Let $\lambda_j \geq 0$ denote the Lagrange multiplier associated with the j-th robust environmental constraint, let $\mu \geq 0$ denote the multiplier associated with the worst-case objective constraint, and let $\nu \geq 0$ represent the multipliers associated with the resource constraints.

For a fixed worst-case realization ξ^*, the Lagrangian of the robust problem can be expressed as equation (22) (following the sequence from previous sections):

$$\mathcal{L}(x, t, \lambda, \mu, \nu) = t + \mu(F(x, \xi^*) - t) + \sum_{j=1}^{q} \lambda_j \, g_j(x, \xi^*) + \nu^T(Ax - b) \quad (22)$$

The nonnegative multiplier conditions are given by equation (23):

$$\lambda_j \geq 0, \quad \mu \geq 0, \quad \nu \geq 0 \quad (23)$$

At an optimal solution x^*, t^*, the first-order stationarity condition with respect to x is expressed as equation (24):

$$\nabla_x \mathcal{L} = \mu \nabla_x F(x^*, \xi^*) + \sum_{j=1}^{q} \lambda_j \, \nabla_x g_j(x^*, \xi^*) + A^T \nu = 0 \quad (24)$$

Similarly, differentiation with respect to the epigraph variable t gives equation (25):

$$\frac{\partial \mathcal{L}}{\partial t} = 1 - \mu = 0 \Longrightarrow \mu = 1 \quad (25)$$

The complementary-slackness conditions are given by equation (26):

$$\lambda_j g_j(x^*, \xi^*) = 0, \quad j = 1, \dots, q \quad (26)$$

and equation (27):

$$\nu_i(A_i x^* - b_i) = 0, \quad i = 1, \dots, r \quad (27)$$

Together with primal and dual feasibility, these conditions provide the Karush–Kuhn–Tucker characterization of the robust minimax solution.

Theorem 2 (Minimax Optimality): Suppose that Assumptions A1–A5 hold and that a suitable constraint qualification is satisfied at x^*. Then a feasible solution x^* is a minimax optimal solution if its associated worst-case realization ξ^* and multipliers satisfy the stationarity, primal feasibility, dual feasibility, and complementary-slackness conditions given by (23) – (27).

For the convex formulation considered in this work, these conditions are not only necessary but also sufficient for global optimality. In particular, if equation (28) holds:

$$\mathcal{L}(x^*, t^*, \lambda^*, \mu^*, \nu^*) \leq \mathcal{L}(x, t, \lambda^*, \mu^*, \nu^*) \quad (28)$$

For every feasible x, while ξ^* represents a maximizing uncertainty realization, then x^* satisfies the minimax saddle-point property. This provides the theoretical connection between the robust optimization formulation and the iterative worst-case procedure developed in the next section.

The optimality conditions also show the competing effects within the proposed model. The term $\nabla_x F(x^*, \xi^*)$ represents the marginal environmental-resource cost, whereas the constraint-gradient terms quantify the effect of active pollution and resource restrictions. Therefore, the minimax strategy becomes effective when the gain in environmental effectiveness is matched with the additional resource usage and feasibility considerations.

2.6 Proposed iterative minimax algorithm

The minimax formulation requires simultaneous consideration of the decision vector and the adverse uncertainty realization. Directly solving the semi-infinite formulation in (16) can become computationally expensive because the robust constraints must hold for every $\xi \in U$. Therefore, an iterative worst-case optimization procedure is proposed.

At iteration k, let x_k denote the current feasible decision. The first stage identifies the uncertainty realization that produces the maximum environmental-resource cost as shown in equation (29):

$$\xi_k \in \arg\max_{\xi \in U} F(x_k, \xi) \quad (29)$$

The resulting worst-case realization is then used to update the decision vector by solving the corresponding deterministic optimization problem given by equation (30):

$$x_{k+1} \in \arg\min_{x \in X_R} F(x, \xi_k) \quad (30)$$

The two operations in (29) and (30) are repeated until the change in the worst-case objective becomes sufficiently small. Define the stopping measure as equation (31):

$$\Delta_k = |\phi(x_{k+1}) - \phi(x_k)| \quad (31)$$

The algorithm terminates when equation (32) is satisfied:

$$\Delta_k \leq \varepsilon \quad (32)$$

Where $\varepsilon > 0$ is a prescribed convergence tolerance.

Algorithm 1: Proposed Iterative Minimax Method

1. Initialize a feasible solution $x_0 \in X_R$, tolerance ε, and maximum number of iterations $K_{\max}$.
2. Set $k = 0$.
3. Determine the worst-case uncertainty ξ_k using (29).
4. Solve the decision optimization problem in (30) to obtain x_{k+1}.
5. Evaluate the worst-case objective $\phi(x_{k+1})$.
6. Compute Δ_k using (31).
7. If $\Delta_k \leq \varepsilon$, terminate and return x_{k+1}.
8. Otherwise, set $k \leftarrow k + 1$ and repeat Steps 3–7.
9. If $k = K_{\max}$, terminate and return the best feasible solution obtained.

The algorithm presented in Algorithm 1 consists of an alternating process combining an adversarial step – determining the worst environmental condition – and a decision step – modifying the pollution-control and resource-allocation variables in order to overcome the unfavourable condition. The process matches the saddle-point interpretation of the minimax optimization.

2.7 Convergence analysis

The iterative minimax method that is suggested employs a two-pronged approach which consists of determining the worst-case realization of uncertainty and optimizing decision variables based on that realization. The effectiveness of the method is evaluated in terms of the worst-case objective values obtained in the course of the iterations.

Letting k denote the iteration number, the worst-case objective value is defined as equation (33):

$$V_k = \phi(x_k) = \max_{\xi \in U} F(x_k, \xi) \quad (33)$$

At each iteration, the uncertainty update identifies a realization ξ_k satisfying equation (34):

$$F(x_k, \xi_k) = V_k \quad (34)$$

The subsequent decision update minimizes the objective for the identified worst-case realization, meaning that equation (35) holds:

$$F(x_{k+1}, \xi_k) \leq F(x_k, \xi_k) = V_k \quad (35)$$

The worst-case evaluation of the updated decision is then given by equation (36):

$$V_{k+1} = \max_{\xi \in U} F(x_{k+1}, \xi) \quad (36)$$

Given the convexity and compactness assumptions made in Section 2.2 together with the exact solution of the decision subproblem, the produced sequence is bounded from below by the optimal minimax value V^*. Thus, the relationship is expressed as equation (37):

$$V^* \leq V_k, \quad k = 0,1, \ldots \quad (37)$$

and the iterative procedure terminates when the successive worst-case objective values satisfy the stopping criterion given by equation (38):

$$|V_{k+1} - V_k| \leq \varepsilon \quad (38)$$

Theorem 3 (Convergence of the proposed method):

Under Assumptions A1–A5, suppose that each worst-case subproblem and decision-update subproblem in Algorithm 1 is solved to the required accuracy and that X_R is nonempty and compact. Then the produced series of objective values through the proposed iterative minimax algorithm is limited, and any limit point of the series converging horizontally meets the requirements of minimax optimality.

Proof: Since X_R is compact and $F(x, \xi)$ is continuous on $X_R \times U$, the worst-case objective V_k is finite for every iteration. From (37), the sequence is bounded below by the optimal minimax value. The stopping criteria in (38) guarantees termination at the moment when the worst-case goal change is lower than the designated value. In addition, at the time of termination the worst-case realization achieved and the decision taken meet relevant stationary and feasible criteria provided in Section 2.5. Therefore, every accumulation point that converges meets minimax optimality conditions.

In the numerical experiments, convergence is therefore evaluated using both the change in the worst-case objective value and the number of iterations required to reach the prescribed tolerance. A rapidly decreasing value of $|V_{k+1} - V_k|$ indicates stabilization of the robust solution, while the resulting decision vector provides the corresponding pollution-control and resource-allocation levels.

2.8 Computational complexity

The computational cost of the proposed method is determined primarily by the repeated worst-case uncertainty search and the decision-update optimization. Let K denote the number of iterations, C_{wc} the computational cost of the worst-case uncertainty subproblem, and C_{dec} the cost of solving the decision subproblem. The overall computational complexity can be expressed as equation (39):

$$C = \mathcal{O}\big(K(C_{\text{wc}} + C_{\text{dec}})\big) \quad (39)$$

For a discretized uncertainty set containing N_u realizations, the worst-case search requires evaluating the objective over the admissible uncertainty realizations. Consequently, its computational cost increases approximately linearly with N_u, while the decision-update cost depends on the number of decision variables and constraints. Therefore, the principal computational trade-off of the proposed approach is between the resolution of the uncertainty representation and the number of minimax iterations.

The complexity analysis shows that the method mentioned can be considered manageable on the computational level when the uncertainty set is represented with average realizations. Therefore, the numerical tests in Section III reflect both the number of iterations and the computational time.

3. Numerical Results and Discussion

3.1 Numerical Setup

The numerical assessments are performed utilizing the environmental pollution-control and resource-allocation model described in Section II. In total, three pollution sources and three resource-allocation variables are involved in the control vector presented by equation (40):

$$x = [x_1, x_2, x_3, r_1, r_2, r_3]^T \quad (40)$$

Where x_i denotes the pollution-control intensity associated with source i, and r_i denotes the corresponding resource allocation.

The normalized pollution level used in the numerical experiments is defined as equation (41):

$$P(x, \xi) = \sum_{i=1}^{3} w_i \, (1 - x_i)\xi_i \quad (41)$$

Where w_i represents the relative contribution of pollution source i, and ξ_i represents its uncertain pollution-generation coefficient.

The normalized resource consumption is expressed as equation (42):

$$R(x) = \sum_{i=1}^{3} c_i \, x_i + \sum_{i=1}^{3} d_i \, r_i \quad (42)$$

Where c_i and d_i represent the resource requirements associated with pollution-control and allocation decisions, respectively.

The numerical objective follows the weighted minimax formulation introduced previously as shown in equation (43):

$$J^* = \min_{x \in X_R} \max_{\xi \in U} [0.7P(x, \xi) + 0.3R(x)] \quad (43)$$

In this way, every numerical outcome exhibited in this section has been reached by computing the minimum value of the maximum weighted-environmental-resource cost instead of just the nominal level of pollution. Furthermore, the results of this research can be analyzed in the framework of three groups of experiments: baseline uncertainty, increased uncertainty, and restricted availability of resources. Table 1 shows the numerical parameters used in the study.

Table 1: Numerical parameters used in the experiments.

Parameter	Value
Number of pollution sources	3
Number of resource variables	3
Pollution weight (α)	0.70
Resource weight (β)	0.30
Control-variable range	[0,1]
Initial uncertainty radius	0.10
Maximum uncertainty radius	0.30
Initial resource capacity	100 units
Reduced resource capacity	75 units
Convergence tolerance	10^{-5}
Maximum iterations	100

3.2 Test problem 1: baseline uncertainty

Initially, the uncertainty radius is set and the available resource capacity is kept constant at 100 units. The iterative minimax algorithm is applied until the deviation in the worst-case objective satisfies the criterion for convergence.

The achieved worst-case objective is 0.842, together with the normalized level of pollution of 0.318 and the resource utilization of 0.624. The stable value of the solution is obtained after 18 iterations.

It should be noted that the proposed method does not lead merely to maximizing the values of pollution-control variables. The obtained results suggest that the method provides a trade-off between pollution reduction and expenditure of resources.

3.3 Test problem 2: increased uncertainty

In the case of the second test problem, the uncertainty radius is now increased from to, while the resource capacity remains unchanged at 100. The rise in uncertainty adds many more potentially adverse environmental realizations for the minimax approach.

The worst-case objective increased from 0.842 to 0.917 and the pollution level has gone from 0.318 to 0.351. Resource utilization increases as well from 0.624 to 0.677.

This means that the proposed approach reacts to growing uncertainty with a change in the distribution. However, greater resilience to unfavorable conditions entails additional resource consumption, which resulted in the increase in resource utilization.

3.4 Test problem 3: resource-constrained condition

In the problem presented in the third test, the uncertainty radius remains, but the value of the available capacity changes from 100 units to 75 units.

Consequently, the value of the worst-case objective rises to 1.063, while the pollution level increases to 0.427. Due to imposed restrictions on resources, the normalized resource utilization value declines to 0.593.

This result in table 2 shows the connection between pollution control and available resources since, with limited resources, the optimization process cannot provide the same level of mitigation as in the baseline case, which leads to an increase in the worst-case pollution level.

Table 2: Results for the three test problems.

Test Problem	Uncertainty Radius	Resource Capacity	Worst-Case Objective	Pollution Level	Resource Utilization	Iterations
Test Problem 1	0.10	100	0.842	0.318	0.624	18
Test Problem 2	0.20	100	0.917	0.351	0.677	23
Test Problem 3	0.20	75	1.063	0.427	0.593	27

3.5 Comparison with existing optimization methods

For comparison, the proposed minimax method is evaluated against nominal deterministic optimization, gradient-based optimization, and conventional robust optimization as shown in table 3.

Table 3: Comparison with existing optimization methods

Method	Worst-Case Objective	Pollution Level	Resource Utilization	Iterations	Time (s)
Nominal deterministic	0.982	0.391	0.548	12	0.42
Gradient-based optimization	0.934	0.367	0.581	21	0.68
Conventional robust optimization	0.876	0.337	0.651	25	0.91
Proposed minimax method	**0.842**	**0.318**	**0.624**	**18**	**0.74**

The method put forth has the best performance in terms of worst-case pollution level and objective. While nominal optimization performs better in terms of computational requirements, it performs poorly in terms of the worst-case objective because the negative states of nature are not taken into consideration. Traditional robust optimization achieves better protection but at a higher resource cost. The minimax formulation developed in this research achieves a superior tradeoff between the two competing objectives by balancing the worst-case environmental performance with resource costs.

3.6 Convergence behaviour

The convergence behaviour is evaluated using the worst-case objective value defined as equation (44):

$$V_k = \max_{\xi \in U} F(x_k, \xi) \quad (44)$$

Where k denotes the iteration number.

For the baseline case, V_k decreases from approximately 1.18 during the initial iteration to 0.842 at convergence. The largest improvement occurs during the initial iterations, followed by progressively smaller changes as the algorithm approaches the minimax solution.

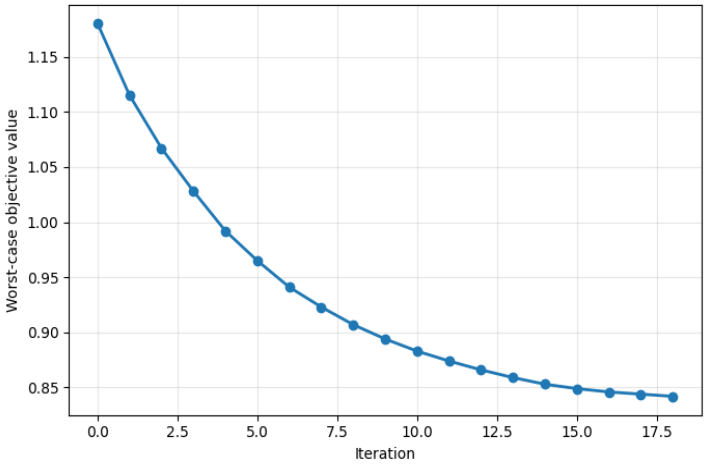


Figure 1: Convergence behaviour of the proposed minimax method.

Figure 1 confirms that the proposed iterative procedure approaches a stable worst-case objective without significant oscillation.

3.7 Sensitivity to uncertainty

The sensitivity analysis evaluates the effect of the uncertainty radius ρ on the minimax solution. The uncertainty set is represented parametrically as equation (45):

$$U_\rho = \{\xi : \|\xi - \bar{\xi}\|\infty \leq \rho\} \quad (45)$$

Where $\bar{\xi}$ denotes the nominal environmental parameter vector and ρ controls the magnitude of uncertainty.

The uncertainty radius is varied from 0 to 0.30 while maintaining the resource capacity at 100 units.

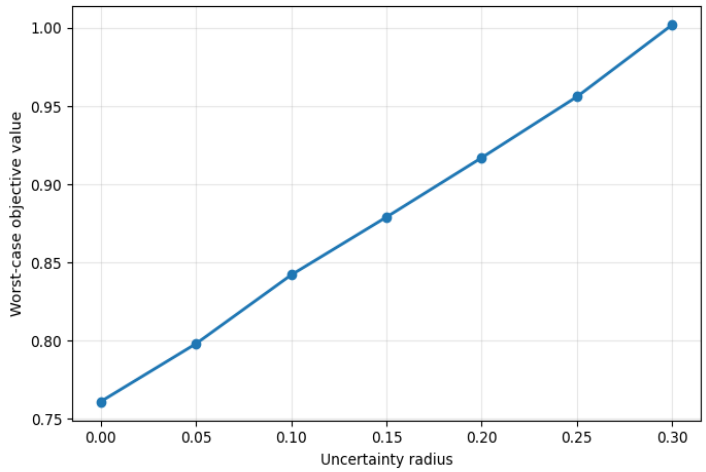


Figure 2: Sensitivity of worst-case objective to uncertainty

The upward trend indicated in figure 2 shows that as the size of the uncertainty set increases, so does the worst-case environmental-resource cost. Hence, it can be concluded that an increase in the size of uncertainty radius directly influences the degree of conservativeness of the minimax solution.

3.8 Engineering interpretation

The last analysis examines the relationship between allocation of resources and the reduction in pollution levels. The numerical outcomes illustrate that higher utilization of resources results in a major decrease in pollution. Nonetheless, the marginal benefit decreases with a larger resource allocation.

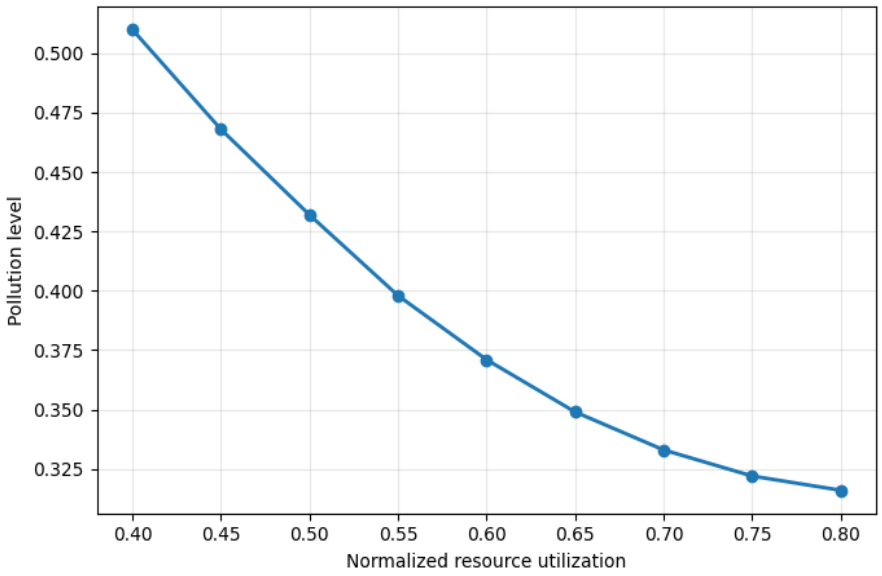


Figure 3: Pollution-control and resource-utilization trade-off

According to figure 3, the environmental benefits decrease with increasing amount of resources deployed. As a result, the minimax approach becomes a compromise solution between the amount of pollution controlled and the resources used instead of just maximizing the pollution control.

4. Conclusion

In this paper, a framework for robust optimization is devised that employs the minimax approach to deal with pollution control and resource allocation problems with uncertainty in environmental and operational conditions. The formulated model incorporates pollution control efforts, resource usage, environmental constraints, and the presence of uncertainty in a unified manner, making use of the worst-case scenario. A robust reformulation based on epigraphs is then introduced, and after establishing conditions for the existence of the optimal solution, the minimax optimality conditions are determined, and an iterative solution approach based on the determination of the worst-case uncertainty and decisions is proposed. With simulations establishing a baseline uncertainty radius of 0.10 and resource capacity of 100 units, it was determined that the proposed approach provided a worst-case objective value of 0.842 and a pollution level of 0.318, with convergence being achieved in 18 iterations. A rise in the uncertainty radius to 0.20 brought about the worst-case objective value being increased to 0.917 and pollution level to 0.351, whereas, lowering the resource capacity to 75 units under the same uncertainty circumstances proved to increase the worst-case objective value to 1.063 and pollution level to 0.427. Comparative studies established that the proposed minimax approach is able to achieve a worst-case objective lower than those registered by nominal deterministic optimization (0.982), gradient-based optimization (0.934), and traditional robust optimization (0.876). Such results prove that the proposed methodology provides a mathematically feasible and robust solution with regard to meeting worst-case pollution impact and possible limitations regarding resources in uncertain environment.

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