Minimax Theory and its Applications
Volume 3 (2018), No. 1, 131–160

A New Minimax Theorem for Linear Operators

Jean Saint Raymond
Sorbonne Universités, Université Pierre et Marie Curie,
Institut de Mathématique de Jussieu, Boîte 186 – 4 place Jussieu,
75252 Paris Cedex 05, France
jean.saint-raymond@imj-prg.fr

Received: December 1, 2017
Accepted: December 23, 2017

The aim of this note is to prove the following minimax theorem which generalizes a result by B. Ricceri and extends a previous result of the author: let E be a infinite-dimensional Banach space, F be a Banach space, X be a convex subset of E whose interior is non-empty for the weak topology on bounded sets, Δ a finite-dimensional convex compact subset of $\mathcal{L}(E,F)$, $\varphi\colon F\to\mathbb{R}$ be a continuous convex coercive map, and $\psi\colon\Delta\to\mathbb{R}$ a convex continuous function. Assume moreover that Δ contains at most one compact operator. Then

$$\sup_{x\in X}\ \inf_{T\in\Delta}\big(\varphi(Tx)+\psi(T)\big)=\inf_{T\in\Delta}\sup_{x\in X}\big(\varphi(Tx)+\psi(T)\big)\ .$$

Keywords: Minimax, Banach spaces, linear operators.

2010 Mathematics Subject Classification: 49J35, 46B04, 46B50

1. Introduction

For any two normed spaces E and F we will denote by $\mathcal{L}(E,F)$ the space of continuous linear mappings from E to F, itself normed by $\|T\|=\sup_{\|x\|\le 1}\|Tx\|$. It was shown by E. Asplund and V. Pták in [1] that for two normed spaces E and F of dimension ≥ 2 and any two A and B in $\mathcal{L}(E,F)$ the following inequality holds

$$\sup_{\|x\|\le 1}\inf_{\lambda\in\mathbb{R}}\|Ax+\lambda Bx\|\le\inf_{\lambda\in\mathbb{R}}\|A+\lambda B\|=\inf_{\lambda\in\mathbb{R}}\sup_{\|x\|\le 1}\|Ax+\lambda Bx\|$$

and that the equality in the above relation is attained for every pair A, B in $\mathcal{L}(E,F)$ if and only if both E and F are inner product spaces. In the sequel the unit ball of E is replaced by a convex set whose interior is non-empty for the weak topology $\sigma(E,E^*)$ or more generally the 'weak topology on bounded sets' and we are interested in proving such a minimax equality.

For a Banach space E, we will denote by $\beta(E,E^*)$ the finest locally convex topology on E which agrees with the weak topology $\sigma(E,E^*)$ on the bounded subsets of E. Converging sequences for $\beta(E,E^*)$ are all weakly converging sequences. If

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

a subset U of E is open for $\beta(E,E^*)$ then every weakly converging sequence in E whose weak limit is in U has all but finitely many terms inside U. Conversely if E^* is separable (e.g. if E is reflexive and separable) this characterizes the $\beta(E,E^*)$-open subsets of E.

The following theorem has been proved by B. Ricceri in [4] (Theorem 3).

Theorem 1.1. *Let E be an infinite-dimensional reflexive Banach space, $T\colon E\to E$ a non-zero linear compact operator, $\varphi\colon E\to\mathbb{R}$ a convex continuous and coercive functional, $J\subset\mathbb{R}$ a compact interval with $0\in J$, $\psi\colon J\to\mathbb{R}$ be a continuous convex function. Then, for each $r>\varphi(0)$, one has*

$$\sup_{x\in X}\inf_{\lambda\in J}(\varphi(Tx-\lambda x)+\psi(\lambda))=r+\psi(0)$$

where $X=\{x\in E:\varphi(Tx)\le r\}$.

Our main theorem extends this latter statement: indeed X has non-empty interior for $\beta(E,E^*)$ because of Theorem 4.4, and for $\Delta=\{T-\lambda I:\lambda\in J\}$, we then have $\sup_{x\in X}\varphi(Tx-\lambda x)=+\infty$ for all $\lambda\in J\setminus\{0\}$ and so $\inf_{\lambda\in J}\sup_{x\in X}\varphi(Tx-\lambda x)+\psi(\lambda)$ is clearly equal to $r+\psi(0)$.

In the previous paper [5], the author proved the same result with the additional hypotheses that the space E does not contain ℓ^1 and Δ is a segment. It will be proved here in Theorem 2.8 that in this case the sets U which are $\beta(E,E^*)$-open are characterized by the fact that every weakly converging sequence in E whose weak limit is in U has all but finitely many terms inside U.

We will also show by providing counter-examples that the hypotheses are necessary: the minimax equality can fail if there are several compact operators in Δ or if X has empty interior for $\beta(E,E^*)$.

The scheme of proof in this paper is quite similar to this of [5] but it has been of course necessary to change a large part of the proofs for getting rid of this geometrical hypothesis and generalizing to the multidimensional case of Δ. We refer to [5] for the proofs which have not been modified.

2. Weak topologies

As said in the introduction, for a Banach space E, we will denote by $\beta(E,E^*)$ the finest locally convex topology on E which agrees with the weak topology $\sigma(E,E^*)$ on the bounded subsets of E. This topology is finer than $\sigma(E,E^*)$ and coarser than the norm topology: so it is compatible with the duality betweeen E and E^*. We will also consider the locally convex topology τ_s for which the convex open sets are the convex subsets $U\subset E$ such that every sequence which converges weakly to some point $a\in U$ has cofinitely many terms in U. Of course such sets are norm-open, and τ_s is coarser than the norm-topology. Since every weakly converging sequence is bounded hence also $\beta(E,E^*)$-converging, it is clear that every $\beta(E,E^*)$-open set is τ_s-open. So τ_s is finer than $\beta(E,E^*)$.

If E is infinite-dimensional and has the Schur property, the weakly converging sequences are norm-converging: so in this case τ_s agrees with the norm-topology and is strictly finer than $\beta(E,E^*)$ since for this latter topology the origin is a cluster point of the unit sphere. But if E is a separable Banach space not containing ℓ^1, e.g. a separable reflexive space, we shall prove that τ_s coincides with $\beta(E,E^*)$.

Definition 2.1. Let E be a metric space, X be a subset of E and $\varepsilon > 0$. Then X will be said to be *ε-distal* if $d(x,y) \geq \varepsilon$ for any two distinct points x and y of X. And E will be said to be *ε-precompact* iff every ε-distal subset of E is finite (in particular E is covered by finitely many open balls of radius ε).

Conversely, in an open ball of radius ε, a 2ε-distal subset has at most one point. So if E is covered by finitely many open balls of radius ε it is 2ε-precompact.

Lemma 2.2. *If $A \subset E^*$ is norm-compact, then its polar set*

$$A^o = \{x \in E : \forall x^* \in A \quad \langle x^*, x\rangle \geq -1\}$$

is a neighborhood of 0 for $\beta(E,E^*)$.

Proof. Let $R > 0$. Then A is covered by finitely many balls of radius $(2R)^{-1}$ centered at x_j^* $(1 \leq j \leq m)$. And the set

$$W = \bigcap_{j=1}^{m} \{x \in E : \left|\langle x_j^*, x\rangle\right| \leq \frac{1}{2}\}$$

is a weak neighborhood of 0 in E which satisfies $A^o \supset W \cap B(0,R)$: indeed for $x \in W$ and $x^* \in A$ there exists $j \leq m$ such that $\left\|x_j^* - x^*\right\| \leq (2R)^{-1}$. If moreover $\|x\| \leq R$ we get

$$|\langle x^*, x\rangle| \leq \left|\langle x^* - x_j^*, x\rangle\right| + \left|\langle x_j^*, x\rangle\right| \leq \|x\| \cdot \left\|x^* - x_j^*\right\| + \frac{1}{2} \leq R \cdot \frac{1}{2R} + \frac{1}{2} = 1$$

hence $\langle x^*, x\rangle \geq -|\langle x^*, x\rangle| \geq -1$, so $x \in A^o$. □

Lemma 2.3. *If $C \subset E$ is a neighborhood of 0 for $\beta(E,E^*)$, then C^o is norm-compact in E^*.*

Proof. For all n, $C \cap B(0,2^n)$ is a weak neighborhood of 0 in $B(0,2^n)$. Thus there exists some finite subset J_n of E^* such that

$$\Big(\|x\| \leq 2^n \text{ and } \forall x^* \in J_n \ |\langle x^*, x\rangle| \leq 1\Big) \Longrightarrow x \in C\ .$$

It follows that for all n:

$$C^o \subset \overline{\text{conv}}\Big(B(0,2^{-n}) \cup \{\pm x^* : x^* \in J_n\}\Big) \subset B(0,2^{-n}) + K\,,$$

where K is the convex norm-compact set $\left\{\sum_{x^*\in J_n}\lambda_{x^*}x^* : \sum|\lambda_{x^*}|\leq 1\right\}$ which is itself covered by finitely many open balls of radius 2^{-n}. We conclude that C^o is covered by finitely many open balls of radius 2^{1-n}, hence is 2^{2-n}-precompact. Thus C^o is totally bounded, and complete since norm-closed in E^*. This completes the proof of compactness of C^o. ☐

Corollary 2.4. *Let C be a closed convex subset of E containing 0. Then C is a neighborhood of 0 for $\beta(E,E^*)$ if and only if C^o is norm-compact in E^*.*

Proof. Indeed we then have $C=(C^o)^o$. ☐

Lemma 2.5. *Let E be a Banach space, C a bounded convex symmetric subset of E and p a continuous linear mapping from E to a finite-dimensional normed space V. Then there exists some constant M such that the inequality $d(x,C\cap\ker p)\leq M\cdot\|p(x)\|$ holds for every $x\in C$.*

Proof. Up to replacing V by its linear subspace generated by $p(C)$, we can assume that the convex symmetric set $W=p(C)$ is a neighborhood of 0 in V. If $(e_1,e_2,\dots,e_k)$ is a basis of V consisting of members of W, we let $a_0=-\dfrac{1}{k}\sum_{j=1}^k e_j\in W$ and $a_j=\dfrac{e_j}{k}$ for $1\leq j\leq k$. We then have $\sum_{j=0}^k a_j=0$ and $a_j\in W$. Then $K=\operatorname{conv}(a_j:0\leq j\leq k)$ is a neighborhood of 0 in V, and there exists $r>0$ such that $B(0,r)\subset K\subset W$ [1]. There are then $(\xi_j)_{0\leq j\leq k}\in C$ such that $p(\xi_j)=a_j$.

If $x\in C$, $w=p(x)$ and $0<\|p(x)\|=\rho\leq r$, then we have $-rw/\rho\in K$ and there exist non-negative $(\lambda_j)_{0\leq j\leq k}$ with sum 1 such that $-rw/\rho=\sum_j\lambda_j a_j$. It follows that

$$0=\frac{\rho}{r+\rho}\cdot\sum_j\lambda_j a_j+\frac{r}{r+\rho}\cdot w\,.$$

This equality still holds if $w=0$ with any (λ_j), hence

$$\hat{x}=\frac{\rho}{r+\rho}\sum_j\lambda_j\xi_j+\frac{r}{r+\rho}\cdot x$$

satisfies $p(\hat{x})=0$ and $\hat{x}\in C$. It follows that

$$\begin{aligned} d(x,C\cap\ker p)\leq\|x-\hat{x}\|&=\frac{\rho}{r+\rho}\left\|x-\sum\nolimits_j\lambda_j\xi_j\right\|\\ &\leq\frac{\rho}{r+\rho}\cdot\operatorname{diam}(C)\leq\|p(x)\|\cdot\frac{\operatorname{diam}(C)}{r}\,.\end{aligned}$$

---

[1] In fact it is possible to prove that if $\dim(V)=k$ and $W\supset B(0,r_0)$ there are $k+1$ elements $a_0,a_1,\dots a_k$ of W such that $\sum_j a_j=0$ and $B(0,r)\subset K=\operatorname{conv}(a_0,a_1,\dots,a_k)$ where $r=k^{-3/2}r_0$: take the vertices of a regular simplex and $r=k^{-1}.r_0$ if V is euclidean, and use Dvoretzky-Rogers theorem [3] to reduce the problem to the euclidean case.

This inequality still holds if $\|p(x)\| \geq r$ since $0 \in C \cap \ker p$ and

$$d(x, C \cap \ker p) \leq \|x\| \leq \operatorname{diam}(C) \leq \frac{\|p(x)\|}{r} \cdot \operatorname{diam}(C) \ .$$

It is then enough to take $M = \operatorname{diam}(C)/r$. □

Lemma 2.6. *Let E be a Banach space, C a non-compact bounded convex symmetric closed subset of E and p a continuous linear mapping from E to a finite-dimensional normed space V. Then $C \cap \ker p$ is not compact. More precisely, if C is not δ-precompact, then $C \cap \ker p$ is δ'-precompact for no $\delta' < \delta/2$.*

Proof. If the complete set C is not compact, there exists $\delta > 0$ and an infinite δ-distal sequence (x_j) in C. Let $\delta' < \delta/2$. Since $W = p(C)$ is a bounded subset of V, $\overline{W}$ is compact and the sequence $(p(x_j))$ has a subsequence $(p(x_{j_\ell}))$ which converges to some $w \in \overline{W}$.

If $w = 0$, it follows from Lemma 2.5 that there are $\xi_\ell \in C_0 = C \cap \ker p$ such that $\|x_{j_\ell} - \xi_\ell\| \leq M \cdot \|p(x_{j_\ell})\|$. Thus we have for $\ell < k$, $\|\xi_\ell - \xi_k\| \geq \delta - M\big(\|p(x_{j_\ell}\| + \|p(x_{j_k})\|\big)$, and in particular $\|\xi_\ell - \xi_k\| \geq \delta/2 > \delta'$ for large ℓ, which contradicts the δ'-precompactness of C_0.

Conversely, if $w \neq 0$, take $t \in \left]\frac{\delta'}{\delta}, \frac{1}{2}\right[$ and $\eta = \dfrac{t}{1-t} \in \left]0, 1\right[$.

We have $-w \in \overline{W}$ and $0 \in \overset{o}{W}$, hence $-\eta w = \eta.(-w) + (1-\eta).0 \in W$. Thus there exists $y \in C$ such that $p(y) = -\eta w$. Letting $x'_j = t x_j + (1-t) y$, we get $x'_j \in C$, $\left\|x'_j - x'_k\right\| = t\left\|x_j - x_k\right\| \geq t\delta > \delta'$ and $p(x'_{j_\ell}) = t p(x_{j_\ell}) - (1-t)\eta w = t\big(p(x_{j_\ell}) - w\big) \to 0$. As in the previous case, we then can find $\xi_\ell \in C_0$ such that $\|\xi_\ell - \xi_k\| \geq \delta'$ for large ℓ and k. □

Theorem 2.7. *Let E be a Banach space and $U \subset E$ be a symmetric convex set. Assume that U contains some ball $B(0, \varepsilon)$ and that U is not a neighborhood of 0 for $\beta(E, E^*)$. Then there exists $R > 0$ such that for all finite-dimensional linear subspace L of E, $U + L$ does not contain the ball $B(0, R)$.*

Proof. Consider the polar set $C = U^o = (\overline{U})^o$ of U, which is a symmetric closed convex subset of E^*. Since $U \supset B(0, \varepsilon)$, we have $C \subset B(0, 1/\varepsilon)$, which shows that C is bounded. Since U is not a neighborhood of 0 for $\beta(E, E^*)$, we know by Corollary 2.4 that C is not compact.

Let L be a finite-dimensional subspace of E. We have $\overline{\operatorname{conv}}(U \cup L) = \overline{U + L}$, whence $(U + L)^o = U^o \cap L^o = C \cap L^o$. Then L^o is a closed linear finite-codimensional subspace of E^* , and the canonic mapping $p\colon E^* \to E^*/L^o$ has finite rank. It follows from Lemma 2.5 that there exists $\delta > 0$ depending only on U such that C is not δ-precompact, and by Lemma 2.6 that $C \cap \ker p = C \cap L^o$ cannot be $(2\delta/5)$-precompact. In particular, we cannot have $C \cap L^o \subset B(0, \delta/5)$. And if we had $\overline{U} + L \supset B(0, R)$, we would have $C \cap L^o \subset B(0, 1/R)$. The statement follows with $R = 5/\delta$. □

Theorem 2.8. *Let E be a separable Banach space not containing ℓ^1 and U be a convex subset of E. If $a \in U$ and if every sequence weakly converging to a has cofinitely many terms in U, then U is a neighborhood of a for $\beta(E, E^*)$. In particular τ_s coincides with $\beta(E, E^*)$.*

Proof. Under these hypotheses, it is clear that U is a norm-neighborhood of a in E. Let $U' = (U - a) \cap (a - U)$; then U' is symmetric and contains a ball centered at 0. If U is not a neighborhood of a for $\beta(E, E^*)$, then U' is not a neighborhood of 0 for $\beta(E, E^*)$ and there exists by previous theorem some $R > 0$ such that, for every finite-dimensional subspace L of E, $\overline{U'} + L$ does not contain $B(0, R)$.

We construct inductively a sequence (x_p) in $B(0, R)$ such that $x_q - x_p \notin U'$ whenever $p < q$. Put $x_0 = 0$. If the x_p's are defined for $p < q$, we take for L the linear space generated by $(x_p)_{p<q}$, and since $B(0, R) \setminus (\overline{U'} + L) \neq \emptyset$, we choose x_q such that $\|x_q\| \leq R$ and that $x_q \notin \overline{U'} + L$. In particular, for $p < q$, $x_q \notin x_p + U'$, it is $x_q - x_p \notin U'$.

The sequence (x_p) has at least a cluster value x^{**} in the ball of E^{**} of radius R equipped with the topology $\sigma(E^{**}, E^*)$ which is compact. And since E does not contain ℓ^1, it follows from Bourgain-Fremlin-Talagrand's theorem (cf. [2]) that there exists a sequence (x_{p_j}) extracted from (x_p) which converges to some x^{**} for $\sigma(E^{**}, E^*)$. Then the sequence $y_j = x_{p_{j+1}} - x_{p_j}$ converges to 0 for $\sigma(E^{**}, E^*)$ it is for the weak topology $\sigma(E, E^*)$. Moreover $y_j \notin U'$ for all j. This means that for all j either $a + y_j \notin U$ or $a - y_j \notin U$. So we can find a sequence $(\varepsilon_j) \in \{-1, 1\}$ such that $a + \varepsilon_j y_j \notin U$ while $(a + \varepsilon_j y_j)$ converges weakly to a. This shows that there exists a sequence which converges to $a \in U$ while none of its terms belongs to U, a contradiction.

It follows that U is a neighborhood of a for $\beta(E, E^*)$. And since this holds for any convex subset U, we conclude that $\tau_s = \beta(E, E^*)$. $\square$

3. A geometric lemma

Let Λ be a finite-dimensional normed space, F be a Banach space, E be an infinite-dimensional normed space and $\Phi \colon \Lambda \times E \to F$ be a continuous bilinear functional. To each element x of E corresponds a linear operator $\Phi_x \in \mathcal{L}(\Lambda, F)$ defined by $\Phi_x(\lambda) = \Phi(\lambda, x)$ and by abuse we will denote for $x \in E$: $\ker(x) := \ker(\Phi_x) \subset \Lambda$ the kernel of the operator Φ_x. It is quite clear that the mapping $x \mapsto \Phi_x$ is continuous from E to $\mathcal{L}(\Lambda, F)$.

The aim of this section is to prove the following:

Theorem 3.1. *Let Λ be a finite-dimensional normed space, F be a Banach space, E be an infinite-dimensional normed space and $\Phi \colon \Lambda \times E \to F$ be a continuous bilinear functional. Assume that there exists some non-empty compact subset K of Λ not containing 0 such that* $\ker(x) \cap K \neq \emptyset$ *for every $x \in E$. Then there exists some $a \in K$ and some closed finite-codimensional subspace $\tilde{E}$ of E such that $\Phi(a, x) = 0$ for every $x \in \tilde{E}$.*

The previous statement will be derived from the following theorem.

Theorem 3.2. *Let* Λ *be a normed space of finite dimension* ℓ*,* K *be a compact subset of* Λ *not containing 0,* F *be a Banach space,* E *be a linear normed space and* $\Phi\colon \Lambda \times E \to F$ *be a continuous bilinear functional. Assume that* $K \cap \ker(x) \neq \emptyset$ *for each* $x \in E$*, that* K *is minimal for this property and spans linearly the space* Λ*. Then there exists a subspace* F_1 *of* F *with dimension at most* $\ell(\ell-1)$ *which contains* $\Phi(a,x)$ *for every* $x \in E$ *and every* $a \in \Lambda$*.*

We need several lemmas for proving this theorem. For each of them we assume silently the conditions of Theorem 3.2 are satisfied.

Lemma 3.3. *The set* U_0 *of those* $x \in E$ *such that the rank* $\mathrm{rk}(\Phi_x)$ *of the linear mapping* Φ_x *is maximal is a dense open subspace of* E*.*

Proof. Since $\mathrm{rk}(\Phi_x) \leq \ell$ there exists some integer $p \leq \ell$ such that we obtain $p = \max_{x\in E} \mathrm{rk}(\Phi_x)$. For every $\alpha = (a_1, a_2, \ldots, a_p) \in \Lambda^p$ the set

$$W_\alpha = \{T \in \mathcal{L}(\Lambda, F) : (Ta_1, Ta_2, \ldots, Ta_p) \text{ is linearly independent}\}$$

is an open subset of $\mathcal{L}(E,F)$ and $U_0 = \bigcup_{\alpha\in\Lambda^p}\{x \in E : \Phi_x \in W_\alpha\}$; so U_0 is open. Furthermore let $y \in U_0$, $x \in E$ and $\varepsilon > 0$. There exists $\alpha \in \Lambda^p$ such that $\Phi_y \in W_\alpha$ and for $t \in \mathbb{R}$ we denote $x_t = (1-t)x + ty \in E$. Of course $\|x_t - x\| = |t|\cdot\|x - y\| < \varepsilon$ for t small enough. The exterior product $P(t) = \bigwedge_{j=1}^p \Phi(a_j, x_t) \in \bigwedge^p F$ is polynomial in t (of degree $\leq p$) and does not vanish at 1. So there are at most p values of t such that $P(t) = 0$, and one can find some t^* such that both $P(t^*) \neq 0$ and $|t^*| < \varepsilon / \|x - y\|$. For such a t^* we have $p \leq \mathrm{rk}(\Phi_{x_{t^*}}) \leq \max_{z\in E} \mathrm{rk}(\Phi_z) = p$, hence $x_{t^*} \in U_0$. This shows that $d(x, U_0) < \varepsilon$. Therefore U_0 is dense in E. ☐

Lemma 3.4. *The set of those* $a \in K$ *such that* $a \in \ker(x)$ *for some* $x \in U_0$ *spans linearly the space* E*.*

Proof. Assume towards a contradiction that there exists an hyperplane H of Λ containing $\bigcup_{x\in U_0} \ker(x)$. We would have $\ker(x) \cap K \subset H$ for all $x \in U_0$. Following Lemma 3.3 for every $x \in E$ there exists a sequence (x_n) in U_0 converging to x. By hypothesis we can find for every n some $a_n \in K \cap \ker(x_n)$. Then $a_n \in K \cap H$ and by compactness of $K \cap H$ there is a subsequence (a_{n_k}) which converges to some $\hat{a} \in K \cap H$. So $\Phi(\hat{a}, x) = \lim_k \Phi(a_{n_k}, x_{n_k}) = 0$. This proves that $\ker(x) \cap (K \cap H) \neq \emptyset$ for every $x \in E$, and contradicts the minimality of K: indeed $K \cap H$ is compact, meets every $\ker(x)$ for $x \in E$ and has linear dimension at most $\ell - 1$ while the linear dimension of K is ℓ. ☐

Lemma 3.5. *If* $x \in U_0$ *and* $a_0 \in \ker(x)$ *then* $\Phi_y(a_0) \in \Phi_x(\Lambda)$ *holds for every* $y \in E$*.*

Proof. Let $T = \Phi_x \in \mathcal{L}(\Lambda, F)$ and $p = \mathrm{rk}(\Phi_x) = \max_{y\in E} \mathrm{rk}(\Phi_y)$. Then there exists $\alpha = (a_1, a_2, \ldots, a_p) \in \Lambda^p$ such that $T\alpha = (Ta_1, Ta_2, \ldots, Ta_p)$ is a basis of the subspace $F_0 = \Phi_x(\Lambda)$ of F. Let $(b_1^*, b_2^*, \ldots, b_p^*)$ be the dual basis of F_0^*

(i.e. $\langle b_i^*, Ta_j\rangle = \delta_i^j$). By Hahn-Banach's theorem we can extend each b_i^* to some $f_i^* \in F^*$.

Let $y \in E$ and consider the matrix $M = (m_{i,j})$ where $m_{i,j} = \langle f_i^*, \Phi_y(a_j)\rangle$ and the vector $X_0 = (z_1, z_2, \dots, z_p) \in \mathbb{R}^p$ such that $z_i = \langle f_i^*, \Phi_y(a_0)\rangle$. For t small enough, $\Phi_x + t\Phi_y$ belongs to the open set W_α and since $\mathrm{rk}(\Phi_x + t\Phi_y) = p$, the vectors $\Phi_{x+ty}(a_j)$ form a basis of $\Phi_{x+ty}(\Lambda)$. So there is a unique $\gamma(t) \in \mathbb{R}^p$ with coordinates $\gamma_j(t)$, such that

$$(*) \qquad \sum_{j=1}^{p} \gamma_j(t)\Phi_x(a_j) + \sum_{j=1}^{p} t\gamma_j(t)\Phi_y(a_j) = \sum_{j=1}^{p} \gamma_j(t).\Phi_{x+ty}(a_j) = \Phi_{x+ty}(a_0)$$

In particular for $t = 0$, $\Phi_x(a_0) = 0$, hence $\gamma_j(0) = 0$ for all $j \leq p$. For all $i \leq p$

$$\begin{aligned}\sum_{j=1}^{p}(\delta_i^j + tm_{i,j})\gamma_j(t) &= \sum_{j=1}^{p} \gamma_j(t)\langle f_i^*, \Phi_{x+ty}(a_j)\rangle = \langle f_i^*, \Phi_{x+ty}(a_0)\rangle \\ &= t\langle f_i^*, \Phi_y(a_0)\rangle = tz_i\,,\end{aligned}$$

hence $(I + tM)\gamma(t) = tX_0$, or $\gamma(t) = t(I + tM)^{-1}X_0$. We conclude that the γ_j's are $\mathcal{C}^1$ functions (even rational fractions) in a neighborhood of 0. Derivating $(*)$ with respect to t at 0 we get:

$$\sum_{j=0}^{p}\Big(\gamma_j'(0)\cdot\Phi_x(a_j) + \gamma_j(0)\cdot\Phi_y(a_j)\Big) = \Phi_y(a_0)$$

hence $\Phi_y(a_0) = \Phi_x(\lambda)$ for $\lambda = \sum_j \gamma_j'(0)\cdot a_j \in \Lambda$. Thus $\Phi_y(a_0) \in \Phi_x(\Lambda)$. □

Proof of Theorem 3.2. It follows from Lemma 3.4 that there are families $\bar{a} = (a_1, a_2, \dots, a_\ell)$ of points of K and $(x_1, x_2, \dots, x_\ell)$ of U_0 such that $a_j \in \ker(x_j)$ and $\bar{a}$ spans the whole Λ. Consider the $\ell(\ell-1)$ vectors $b_{i,j} = \Phi_{x_i}a_j$ for $i \neq j$ in $[1, \ell]$ and the linear subspace F_1 they span, which has obviously dimension at most $\ell(\ell-1)$.

If $x \in E$ and $a \in \Lambda$ there are real coefficients $(\beta_i)_{i\leq\ell}$ such that $a = \sum_i \beta_i a_i$. So $\Phi(a, x) = \sum_i \beta_i\Phi_x(a_i)$, and by Lemma 3.5 $\Phi_x(a_i) \in \Phi_{x_i}(\Lambda)$. Therefore there are real numbers $\mu_{i,j}$ such that $\Phi_x(a_i) = \sum_j \mu_{i,j}\Phi_{x_i}(a_j) = \sum_{j\neq i}\mu_{i,j}b_{i,j}$ since $\Phi_{x_i}(a_i) = 0$. Thus $\Phi(a, x) = \Phi_x(a) = \sum_{i\neq j}\lambda_i\mu_{i,j}b_{i,j} \in F_1$. □

Proof of Theorem 3.1. We now intend to derive Theorem 3.1 from Theorem 3.2. Clearly there exists a minimal compact subset K_0 of K which meets $\ker(x)$ for every $x \in E$. Define $\Lambda_0 = \mathrm{lin}(K_0) \subset \Lambda$ and $\ell_0 \leq \ell$ the dimension of Λ_0. Since $0 \notin K_0$ and $K_0 \neq \emptyset$, we have $\ell_0 \geq 1$. Replacing Λ by Λ_0 and K by K_0 we can apply Theorem 3.2 and find a subspace F_1 of F of dimension $N \leq \ell_0(\ell_0 - 1) \leq \ell(\ell-1)$ containing $\Phi(\lambda, x)$ for every $x \in E$ and every $\lambda \in \Lambda_0$, hence $\Phi(a, x)$ for $x \in E$ and $a \in K_0$.

Choose a basis $(b_1, b_2, \dots, b_N)$ of F_1 and denote $(b_1^*, b_2^*, \dots, b_N^*)$ the dual basis in F_1^*. Extend by Hahn-Banach's theorem each b_j^* into a $\varphi_j \in F^*$.

So we get a finite family $(\varphi_j)_{1\le j\le N}$ of continuous linear functionals on F such that $F_1 \cap \bigcap_{j\le N} \ker(\varphi_j) = \bigcap_j \ker(b_j^*) = \{0\}$. Choose any $a \in K_0$. For $1 \le j \le N$ consider the continuous linear functional on E defined by $\tau_j \colon x \mapsto \langle \varphi_j, \Phi(a,x)\rangle$ and

$$\tilde{E} = \{x \in E : \forall j \in [1,N] \quad \tau_j(x) = 0\,\},$$

which is a closed finite-codimensional subspace of E. Then for every $x \in \tilde{E}$ and $b = \Phi_x(a)$, we have $\langle \varphi_j, b\rangle = \tau_j(x) = 0$ and $b \in F_1$, hence $b = 0$. □

Notice that in order to get Theorem 3.1 we have to restrict the conclusion to some subspace $\tilde{E}$ of E. In fact we can construct the following example:

Example 3.6. There exists a compact subset Δ of $\mathbb{R}^3$ not containing 0 and a 2-dimensional subspace E of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ such that $\Delta \cap \ker U \neq \emptyset$ for every $U \in E$ but $\Delta \cap \bigcap_{U\in E} \ker(U) = \emptyset$.

Proof. Take $\Delta = \{(x,y,z) \in \mathbb{R}^3 : z = 1 \text{ and } x^2 + y^2 \le 1/4\}$ which is a compact disc in the affine hyperplane $H = \{(x,y,z) : z = 1\}$. Define the linear mappings S, T in $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ having respective matrices

$$\begin{pmatrix} 0 & 1 & -1/2 \\ -1 & 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

and let E be the linear space spanned by S and T. For $\lambda \in \mathbb{R}$ and $X = (x,y,z)$, we have $X \in H \cap \ker(S + \lambda T)$ iff

$$\begin{cases} \lambda x + (y + z/2) = z \\ -x + \lambda(y + z/2) = 0 \\ z = 1 \end{cases}$$

hence $x = \lambda(y + 1/2)$ and $(y + 1/2)(1 + \lambda^2) = 1$, thus $x = \dfrac{\lambda}{1+\lambda^2}$, $y = \dfrac{1-\lambda^2}{2(1+\lambda^2)}$.

So $H \cap \ker(S + \lambda T) = \Big\{\big(\dfrac{\lambda}{1+\lambda^2}, \dfrac{1-\lambda^2}{2(1+\lambda^2)}, 1\big)\Big\}$ and $H \cap \ker(T) = \{(0, -1/2, 1)\}$.

For every non-zero $V \in E$ there is a unique point X_V in $H \cap \ker(V)$ satisfying $X_V \in \Delta$ and $x^2 + y^2 = 1/4$. It appears that this unique point X_V depends on V, so that $\Delta \cap \bigcap_{V\in E} \ker(V)$ is empty. □

Corollary 3.7. *Let E and F be infinite-dimensional Banach spaces and $(S_j)_{1\le j\le \ell}$ a linearly independent finite family of continuous linear operators from E to F. Then either there exists some $a \in E$ such that the family $(S_j a)_{j\le \ell}$ is independent, or there is some non-zero $\mu = (\mu_j) \in \mathbb{R}^\ell$ such that $\sum_j \mu_j S_j$ has finite rank.*

Proof. Suppose that the family $(S_j a)_{j\le \ell}$ is independent for none $a \in E$ and consider the bilinear mapping $\Phi \colon \mathbb{R}^\ell \times E \to F$ defined by $\Phi(\lambda, x) = \sum_{j=1}^{\ell} \lambda_j S_j x$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Let $K = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{R}^\ell : \sum_j \lambda_j^2 = 1\}$. Denote $S_\lambda = \sum_j \lambda_j S_j$ for $\lambda \in \mathbb{R}^\ell$. Then for every $x \in E$ there is $\lambda \in K$ such that $\Phi(\lambda, x) = S_\lambda(x) = 0$, and it follows from Theorem 3.1 that there are $\tilde{E} \subset E$ finite-codimensional and $\mu \in K$ such that $\Phi(\mu, x) = 0$ for all $x \in \tilde{E}$. Thus $\mathrm{rk}(S_\mu) \leq \mathrm{codim}(\tilde{E})$. ☐

4. E'-convergence and compact operators

Definition 4.1. Let E be a Banach space, E^* its dual and E' a linear subspace of E^*. We will say that *the sequence* (x_n) *E'-converges to* $a \in E$ iff it is bounded in E and converges to a for the weak topology $\sigma(E, E')$, it is $\lim_n \langle \theta, x_n - a \rangle = 0$ for every $\theta \in E'$. We also will say that *a sequence* (x_n) *is E'-null* if it E'-converges to 0. We will denote by $\mathcal{Z}$ the linear space of E'-null sequences.

Of course if $E' = E^*$, the sequence E^*-converges to a iff it converges weakly to a.

Lemma 4.2. *Assume that E' is a separable linear subspace of E^* and that (x_n) is a bounded sequence in E. Then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} \langle \theta, x_{n_k} \rangle$ exists in $\mathbb{R}$ for all $\theta \in E'$.*

Proof. If $(\theta_j)_{j\in\mathbb{N}}$ is a norm-dense sequence in E', each sequence $(\langle \theta_j, x_n \rangle)_n$ is bounded in $\mathbb{R}$. Thus there exists a subsequence (x_{n_k}) such that : $\lim_{k\to\infty} \langle \theta_j, x_{n_k} \rangle$ exists for all j. And it is easy to see that the sequence $(\langle \theta, x_{n_k} \rangle)_k$ is then Cauchy in $\mathbb{R}$ for every $\theta \in E'$. ☐

Lemma 4.3. *If T is not compact, there exists an $\varepsilon > 0$ and a sequence (a_n) in the unit ball B of E such that : $d(Ta_n, \mathrm{lin}(\{Ta_p : p < n\})) \geq \varepsilon$ holds for all $n \in \mathbb{N}$.*

Proof. Let B be the unit ball of E. Define inductively $\varepsilon_n \geq 0$ and $a_n \in B$ such that we have

$$\varepsilon_n = \sup_{a\in B} d(Ta, L_n) \quad \text{and} \quad d(Ta_n, L_n) \geq \frac{\varepsilon_n}{2}$$

where $L_n = \mathrm{lin}(Ta_1, Ta_2, \ldots, Ta_{n-1})$. Of course $L_0 = \{0\}$ and $\varepsilon_0 = \|T\|$.

It is clear that the sequence (ε_n) is non-increasing and does not vanish if $\mathrm{rk}(T) = +\infty$. Of course T is compact if it has finite rank.

Suppose that T has infinite rank. We first show that if $\varepsilon = \inf_n(\varepsilon_n/2) = 0$ then T is compact. Indeed let $\eta > 0$; there exists n such that $\varepsilon_n < \eta' = \eta/2$. Then $C = \{y \in L_n : \|y\| \leq \|T\| + \eta'\}$ is compact, hence covered by finitely many balls $(B(y_j, \eta'))_{j\leq N}$. For all $a \in B$, we have by definition $d(Ta, L_n) \leq \varepsilon_n < \eta'$. Then there exists $y \in L_n$ such that $\|Ta - y\| < \eta'$, thus $\|y\| \leq \|Ta\| + \|y - Ta\| < \|T\| + \eta'$ and there exists $j \leq N$ such that $\|y - y_j\| < \eta'$, hence $\|Ta - y_j\| < 2\eta' = \eta$, what shows that $T(B) \subset \bigcup_{j\leq N} B(y_j, \eta)$. It follows that $T(B)$ is totally bounded, hence that T is a compact operator.

Conversely if $\varepsilon = \inf_n \dfrac{\varepsilon_n}{2} > 0$ we have for all n: $d(Ta_n, L_n) \geq \dfrac{\varepsilon_n}{2} \geq \varepsilon$. ☐

Theorem 4.4. *Let E be a separable Banach space, V be a Banach space and $T \in \mathcal{L}(E,V)$. Then the operator T is compact if and only if there exists a separable subspace E' of E^* such that for every E'-null sequence (x_n) the sequence (Tx_n) converges in norm to 0 in V. More precisely, if T is not compact, E' is a separable subspace of E^* and m is any integer, then there is a linear subspace $\mathcal{Y}$ of $\mathcal{Z}$ of dimension m such that for every non-zero $v = (v_n) \in \mathcal{Y}$ the inequality $\inf_n \|Tv_n\| > 0$ holds.*

Proof. Suppose T is compact. Then $T^*\colon V^* \to E^*$ is compact too, so $T^*(B^*)$ is relatively compact in E^* hence norm-separable. Let (φ_k) be a sequence in B^* such that $(\psi_k) = (T^*\varphi_k)$ is norm-dense in $T^*(B^*)$ and denote by E' the linear span of the ψ_k's. Suppose that (x_n) is a bounded sequence in E and that $\lim_n \langle \psi_k, x_n\rangle = 0$ for all k. Let $M = \sup_n \|x_n\|$. Then the sequence (Tx_n) has at least one cluster point b (for the norm-topology), and if this cluster point is unique, then (Tx_n) converges to b. Assume $b \neq 0$; there is some $\varphi \in B^*$ such that $\langle \varphi, b\rangle > \delta = \|b\|/2$, then some k such that $\|\psi_k - T^*\varphi\| < \delta/2M$. There exists an increasing sequence (n_j) such that $\langle \varphi, Tx_{n_j}\rangle > \delta$ for all j. So

$$\begin{aligned}\langle \psi_k, x_{n_j}\rangle &\geq \langle T^*\varphi, x_{n_j}\rangle - \|\psi_k - T^*\varphi\| \cdot \|x_{n_j}\| = \langle \varphi, Tx_{n_j}\rangle - \|\psi_k - T^*\varphi\| \cdot \|x_{n_j}\| \\ &> \delta - M \cdot \frac{\delta}{2M} = \frac{\delta}{2}\end{aligned}$$

contradicting the hypothesis $\langle \psi_k, x_n\rangle \to 0$.

Conversely, assume that E' is a separable subspace of E^* and T is not compact. Choose first following Lemma 4.3 some $\varepsilon > 0$ and some sequence (a_n) in the unit ball B of E such that $d(Ta_n, L_n) \geq \varepsilon$ for all n where $L_n = \mathrm{lin}(Ta_1, Ta_2, \ldots, Ta_{n-1})$. By Lemma 4.2 we can and do assume that $\lambda(\theta) := \lim_{n\to\infty}\langle \theta, a_n\rangle$ exists for all $\theta \in E'$. Then put for $1 \leq j \leq m$: $b_j = (b_n^{(j)})$, where $b_n^{(j)} = a_{n+j} - a_n$, and let $\mathcal{Y}$ be the linear space of sequences spanned by the b_j's. We clearly have $\sup_n \|b_n^{(j)}\| \leq 2$, what shows that the sequences b_j are bounded and so are all members of $\mathcal{Y}$. Moreover for $\theta \in E'$ we have

$$\lim_{n\to\infty} \langle \theta, b_n^{(j)}\rangle = \lim_{n\to\infty} \langle \theta, a_{n+j}\rangle - \lim_{n\to\infty} \langle \theta, a_n\rangle = \lambda(\theta) - \lambda(\theta) = 0\,.$$

It follows that $\mathcal{Y} \subset \mathcal{Z}$. Finally if $v = (v_n) \in \mathcal{Y}$, there are real coefficients $(\alpha_j)_{1\leq j\leq m}$ such that $v = \sum_{j=1}^m \alpha_j b_j$. If these coefficients are not all zero there exists $p \leq m$ such that $\alpha_p \neq 0$ and $\alpha_j = 0$ for $p < j \leq m$. We then have

$$Tv_n = \sum_{j=1}^{m} \alpha_j \big(Ta_{n+j} - Ta_n\big) = \sum_{j=1}^{p} \alpha_j \big(Ta_{n+j} - Ta_n\big) = \alpha_p.Ta_{n+p} + y,$$

where $y \in L_{p+n}$, hence $\|v_n\| \geq |\alpha_p|.\varepsilon$, and $\inf_n \|Tv_n\| \geq |\alpha_p|.\varepsilon > 0$. In particular this shows that $v \neq 0$, thus that the $(b_j)_{j\leq m}$ form a basis of $\mathcal{Y}$ and that $\dim(\mathcal{Y}) = m$.

Applying what precedes for $m = 1$, we get a E'-null sequence (x_n) in E such that $\inf_n \|Tx_n\| > 0$, hence (Tx_n) does not converge to 0 in F. $\square$

Lemma 4.5. *Let E and V be Banach spaces, $\pi \in \mathcal{L}(E,V)$ be a compact operator and Y be a convex non-empty open subset of V. Then $\pi^{-1}(Y)$ is open for $\beta(E,E^*)$.*

Proof. Up to a translation in E we can assume that $0 \in Y$ hence that $Y \supset B(0,2^{-m})$ for some $m \in \mathbb{N}$. Then the polar set Y^o is contained in $2^m \cdot B^*$ (where B^* denotes the unit ball of E^*) and

$$\pi^{-1}(Y)^o = \{\varphi \in E^* : \forall x \in \pi^{-1}(Y)\ \langle \varphi, x\rangle \geq -1\}$$

Since εY is a convex neighborhood of 0 we have for every $\varepsilon > 0$:

$$\overline{Y} \subset Y + \varepsilon Y = (1+\varepsilon)Y,$$

hence $\pi^{-1}(\overline{Y}) \subset (1+\varepsilon)\pi^{-1}(Y)$ and $\pi^{-1}(Y)^o \supset \pi^{-1}(\overline{Y})^o \supset (1+\varepsilon)^{-1}\pi^{-1}(Y)^o$. Therefore $\pi^{-1}(Y)^o = \pi^{-1}(\overline{Y})^o$.

For $z \in \big(\pi^*(Y^o)\big)^o$ and any $\xi \in Y^o$ we have $\langle \xi, \pi z\rangle = \langle \pi^*\xi, z\rangle \geq -1$, that means $\pi z \in (Y^o)^o = \overline{Y}$, whence $z \in \pi^{-1}(\overline{Y})$. Thus $\big(\pi^*(Y^o)\big)^o \subset \pi^{-1}(\overline{Y})^o = \big(\pi^{-1}(Y)\big)^o$ and it follows that $2^m \cdot \overline{\pi^*(B^*)} \supset \overline{\pi^*(Y^o)} \supset \pi^{-1}(Y)^o$.

Since π is compact so is π^*, and we conclude that the convex weakly closed set $\pi^{-1}(Y)^o$ is norm-compact thus that $\pi^{-1}(Y)$ is a neighborhood of 0 in E for $\beta(E,E^*)$. □

Lemma 4.6. *Let E be a infinite-dimensional Banach space and X be a convex subset of E whose interior for $\beta(E,E^*)$ is non-empty. Then there exists a Banach space V, a compact linear mapping $\pi\colon E \to V$ with dense range and a convex open subset Y of V such that $\pi^{-1}(Y) \subset X \subset \overline{\pi^{-1}(Y)}$.*

Proof. Let X_0 be the interior of X for $\beta(E,E^*)$ and $a \in X_0$. Put $W = (X_0 - a) \cap (a - X_0)$ which is a symmetric convex subset of E. Then W is open for $\beta(E,E^*)$ and contains 0. Since $\beta(E,E^*)$ is coarser than the norm-topology, W is absorbing and the Minkowski functional $p_W\colon x \mapsto \inf\{r > 0 : r^{-1}x \in W\}$ is a semi-norm on E such that $W = \{x : p_W(x) < 1\}$. Denote by V the separated completion of (E, p_W) and π the canonical mapping from E to V. By definition $\pi(E)$ is dense in the Banach space V, and $\pi(W) = \pi(E) \cap B(0,1)$. We will show that $\pi(X_0)$ is open in $\pi(E)$ and that if Y denotes the interior of $\overline{\pi(X_0)} = \overline{\pi(X)}$ then we have $X_0 = \pi^{-1}(Y)$. Then since $X_0 \neq \emptyset$ we get $\pi^{-1}(Y) = X_0 \subset X \subset \overline{X_0} = \overline{\pi^{-1}(Y)}$.

Indeed let $b \in \pi(X_0)$ and $u \in X_0$ such that $\pi u = b$. The set $\{t \in \mathbb{R} : u - t(a-u) \in X_0\}$ is open and contains 0. Hence it contains also some $\varepsilon > 0$ and we let $v = u - \varepsilon(a-u)$. The homothety with center $v \in X_0$ and ratio $\eta = \varepsilon(1+\varepsilon)^{-1} < 1$ transforms X_0 into itself and in particular a into u, hence $a + W \subset X_0$ into $u + \eta \cdot W$. Thus $\pi(X_0) \supset \pi(E) \cap B(b,\eta)$ and $\pi(X_0)$ is a neighborhood of b in $\pi(E)$. Finally if Y is the interior of the convex set $\overline{\pi(X_0)}$ then $Y \cap \pi(E)$ is a convex open subset of $\pi(E)$ contained in $\pi(E) \cap \overline{\pi(X_0)}$ and containing $\pi(X_0)$: indeed if $u \in \pi(X_0)$ we have shown the existence of a ball $B(u,r)$ such that $\pi(E) \cap B(u,r) \subset \pi(X_0)$, hence that $\overline{\pi(E) \cap B(u,r)} \subset B(u,r) \subset \overline{\pi(X_0)}$ and $u \in Y$.

It remains to show the compactness of π. By definition of the topology $\beta(E, E^*)$, for each $q \in \mathbb{N}$ there is a finite set $J_q \subset E^*$ such that

$$\|x\| \leq 2^q \text{ and } \big(\forall \varphi \in J_q \ \ |\langle \varphi, x\rangle| < 1\big) \Longrightarrow x \in W\,.$$

Let E' be the linear span of $\bigcup_q J_q$. We show that any sequence (x_n) in B which E'-converges to 0 satisfies $p_W(x_n) \to 0$, hence $\pi x_n \to 0$. Indeed for any integer q there is some N such that for all $\ell \in J_q$ and all $n \geq N$ we have $|\langle \ell, x_n\rangle| < 2^{-q}$. So for $n \geq N$,

$$\|2^q x_n\| \leq 2^q \text{ and } \big(\forall \varphi \in J_q \ \ |\langle \varphi, 2^q x_n\rangle| < 1\big)$$

what implies $2^q x_n \in W$, it is $p_W(x_n) \leq 2^{-q}$. It follows that $\pi x_n \to 0$ in V. And by Theorem 4.4 this shows the compactness of π. □

Theorem 4.7. *Let E and V be Banach spaces, U a bounded convex open set in V and $T \in \mathcal{L}(E, V)$. Then $T^{-1}(U)$ has non-empty interior for $\beta(E, E^*)$ iff T is compact.*

Proof. It follows from Lemma 4.5 that $T^{-1}(U)$ has non-empty interior for $\beta(E, E^*)$ if T is compact.

Conversely, denote $M := \sup_{v\in U} \|v\|$. If $T^{-1}(U)$ has non-empty interior it follows from Lemma 4.6 that there exists a Banach space W, a compact operator $S \in \mathcal{L}(E, W)$ with dense range and an open set $Y \subset W$ such that $T^{-1}(U) \supset S^{-1}(Y)$. Then there is $x_0 \in E$ and $\varepsilon > 0$ such that $Y \supset B(Sx_0, \varepsilon)$. We claim that T factorizes through S : there exists some linear continuous operator $R\colon W \to V$ such that $T = R \circ S$, and this will imply that T is compact.

Indeed let x and z be points in E such that $Sx = Sz$. Then for all integer n we have $S(n(x - z)) = 0 \in B(0, \varepsilon)$ hence $S(x_0 + n(x - z)) \in Y$ and $x_0 + n(x - z) \in S^{-1}(Y) \subset T^{-1}(U)$. So we get $Tx_0 + n \cdot T(x - z) \in U$ and

$$\sup_n \big(n \cdot \|T(x - z)\|\big) \leq M + \|Tx_0\| < +\infty\ .$$

Thus $Tx - Tz = T(x - z) = 0$. It follows that there exists a mapping $r\colon S(E) \to V$ such that $Tx = r(Sx)$ for every $x \in E$. One readily checks that r is linear. Moreover, if $w \in S(E)$ and $\|w\| < 1$, we have $w = Sx$ for some $x \in E$ and $S(x_0 + \varepsilon x) = Sx_0 + \varepsilon Sx \in B(Sx_0, \varepsilon) \subset Y$ hence $x_0 + \varepsilon x \in S^{-1}(Y) \subset T^{-1}(U)$ and $Tx_0 + \varepsilon Tx \in U$. So

$$\|r(w)\| = \|r(Sx)\| = \|Tx\| \leq M_1 = \frac{1}{\varepsilon}\big(M + \|Tx_0\|\big)$$

what implies that r is continuous on $S(E)$ with norm $\leq M_1$. So r extends to a continuous linear operator R from $\overline{S(E)} = W$ to V and we get $T = S \circ R$. □

Corollary 4.8. *Let E be a Banach space, X be a non-empty convex symmetric open subset of E and $p_X\colon E \to \mathbb{R}^+$ be the Minkowski functional: $x \mapsto \inf\{t > 0 : x \in t \cdot X\}$. Then p_X is a semi-norm on E, the separated completion V of (E, p_X) is a Banach space, and the canonical mapping $\pi\colon E \to V$ is compact iff X has non-empty interior for $\beta(E, E^*)$.*

Proof. Indeed $\pi^{-1}(B_V) = X$, where B_V denotes the open unit ball of V. □

The following lemma will not be used in the sequel but it is interesting by itself.

Lemma 4.9. *Let E be a separable Banach space, $N \subset E$ be a closed subspace, $q\colon E \to Q = E/N$ be the canonical projection, X be a convex subset of E with non-empty interior for the topology $\beta(E, E^*)$. Then $q(X)$ is a convex subset of Q with non-empty interior for $\beta(Q, Q^*)$.*

Proof. Following Lemma 4.6, there are a Banach space V, a compact linear mapping $\pi\colon E \to V$, and an open convex subset $Y \subset V$ such that $X_0 = \pi^{-1}(Y) \subset X \subset \overline{\pi^{-1}(Y)}$. Then $M = \overline{\pi(N)}$ is a closed linear subspace of V. Consider the canonical projection $\tilde{q}\colon V \to \tilde{Q} = V/M$, which is an open mapping. Then $\tilde{Y} = \tilde{q}(Y)$ is open in $\tilde{Q}$. Since $\tilde{q} \circ \pi$ is 0 on $N = \ker q$, there is a linear mapping $\tilde{\pi}\colon Q = E/N \to \tilde{Q}$ such that $\tilde{\pi} \circ q = \tilde{q} \circ \pi$. Moreover if U is any open subset of $\tilde{Q}$ then $\tilde{\pi}^{-1}(U) = q\big((\tilde{q} \circ \pi)^{-1}(U)\big)$, and this set is open since $\tilde{q} \circ \pi$ is continuous and q open. So $\tilde{\pi}$ is continuous. We now show that $\tilde{\pi}$ is compact. Let (z_n) be a bounded sequence in Q, and suppose $\sup_n \|z_n\| < M$. Then for each n we can find $x_n \in E$ such that $\|x_n\| < M$ and $qx_n = z_n$. Since $\tilde{q} \circ \pi$ is compact there is a convergent subsequence $(\tilde{q} \circ \pi x_{n_k})$ of the sequence $(\tilde{q} \circ \pi x_n)$. Finally $(\tilde{\pi} z_{n_k}) = (\tilde{\pi} \circ q x_{n_k}) = (\tilde{q} \circ \pi x_{n_k})$ is a convergent subsequence of $(\tilde{\pi} z_n)$.

We now prove that the convex set $q(X) \subset Q$ has non-empty interior for the topology $\beta(Q, Q^*)$. Indeed $\tilde{Y}$ is a non-empty convex subset of $\tilde{Q}$ and we claim that

$$q(X) \supset q(X_0) = \tilde{\pi}^{-1}(\tilde{Y}) .$$

Since $\tilde{T}$ is compact, this will show that $q(X_0)$ is open for $\beta(\tilde{Q}, \tilde{Q}^*)$. In fact

$$\begin{aligned} z \in q(X_0) &\iff \exists x \ \big(q(x) = z \text{ and } \pi(x) \in Y\big) \\ &\implies \exists x \in \pi^{-1}(Y) \quad \tilde{\pi}(z) = \tilde{q} \circ \pi(x) \implies \tilde{\pi}(z) \in \tilde{q}(Y) = \tilde{Y} . \end{aligned}$$

Conversely, if $\tilde{\pi}(z) \in \tilde{Y}$, there are $x_0 \in E$ such that $z = qx_0$ and $y \in Y$ such that $\tilde{\pi}(z) = \tilde{q}(y)$. For each $x \in E$ such that $\pi(x) = y$ we have $\tilde{\pi}(z) = \tilde{q} \circ \pi(x) = \tilde{\pi} \circ q(x)$, so

$$\begin{aligned} &0 = \tilde{\pi} \circ q(x - x_0) = \tilde{q} \circ \pi(x - x_0) \\ &\implies y - \pi(x_0) = \pi(x - x_0) \in M = \overline{\pi(N)} \implies y \in \overline{\pi(x_0) + \pi(N)}. \end{aligned}$$

And we deduce that $\overline{\pi(x_0 + N)} \cap Y \neq \emptyset$, hence that $\pi(x_0 + N) \cap Y \neq \emptyset$ since Y is open. It follows that $(x_0 + N) \cap X_0 \neq \emptyset$, and finally that $z = q(x_0) \in q(X_0)$. □

5. Unboundedness theorems

Definition 5.1. Let E be a Banach space and E^* be its dual. A subspace E' of E^* will be called *norming* if the equality $\|x\| = \sup\{\langle \varphi, x\rangle : \varphi \in E' \text{ and } \|\varphi\| \leq 1\}$ holds for all $x \in E$.

Of course, by Hahn-Banach's theorem, E^* itself is norming.

Lemma 5.2. *Let F be a separable Banach space. Then there exists a norming separable subspace F' of F^*.*

Proof. For every φ in the unit sphere Σ^* of F^*, the set

$$U_{k,\varphi} = \{y \in F : \|y\| = 1 \text{ and } \langle \varphi, y\rangle > 2^{-1/k}\}$$

is an open subset of the unit sphere S of F. By Hahn-Banach's theorem $S = \bigcup_{\varphi\in\Sigma^*} U_{k,\varphi}$ for every k. Since S is separable there is a countable set $A_k \subset \Sigma^*$ such that $S = \bigcup_{\varphi\in A_k} U_{k,\varphi}$. So the subspace F' spanned by $\bigcup_k A_k$ is norming. □

Denote by $\mathfrak{U}$ some free ultrafilter on $\mathbb{N}$, $\mathcal{Z}$ the closed subspace of $\ell^\infty(E)$ consisting of all E'-null sequences in E, $c_0^{\mathfrak{U}}(F) \subset \ell^\infty(F)$ the space of bounded sequences (z_n) in F such that $\lim_{n,\mathfrak{U}} z_n = 0$ and by $\mathcal{F}$ the quotient space $\ell^\infty(F)/c_0^{\mathfrak{U}}(F)$, which is an ultrapower of F.

Theorem 5.3. *Let E be a separable infinite-dimensional Banach space, E' a separable subspace of E^*, F be a separable Banach space, Δ a finite-dimensional compact subset of $\mathcal{L}(E,F)$ not containing 0. If there is no compact operator $T \in \Delta$ then there exists a sequence (w_n) in $\mathcal{Z}$ and $\varepsilon > 0$ such that $\|Tw_n\| \geq \varepsilon$ for all $T \in \Delta$ and all $n \in \mathbb{N}$.*

For every $\hat{w} = (w_n) \in \mathcal{Z}$ and $T \in \Delta$, the sequence (Tw_n) is bounded in F and we define $\hat{T}\hat{w} \in \mathcal{F}$ as the class of this sequence modulo $c_0^{\mathfrak{U}}(F)$. So $\hat{T}$ is a continuous linear operator from $\mathcal{Z}$ to $\mathcal{F}$: clearly $\|\hat{T}\| \leq \|T\|$.

Lemma 5.4. *Let $\hat{w} = (w_n) \in \mathcal{Z}$. If $\lim_{n\to\infty} \inf_{T\in\Delta} \|Tw_n\| = 0$ then there exists an $S \in \Delta$ such that $\hat{S}\hat{w} = 0$.*

Proof. Let $\varepsilon_n = \inf_{T\in\Delta} \|Tw_n\|$. By compactness of Δ there is some $T_n \in \Delta$ such that $\|T_n w_n\| = \varepsilon_n$. Define $S = \lim_{n,\mathfrak{U}} T_n \in \Delta$. Since

$$\|Sw_n\| \leq \|Sw_n - T_n w_n\| + \|T_n w_n\| \leq \|T_n - S\| \,.\, \sup_{n\in\mathbb{N}} \|w_n\| + \varepsilon_n$$

we get $(Sw_n) \in c_0^{\mathfrak{U}}$, hence $\hat{S}\hat{w} = 0 \in \mathcal{F}$. □

Proof of Theorem 5.3. If there are $\varepsilon > 0$ and $\hat{w} = (w_n) \in \mathcal{Z}$ such that $\limsup_n \inf_{T\in\Delta} \|Tw_n\| \geq 2\varepsilon$ then the set $D = \{n \in \mathbb{N} : \forall T \in \Delta \quad \|Tw_n\| \geq \varepsilon\}$ is infinite. Enumerate D into an increasing sequence (n_k) and put $w'_k = w_{n_k}$. Then $\widehat{w'} = (w'_k) \in \mathcal{Z}$ and we get $\|Tw'_k\| \geq \varepsilon$ for all k and all $T \in \Delta$ as wanted.

Assume towards a contradiction that there is no $\varepsilon > 0$ and $\hat{w} = (w_n) \in \mathcal{Z}$ such that $\limsup_n \|Tw_n\| \geq 2\varepsilon$ for all $T \in \Delta$. Then for every $\hat{w} = (w_n) \in \mathcal{Z}$ and $\varepsilon > 0$ we have $\limsup_{n\to\infty} \inf_{T\in\Delta} \|Tw_n\| < 2\varepsilon$, it is $\lim_{n\to\infty} \inf_{T\in\Delta} \|Tw_n\| = 0$ hence by Lemma 5.4: $\hat{S}\hat{w} = 0$ for some $S \in \Delta$. It follows that for each $\hat{w} \in \mathcal{Z}$ we can find $S \in \Delta$ such that $\hat{S}\hat{w} = 0$.

Consider the finite-dimensional subspace Λ of $\mathcal{L}(E,F)$ spanned by Δ and the continuous bilinear functional $\Phi\colon \Lambda \times \mathcal{Z} \to \mathcal{F}$ defined by $\Phi(T,\hat{w}) = \hat{T}\hat{w}$. It follows from what precedes that for every $\hat{w} \in \mathcal{Z}$ there exists $T \in \Delta$ such that $\Phi(T,\hat{w}) = 0$. Apply Theorem 3.1 to Λ, $\mathcal{F}$ and Δ: we can find some $\Theta \in \Delta$ and some subspace $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$ of finite codimension m such that $\Phi(\Theta,\tilde{w}) = 0$ for every $\tilde{w} \in \tilde{\mathcal{Z}}$.

If Θ was not compact we could find by Theorem 4.4 a subspace $\mathcal{Y}$ of dimension $m+1$ of $\mathcal{Z}$ such that $\inf_n \|\Theta v_n\| > 0$ for every non-zero element $v = (v_n)$ of $\mathcal{Y}$. Thus $\tilde{\mathcal{W}} \cap \mathcal{Y} \neq \{0\}$, and there exists a non-zero $\hat{v} \in \tilde{\mathcal{W}} \cap \mathcal{Y}$. So we would get

$$0 = \|\Phi(\Theta,\hat{v})\| = \lim_{n,\mathfrak{U}} \|\Theta v_n\| \geq \inf_n \|\Theta v_n\| > 0\ ,$$

a contradiction.

Notice that this necessary condition is also sufficient if it holds for all separable subspaces E' of E^*: indeed if some $S \in \Delta$ was compact it would exist by Theorem 4.4 some such E' such that $Sw_n \to 0$ for all $w = (w_n) \in \mathcal{Z}$; then for all sequence $(w_n) \in \mathcal{Z}$ we should have

$$\limsup_{n\to\infty} \big(\inf_{T\in\Delta} \|Tw_n\|\big) \leq \lim_{n\to\infty} \|Sw_n\| = 0\ . \qquad \square$$

Lemma 5.5. *Let E be a separable infinite-dimensional Banach space, F a separable Banach space, Δ a convex compact finite-dimensional subset of $\mathcal{L}(E,F)$ containing no compact operator. Assume that Y is a non-empty open subset of the normed space V, $\pi \in \mathcal{L}(E,V)$ is a compact operator with dense range and $X = \pi^{-1}(Y)$. Then $\sup_{x\in X} \inf_{T\in\Delta} \|Tx\| = +\infty$.*

Proof. Let $y_0 \in Y \cap \pi(E)$, $x_0 \in \pi^{-1}(y_0)$ and $r > 0$ such that the closed ball $\tilde{B}(y_0,r)$ is contained in Y. Choose, following Theorem 4.4, some separable subspace $E' \subset E^*$ such that every E'-null sequence (x_n) satisfies $\|\pi x_n\| \to 0$.

It follows from Theorem 5.3 (notice that Δ does not contain the compact operator 0) that one can find in B_E an E'-null sequence (w_n) and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ and all $T \in \Delta$ the inequality $\|Tw_n\| \geq \varepsilon$ holds. Then the sequence (πw_n) converges to 0 in V and we put for $n \in \mathbb{N}$:

$$x_n = x_0 + \frac{r}{2^{-n} + \|\pi w_n\|} \cdot w_n\ .$$

We then have $\|\pi x_n - y_0\| = \|\pi x_n - \pi x_0\| = \dfrac{r.\,\|\pi w_n\|}{2^{-n} + \|\pi w_n\|} < r$, hence $\pi x_n \in B(y_0,r) \subset Y$. For $x_n \in X$ and $T \in \Delta$ we get

$$\|Tx_n\| \geq \frac{r}{2^{-n} + \|\pi w_n\|} \cdot \|Tw_n\| - \|Tx_0\| \geq \frac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \|T\| \cdot \|x_0\|\ ,$$

hence $\inf_{T\in\Delta} \|Tx_n\| \geq \dfrac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \sup_{T\in\Delta} \|T\| \cdot \|x_0\|$ and

$$\sup_{x\in X} \inf_{T\in\Delta} \|Tx\| \geq \sup_n \frac{r.\varepsilon}{2^{-n} + \|\pi w_n\|} - \sup_{T\in\Delta} \|T\| \cdot \|x_0\| = +\infty\,,$$

and the proof of the lemma is complete. $\square$

Corollary 5.6. *Let E be a separable infinite-dimensional Banach space, F a separable Banach space, Δ a convex compact finite-dimensional subset of $\mathcal{L}(E,F)$ containing no compact operator and $\varphi\colon F \to \mathbb{R}$ a continuous and coercive function. Assume that Y is a non-empty convex open subset of the normed space V, $\pi \in \mathcal{L}(E,V)$ a compact operator with dense range and $X = \pi^{-1}(Y)$.*
Then $\sup_{x\in X} \inf_{T\in\Delta} \varphi(Tx) = +\infty$.

Proof. Since φ is coercive, for each $M \in \mathbb{R}^+$, there is some real $R > 0$ such that for all $u \in F$ $\varphi(u) < M \Longrightarrow \|u\| < R$. Following lemma 5.5, there exists $x \in X$ such that $\|Tx\| \geq R$ for all $T \in \Delta$; then we have for this vector x: $\inf_{T\in\Delta} \varphi(Tx) \geq M$. □

Lemma 5.7. *Let E be a separable infinite-dimensional Banach space, F and V be Banach spaces, $\Delta \subset \mathcal{L}(E,F)$ a compact convex set containing exactly one compact operator H, Y be a non-empty convex open subset of V, $\pi \in \mathcal{L}(E,V)$ a compact operator with dense range and $X = \pi^{-1}(Y)$. Then, for $T \neq H$ in Δ, we have* $\sup_{x\in X} \|Tx\| = +\infty$.

Proof. By hypothesis, for $T \in \Delta$ distinct from H, $\{T\}$ is a convex compact subset of $\mathcal{L}(E,F)$ and contains no compact operator: so it follows from Lemma 5.5 that

$$\sup_{x\in X} \|Tx\| = \sup_{x\in X} \inf_{S\in\{T\}} \|Sx\| = +\infty\,. \qquad \square$$

Corollary 5.8. *Let E be a separable infinite-dimensional Banach space, F and V be Banach spaces, $\Delta \subset \mathcal{L}(E,F)$ a compact convex set containing exactly one compact operator H, Y be a non-empty convex open subset of V, $\pi \in \mathcal{L}(E,V)$ be a compact operator with dense range and $X = \pi^{-1}(Y)$ and $\varphi\colon F \to \mathbb{R}$ a convex continuous and coercive function. Then, for $T \neq H$ in Δ,* $\sup_{x\in X} \varphi(Tx) = +\infty$.

Proof. Since φ is coercive, for each $M \in \mathbb{R}^+$, there is $R > 0$ such that for all $u \in F$ $\varphi(u) < M \Longrightarrow \|u\| < R$. Following Lemma 5.7, there exists $x \in X$ such that $\|Tx\| \geq R$; then we have for this vector x: $\varphi(Tx) \geq M$. □

6. Approximation theorems

The first lemma here is a variation of Lemma 10 in [5].

Lemma 6.1. *Let φ be a convex continuous and coercive function on the separable Banach space F. Then the conjugate function φ^* has a proper domain D which is a convex neighborhood of 0 in F^*. Moreover there exists a separable subspace F' of F^* such that for any dense subset Z of F', we have for all $x \in F$:*

$$\varphi(x) = \sup_{\xi\in Z} \langle \xi, x\rangle - \varphi^*(\xi).$$

Proof. Since φ^* is convex and the proper domain of φ^* is

$$D = \{\xi : \varphi^*(\xi) < +\infty\} = \bigcup_n \{\xi : \varphi^*(\xi) \leq n\}$$

it is clear that D is convex. Since φ is coercive the set $\{x : \varphi(x) \leq 1 + \varphi(0)\,\}$ is bounded in F, and there exists $R > 0$ such that $\varphi(x) < 1+\varphi(0) \Longrightarrow \|x\| < R$. By convexity we deduce that $\varphi(x) \geq \varphi(0)+\|x\|/R$ if $\|x\| \geq R$ hence that $\varphi(x)-\langle\xi, x\rangle$ is bounded from below outside the ball of radius R if $\xi \in F^*$ and $\|\xi\| < 1/R$.

The convex continuous function $x \mapsto \varphi(x) - \langle\xi, x\rangle$ is necessarily bounded from below on the bounded convex complete set $\widetilde{B}(0, R)$; it follows that $D \supset B(0, 1/R)$ hence that the interior D^o of D is non-empty. The convex l.s.c. function φ^* is finite on D. Hence, if L_m denotes the closed subset of D^o defined by $\{\xi \in D^o : \varphi^*(\xi) \leq m\}$ we have $D^o = \bigcup_{m\in\mathbb{N}} L_m$ and some L_m has non-empty interior by Baire's Category Theorem. So φ^* is bounded from above on a neighborhood of some point of D^o , hence is continuous on D^o. Since $\langle\xi, x\rangle \leq \varphi(x) + \varphi^*(\xi)$ for all $x \in F$ and all $\xi \in F^*$, we have $\varphi(x) \geq \sup_{\xi\in Z}\,[\langle\xi, x\rangle - \varphi^*(\xi)]$ for any $Z \subset F^*$.

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of open balls in F which constitutes a basis for the topology. Assume $F' \subset F^*$ and Z is dense in F'. Since D^o is open, $D^o \cap Z$ is dense in $D^o \cap F'$. By continuity of φ^* on D^o we necessarily have

$$\varphi(x) \geq \sup_{\xi\in D^o\cap Z} [\langle\xi, x\rangle - \varphi^*(\xi)] = \sup_{\xi\in D^o\cap F'} [\langle\xi, x\rangle - \varphi^*(\xi)] \ .$$

If $\alpha < \varphi(x)$, the point (x, α) does not belong to the epigraph G of φ, which is the closed convex subset $G = \{(u, t) : t \geq \varphi(u)\}$ of $F \times \mathbb{R}$. And there exists $(q, n) \in \mathbb{Q} \times \mathbb{N}$ such that $\alpha < q$, $x \in B_n$ and $\inf_{y\in B_n} \varphi(y) > q$.

Denote by A the countable set $\{(q, n) \in \mathbb{Q} \times \mathbb{N} : \inf_{y\in B_n} \varphi(y) > q\}$. Then it follows from Hahn-Banach's theorem that for $(q, n) \in A$ there exists $\xi_{q,n} \in F^*$ such that

$$\langle\xi_{q,n}, x\rangle - q > \sup_{(u,s)\in G} [\langle\xi_{q,n}, u\rangle - s] = \sup_{u\in F} [\langle\xi_{q,n}, u\rangle - \varphi(u)] = \varphi^*(\xi_{q,n})$$

what implies $\xi_{q,n} \in D$ and $\langle\xi_{q,n}, x\rangle - \varphi^*(\xi_{q,n}) > q$. Let F' be the linear space spanned by the linear functionals $(\xi_{q,n})_{(q,n)\in A}$. Thus

$$\varphi(x) = \sup_{\xi\in D\cap F'} [\langle\xi, x\rangle - \varphi^*(\xi)]\,.$$

For $\xi \in D \cap F'$, we have $t\xi \in D^o \cap F'$ for all $t \in [0, 1[$ since $0 \in D^o$. The function $t \mapsto \varphi^*(t\xi)$ is convex and l.s.c. on $[0, 1]$, hence continuous at 1, and it follows:

$$\langle\xi, x\rangle - \varphi^*(\xi) = \lim_{t\to1,t<1} \langle t\xi, x\rangle - \varphi^*(t\xi) \leq \sup_{\xi\in D^o\cap F'} [\langle\xi, x\rangle - \varphi^*(\xi)] \ ,$$

hence that

$$\begin{aligned}\varphi(x) = \sup_{\xi\in D\cap F'} [\langle\xi, x\rangle - \varphi^*(\xi)] &\leq \sup_{\xi\in D^o\cap F'} [\langle\xi, x\rangle - \varphi^*(\xi)] \\ &\leq \sup_{\xi\in Z} [\langle\xi, x\rangle - \varphi^*(\xi)] \leq \varphi(x)\,,\end{aligned}$$

the desired conclusion. □

From now on we are interested in the following setting: Δ is a finite-dimensional convex compact subset of $\mathcal{L}(E,F)$ *containing only one compact operator* H and D is the linear subspace of $\mathcal{L}(E,F)$ generated by $\Delta - H = \{T - H : T \in \Delta\}$. We denote by $\mathcal{K} = \mathcal{K}(E,F)$ the set of compact operators from E to F which is a closed linear subspace of $\mathcal{L}(E,F)$ and $D_0 = D \cap \mathcal{K}$. Of course it can happen that $D_0 \neq \{0\}$ even though $(H + \mathcal{K}) \cap \Delta = \{H\}$. We are also given a convex continuous function $\psi\colon \Delta \to \mathbb{R}$.

Definition 6.2. A finite family $(S_j)_{1 \le j \le \ell}$ will be said to be *adapted to* Δ if it is a basis of D and $(S_j)_{r < j \le \ell}$ is a basis of D_0.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \mathbb{R}^\ell$ denote $S_\lambda = \sum_j \lambda_j S_j$. We equip $\mathbb{R}^\ell$ with its canonical euclidean structure induced by the scalar product $\langle \lambda, \mu \rangle = \sum_{j \le \ell} \lambda_j \mu_j$, and define Σ its unit sphere. Moreover for $\xi \in F^*$ define $T_\xi \in \mathcal{L}(E, \mathbb{R}^\ell)$ by

$$T_\xi(x) = \big(\langle \xi, S_1 x\rangle, \langle \xi, S_2 x\rangle, \dots, \langle \xi, S_\ell x\rangle\big) = \big(\langle S_1^* \xi, x\rangle, \langle S_2^* \xi, x\rangle, \dots, \langle S_\ell^* \xi, x\rangle\big) \in \mathbb{R}^\ell$$

for $x \in E$. Let Γ be the compact convex subset $\{\lambda \in \mathbb{R}^\ell : H + S_\lambda \in \Delta\}$ of $\mathbb{R}^\ell$ and $\chi\colon \Gamma \to \mathbb{R}$ be then defined by $\chi(\lambda) = \psi(H + S_\lambda)$.

Let $\varpi\colon \mathbb{R}^\ell \to \mathbb{R}^r$ be the mapping $\lambda \mapsto (\lambda_1, \lambda_2, \dots, \lambda_r)$, Γ_0 be the subspace $\ker(\varpi)$ of $\mathbb{R}^\ell$, and $\Gamma_1 = \Gamma_0^\perp$ be the subspace $\varpi^*(\mathbb{R}^r) = \{\lambda \in \mathbb{R}^\ell : \forall j > r\ \lambda_j = 0\}$ of $\mathbb{R}^\ell$. Then $q = \varpi^* \varpi\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ is the orthogonal projection onto Γ_1.

Lemma 6.3. *Let f be a linear functional on Γ_0 and g be a convex continuous real function on Γ such that $g(0) > 0$. Then there exists a $\mu \in \mathbb{R}^\ell$ such that $f(\lambda) = \langle \mu, \lambda \rangle$ for all $\lambda \in \Gamma_0$ and $\inf_{\lambda \in \Gamma}\big(g(\lambda) - \langle \mu, \lambda \rangle\big) > 0$.*

Proof. Consider the two convex subsets of $\mathbb{R}^\ell \times \mathbb{R}$ defined by

$$\begin{cases} \Gamma_0' = \{(\lambda, t) : \lambda \in \Gamma_0 \text{ and } t = f(\lambda)\,\} \\ C_+ = \{(\lambda, t) : \lambda \in \Gamma \text{ and } g(\lambda) \le t \le t^*\,\}\ . \end{cases}$$

where $t^* = \sup_{\lambda \in \Gamma} g(\lambda)$. The first one is a linear subspace and the second one is compact. Moreover they are disjoint: indeed if $(\lambda, t) \in \Gamma_0' \cap C_+$ we have $H + S_\lambda \in D_0 \cap \Delta$ hence $\lambda = 0$, thus $t = f(\lambda) = 0 < g(0) \le t = 0$. Notice also that $\big(\lambda, g(\lambda)\big) \in C_+$ for all $\lambda \in \Gamma$.

Then it follows from Hahn-Banach's theorem that there exists a linear functional σ on $\mathbb{R}^\ell \times \mathbb{R}$, of the form $\sigma(\lambda, t) = \rho.t - \langle \mu, \lambda \rangle$ for some $\rho \in \mathbb{R}$ and $\mu \in \mathbb{R}^\ell$, which separates Γ_0' from C_+:

$$\sup_{(\lambda,t) \in \Gamma_0'} (\rho.t - \langle \mu, \lambda \rangle) < \inf_{(\lambda,t) \in C_+} (\rho.t - \langle \mu, \lambda \rangle)\ .$$

Since Γ_0' is a linear subspace we get $\sigma(\lambda, t) = 0$ for all $(\lambda, t) = (\lambda, f(\lambda)) \in \Gamma_0'$ and since $(\lambda, g(\lambda)) \in C_+$ for $\lambda \in \Gamma$ we also get $0 < \rho.g(0)$. So $\rho > 0$ and replacing μ by $\rho^{-1}.\mu$ we get: $\sup_{(\lambda,t) \in \Gamma_0'} t - \langle \mu, \lambda \rangle = 0 < \eta := \inf_{\lambda \in \Gamma} g(\lambda) - \langle \mu, \lambda \rangle$, hence $g(\lambda) \ge \langle \mu, \lambda \rangle + \eta$ for $\lambda \in \Gamma$, and $f(\lambda) = \langle \mu, \lambda \rangle$ for $\lambda \in \Gamma_0$. $\square$

Lemma 6.4. *If* $\mu_0 \in \mathbb{R}^\ell$ *and* $\delta > 0$ *there exists* $\nu \in \Gamma_1$ *such that:*

$$\inf_{\lambda\in\Gamma} \langle \nu + \mu_0, \lambda\rangle \geq -\delta\,.$$

Proof. Apply Lemma 6.3 with $f\colon \lambda \mapsto -\langle \mu_0, \lambda\rangle$ on Γ_0 and g constantly equal to δ on Γ. We get $\mu \in \mathbb{R}^\ell$ such that $\langle \mu, \lambda\rangle = -\langle \mu_0, \lambda\rangle$ for $\lambda \in \Gamma_0$ and $\langle \mu, \lambda\rangle > \delta$ for $\lambda \in \Gamma$. Then let $\nu = -(\mu + \mu_0)$: we get $\nu + \mu_0 = -\mu$, so $\langle \nu, \lambda\rangle = 0$ for $\lambda \in \Gamma_0$, hence $\nu \in \Gamma_0^\perp = \Gamma_1$. Moreover for $\lambda \in \Gamma$: $\delta = g(\lambda) > \langle \mu, \lambda\rangle = -\langle \nu + \mu_0, \lambda\rangle$. Thus $\inf_{\lambda\in\Gamma} \langle \nu + \mu_0, \lambda\rangle \geq -\delta$. □

Lemma 6.5. *For fixed* $\varepsilon > 0$ *there exists* $\hat\mu \in \Gamma_1$ *such that* $\langle \lambda, \hat\mu\rangle \leq \chi(\lambda) - \psi(H) + \varepsilon$ *for all* $\lambda \in \Gamma$.

Proof. Apply Lemma 6.3 with $f = 0$ and $g(\lambda) = \chi(\lambda) - \psi(H) + \varepsilon$. Notice that $g(0) = \varepsilon > 0$. We get $\hat\mu \in \mathbb{R}^\ell$ such that $\langle \hat\mu, \lambda\rangle < g(\lambda) = \chi(\lambda) - \psi(H) + \varepsilon$ for $\lambda \in \Gamma$, and $\langle \hat\mu, \lambda\rangle = 0$ for $\lambda \in \Gamma_0$, hence $\hat\mu \in \Gamma_0^\perp = \Gamma_1$. □

Theorem 6.6. *Let* E *and* F *be infinite-dimensional separable Banach spaces,* Δ *a finite-dimensional convex compact subset of* $\mathcal{L}(E, F)$ *containing only one compact operator* H *and* $(S_j)_{1\leq j\leq \ell}$ *an adapted family,* $K \in \mathcal{L}(E, V)$ *a compact operator,* F' *a separable linear subspace of* F^**. Then there exists a separable closed subspace* F_1 *of* F^* *containing* F' *such that for comeagerly many* $\xi \in F_1$ *there exists for all* $\lambda \in \Gamma_1$ *a sequence* (w_n) *in* E *such that* $Kw_n \to 0$ *and* $T_\xi(w_n) \to \lambda$.

Proof. It is clear that $\xi \mapsto T_\xi$ is linear and continuous from F^* to $\mathcal{L}(E, \mathbb{R}^\ell)$. For $x \in E$ denote $|||x||| = \|Kx\|$; this defines on E a coarser norm whose unit ball will be denoted by B_K. And we will say that a sequence (x_n) is K-null if $\sup_n \|x_n\| < +\infty$ and $\lim_n |||x_n||| = 0$. Since F' is separable we can replace F' by some larger separable space F_1 which is norming and closed in F^*, and find a dense sequence (ξ_m) in F_1, so that $\|y\| = \sup_m\{|\langle \xi_m, y\rangle| : \|\xi_m\| \leq 1\}$ for all $y \in F$. Thus the set $\{S_j^*\xi_m : 1 \leq j \leq \ell,\ m \in \mathbb{N}\} \subset E^*$ is countable and can be enumerated in a sequence $(\theta_n)_{n\in\mathbb{N}}$.

Denote F_0 the linear space generated by the ξ_m's and U the compact linear operator $E \to \ell^2$ defined by

$$Ux = \Big(\frac{\langle \theta_n, x\rangle}{1 + 2^n\,\|\theta_n\|}\Big)_n\,.$$

Define $V' = V \times F^{\ell-r} \times \ell^2$, equipped with the norm: $\|(v, (y_j), u)\| = \|v\| + \sum_j \|y_j\| + \|u\|_2$. Replacing K by $K'\colon x \mapsto (Kx, (S_jx)_{j>r}, Ux)$ from E to V' which is compact too, we see that any K'-null sequence is K-null and that for all $x \in E$: $|\langle \theta_n, x\rangle| \leq 2^n \cdot \|Ux\| \leq 2^n \cdot \|K'x\|$. It follows that, for all $\xi \in F_0$ and all $j \leq \ell$, $S_j^*\xi$ is continuous on $(E, |||.|||)$. Moreover for $j > r$ and $\xi \in F^*$ $S_j^*\xi$ is also continuous on $(E, |||.|||)$.

Then for $\xi \in F_0$ there exists some constant β (depending on ξ) such that for all $\mu \in \mathbb{R}^\ell$ and all x we have $\big|\langle S_\mu^*\xi, x\rangle\big| \leq \beta.\,\|\mu\|\,.\,\|Kx\|$. We also have $\|S_jx\| \leq \|K'x\|$ for $j > r$. Let E' be the closed subspace of E^* spanned by the θ_n's.

Denote $R = \big(\sum_j \|S_j\|^2\big)^{1/2}$: we have for all $x \in E$ and $\lambda \in \mathbb{R}^\ell$:

$$\begin{aligned} \|S_\lambda x\| = \Big\|\sum\nolimits_j \lambda_j S_j x\Big\| &\le \sum |\lambda_j| \,.\, \|S_j\| \,.\, \|x\| \\ &\le (\sum \lambda_j^2)^{1/2} (\sum \|S_j\|^2)^{1/2} \, \|x\| = R.\, \|\lambda\| \,.\, \|x\| \;. \end{aligned}$$

Since $\Sigma_1 = \Sigma \cap \Gamma_1$ is compact and none S_λ is compact for $\lambda \in \Sigma_1$ it follows from Theorem 5.3 that there exists $\delta > 0$ and an E'-null sequence (x_n) in the unit sphere of E such that $\|S_\lambda x_n\| \ge \delta$ for all n and all $\lambda \in \Sigma_1$.

It follows from Theorem 4.4 that $K' x_n \to 0$, and a fortiori $K x_n \to 0$. Passing to a subsequence we can assume moreover that $|||x_n||| = \|K x_n\| \le 2^{-n}$.

For the further argumentation within the proof of Theorem 6.6 the following Lemma is helpful.

Lemma 6.7. *Let $\xi \in F_1$. If $q \circ T_\xi(B_K)$ is not the whole Γ_1, there exists $\mu \in \Sigma_1$ and $\beta > 0$ such that $\big|\langle S_\mu^* \xi, x\rangle\big| \le \beta.\, |||x|||$ holds for all $x \in E$.*

Proof. Consider the symmetric convex set $C = \varpi \circ T_\xi(B_K) \subsetneq \mathbb{R}^r$. If we had $\varpi^{-1}\big(\overline{C}\big) = \mathbb{R}^\ell$, we would necessarily have $\overline{C} = \mathbb{R}^r$ hence $C = \mathbb{R}^r$. Thus there exists $\lambda_0 \in \mathbb{R}^\ell \backslash \varpi^{-1}(\overline{C})$, and following Hahn-Banach's theorem, some $\mu \in \mathbb{R}^\ell$ such that $\sup_{\varpi(\lambda)\in\overline{C}} \langle \mu, \lambda\rangle < \beta := \langle \mu, \lambda_0\rangle$. We necessarily have $\mu \ne 0$ and $\varpi(\mu) = 0$. Replace μ by $\mu/\|\mu\|$ and β by $\beta/\|\mu\|$: then $\mu \in \Sigma_1$.

It follows that, for all $x \in B_K$ we have: $\langle S_\mu^* \xi, x\rangle = \sum_j \mu_j \langle \xi, S_j x\rangle = \langle \mu, T_\xi(x)\rangle \le \beta$, and by homogeneity $\big|\langle S_\mu^* \xi, x\rangle\big| \le \beta.\, \|Kx\| = \beta.\, |||x|||$. ☐

We now continue the proof of Theorem 6.6. By the compactness of Σ_1, one can find a finite family $(\mu_k)_{k\le N}$ in Σ_1 such that Σ_1 is covered by the N balls $B(\mu_k, \delta/(4R))$. Then consider the set

$$M_{k,m} = \{\xi \in F_1 : \exists \mu \in \Sigma_1 \;\; \|\mu - \mu_k\| \le \frac{\delta}{4R} \text{ and } \forall x \in E \;\; \big|\langle S_\mu^* \xi, x\rangle\big| \le m \, \|Kx\| \,\} \;.$$

The set

$$N_{k,m} = \{(\mu, \xi) \in \Sigma_1 \times F_1 : \|\mu - \mu_k\| \le \frac{\delta}{4R} \text{ and } \forall x \in B_K \;\; |\langle \xi, S_\mu x\rangle| \le m.\, \|Kx\| \,\}$$

is clearly closed in $\Sigma_1 \times F_1$ and the second projection $\pi_2 : (\mu, \xi) \to \xi$ is a perfect mapping since Σ_1 is compact. Thus $M_{k,m} = \pi_2(N_{k,m})$ is closed in F_1. Let M be the set of those $\xi \in F_1$ for which $\varpi \circ T_\xi(B_K) \ne \mathbb{R}^r$. It follows from what precedes that $M \subset \bigcup_{k,m} M_{k,m}$.

We now show that each $M_{k,m}$ has empty interior in $F_1 = \overline{F_0}$: indeed if not it would exist some $\xi_0 \in F_0$ and $\rho > 0$ such that $F_1 \cap B(\xi_0, \rho) \subset M_{k,m}$. For any χ in the unit ball of F_1 let $\xi = \xi_0 + \rho.\chi$: then $\xi \in B(\xi_0, \rho)$. Since $\xi_0 \in F_0$, there exists some constant β such that $\left|\langle S_\mu^*\xi_0, x\rangle\right| \le \beta\,|||x|||$ for all $\mu\in\Sigma$ and $x\in E$, hence

$$\begin{aligned}\langle S_\mu^*\xi, x_n\rangle &\ge \rho.\langle\chi, S_{\mu_k}x_n\rangle - \rho.\left|\langle\chi, S_{\mu-\mu_k}x_n\rangle\right| - \left|\langle S_\mu^*\xi_0, x_n\rangle\right|\\ &\ge \rho.\langle\chi, S_{\mu_k}x_n\rangle - \rho\left\|S_{\mu-\mu_k}\right\|.\left\|x_n\right\| - \beta\,|||x_n|||\\ &\ge \rho.\langle\chi, S_{\mu_k}x_n\rangle - \rho R\left\|\mu-\mu_k\right\| - \beta\,|||x_n|||\\ &\ge \rho.\langle\chi, S_{\mu_k}x_n\rangle - \rho\delta/4 - \beta.2^{-n}\end{aligned}$$

Choose $n\in\mathbb{N}$ such that $(m+\beta).2^{-n} < \rho\delta/4$. We can then find $\chi\in F_1$ such that $\|\chi\|\le 1$ and $\langle\chi, S_{\mu_k}x_n\rangle > (3/4)\left\|S_{\mu_k}x_n\right\|$, hence

$$\rho.\langle\chi, S_{\mu_k}x_n\rangle \ge \frac{3\rho}{4}.\left\|S_{\mu_k}x_n\right\| \ge \frac{3\rho\delta}{4}\,.$$

We then get

$$\langle S_\mu^*\xi, x_n\rangle - m.\left\|Kx_n\right\| \ge \langle S_\mu^*\xi, x_n\rangle - m.2^{-n} \ge \frac{3\rho\delta}{4} - \frac{\rho\delta}{4} - (m+\beta).2^{-n} > \frac{\rho\delta}{4} > 0$$

what shows that $\xi\notin M_{k,n}$, a contradiction. It follows that each $M_{k,m}$ is nowhere dense and that M is meager. In particular $Z = F_1\setminus M$ is dense (and even comeager) in F_1. And for every $\xi\in Z$, $n\in\mathbb{N}$ and $\lambda\in\Gamma_1$, $2^n.\lambda\in\Gamma_1\subset T_\xi(B_k)$.

So there is some $x_n\in B_K$ such that $\varpi\circ T_\xi(x_n) = \varpi(2^n.\lambda)$. Then $w_n = 2^{-n}x_n$ satisfies $|||w_n|||\le 2^{-n}$ and $\varpi(T_\xi w_n - \lambda) = 0$, hence $S_j^*\xi = \lambda_j$ for $j\le r$. Moreover, for $j>r$, $\left\|S_j^*\xi w_n\right\| \le |||w_n||| \to 0$. Thus $T_\xi w_n\to\lambda$. And this completes the proof of Theorem 6.6. ☐

7. The minimax theorem

Lemma 7.1. *Let E be a separable infinite-dimensional Banach space, F be a Banach space, Δ be a finite-dimensional convex compact subset of $\mathcal{L}(E,F)$ containing only one compact operator H, $(S_j)_{1\le j\le\ell}$ be an adapted family, Y be a non-empty convex open subset of the normed space V and $\pi\colon E\to V$ be a compact operator with dense range, $X_0 = \pi^{-1}(Y)$, F' be a separable subspace of F^* and $\psi\colon\Delta\to\mathbb{R}$ be a convex continuous function. Then there exists a separable closed subspace F_1 of F^* containing F' and a dense subset Z of F_1 such that for all $\xi\in Z$*

$$\sup_{x\in X_0}\inf_{\lambda\in\Gamma}\Big(\langle\xi,(H+S_\lambda)x\rangle + \chi(\lambda)\Big) \ge \sup_{x\in X_0}\langle\xi, Hx\rangle + \psi(H)\ .$$

Proof. Let $\xi\in Z$, $\lambda\in\Gamma$, $x_0\in X_0$, $y_0=\pi(x_0)\in Y$ and $\varepsilon>0$, and fix $\hat\mu\in\Gamma_1$ such that $\langle\lambda,\hat\mu\rangle\le\chi(\lambda)-\psi(H)+\varepsilon/2$ for all $\lambda\in\Gamma$, following Lemma 6.5. Choose also by Lemma 6.4 some $\nu\in\Gamma_1$ such that $\inf_{\lambda\in\Gamma}\langle T_\xi(x_0)+\nu,\lambda\rangle\ge -\varepsilon/2$.

Define K the operator $x\mapsto(Hx,\pi x)$ from E to the product space $W = F\times V$ normed by $\|(y,u)\| = \|y\| + \|u\|$ and recall that $\chi(\lambda) = \psi(H+S_\lambda)$; Since H and π are compact, K is compact too and by Theorem 6.6 one can find a separable closed subspace F_1 of F^* containing F' and a dense subset Z of F_1 such that for fixed $\xi \in Z$ and $\mu \in \Gamma_1$ there is a sequence (w_n) in E such that $\|Kw_n\| \to 0$ and that $\mu_n = T_\xi(w_n) \to \nu - \hat{\mu}$.

Then with $\theta = S_\lambda^* \xi$, $x_n = x_0 + w_n$ satisfies

$$\begin{aligned}\langle \theta, x_n \rangle = \langle S_\lambda^* \xi, x_n \rangle = \langle \xi, S_\lambda(x_0 + w_n) \rangle &= \langle \lambda, T_\xi(x_0) \rangle + \langle \lambda, T_\xi(w_n) \rangle \\ = \langle \lambda, T_\xi(x_0) \rangle + \langle \lambda, \mu_n \rangle &\to \langle \lambda, T_\xi(x_0) + \nu \rangle - \langle \lambda, \hat{\mu} \rangle \geq -\langle \lambda, \hat{\mu} \rangle - \frac{\varepsilon}{2}\end{aligned}$$

and $\pi x_n = y_0 + \pi w_n \to y_0$. Because Y is open in V we have $\pi x_n \in Y$ for n large enough, hence $x_n \in X_0$. Moreover

$$\langle \xi, Hx_n \rangle = \langle \xi, Hx_0 \rangle + \langle \xi, Hw_n \rangle \to \langle \xi, Hx_0 \rangle \ ,$$

from what it follows that for $\lambda \in \Gamma$,

$$\begin{aligned}\langle \xi, (H + S_\lambda)x_n \rangle + \chi(\lambda) &= \langle \xi, Hx_n \rangle + \langle \xi, S_\lambda x_n \rangle + \chi(\lambda) \\ &\geq \langle \xi, Hx_n \rangle + \langle \theta, x_n \rangle + \langle \lambda, \hat{\mu} \rangle + \psi(H) - \frac{\varepsilon}{2} \\ &\geq \langle \xi, Hx_n \rangle + \psi(H) - \varepsilon \to \langle \xi, Hx_0 \rangle + \psi(H) - \varepsilon\end{aligned}$$

thus

$$\sup_{x \in X_0} \inf_{\lambda \in \Gamma} \Big(\langle \xi, (H + S_\lambda)x \rangle + \chi(\lambda) \Big) \geq \langle \xi, Hx_0 \rangle + \psi(H) - \varepsilon$$

and since this holds for all $x_0 \in X_0$ and $\varepsilon > 0$ we get

$$\sup_{x \in X_0} \inf_{\lambda \in \Gamma} \Big(\langle \xi, (H + S_\lambda)x \rangle + \chi(\lambda) \Big) \geq \sup_{x \in X_0} \langle \xi, Hx \rangle + \psi(H) \, ,$$

the desired inequality. ☐

Lemma 7.2. *Let E and F be separable infinite-dimensional Banach spaces, Δ be a finite-dimensional convex compact subset of $\mathcal{L}(E, F)$ containing only one compact operator H, $(S_j)_{1 \leq j \leq \ell}$ be an adapted family, Y be a non-empty convex open subset of the normed space V and $\pi\colon E \to V$ be a compact operator with dense range, $X_0 = \pi^{-1}(Y)$, $\varphi\colon F \to \mathbb{R}$ be a convex continuous and coercive function and $\psi\colon \Delta \to \mathbb{R}$ be a convex continuous function. Then*

$$\sup_{\pi(x) \in Y} \inf_{\lambda \in \Gamma} \varphi\big(Hx + S_\lambda x\big) + \psi(H + S_\lambda) \geq \psi(H) + \sup_{\pi(x) \in Y} \varphi(Hx) \ .$$

Proof. Put $\Phi(\lambda, x) = \varphi\big(Hx + S_\lambda x\big)$ and $\chi(\lambda) = \psi(H + S_\lambda)$. It follows from lemma 6.1 that, for some dense subset Z of some closed separable linear subspace F_1 of F^*, we have

$$\Phi(\lambda, x) = \sup_{\xi \in Z} \langle \xi, Hx + S_\lambda x \rangle - \varphi^*(\xi) \ ,$$

hence

$$\begin{aligned}\inf_{\lambda\in\Gamma}\big(\Phi(\lambda,x)+\chi(\lambda)\big) &= \inf_{\lambda\in\Gamma}\sup_{\xi\in Z}\Big(\langle\xi,Hx+S_\lambda x\rangle-\varphi^*(\xi)+\chi(\lambda)\Big)\\ &\geq \sup_{\xi\in Z}\inf_{\lambda\in\Gamma}\Big(\langle\xi,Hx+S_\lambda x\rangle-\varphi^*(\xi)+\chi(\lambda)\Big)\\ &\geq \sup_{\xi\in Z}\Big(-\varphi^*(\xi)+\inf_{\lambda\in\Gamma}\big(\langle\xi,(H+S_\lambda)x\rangle+\chi(\lambda)\big)\Big)\,,\end{aligned}$$

and by Lemma 7.1,

$$\begin{aligned}\sup_{\pi(x)\in Y}\inf_{\lambda\in\Gamma}\big(\Phi(\lambda,x)+\chi(\lambda)\big) &\geq \sup_{\pi(x)\in Y}\sup_{\xi\in Z}\Big(-\varphi^*(\xi)+\inf_{\lambda\in\Gamma}\big(\langle\xi,(H+S_\lambda)x\rangle+\chi(\lambda)\big)\Big)\\ &\geq \sup_{\xi\in Z}\Big(-\varphi^*(\xi)+\sup_{\pi(x)\in Y}\inf_{\lambda\in\Gamma}\big(\langle\xi,(H+S_\lambda)x\rangle+\chi(\lambda)\big)\Big)\\ &\geq \sup_{\xi\in Z}\big(-\varphi^*(\xi)+\psi(H)+\sup_{\pi(x)\in Y}\langle\xi,Hx\rangle\big)\\ &\geq \psi(H)+\sup_{\pi(x)\in Y}\big(\sup_{\xi\in Z}\langle\xi,Hx\rangle-\varphi^*(\xi)\big)\,,\\ &= \psi(H)+\sup_{\pi(x)\in Y}\varphi(Hx)\end{aligned}$$

the desired inequality. □

Theorem 7.3. *Let E and F be separable infinite-dimensional Banach spaces, Δ be a finite-dimensional compact convex subset of $\mathcal{L}(E,F)$, $\varphi\colon F\to\mathbb{R}$ be a convex continuous and coercive function, X be a convex subset of E whose interior is non-empty for the topology $\beta(E,E^*)$ and $\psi\colon\Delta\to\mathbb{R}$ be a convex continuous function. Assume that Δ contains at most one compact operator. Then the following holds:*

$$\sup_{x\in X}\inf_{T\in\Delta}\big(\varphi(Tx)+\psi(T)\big)=\inf_{T\in\Delta}\sup_{x\in X}\big(\varphi(Tx)+\psi(T)\big)\ .$$

Proof. By lemma 4.6, we know that there exists a normed space V, a compact linear mapping $\pi\colon E\to V$ with dense range and a convex open subset Y of V such that $\pi^{-1}(Y)\subset X\subset\pi^{-1}(\overline{Y})$. It is then easy to see that the supremum over X is equal to the supremum over $\pi^{-1}(Y)$, and we will only prove the statement when $X=\pi^{-1}(Y)$. The inequality

$$\sup_{x\in X}\inf_{T\in\Delta}\big(\varphi(Tx)+\psi(T)\big)\leq\inf_{T\in\Delta}\sup_{x\in X}\big(\varphi(Tx)+\psi(T)\big)$$

is standard. So we have only to prove the converse inequality.

Following Corollary 5.6, it is enough to consider the case where the convex compact set Δ contains exactly one compact operator H.

Then by Corollary 5.8, we have

$$\inf_{T\in\Delta}\sup_{x\in X}\big(\varphi(Tx)+\psi(T)\big)=\sup_{x\in X}\big(\varphi(Hx)+\psi(H)\big)\ ,$$

and it follows from Lemma 7.2 that

$$\begin{aligned}\sup_{\pi(x)\in Y} \varphi(Hx)+\psi(H) &\leq \sup_{\pi(x)\in Y} \inf_{\lambda\in\Gamma} \varphi\big(Hx+S_\lambda x\big)+\psi(H+S_\lambda)\\ &= \sup_{\pi(x)\in Y} \inf_{T\in\Delta} \varphi(Tx)+\psi(T),\end{aligned}$$

and this completes the proof. □

The following statement extends Theorem 7.3 by removing the hypotheses of separability on E and F.

Theorem 7.4. *Let E and F be infinite-dimensional Banach spaces, Δ be a compact convex finite-dimensional subset of $\mathcal{L}(E,F)$, $\varphi\colon F\to\mathbb{R}$ be a convex continuous and coercive function, X be a convex subset of E whose interior is non-empty for the topology $\beta(E,E^*)$ and $\psi\colon \Delta\to\mathbb{R}$ be a convex continuous function. Assume that Δ contains at most one compact operator. Then the following holds:*

$$\sup_{x\in X} \inf_{T\in\Delta} \big(\varphi(Tx)+\psi(T)\big) = \inf_{T\in\Delta} \sup_{x\in X} \big(\varphi(Tx)+\psi(T)\big)\ .$$

Proof. If E is separable, we can assume F is also separable: indeed $G=\overline{\sum_j S_j(E)}$ is a separable Banach space, and $\varphi_{|G}$ is convex continuous and coercive.

From now on assume E is not separable. The function $(x,T)\mapsto\varphi(Tx)+\psi(T)$ is continuous on $X\times\Delta$ and Δ is compact.

It follows that the function $\Phi\colon x\mapsto \inf_{T\in\Delta}\big(\varphi(Tx)+\psi(T)\big)$ is continuous on X. So there is some countable set $D\subset X$ such that $\sup_{x\in X}\Phi(x)=\sup_{x\in D}\Phi(x)$. And for any separable subspace $E_0\subset E$ containing D we also have

$$\sup_{x\in X}\Phi(x)=\sup_{x\in D}\Phi(x)\leq \sup_{x\in X\cap E_0}\Phi(x)\leq \sup_{x\in X}\Phi(x)$$

hence $\sup_{x\in X\cap E_0}\Phi(x)=\sup_{x\in X}\Phi(x)$.

Assume that for all $T\neq H$ in Δ, T is not compact: then for all $T\in\Delta$ there is $\delta>0$ and a sequence (x_n) in the unit ball of E such that $\|Tx_n-Tx_p\|\geq 3\delta$ if $n\neq p$. If $S\in\Delta$ and $\|S-T\|<\delta$ we have for $n\neq p$:

$$\|Sx_n-Sx_p\|\geq\|Tx_n-Tx_p\|-\|S-T\|\,.\,\|x_n\|-\|S-T\|\,.\,\|x_p\|>3\delta-\delta-\delta=\delta\ .$$

It follows that $\Delta\setminus\{H\}$ is covered by balls $B(T,\delta_T)$ such that exists a sequence $x_n^{(T)}$ in the unit ball of E with $\left\|Sx_n^{(T)}-Sx_p^{(T)}\right\|>\delta_T$ for $n\neq p$ and $S\in B(T,\delta_T)$. By Lindelöf's theorem $\Delta\setminus\{H\}$ is covered by some countable subfamily: so there is some countable set $\Gamma_1\subset E$ such that for each $T\in\Delta\setminus\{H\}$ one can find an infinite subset J of Γ_1 satisfying $\inf_{x\neq y,\ x,y\in J}\|Tx-Ty\|>0$. It follows that for every separable subspace E_1 of E containing Γ_1 and every $T\in\Delta\setminus\{H\}$ the restriction $T_{|E_1}$ is not compact.

There is a countable $D' \subset X$ such that

$$\sup_{x\in D'} \varphi(Hx) + \psi(H) = \sup_{x\in X} \varphi(Hx) + \psi(H)$$

Choosing a separable subspace E_1 containing $D \cup \Gamma_1 \cup D'$, and applying Theorem 7.3 to E_1, we get

$$\begin{aligned}\sup_{x\in X} \inf_{T\in\Delta} \big(\varphi(Tx) + \psi(T)\big) &= \sup_{x\in X\cap E_1} \inf_{T\in\Delta} \big(\varphi(Tx) + \psi(T)\big) \\ &= \inf_{T\in\Delta} \sup_{x\in X\cap E_1} \big(\varphi(Tx) + \psi(T)\big) = \inf_{T\in\Delta} \sup_{x\in X} \big(\varphi(Tx) + \psi(T)\big)\end{aligned}$$

what completes the proof of the general case. □

8. Counter-examples

The hypothesis made on the set Δ to contain at most one compact operator could seem quite artificial. In fact without this hypothesis on Δ the statement of Theorem 7.3 becomes false, as shown by the following example, extracted from [5], where S and T are compact.

Example 8.1. There exist two continuous linear operators S and T from the Hilbert space $\mathcal{E} = \ell^2$ to a Banach space $\mathcal{F}$ isomorphic to ℓ^2, X a convex subset of $\mathcal{E}$ whose interior is non-empty for $\beta(\mathcal{E}, \mathcal{E}^*)$, a compact interval $J = [0, 2]$ and a convex continuous and coercive function $\varphi\colon \mathcal{F} \to \mathbb{R}$ such that

$$\sup_{x\in X} \inf_{\lambda\in J} \big(\varphi(Tx - \lambda Sx)\big) < \inf_{\lambda\in J} \sup_{x\in X} \big(\varphi(Tx - \lambda Sx)\big)\,. \qquad \square$$

If one replaces the segment $\{T - \lambda S : \lambda \in J\} \subset \mathcal{L}(E, F)$ by some 2-dimensional simplex $\Delta = \{\lambda_0 T_0 + \lambda_1 T_1 + \lambda_2 T_2 : \lambda_j \geq 0,\ \sum \lambda_j = 1\}$, even adding the hypotheses that the mapping $x \mapsto (T_j x)_{j\leq 2}$ is an isomorphism from E onto its range in $\mathcal{F}^3$ and that X is a convex subset of E with non-empty interior for $\beta(E, E^*)$, the minimax equality can no longer hold. The construction is similar as in Example 8.1:

Example 8.2. There exist two continuous linear operators S and T from the Hilbert space $\mathcal{E} = \ell^2$ to a Banach space $\mathcal{F}$, X a convex subset of $\mathcal{E}$ whose interior is non-empty for $\beta(\mathcal{E}, \mathcal{E}^*)$, a compact interval $J \subset \mathbb{R}$ and a convex continuous and coercive function $\varphi\colon \mathcal{F} \to \mathbb{R}$ such that

$$\sup_{x\in X} \inf_{\lambda\in J} \big(\varphi(Tx - \lambda Sx)\big) < \inf_{\lambda\in J} \sup_{x\in X} \big(\varphi(Tx - \lambda Sx)\big)\,. \qquad \square$$

We begin by exhibiting an explicit example (the same as in [5] of a pair A, B of operators between normed spaces which do not satisfy the minimax equality of Asplund and Pták.

Define the two-dimensional normed spaces E and F as the linear space $\mathbb{R}^2$ equipped respectively with the norms $\|.\|_1 : (x, y) \mapsto |x| + |y|$ and $\|.\|_\infty : (x, y) \mapsto$

$\max(|x|, |y|)$. Denote by $A \in \mathcal{L}(E, F)$ the operator whose matrix in the canonical bases is $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ and by I the identity mapping.

Lemma 8.3. *We have:* $\inf_{\lambda\in\mathbb{R}} \sup_{\|z\|_1\leq 1} \|Az - \lambda z\|_\infty = \inf_{\lambda\in\mathbb{R}} \|A - \lambda I\| = \dfrac{3}{2}$.

Lemma 8.4. *We have:* $\sup_{\|z\|_1\leq 1} \inf_{\lambda\in\mathbb{R}} \|Az - \lambda z\|_\infty \leq \dfrac{5}{4} < \dfrac{3}{2}$. *More precisely, for each z in the unit ball of E there is $\lambda^* \in [0, 2]$ such that* $\|Az - \lambda^* z\|_\infty \leq \dfrac{5}{4}$.

Lemma 8.5. *There exist two Banach spaces E_2 and F_2 which are isomorphic to the Hilbert space ℓ^2 and operators A_2 and B_2 in $\mathcal{L}(E_2, F_2)$ such that*

$$\sup_{\|z\|_{E_2}\leq 1} \inf_{\lambda\in[0,2]} \|A_2 z - \lambda B_2 z\|_{F_2} \leq \frac{5}{4} < \frac{3}{2} \leq \inf_{\lambda\in\mathbb{R}} \sup_{\|z\|_{E_2}\leq 1} \|A_2 z - \lambda B_2 z\|_{F_2} .$$

They are Lemmas 16 to 18 in [5].

Proof. Since E_2 is isomorphic to ℓ^2, we can find a one-to-one compact operator $\pi\colon \mathcal{E} = \ell^2 \to E_2$ with dense range, define $\mathcal{F} = F_2$, $T = A_2 \circ \pi$ and $S = B_2 \circ \pi$ and $\varphi = \|.\|_{F_2}$, choose $J = [0, 2]$ and Y as the unit ball of E_2, then define $X = \pi^{-1}(Y)$. So X has non-empty interior for $\beta(\mathcal{E}, \mathcal{E}^*)$ and for any $\lambda \in J$ we have

$$\sup_{x\in X} \|Tx - \lambda Sx\| = \sup_{y\in Y} \|(A_2 - \lambda B_2)y\| = \sup_{y\in Y\cap\pi(\ell^2)} \|(A_2 - \lambda B_2)y\|$$

Take $T_0 = T$, $T_1 = T - 2S$, and choose for T_2 any isomorphism from $\mathcal{E} = \ell^2$ onto $\mathcal{F}$. Since T_2 is an isomorphism, it is clear that $x \mapsto (T_j x)_{j\leq 2}$ is an isomorphism from $\mathcal{E}$ onto its range in $\mathcal{F}^3$.

So any $U \in \Delta$ has the form $U = \lambda_2 T_2 + (1 - \lambda_2)(T - \lambda S)$ for some $\lambda \in J = [0, 2]$ and $\lambda_2 \in [0, 1]$. It follows that

$$\inf_{U\in\Delta} \varphi(Ux) \leq \inf_{\lambda\in J} \varphi(Tx - \lambda Sx)$$

hence

$$\sup_{x\in X} \inf_{U\in\Delta} \varphi(Ux) \leq \sup_{x\in X} \inf_{\lambda\in J} \varphi(Tx - \lambda Sx) \leq \frac{5}{4} .$$

And by Lemma 5.5, since $T - \lambda S$ is compact for any $\lambda \in J$,

$$\sup_{x\in X} \varphi\big(\lambda_2 T_2 x + (1 - \lambda_2)(Tx - \lambda Sx)\big) = +\infty$$

for every $\lambda_2 \in]0, 1]$. Thus

$$\inf_{U\in\Delta} \sup_{x\in X} \varphi(Ux) = \inf_{\lambda\in J} \sup_{x\in X} \varphi(Tx - \lambda Sx) \geq \frac{3}{2}$$

and this shows that $\sup_{x\in X} \inf_{U\in\Delta} \varphi(Ux) < \inf_{U\in\Delta} \sup_{x\in X} \varphi(Ux)$. □

The previous example shows that Δ should contain at most one compact operator for granting the validity of the minimax equality.

We now prove that the interior of X for $\beta(E, E^*)$ has to be non-empty for guaranteeing the validity of Theorem 7.3 for all (F, Δ, φ), at least for X being symmetric.

Assume E is an infinite-dimensional Banach space and X is a convex symmetric open subset of E. Then, as in Lemma 4.6, the Minkowski functional p of X is a semi-norm on E ; we denote V the separated completion of (E, p) and $\pi\colon E \to V$ the canonical mapping. It follows from Theorem 4.7 that π is compact if and only if X is a neighborhood of 0 for $\beta(E, E^*)$.

Assume now that X is not a neighborhood of 0 for $\beta(E, E^*)$: so π is not compact and in particular V is infinite-dimensional. Choose two linearly independent continuous linear functionals g_1 and g_2 on V: the continuous linear mapping $g\colon V \to \mathbb{R}^2$ defined by $g(u) = \big(\langle g_1, u\rangle, \langle g_2, u\rangle\big)$ is onto. Let $V_0 = \ker(g)$ and F be $V_0 \times \mathbb{R}^2$ equipped with the norm $(u, \lambda) \mapsto \|(u, \lambda)\|_F = \alpha\,\|u\| + \|\lambda\|_1$ where $\|\lambda\|_1 = |\lambda_1| + |\lambda_2|$ and α a constant to be defined later on. Denote $P\colon V \to V_0$ any linear projection.

Then the mapping $j\colon V \to F$ defined by $x \mapsto (Px, gx)$ is an isomorphism. If B_V denotes the unit ball of V then $C = \overline{g(B_V)}$ is a compact convex symmetric subset of $\mathbb{R}^2$, with non-empty interior since its spans $\mathbb{R}^2$, hence the unit ball for some norm $|||.|||$ on $\mathbb{R}^2$. Since $|||.|||$ and $\|.\|_1$ are not both euclidean on $\mathbb{R}^2$, it follows from [1] that the minimax inequality is strict for $\mathcal{L}((\mathbb{R}^2, |||.|||), (\mathbb{R}^2, \|.\|_1)$ it is:

$$\beta = \sup_{|||u|||\le 1} \inf_{t\in\mathbb{R}} \|S_0u + tS_1u\|_1 < \inf_{t\in\mathbb{R}} \sup_{|||u|||\le 1} \|S_0u + tS_1u\|_1 = \gamma$$

for some two linear operators S_0 and S_1 from $\mathbb{R}^2$ to $\mathbb{R}^2$. Define now two linear operators from V to F by:

$$\hat{S}_0x = (Px, S_0gx)\ , \qquad \hat{S}_1x = (0, S_1gx)$$

and two operators from E to F by $\tilde{S}_0 = \hat{S}_0 \circ \pi$ and $\tilde{S}_1 = \hat{S}_1 \circ \pi$. The continuous convex coercive function φ on F will be the norm $\|.\|_F$

Lemma 8.6. *For every $t \in \mathbb{R}$ the mapping $\tilde{S}_0 + t\tilde{S}_1$ is not compact.*

Proof. The projection $q : (u, \lambda) \mapsto u$ is continuous from F to V_0. If $\tilde{S}_0 + t\tilde{S}_1$ was compact for some $t \in \mathbb{R}$, so would be $q \circ (\tilde{S}_0 + t\tilde{S}_1) = P \circ \pi$ and so would be also π since $(I - P) \circ \pi$ has finite rank 2, but this contradicts the assumption that X has empty interior for $\beta(E, E^*)$. □

Lemma 8.7. *There exists an integer n_0 such that*

$$\beta' = \sup_{|||u|||\le 1} \inf_{|t|\le n_0} \|S_0u + tS_1u\|_1 < \gamma\,.$$

Proof. For all $t \in \mathbb{R}$ the function $u \mapsto \|S_0u + tS_1u\|_1$ is $(|t|\,.\,\|S_1\|)$-lipschitz on C: it follows that the function $f_n\colon u \mapsto \inf_{|t|\le n} \|S_0u + tS_1u\|_1$ is also $(n.\,\|S_1\|)$-lipschitz on C hence continuous on the compact set C. Moreover the sequence (f_n)

is obviously non-increasing and for each $u \in C$ we have $\lim_n f_n(u) = \inf_n f_n(u) = \beta < \gamma$. It follows then from Dini's theorem that the convergence is uniform on C and that for some n_0 we have $f_{n_0}(u) \leq \beta' = (\beta + \gamma)/2 < \gamma$ for all $u \in C$. ☐

Let $\Delta = \{\tilde{S}_0 + t\tilde{S}_1 : |t| \leq n_0\}$. This set is a segment, hence compact and convex, and contains none compact operator.

Lemma 8.8. *We have* $\inf_{T\in\Delta} \sup_{x\in X} \varphi(Tx) \geq \gamma$.

Proof. Indeed if $T \in \Delta$ there is $t \in [-n_0, n_0]$ such that $T = \tilde{S}_0 + t\tilde{S}_1$, and if $x \in X$ then $\|\pi x\| \leq 1$; let $v = \pi x$. Thus

$$\varphi(Tx) = \left\|\hat{S}_0 v + t\hat{S}_1 v\right\|_F = \alpha \left\|P(\hat{S}_0 v + t\hat{S}_1 v)\right\| + \|S_0 v + tS_1 v\|_1 \geq \|S_0 v + tS_1 v\|_1 .$$

So

$$\begin{aligned}\sup_{x\in X} \varphi(Tx) &\geq \sup_{\|v\|\leq 1} \|S_0 v + tS_1 v\|_1 \geq \inf_{|t|\leq n_0} \sup_{v\in B_V} \|S_0 v + tS_1 v\|_1 \\ &\geq \inf_{t\in\mathbb{R}} \sup_{v\in B_V} \|S_0 v + tS_1 v\|_1 = \gamma ,\end{aligned}$$

and we get $\inf_{T\in\Delta} \sup_{x\in X} \varphi(Tx) \geq \gamma$. ☐

Lemma 8.9. *We have* $\sup_{x\in X} \inf_{T\in\Delta} \varphi(Tx) < \gamma$.

Proof. Again for $x \in X$ and $T \in \Delta$ we have $\|v\| \leq 1$ for $v = \pi x$ and $T = \tilde{S}_0 + t\tilde{S}_1$ for some $t \in [-n_0, n_0]$. Thus

$$\begin{aligned}\varphi(Tx) &= \left\|\hat{S}_0 v + t\hat{S}_1 v\right\|_F = \alpha \left\|P(\hat{S}_0 v + t\hat{S}_1 v)\right\| + \|S_0 v + tS_1 v\|_1 \\ &\leq \alpha \|P\| \left(\left\|\hat{S}_0\right\| + n_0 \left\|\hat{S}_1\right\|\right) + \|S_0 v + tS_1 v\|_1 .\end{aligned}$$

Let $M = \|P\| \left(\left\|\hat{S}_0\right\| + n_0 \left\|\hat{S}_1\right\|\right)$ and fix $\alpha < \dfrac{\gamma - \beta'}{M}$. Thus

$$\inf_{T\in\Delta} \varphi(Tx) \leq M\alpha + \inf_{|t|\leq n_0} \|S_0 v + tS_1 v\|_1 ,$$

hence

$$\begin{aligned}\sup_{x\in X} \inf_{T\in\Delta} \varphi(Tx) = \sup_{v\in B_V} \inf_{T\in\Delta} \varphi(Tx) &\leq M\alpha + \sup_{v\in B_V} \inf_{|t|\leq n_0} \|S_0 v + tS_1 v\|_1 \\ &\leq M\alpha + \beta' < \gamma ,\end{aligned}$$

the desired inequality. ☐

The two previous lemmas show that the minimax equality does not hold as soon as X is not assumed to be a $\beta(E, E^*)$-neighborhood of 0 in E (at least for X symmetric).

References

[1] E. Asplund, V. Pták: *A minimax inequality for operators and a related numerical range*, Acta Math. 126 (1971) 53–62.

[2] J. Bourgain, D. H. Fremlin, M. Talagrand: *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100(4) (1978) 845–886.

[3] A. Dvoretzky, C. A. Rogers: *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. 36 (1950) 192–197.

[4] B. Ricceri: *A minimax theorem in infinite-dimensional topological vector spaces*, Linear and Nonlinear Analysis 2(1) (2016) 47–52.

[5] J. Saint Raymond: *A minimax theorem for linear operators*, Minimax Theory Appl. 1(2) (2016) 291–305.