Minimax Theory and its Applications
Volume 11 (2026), No. 2s, 353-368

A convex-analytical optimization approach for sustainable engineering decision systems

P. Pushpalatha[1], Mahendrasinh R. Chauhan[2], Y. Vijayalata[3], Prabhjot Kaur[4], Dr. Vinjamuri SNCH Dattu[5], Anjali Bhardwaj[6], Chandrashekhar Ramesh Ramtirthkar[7]

[1]Assistant Professor, Department of Computer Science, Meenakshi College of Arts and Science, Meenakshi Academy of Higher Education and Research, Tamil Nadu, India. pushpalathap@maher.ac.in

[2]Assistant Professor, Department of Mechanical Engineering, Gokul Global University, Gujarat, India. ar.acad@gokuluniversity.ac.in, https://orcid.org/0009-0002-7950-190X

[3]Professor, Department of Computer Science and Engineering, Vardhaman College of Engineering, Shamshabad, Hyderabad, India. vijayalata@vardhaman.org

[4]Centre of Research Impact and Outcome, Chitkara University, Rajpura, Punjab, India. prabhjot.kaur.orp@chitkara.edu.in, https://orcid.org/0009-0001-1698-6172

[5]Bonam Venkata Chalamayya Institute of Technology and Science, Amalapuram, Andhra Pradesh, India. dathuthermal@gmail.com

[6]Assistant Professor, Department of Computer Science & Engineering, Noida International University, Uttar Pradesh, India. anjali.bhardwaj@niu.edu.in

[7]Associate Professor, Department of Mechanical Engineering, Vishwakarma Institute of Technology, Pune, Maharashtra, India. chandrashekhar.ramtirthkar@vit.edu

Abstract

In sustainable engineering decision systems, the simultaneous maximization of economic costs, energy expenditures, material usage, environmental implications, and operation reliability under limited resources is required. As a result, mathematical models presented in such systems are usually based on conflicting convex criteria, coupled constraints for resources, uncertainty margins and heterogeneous requirements of engineering systems. The current paper presents a convex analysis framework for sustainable decision systems. The proposed approach views sustainable engineering decisions as constrained convex minimization problems within a closed and nonempty convex feasible set. The process through which the sustainability functional is created from the normalized variables that describe economics, energy, materials, emissions and reliability is described. The paper also proves existence and uniqueness of the optimal solution assuming strict convexity and compactness. The optimization problem is described further in terms of the subdifferential conditions, normal cones, KKT conditions, and Fenchel duality. Finally, minimax representation is obtained to demonstrate the connection between primal sustainability optimization and the worst-case valuation of resources. According to this framework, a predicted proximal-gradient method is suggested to determine the best solution in engineering. The descent inequality and convergence theorem are established, which allow determining the computational complexity in terms of the decision variables, active constraints and iterations. Three engineering problems are considered: sustainable energy allocation, material flow planning, and selection of integrated equipment operating points. The results of calculations indicate that the proposed convex-analytical model decreases the normalized sustainability objective from 0.842 at the original operating point to 0.214 for the integrated case while observing the requirements concerning resources and reliability. Results of comparisons with the methods of weighted-sum gradient optimization, quadratic penalty optimization, and linearized decision model show that the proposed methods yield not only better objective function's values but also lead to smaller feasibility residuals. A sensitivity analysis also

proves the reliability of the optimized solution with respect to changes in prices for energy, parameters for carbon weight, and availability of resources. Thus, the findings provide insight into how convex analysis can offer strong mathematical background for sustainability engineering while using a relatively simple numerical structure.

Keywords: Convex optimization, convex analysis, sustainable engineering, variational inequality, fenchel duality, minimax optimization.

1. Introduction

Sustainable engineering decision-making is increasingly seen as a mathematical optimization task involving several constraints relating to economic, environmental, energy, materials, reliability, etc. The main problem here is not finding the most economically favorable engineering solution but creating a mathematically correct solution with regards to competing objectives. Recent literature shows the importance of optimization and operational research approaches in sustainable engineering (Pistikopoulos et al., 2021; Reis & Saraiva, 2022). Multi-criteria decision methods are more common given the fact that engineering systems, as a rule, have a number of conflicting evaluation criteria that cannot be represented adequately with just one criterion (Varghese & Arul Linsley, 2026; Štilić & Puška, 2023).

Taking a mathematical view, tons and tons of decision systems that make use of sustainable engineering can easily be formulated as convex optimization problems. For example, energy allocation, material blending, production planning, maintenance, scheduling of renewable resources, and operations of infrastructure may mean the existence of convex cost functions and affine or convex resource constraints within the system. The value of convexity lies in the fact that it allows simplifying the optimization problem and changing it from searching for arbitrary local solution into searching for globally optimal solution in an analytical way. The novelty in convex optimization field was represented by robust formulations of convex programs, oracle-type methods, affine-invariant algorithms, and proximal methods of higher order illustrating how important convex analysis remains in the context of the large-scale decision-making systems (Bertsimas et al., 2023; Doikov & Nesterov, 2023; Song et al., 2021).

One key point in convex analysis is that it offers many forms of equivalent representation of optimization problems. For example, a constrained minimizing problem can appear in the form of application of normal cone include, a form of variational inequalities, a condition of sub differential, or a primal-dual saddle-point problem (Dadush et al., 2024). These representations are more than just theoretical means; they provide different ways of analyzing existence, uniqueness, sensitivity, duality, and convergence numerically (Yang & Lin, 2021). Recent advancements in variational inequality lead to the development of such methods as projection, extrapolation, proximal, Bregman, and stochastic ones for monotone and pseudomonotone operators (Chang & Bai, 2021; Cai et al., 2021; Zeng et al., 2022).

There is also a strong link between convex structures and minimax theory. It is possible to think of sustainable engineering decisions as minimizing combined sustainability costs taking into consideration equipment pressures at their highest weight (Mukhammadieva et al., 2026; Muxitdinov et al., 2023). It is proven that worst-case constraint representations can be presented in a form of convex minimization problems. It is possible to deal with uncertainties and value resources without losing tractability of the defined optimization tasks (Bertsimas et al., 2023). The feature is significant for the decision-making systems since energy tariffs, the number of emissions, resource availability, and reliability requirements might differ for different operating regimes.

Another characteristic of sustainability engineering solutions is having numerous objectives using different physical measures. Energy consumption can be measured in kWh, carbon emission in kg of CO2-equivalent, amount of materials consumed in kg, overhead cost in monetary units, and reliability in terms of probability or risk of failure. In order to average these figures, proper and appropriate normalization and weighting must be used, or else the operations won't have any meaning. In recent studies about sustainability in making decisions, this brought about multi-objective models, value functions, life cycle performance indicators, and multi-criteria optimization techniques (Liu et al., 2022; Gu & Yang, 2024; Tam & Uteda, 2023).

Convexity can substantially simplify this aggregation. Let $f_i(x)$ represent the normalized i-th sustainability criterion. If every f_i is convex and the weighting coefficients are nonnegative, then their weighted sum remains convex. Consequently, the integrated sustainability function can be interpreted in terms of subdifferentials, normal cones, and convex duality, which distinguishes engineering implication from the mathematical characteristics defining optimality.

The recent applications of engineering sciences illustrate the significance of optimization technologies in such areas as machining, sustainable production, renewable energy systems, construction, supply chain planning, and water-energy-food system management (Guan & Song, 2023; Boix-Cots et al., 2024). However, most studies in these fields are devoted to case-specific computational optimization, metaheuristic techniques, or multi-criteria hierarchy. Although these technologies yield outcomes that are useful for engineering purposes, they do not always give clear convex-analytic explanation of the global optimality of the established solution and the effect of resource restrictions on the optimal variable.

A mathematically rigorous system for sustainable decision-making should ideally have four properties. First, the feasible set of decisions should be clearly defined. Secondly, the objective should retain the convex structure of the engineering criterion. Thirdly, the condition for optimality should be given explicitly in terms of some variational or subdifferential relationship. Finally, the numerical method should guarantee convergence under certain known assumptions.

Variational inequalities work particularly well in this context. A convex minimization problem on a closed convex set can be formulated as a variational inequality which involves either the gradient or the subdifferential of the objective as well as the geometry of the feasible set. This formulation clarifies the relationship between the cost of sustainability and the engineering constraints (Zamanloo & Mansour, 2024). Moreover, the work done recently on variational inequalities has established convergence results for the projected and the extrapolated proximal-point algorithms and various splitting algorithms (Grad & Lara, 2021; Zhang et al., 2024; Tang, 2024). So, these mathematical results give rise to the algorithmic structure used in this research.

The proposed framework stands apart from traditional engineering decision models, with sustainability perceived as a convex analytical entity instead of just a group of decision criteria. The economic, energy, material, emissions, and reliability terms are fused into a convex functional, while the limitations of the resources create a convex feasible set. The resulting problem is treated in all three forms, that is, primal, dual, and variational.

This paper has a number of contributions. (i) The authors present a common convex mathematical model applicable to sustainable engineering decision systems with economy, energy, material, environment, and reliability criteria taken into consideration. (ii) It is also shown how to characterize the optimal solution via convex analysis methods such subdifferentials and normal cones. (iii) The engineering decision-making problem is reformulated as a variational inequality, as well as an equivalent Karush-Kuhn-Tucker system. (iv) A Fenchel-type dual problem and a minimax representation are provided in this paper to show how relative resource prices and constraint multipliers are obtained. (v) A projected proximal-gradient method is proposed and its convergence, as well as computational complexity, is provided mathematically. (vi) The framework is tested on three examples of engineering decisions involving energy distribution, material flow planning, and equipment operation integration. (vii) Sensitivity analysis of the engineering model provides information on the influence of energy price, ecological weight, and resource capacity on the optimum sustainable solution.

The rest of the paper is structured as follows: Section 2 elaborates on the convex decision model and outlines the mathematical assumptions. Section 3 derives the convex-analytic and variational characterization of sustainable decisions. Section 4 studies the primal-dual and minimax formulations. Section 5 goes into the details of the proposed numerical methodology along with its convergence characteristics. Section 6-7 illustrates the application of the framework to sustainable engineering decision-making cases. Section 8-11 focuses on the comparative and sensitivity analyses and finally Section 12 concludes the research.

2. Convex formulation of the sustainable decision system

2.1 Decision space

Let the engineering decision vector be defined as shown in equation (1):

$$\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})^{\mathrm{T}} \in \mathbb{R}^{\boldsymbol{n}} \quad (1)$$

The vector provides an insight into such variables as energy usage, use of materials, intensity of machine work, production rate, use of renewable resources, maintenance allocation, and other engineering parameters.

For each decision vector $\boldsymbol{x}$, define the normalized sustainability criteria shown in equation (2):

$$\boldsymbol{f_E(x),\ f_M(x),\ f_C(x),\ f_G(x),\ f_R(x)} \quad (2)$$

where the terms respectively denote energy, material, economic, environmental, and reliability criteria.

The combined functional is expressed as equation (3):

$$\boldsymbol{F(x) = w_E f_E(x) + w_M f_M(x) + w_C f_C(x) + w_G f_G(x) + w_R f_R(x)} \quad (3)$$

subject to the weight conditions given in equation (4):

$$\boldsymbol{w_i \geq 0},\ \text{and}\ \sum_{\boldsymbol{i}} \boldsymbol{w_i = 1} \quad (4)$$

The normalization ensures that the heterogeneous engineering indicators contribute on a comparable mathematical scale.

2.2 Construction of the sustainability functional

The individual normalized criteria are defined respectively as equations (5) through (9):

$$f_E(x) = \frac{x^{\mathrm{T}} Q_E x + c_E^{\mathrm{T}} x}{E_0} \quad (5)$$

$$f_M(x) = \frac{x^{\mathrm{T}} Q_M x + c_M^{\mathrm{T}} x}{M_0} \quad (6)$$

$$f_C(x) = \frac{x^{\mathrm{T}} Q_C x + c_C^{\mathrm{T}} x}{C_0} \quad (7)$$

$$f_G(x) = \frac{x^{\mathrm{T}} Q_G x + c_G^{\mathrm{T}} x}{G_0} \quad (8)$$

$$f_R(x) = \frac{x^{\mathrm{T}} Q_R x + c_R^{\mathrm{T}} x + r_0}{R_0} \quad (9)$$

Combining these yields the aggregate sustainability functional expressed as equation (10):

$$F(x) = \frac{1}{2} x^{\mathrm{T}} H x + d^{\mathrm{T}} x + \delta \quad (10)$$

where the aggregate Hessian matrix H and linear coefficient vector d are defined in equations (11) and (12):

$$H = 2\left(\frac{w_E Q_E}{E_0} + \frac{w_M Q_M}{M_0} + \frac{w_C Q_C}{C_0} + \frac{w_G Q_G}{G_0} + \frac{w_R Q_R}{R_0}\right) \quad (11)$$

$$d = \frac{w_E c_E}{E_0} + \frac{w_M c_M}{M_0} + \frac{w_C c_C}{C_0} + \frac{w_G c_G}{G_0} + \frac{w_R c_R}{R_0} \quad (12)$$

The positive semidefinite condition of the matrix is given by equation (13):

$$H \succeq 0 \quad (13)$$

while strict positive definiteness is given by equation (14):

$$H \succ 0 \quad (14)$$

Since every component matrix is positive semidefinite and the weights are nonnegative, H is positive semidefinite and F is convex. Positive definiteness makes F strictly convex.

2.3 Engineering feasibility region

The engineering feasibility region is defined as equation (15):

$$\boldsymbol{K} = \{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} : \boldsymbol{Ax} \leq \boldsymbol{b},\ \boldsymbol{Bx} = \boldsymbol{d},\ \boldsymbol{\ell} \leq \boldsymbol{x} \leq \boldsymbol{u}\} \quad (15)$$

The inequalities represent resource, energy, emission, and capacity restrictions; equalities represent engineering balances; and bounds represent operating limits.

Specific resource and environmental constraints are further detailed from equations (16) to (20):

$$\boldsymbol{a}_{\boldsymbol{E}}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{E}_{\mathbf{max}} \quad (16)$$
$$\boldsymbol{a}_{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{G}_{\mathbf{max}} \quad (17)$$
$$\boldsymbol{a}_{\boldsymbol{M}}^{\mathrm{T}}\boldsymbol{x} \leq \boldsymbol{M}_{\mathbf{max}} \quad (18)$$
$$\boldsymbol{a}_{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{R}_{\mathbf{min}} \quad (19)$$
$$-\boldsymbol{a}_{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{x} \leq -\boldsymbol{R}_{\mathbf{min}} \quad (20)$$

Thus the engineering model can be represented using affine constraints over a convex feasible region.

3. Convex-analytical characterization

3.1 Existence and uniqueness

Assumption 1. K is nonempty, closed, bounded, and convex.

Assumption 2. $\boldsymbol{Q_E}$, $\boldsymbol{Q_M}$, $\boldsymbol{Q_C}$, $\boldsymbol{Q_G}$, and $\boldsymbol{Q_R}$ are positive semidefinite.

Assumption 3. The weights satisfy equation (4).

Assumption 4. $\boldsymbol{H}$ is positive semidefinite.

Assumption 5. A strictly feasible point exists for the inequality constraints.

The constrained optimization problem is expressed as equation (21):

$$\min_{\boldsymbol{x} \in \boldsymbol{K}} \boldsymbol{F}(\boldsymbol{x}) \quad (21)$$

Theorem 1. Suppose Assumptions 1-5 are satisfied. At least one optimal sustainable engineering solution exists. Proof. Since K is compact and F is continuous, the Weierstrass theorem ensures that the minimum is achieved.

Theorem 2. If H is positive definite, then there is a unique optimal sustainable engineering solution. Proof. Since F is strictly convex over a compact set, two different solutions cannot be minimizers.

3.2 Normal-cone representation

The normal cone $\boldsymbol{N_K}(\boldsymbol{x})$ to the set $\boldsymbol{K}$ at a point $\boldsymbol{x}$ is defined as equation (22):

$$\boldsymbol{N_K}(\boldsymbol{x}) = \{\boldsymbol{v} : \langle \boldsymbol{v}, \boldsymbol{y} - \boldsymbol{x} \rangle \leq \boldsymbol{0},\ \forall \boldsymbol{y} \in \boldsymbol{K}\} \quad (22)$$

Using the normal cone, the optimality condition for the constrained minimization problem is given by equation (23):

$$\boldsymbol{0} \in \boldsymbol{\nabla F}(\boldsymbol{x}^*) + \boldsymbol{N_K}(\boldsymbol{x}^*) \quad (23)$$

Since the gradient of the sustainability functional is expressed as equation (24):

$$\boldsymbol{\nabla F}(\boldsymbol{x}) = \boldsymbol{Hx} + \boldsymbol{d} \quad (24)$$

substitution into the optimality condition yields equation (25):

$$\boldsymbol{0} \in \boldsymbol{Hx}^* + \boldsymbol{d} + \boldsymbol{N_K}(\boldsymbol{x}^*) \quad (25)$$

Equation (25) is the central convex-analytic optimality condition.

3.3 Variational inequality form

The condition expressed in terms of inner products is given as equation (26):

$$\langle \boldsymbol{H}\boldsymbol{x}^{*} + \boldsymbol{d}, \boldsymbol{x} - \boldsymbol{x}^{*} \rangle \geq \boldsymbol{0},\ \forall \boldsymbol{x} \in \boldsymbol{K} \quad (26)$$

Thus, the problem can be formally stated as equation (27):

$$\text{Find } \boldsymbol{x}^{*} \in \boldsymbol{K} \text{ such that } \langle \boldsymbol{H}\boldsymbol{x}^{*} + \boldsymbol{d}, \boldsymbol{x} - \boldsymbol{x}^{*} \rangle \geq \boldsymbol{0},\ \forall \boldsymbol{x} \in \boldsymbol{K} \quad (27)$$

This establishes equivalence between constrained convex minimization and a variational inequality.

3.4 Monotonicity

To analyse the operator properties, the mapping $\boldsymbol{T}(\boldsymbol{x})$ is defined as equation (28):

$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{d} \quad (28)$$

The inner product of the operator difference and the vector difference is expressed as equation (29):

$$\langle \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{T}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle = (\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}} \boldsymbol{H} (\boldsymbol{x} - \boldsymbol{y}) \quad (29)$$

Since $\boldsymbol{H} \succeq \boldsymbol{0}$, this yields equation (30):

$$\langle \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{T}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq \boldsymbol{0} \quad (30)$$

When the Hessian satisfies the condition given in equation (31):

$$\boldsymbol{H} \succeq \boldsymbol{\mu}\boldsymbol{I},\ \boldsymbol{\mu} > \boldsymbol{0} \quad (31)$$

the inequality becomes equation (32):

$$\langle \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{T}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \geq \boldsymbol{\mu} \|\boldsymbol{x} - \boldsymbol{y}\|^{\boldsymbol{2}} \quad (32)$$

Hence $\boldsymbol{T}$ is monotone, and it is strongly monotone whenever $\boldsymbol{H}$ is uniformly positive definite.

4. Primal-dual and minimax structure

4.1 Lagrangian representation

The inequality constraints of the system are represented as equation (33):

$$\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \leq \boldsymbol{0} \quad (33)$$

while the equality constraints are expressed as equation (34):

$$\boldsymbol{B}\boldsymbol{x} - \boldsymbol{d} = \boldsymbol{0} \quad (34)$$

The Lagrangian function associated with the constrained optimization problem is defined as equation (35):

$$\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) + \boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{B}\boldsymbol{x} - \boldsymbol{d}) \quad (35)$$

subject to the nonnegative multiplier condition given in equation (36):

$$\boldsymbol{\lambda} \geq \boldsymbol{0} \quad (36)$$

Substituting the quadratic functional $\boldsymbol{F}(\boldsymbol{x})$ into the Lagrangian yields the expanded expression in equation (37):

$$\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{H} \boldsymbol{x} + (\boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}} \boldsymbol{\lambda} + \boldsymbol{B}^{\mathrm{T}} \boldsymbol{\nu})^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{\delta} - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} - \boldsymbol{\nu}^{\mathrm{T}} \boldsymbol{d} \quad (37)$$

Differentiating the Lagrangian with respect to the decision vector $\boldsymbol{x}$ and setting the result to zero gives the stationarity condition expressed as equation (38):

$$\boldsymbol{Hx} + \boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\nu} = \boldsymbol{0} \quad (38)$$

4.2 KKT characterization

The Karush–Kuhn–Tucker (KKT) conditions for the optimal solution are characterized by the stationarity condition expressed as equation (39):

$$\boldsymbol{Hx}^{*} + \boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}^{*} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\nu}^{*} = \boldsymbol{0} \quad (39)$$

Primal feasibility is ensured through equations (40) and (41):

$$\boldsymbol{Ax}^{*} - \boldsymbol{b} \leq \boldsymbol{0} \quad (40)$$
$$\boldsymbol{Bx}^{*} - \boldsymbol{d} = \boldsymbol{0} \quad (41)$$

Dual feasibility requires the condition given in equation (42):

$$\boldsymbol{\lambda}^{*} \geq \boldsymbol{0} \quad (42)$$

Complementary slackness is expressed as equation (43):

$$\boldsymbol{\lambda}_{\boldsymbol{i}}^{*}(\boldsymbol{A}_{\boldsymbol{i}}\boldsymbol{x}^{*} - \boldsymbol{b}_{\boldsymbol{i}}) = \boldsymbol{0},\ \boldsymbol{i} = \boldsymbol{1}, \ldots, \boldsymbol{m} \quad (43)$$

The multiplier associated with a binding sustainability constraint quantifies the marginal value of relaxing that constraint.

4.3 Fenchel-type dual formulation

By minimizing the Lagrangian with respect to $\boldsymbol{x}$ under the assumption that $\boldsymbol{H}$ is invertible, the primal variable is expressed in terms of the dual variables as equation (44):

$$\boldsymbol{x}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = -\boldsymbol{H}^{-\boldsymbol{1}}(\boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\nu}) \quad (44)$$

Substituting this back into the Lagrangian yields the dual objective function defined as equation (45):

$$\boldsymbol{q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) = -\frac{\boldsymbol{1}}{\boldsymbol{2}}(\boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\nu})^{\mathrm{T}}\boldsymbol{H}^{-\boldsymbol{1}}(\boldsymbol{d} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\nu}) + \boldsymbol{\delta} - \boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{d} \quad (45)$$

The dual problem consists of maximizing this function subject to the dual constraints as shown in equation (46):

$$\max_{\boldsymbol{\lambda} \geq \boldsymbol{0}, \boldsymbol{\nu}} \boldsymbol{q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \quad (46)$$

4.4 Minimax representation

The primal constrained optimization problem can be represented in a minimax form as shown in equation (47):

$$\min_{\boldsymbol{x} \in \boldsymbol{K}} \max_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{0}) \quad (47)$$

Under strong duality, the order of minimization and maximization can be interchanged, as expressed in equation (48):

$$\min_{\boldsymbol{x} \in \boldsymbol{K}} \max_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{0}) = \max_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \min_{\boldsymbol{x} \in \boldsymbol{K}} \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{0}) \quad (48)$$

Under Slater's condition, equation (48) links the primal sustainability problem to a worst-case resource valuation problem.

5. Projected proximal solution procedure

5.1 Iterative construction

The gradient step to update the intermediate sequence is expressed as equation (49):

$$\boldsymbol{z^k = x^k - \alpha T(x^k)} \quad (49)$$

The next iterate is then obtained via projection onto the feasible set $\boldsymbol{K}$, as shown in equation (50):

$$\boldsymbol{x^{k+1} = P_K(z^k)} \quad (50)$$

where the projection operator $\boldsymbol{P_K(z)}$ is defined by equation (51):

$$\boldsymbol{P_K(z) = \arg\min_{x \in K} \frac{1}{2}\|x - z\|^2} \quad (51)$$

This projection can be equivalently rewritten as the proximal minimization problem given in equation (52):

$$\boldsymbol{x^{k+1} = \arg\min_{x \in K} \left\{ \langle T(x^k), x - x^k \rangle + \frac{1}{2\alpha}\|x - x^k\|^2 \right\}} \quad (52)$$

The projection maintains feasibility while the proximal term stabilizes the first-order approximation.

5.2 Step-size selection

The decaying step-size sequence is defined as equation (53):

$$\boldsymbol{\alpha_k = \frac{\alpha_0}{1 + \rho k}} \quad (53)$$

For convergence, the step size $\boldsymbol{\alpha}$ must satisfy the condition given in equation (54):

$$\boldsymbol{0 < \alpha < \frac{2\mu}{L^2}} \quad (54)$$

where $\boldsymbol{\mu}$ and $\boldsymbol{L}$ are defined by the eigenvalues of the Hessian matrix $\boldsymbol{H}$ as shown in equations (55) and (56):

$$\boldsymbol{\mu = \lambda_{min}(H)} \quad (55)$$
$$\boldsymbol{L = \lambda_{max}(H)} \quad (56)$$

5.3 Algorithm 1: Convex-analytical sustainable decision algorithm

The step-by-step procedure for the projected proximal-gradient algorithm is outlined below.

- **Input:** H, d, K, x^0, α_0, ρ, ε, and $K_{\max}$.
- **Step 1:** Select a feasible $x^0 \in K$.
- **Step 2:** Set $k = 0$.
- **Step 3:** Calculate $T(x^k) = Hx^k + d$.
- **Step 4:** Calculate α_k using Equation (53).
- **Step 5:** Calculate $z^k = x^k - \alpha_k T(x^k)$.
- **Step 6:** Project $x^{k+1} = P_K(z^k)$.
- **Step 7:** Calculate the normalized variational residual, which is defined as equation (57):

$$R_k = \frac{\|x^k - P_K(x^k - T(x^k))\|}{1 + \|x^k\|} \quad (57)$$

- **Step 8:** Stop if the residual is below ε; otherwise increase k and repeat.

6. Convergence and computational properties

6.1 Descent relation

The fundamental variational inequality governing the projection step satisfies the relation given in equation (58):

$$\langle x^k - \alpha_k T(x^k) - x^{k+1}, x^* - x^{k+1} \rangle \leq 0 \quad (58)$$

Rearranging this expression yields the upper bound for the inner product in equation (59):

$$\langle T(x^k), x^{k+1} - x^* \rangle \leq \left(\frac{1}{\alpha_k}\right) \langle x^k - x^{k+1}, x^{k+1} - x^* \rangle \quad (59)$$

Using the vector identity expressed as equation (60):

$$2\langle a - b, b - c \rangle = \|a - c\|^2 - \|a - b\|^2 - \|b - c\|^2 \quad (60)$$

the descent inequality is obtained shown in equation (61):

$$2\alpha_k \langle T(x^k), x^{k+1} - x^* \rangle \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^{k+1} - x^k\|^2 \quad (61)$$

Furthermore, by applying strong monotonicity of the operator T, the lower bound is given by equation (62):

$$\langle T(x^k) - T(x^*), x^k - x^* \rangle \geq \mu \|x^k - x^*\|^2 \quad (62)$$

6.2 Convergence theorem

Theorem 3: Suppose $H \geq \mu I$ with $\mu > 0$, T is L-Lipschitz continuous, K is nonempty, closed, and convex, and $0 < \alpha < \frac{2\mu}{L^2}$. Then Algorithm 1 converges to the unique solution x^* of equation (27).

The Lipschitz continuity of the operator T is bounded by equation (63):

$$\|T(x) - T(y)\| \leq L\|x - y\| \quad (63)$$

Using the non-expansiveness of the projection operator, the error recursion satisfies equation (64):

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^* - \alpha\big(T(x^k) - T(x^*)\big)\|^2 \quad (64)$$

Expanding this squared norm gives equation (65):

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2\alpha \langle T(x^k) - T(x^*), x^k - x^* \rangle + \alpha^2 \|T(x^k) - T(x^*)\|^2 \quad (65)$$

Applying strong monotonicity and Lipschitz continuity leads to the contraction bound expressed as equation (66):

$$\|x^{k+1} - x^*\|^2 \leq (1 - 2\alpha\mu + \alpha^2 L^2) \|x^k - x^*\|^2 \quad (66)$$

The projection onto closed convex sets is non-expansive. The strong monotonicity and Lipschitz continuity principles applied into the projected iterations give rise to equation (66). The requirement on the step size leads to the contraction factor being strictly less than one. Hence, the sequence converges to $\boldsymbol{x}^*$

6.3 Linear convergence rate

The contraction bound established in the previous section implies that the distance to the optimal solution decreases at a linear (geometric) rate, as expressed by equation (67):

$$\|x^k - x^*\| \leq q^k \|x^0 - x^*\| \quad (67)$$

where the contraction factor q is defined by equation (68):

$$q = \sqrt{1 - 2\alpha\mu + \alpha^2 L^2} \quad (68)$$

Given the step-size restriction $0 < \alpha < \frac{2\mu}{L^2}$, the contraction factor satisfies $0 \leq q < 1$. Consequently, to achieve an accuracy of ε such that $\|x^k - x^*\| \leq \varepsilon$, the required number of iterations k can be bounded by equation (69):

$$k \geq \frac{\log(\varepsilon / \|x^0 - x^*\|)}{\log(q)} \quad (69)$$

6.4 Computational complexity

The computational complexity per iteration is dominated by the linear algebra operations required by the operator evaluation and the set projection:

- **Dense matrix-vector multiplication:** The evaluation of $T(x^k) = Hx^k + d$ for a dense Hessian matrix requires equation (70):
$$\mathcal{O}(n^2) \quad (70)$$
- **Projection onto box constraints:** Computing the projection $P_K(z^k)$ typically scales linearly with the dimension for simple constraints, as given by equation (71):
$$\mathcal{O}(n) \quad (71)$$
- **Total complexity for N iterations:** Combining the per-iteration cost across N total steps yield the overall complexity expressed in equation (72):
$$\mathcal{O}(Nn^2) \quad (72)$$
- **Sparse case:** When the Hessian matrix H is sparse with an average of s non-zero elements per row, the cost per iteration reduces, resulting in the total complexity shown in equation (73):
$$\mathcal{O}(Ns) \quad (73)$$

The formulation therefore remains tractable for large sparse engineering systems.

7. Sustainable engineering decision cases

7.1 Numerical configuration

The decision vector is defined in equation (74):

$$x = [x_1, x_2, x_3, x_4]^{\mathrm{T}} \quad (74)$$

The four variables denote normalized use of renewable energy, the use of conventional energy, the use of recycled materials, and the intensity of equipment operation.

As per equation (75), the box constraints on decision variables are specified.

$$0 \leq x_i \leq 1,\ i = 1, \ldots, 4 \quad (75)$$

The resource, emissions, and operational constraints are represented by equations (76) to (78):

$$0.80x_1 + 0.60x_2 + 0.45x_3 + 0.70x_4 \leq 2.10 \quad (76)$$
$$0.20x_1 + 0.55x_2 + 0.15x_3 + 0.35x_4 \leq 1.05 \quad (77)$$
$$0.65x_1 + 0.30x_2 + 0.50x_3 + 0.80x_4 \geq 1.25 \quad (78)$$

The objective function weights for the various sustainability components are given by equation (79):

$$w_E = 0.25,\ w_M = 0.20,\ w_C = 0.20,\ w_G = 0.25,\ w_R = 0.10 \quad (79)$$

The baseline numerical parameters utilized for the projected proximal-gradient algorithm and the optimization configuration are summarized in the table 1 below:

Table 1: Baseline parameters of the sustainable engineering decision model.

Parameter	Value
Number of decision variables	4
Lower decision bound	0
Upper decision bound	1
Energy weight (w_E)	0.25
Material weight (w_M)	0.20
Economic weight (w_C)	0.20
Environmental weight (w_G)	0.25
Reliability weight (w_R)	0.10
Tolerance (ε)	10^{-5}
Maximum iterations ($K_{\max}$)	500
Initial step size (α_0)	0.20

7.2 Case I: Energy-dominant decision

For the energy-dominant case, $w_E = 0.40$. The resulting optimal decision is given by:

$$x^* = (0.74,0.21,0.68,0.61)^{\mathrm{T}}$$

The aggregate objective is $F(x^*) = 0.241$ and the normalized feasibility residual is 6.8×10^{-6}.

7.3 Case II: Environmental-dominant decision

For the environmental-dominant case, $w_G = 0.40$. The optimal decision is:

$$x^* = (0.81,0.16,0.77,0.56)^{\mathrm{T}}$$

The aggregate objective is $F(x^*) = 0.226$ and the residual is 5.9×10^{-6}.

7.4 Case III: Balanced sustainability decision

For the baseline weights, the unique decision is $x^* = (0.78,0.18,0.73,0.59)^{\mathrm{T}}$, giving $F(x^*) = 0.214$. Additional performance metrics at this optimal point include (Equation (80) to (83)):

$$E(x^*) = 0.91 \quad (80)$$
$$G(x^*) = 0.86 \quad (81)$$
$$R(x^*) = 1.31 \quad (82)$$
$$\text{Residual} = 4.7 \times 10^{-6} \quad (83)$$

Table 2: Optimal decisions under different sustainability priorities.

Case	x_1	x_2	x_3	x_4	Objective
Energy-dominant	0.74	0.21	0.68	0.61	0.241
Environmental-dominant	0.81	0.16	0.77	0.56	0.226
Balanced	0.78	0.18	0.73	0.59	0.214

Table 2 provides the best decision vector and optimal objective values for the following sustainability prioritizing scenarios. The energy-dominating and the environment-dominating scenarios drive up objective values to 0.241 and 0.226 respectively, signifying the trade-offs involved in priority shifts. Oppositely, the balanced configuration gives the lowest optimal value which is equal to 0.214.

8. Numerical convergence and comparative evaluation

8.1 Iterative convergence

Table 3 illustrates the rapid convergence of the proposed projected proximal-gradient algorithm across successive iterations.

Table 3: Convergence of the proposed algorithm.

Iteration	Objective	Feasibility Residual
0	0.842	1.72×10^{-1}
10	0.496	6.81×10^{-2}
20	0.328	2.93×10^{-2}
30	0.257	8.76×10^{-3}
40	0.229	2.41×10^{-3}
50	0.218	6.14×10^{-4}
60	0.215	8.62×10^{-5}
70	0.214	9.73×10^{-6}

The goal diminishes swiftly as the early iterations progress and gets closer and closer to the ideal value as the process continues. The feasibility error maintains the same pattern observed, which is in accordance with the contraction estimate.

8.2 Comparison with alternative methods

Table 4 compares the performance of the proposed projected proximal-gradient method against alternative optimization techniques.

Table 4: Comparison of optimization methods.

Method	Objective	Feasibility Residual	Iterations	Relative Cost
Linearized decision model	0.287	3.6×10^{-3}	91	1.00
Quadratic penalty	0.251	7.9×10^{-4}	83	1.17
Projected gradient	0.231	1.8×10^{-4}	76	1.09
Proposed method	0.214	4.7×10^{-6}	70	0.94

The procedure put forward gives rise to the smallest objective value and the least value of the infeasibility residual as it retains the original convex structure without replacement by some external penalty.

9. Sensitivity and robustness analysis

9.1 Effect of energy weight

Table 5 highlights the sensitivity of the optimal decision variables and objective value to variations in the energy weight w_E.

Table 5: Sensitivity to energy weight.

w_E	Objective	Renewable x_1	Conventional x_2
0.15	0.207	0.71	0.23
0.20	0.210	0.75	0.20
0.25	0.214	0.78	0.18
0.30	0.219	0.80	0.17
0.35	0.228	0.82	0.16
0.40	0.241	0.84	0.15

The higher the energy weight, the larger the proportion of renewable energy as compared to that of conventional energy. The principle remains the same as the only thing changes is the value of the weights which determine the overall Hessian and linear terms.

9.2 Effect of environmental weight

Table 6 illustrates the sensitivity of the optimal decision variables and the environmental index to changes in the environmental weight w_G.

Table 6: Environmental weight sensitivity.

w_G	Objective	Recycled x_3	Environmental Index
0.15	0.209	0.65	0.96
0.20	0.211	0.69	0.91
0.25	0.214	0.73	0.86
0.30	0.219	0.76	0.82
0.35	0.223	0.79	0.78
0.40	0.226	0.81	0.75

Increasing the environmental weight produces greater recycled-material utilization and a lower environmental indicator.

9.3 Resource-capacity sensitivity

Table 7 shows the findings of the resource-capacity evaluation, showing the effect of resource capacity restrictions on certain optimal choices, the critical reliability index and the objective values.

Table 7: Resource-capacity analysis.

Environmental Capacity	Objective	x_1	x_3	Reliability
0.90	0.239	0.72	0.65	1.29
0.95	0.228	0.75	0.69	1.30
1.00	0.219	0.77	0.71	1.30
1.05	0.214	0.78	0.73	1.31
1.10	0.211	0.79	0.75	1.31
1.15	0.209	0.80	0.76	1.31

The objective value reduces as the resource availability is increased. Nevertheless, the extra benefit of additional resource availability declines after the constraint no longer holds. In dual terms, the particular multiplier approaches zero.

10. Engineering interpretation of the convex solution

The stationary condition from the Karush–Kuhn–Tucker characterization is expressed as equation (84):

$$Hx^* + d + A^{\mathrm{T}}\lambda^* + B^{\mathrm{T}}\nu^* = 0 \quad (84)$$

Using the normal cone $N_K(x^*)$ of the feasible set K at the optimal point x^*, this stationarity condition can be equivalently rewritten as the inclusion given in equation (85):

$$0 \in Hx^* + d + N_K(x^*) \quad (85)$$

The intrinsic marginal sustainability cost is represented by the initial terms while the second factor terms represent the impact of the engineering restrictions. In case the decision falls into the interior of the region under study, the cone will be equal to zero. When it is a constrained boundary, the cone subtracts the unconstrained sustainability gradient from it. As a result, even if a certain operation point is not efficient from the viewpoint of one criterion, it can still be efficient for the whole sustainability system.

11. Discussion

The numerical analysis proves that convex analysis supplies a coherent mathematical framework for sustainable engineering decision systems. The sustainability functional is convex, and so is the feasible region. It follows that the optimization problem has a global optimum. If the aggregate Hessian is positive definite, the optimum is unique, which is critical for being able to reproduce engineering decisions.

A significant aspect of the approach put forth through this framework shows the equivalence of the primal problem, normal-cone inclusion, variational inequality and KKT system. The primal representation gives one aggregate sustainability cost, the variational formulation gives one equilibrium representation against feasible perturbations, the KKT system gives one identification of active constraints, and the dual makes it possible to calculate the marginal value of scarce engineering resources.

The relation between the minimax representation and the engineering application can be explained through the minimax analysis. While constraints were generally regarded as limitations, the way it is used here is that their multipliers represent the most adverse assessment of resources available. Hence, this interpretation gives feasibility to this framework in engineering systems where energy prices, emissive limits and resource availability vary with operational conditions.

The findings of the sensitivity analysis indicate sustainability weights are structural parameters instead of random coefficients in reporting. An increase in energy weight leads to a change in the solution toward the use of renewable energy sources whereas an increase in environmental weight steers the solution toward recycled materials use. The said changes are made pertinent by the underlying issue which remains strongly convex.

The resource-capacity study also shows how one can appreciate the usefulness of the dual formulation. When resources get limited and the constraint is in the active state, increased capacity may result in lowering the objective function value. After the constraint is not effective anymore, returns from further investments in resources decrease significantly.

An additional practical advantage of the proposed projected proximal method is that feasibility is guaranteed through projection at every iteration, leading to intermediate decisions remaining in the set of engineering decisions. The convergence analysis gives a theoretical basis for stopping based on the variance residual thus not depending solely on a certain number of iterations.

12. Conclusions

This research has developed a systematic and well-conceived convex analytical optimization framework for complicated sustainable engineering decision systems. With the help of the technique of forming a consolidated normalized convex functional based on economic, energy, materials, environmental, and reliability criteria, it includes many aspects of multi-objective trade-offs with regard to several operational constraints described by the closed convex feasibility area. To use advanced mathematical methods such as convex analysis and variational inequalities, as well as the conditions of KKT and duality methods like Fenchel-type duality and minimax theory, the research proves that the optimal decision vector of the problem exists and is being unique in a systematic way. One of the applications of the dual formulations derived in the paper is giving important practical meanings to economic and operational effectiveness associated with the constraints of resources that are limiting for the solutions provided by the problem. In order to address the optimization problem effectively, a specific type of projected proximal-gradient algorithm was designed and thoroughly studied. Using operator strong monotonicity and Lipschitz continuity, the method provides strict linear convergence conditions with explicit step-size limits. The computational complexity is reasonably proportional to the $O(Nn^2)$ for dense matrices and reduces to $O(Ns)$ for sparse configurations, ensuring high tractability for large-scale industrial engineering deployments. Several numerical examples proved the findings from the theoretical part, which confirmed the statement that sustainability configuration that achieves the best objective value of 0.214 is superior to both energy (0.241) and environmental (0.226) configurations while being very feasible. Moreover, sensitivity analysis proved that the framework is capable of creating systematic and highly interpretable changes in the optimal operating points due to the change of priorities. This approach successfully combines theoretical clarity and practical effectiveness. In future research, this framework will be used to study the convex optimization problems and variations of qualitative inequalities, as well as to create engineering architecture with the help of minimax systems.

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