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A generalized equilibrium optimization framework for multi-objective engineering systems

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Abstract

Multi-objective engineering design refers to the necessity of simultaneously optimizing contradicting performance parameters while being subjected to physical, operational, and resource constraint requirements. Most classical methods of scalarization mostly lead to compromise solutions without any sound equilibrium interpretation and convergence guarantees. A generalized equilibrium optimization framework is proposed that enables the representation of vector-valued engineering objectives by a regularized equilibrium operator defined over the closed convex feasible set. In the specific formulation, weighted and achievement-based scalarization methods can be incorporated via variational inequalities, normal cone inclusions, and Karush–Kuhn–Tucker conditions, allowing the definition of Pareto-optimal engineering decisions. Conditions for the existence of equilibrium solutions can be defined based on compactness and continuity conditions, while uniqueness can be proven based on the strict monotonicity of the equilibrium operator. A projected generalized equilibrium algorithm is proposed having stopping criteria based on the residuals and perform a convergence analysis. Finally, the execution of the proposed method is illustrated using three engineering examples, namely thermal equipment sizing, constrained power dispatching, and structural design. The thermal optimization achieved a compromise objective value of 0.218 with an equilibrium residual of 7.4×10^{-7}, while the power dispatch model maintained exact generation balance with a residual of 8.1×10^{-7}. The structural design case obtained a compromise value of 0.226 and residual of 6.9×10^{-7}. According to the convergence analysis, it was revealed that during the 120 iterations the equilibrium objective was reduced from 0.842 to 0.218. Using either weighted projected gradient method or penalty projected gradient method the resulting level attained with the proposed approach was 0.218 achieved in only 120 iterations thus proving the least effective performance of the new framework due to its capacity to provide mathematically provable equilibrium consistency, stable convergence and interpretable trade-offs.

Keywords: Multi-objective optimization, equilibrium problem, variational inequality, pareto optimality, strong monotonicity, convex analysis, engineering design, projected algorithm.

1. Introduction

In recent years, the field of engineering systems has seen a shift towards designing them based on trade-offs among different performance requirements instead of based on optimizing a single scalar parameter. Improvement in one performance requirement such as cost, energy consumption, velocity and weight, reliability, and thermal properties can lead to declines in other performance requirements. This is why multi-objective optimization is seen as a natural way to formulate a problem in mathematical terms in the field of engineering projects, engineering planning, and engineering control (Eichfelder et al., 2021; Kasiyanto et al., 2025; Mishra & Soy, 2025). The recent studies show a growing need for the creation of formulations that meet the existing requirements and are able to deal with conflicting objectives, mixed variables and parameters, restrictions, and expensive engineering calculations (Manson et al., 2021; Premkumar et al., 2022; Cai et al., 2021).

A classic multi-objective issue is displayed as a feasible decision space and a mapping in vector form. Even though the central ordering principle is based on the Pareto optimum, the whole collection of admissible alternatives, or Pareto set leads to no unique operating engineering point (Chang & Bai, 2021; Tam & Uteda, 2023). The problem of selecting a suitable compromise point is often accomplished by scalarization, although the result is considerably influenced by weights, normalization, and the curvilinear shape of the domain of ideal objectives. This problem initiated a series of works in mixed-variable multi-objective optimization, global multi-objective search, constrained evolutionary optimization, and decision-oriented engineering design (Lei & He, 2021; Gu & Yang, 2024; Zeng et al., 2022).

The second line of development pertains to equilibrium and variational formulations. In an equilibrium problem, a solution has to be found that is steady; that is, any perturbation in that solution does not work in favor of improving both the gap and the bifunction. It can be said that these formulations have a lot in common with the concept of variational inequalities, complementarity systems, saddle-point problems, and constrained optimization (Fan et al., 2021). In recent years, more mathematicians have developed the theory of convergence of projected, extrapolated, proximal, and Bregman-type approaches to variational inequalities, which is applicable for pseudo-monotone and non-smooth types of operators (Pan et al., 2022; Luo et al., 2023; Abdel-Basset et al., 2021). These developments provide evidence that an equilibrium approach can not only serve as a way to find the optimal solution but also indicate the mathematical requirements for a solution that has to be accepted by the engineering community (Minocha & Singh, 2022; Chalabi et al., 2023).

Alongside this development, the utilization of equilibrium-optimal computational techniques has gained momentum in engineering applications. The multi-objective equilibrium optimizers have used Pareto dominance, archives, diversity techniques, and equilibrium-inspired update rules in order to solve engineering design problems that are nonlinear in nature. Other noteworthy equilibrium studies have enhanced the process of exploration, exploitation, and convergence in many engineering disciplines including heat exchangers and constrained design (Rizk-Allah et al., 2024; Kalita et al., 2024; Cheng & De Waele, 2025). Although these computational methods are very efficient, the terminology of equilibrium is generally synonymous with population update methods rather than with a variational equilibrium condition.

Work in multi-objective engineering design has shown that there can be challenges in solving costly, black-box, and constrained problems. For example, the resulting Pareto front can be sparse with objectives all measured in different physical units (Wang et al., 2025; Dang et al., 2025). Consequently, there is the emergence of various knowledge-based, auxiliary-task, and multi-objective metaheuristic methods designed to increase convergence and diversity in solutions. Although these innovations can be effective, there can be a need for an analytical framework if the requirement is to achieve a clear mathematical understanding of a trade-off solution.

The current research creates an optimization framework based on general equilibrium for engineering systems using various objectives (Nathiya et al., 2023). The main assumption behind this approach is the definition of an equilibrium operator that is based on the gradients of normalized engineering objectives and a preference vector. The next step is to introduce a regularization term that ensures uniqueness and stability of the numerical solution, and finally apply the physical constraints that establishes the normal cone. Consequently, according to the reasonable assumptions it can be stated that the equilibrium condition is mathematically equivalent to the class of constrained scalarized problems.

The system has been developed on five premises. The first requirement is that engineering goals should be clear and obvious instead of hidden in the black-box fitness score. The second demand is that the feasible set should

reflect physical and operational constraints. The third need is that the ideal compromise must be consistent in the equilibrium relation to validate through residuals. The fourth condition states that there must be sufficient criteria for existence and uniqueness. The fifth element states that the computational loop must be proved to be convergent and not rely only on empirical termination. All five points is considered when working out a new approach further in the paper.

The key findings include: (i) a generalized equilibrium formulation for the constrained multi-objective engineering systems; (ii) a preference-regularized equilibrium operator to unify weighted-sum and achievement-type compromise models; (iii) existence, Pareto efficiency, uniqueness, and sensitivity results formulated in terms of variational and normal-cone conditions; (iv) a projected equilibrium method with an explicit residual-based stopping criterion; (v) convergence and complexity bounds based on Lipschitz continuity and strong monotonicity; and (vi) validation of the numerical results for thermal, electrical, and structural engineering systems.

The rest of this paper is organized in the following manner. In Section 2, the mathematical framework is defined and the multi-objective engineering model. Section 3 constructs the generalized equilibrium operator and establishes its regularized version. In Section 4, the existence and optimality conditions are established. Section 5 gives the primal-dual and KKT characterization. In Section 6, the projected equilibrium method is presented. In Section 7, convergence and computational complexity is proven. In Section 8, engineering applications are provided and numerical examples. In Section 9, the comparative analysis and sensitivity analysis is studied. Finally, in Section 10, the engineering interpretation of results is given; and in Section 11, paper is concluded.

2. Mathematical setting for engineering trade-offs

2.1 Decision variables and feasible engineering set

Let the engineering design or operating vector be represented as given in equation (1):

$$x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \tag{1}$$

The dimensions, rates of flow, power output, temperature, quantity of materials, inputs used in control systems, and other engineering quantities can all be represented in terms of components of x. The feasible set is represented by constraints relating to inequalities, equalities, and bounds. The separation is used because the equilibrium operator yields the details of trade-offs, while the feasible set ensures physical feasibility.

Let $g(x) \in \mathbb{R}^p$ denote inequality functions, $h(x) \in \mathbb{R}^q$ equality functions, and $\ell, u \in \mathbb{R}^n$ be lower and upper bounds. The feasible engineering region is defined by equation (2):

$$K = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, \ell \leq x \leq u\} \tag{2}$$

As the theoretical development proceeds, it is presumed that K is nonempty, closed, and convex. This is a premise satisfied by linear balances, convex limitations of resources, box ranges of operation, and convexified physical constraints in engineering applications. The framework may be implemented with linear official engineering operating regime or locally convex admissible areas in case the original model is nonlinear in nature.

$$\langle y - x, z - x \rangle = (y - x)^{\mathrm{T}}(z - x), \quad x, y, z \in \mathbb{R}^n \tag{3}$$

The Euclidean inner product in equation (3) is used throughout the equilibrium and convergence analysis. The induced norm is denoted by $\|x\| = \sqrt{\langle x, x \rangle}$

2.2 Vector-Valued engineering objectives

Suppose that m conflicting engineering criteria are represented by continuously differentiable functions $f_i : K \to \mathbb{R}, i = 1, \dots, m$. For a minimization problem, lower values are preferred for every criterion. Because engineering criteria generally have different units, direct aggregation is not meaningful. Let f_i^0 and f_i^r denote reference and scaling values, respectively, with $f_i^r > 0$. The normalized objective is given by equation (4):

$$\varphi_i(x) = \frac{f_i(x) - f_i^0}{f_i^r}, \quad i = 1, \dots, m \tag{4}$$

The normalized objective vector is written as equation (5):

$$\Phi(x) = \left(\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)\right)^{\mathrm{T}} \qquad (5)$$

A preference vector $w \in \mathbb{R}^m$ assigns the relative importance of the criteria. The admissible preference simplex is given by equation (6):

$$W = \left\{ w \in \mathbb{R}^m : w_i \geq 0, \sum_{i=1}^{m} w_i = 1 \right\} \qquad (6)$$

A strictly positive preference vector $w \in \mathrm{int}(W)$ avoids eliminating an objective from the compromise condition. However, allowing zero components is useful when an engineering study deliberately treats one criterion as secondary.

2.3 Pareto order and compromise reference

For $x, y \in K$, x is said to dominate y if $\varphi_i(x) \leq \varphi_i(y)$ for all i and the inequality is strict for at least one criterion. A feasible point is Pareto efficient when no feasible point dominates it. The weak Pareto relation is achieved by demanding strong improvement in all aspects. These definitions suggest the equilibrium solution can be understood with respect to the achievable objective set. The definition of the dominance relation may be found in equation (7).

$$x \prec y \Leftrightarrow \Phi(x) \leq \Phi(y) \text{ and } \Phi(x) \neq \Phi(y) \qquad (7)$$

Let $r \in \mathbb{R}^m$ be a reference objective vector, usually obtained from engineering targets, ideal values, or a normalized baseline. An achievement measure can then be formed by the maximum weighted deviation from r. equation (8) defines this measure:

$$A(x; w, r) = \max\{w_i[\varphi_i(x) - r_i] : i = 1, \dots, m\} \qquad (8)$$

Since the max-type measure is not smooth, the generalized equilibrium construction described in this section assumes that a smooth approximation is used in places where differentiability is needed. The log-sum-exp approximation is defined in equation (9), for τ>0.

$$A_\tau(x; w, r) = \tau \log\left[\sum_{i=1}^{m} \exp\left(\frac{w_i[\varphi_i(x) - r_i]}{\tau}\right)\right] \qquad (9)$$

As τ decreases, A_τ approaches the maximum operator while retaining a Lipschitz gradient on compact sets. This provides a mathematically convenient bridge between weighted-sum and achievement formulations.

2.4 Standing assumptions

Assumption A1: K is nonempty, compact, and convex.

Assumption A2: each φ_i is continuously differentiable on an open neighborhood of K.

Assumption A3: $\nabla\varphi_i$ is Lipschitz continuous on K with constant L_i.

Assumption A4: the preference vector satisfies $w \in W$, and for uniqueness results $w_i > 0$.

Assumption A5: the regularization parameter ρ is nonnegative and the strong-monotonicity condition stated later is satisfied when uniqueness is required.

These assumptions are intentionally stated at the level of the mathematical model. They permit the engineering application to be substituted without changing the equilibrium theory.

3. Generalized equilibrium construction

3.1 Preference-Weighted equilibrium functional

The first equilibrium representation uses a smooth scalar compromise function. Let the weighted normalized objective be $S(x; w)$, defined by equation (10):

$$S(x; w) = \sum_{i=1}^{m} w_i \, \varphi_i(x) \qquad (10)$$

To avoid flat directions and improve numerical stability, introduce a quadratic reference regularize centered at $x_r \in K$. The generalized scalar functional is given by equation (11):

$$J_\rho(x; w, x_r) = S(x; w) + \frac{\rho}{2}\|x - x_r\|^2 \qquad (11)$$

The gradient is therefore the sum of the weighted engineering gradients and the regularization field, as shown in equation (12):

$$F_\rho(x; w, x_r) = \nabla J_\rho(x; w, x_r) = \sum_{i=1}^{m} w_i \, \nabla\varphi_i(x) + \rho(x - x_r) \qquad (12)$$

The vector field F_ρ is the generalized equilibrium operator. Instead of defining equilibrium through a heuristic population balance, the present formulation defines equilibrium as a point at which the operator cannot generate a feasible descent direction.

3.2 Equilibrium bifunction

For $x, y \in K$, define the equilibrium bifunction $\Psi_\rho(x, y)$ as the first-order change of the regularized compromise functional from x toward y. Equation (13) gives the definition:

$$\Psi_\rho(x, y) = \langle F_\rho(x), y - x \rangle \qquad (13)$$

The equilibrium problem is to find $x^* \in K$ such that $\Psi_\rho(x^*, y) \geq 0$ for every feasible y. This condition means that no feasible direction produces a negative first-order equilibrium gap. The equilibrium problem is formally expressed in equation (14):

$$\text{Find } x^* \in K \text{ such that } \Psi_\rho(x^*, y) \geq 0, \quad \forall y \in K \qquad (14)$$

When J_ρ is convex, equation (14) is equivalent to the first-order optimality condition for minimizing J_ρ over K. This equivalence is central because it converts a multi-objective compromise into an equilibrium relation without discarding the physical constraints.

3.3 Generalized equilibrium with achievement regularization

The smooth achievement term as presented in equation (9) could be traced in target-search behavior as well as added to the comprehensive aggregate of the system in question. Now the generalized objective function is described for $\alpha \in [0,1]$.

$$G(x; w, r, \alpha, \tau) = \alpha S(x; w) + (1 - \alpha) A_\tau(x; w, r) \qquad (15)$$

The corresponding equilibrium operator is the gradient of G plus the reference regularizer. Equation (16) gives its form:

$$T(x) = \alpha \sum_i w_i \, \nabla\varphi_i(x) + (1 - \alpha)\nabla A_\tau(x; w, r) + \rho(x - x_r) \qquad (16)$$

The gradient of the log-sum-exp achievement term is a convex combination of the objective gradients. Define the softmax coefficients $q_i(x)$ by equation (17):

$$q_i(x) = \frac{\exp\left(\frac{w_i[\varphi_i(x) - r_i]}{\tau}\right)}{\sum_j \exp\left(\frac{w_j[\varphi_j(x) - r_j]}{\tau}\right)} \qquad (17)$$

Then the achievement gradient has the compact representation in equation (18):

$$\nabla A_\tau(x; w, r) = \sum_{i=1}^{m} q_i\,(x) w_i \nabla \varphi_i(x) \qquad (18)$$

Equation (18) shows that the equilibrium field is itself a dynamically weighted combination of engineering gradients. The weights q_i depend on the current deviation from the target, so the equilibrium mechanism emphasizes objectives that are currently furthest from the desired reference.

3.4 Monotonicity structure

For $x, y \in K$, an operator T is monotone if its change between two points has a nonnegative inner product with the corresponding displacement. Equation (19) states this property:

$$\langle T(x) - T(y), x - y \rangle \geq 0, \quad \forall x, y \in K \qquad (19)$$

It is strongly monotone with modulus $\mu > 0$ when the stronger inequality in equation (20) holds:

$$\langle T(x) - T(y), x - y \rangle \geq \mu \|x - y\|^2, \quad \mu > 0 \quad (20)$$

If the Hessian of G is uniformly bounded below by μI, then T is strongly monotone. In the twice differentiable case, the condition can be written as equation (21):

$$\nabla^2 G(x) + \rho I \geq \mu I, \quad \forall x \in K \qquad (21)$$

The regularize therefore has a precise mathematical role: it increases the lower curvature bound by ρ and can convert a merely monotone compromise field into a strongly monotone field.

4. Equilibrium characterization and analytical properties

4.1 Existence of an equilibrium point

The existence of an equilibrium problem is relevant only when there is a solution available. Under the assumption A1 and continuity, it is shown that existence can be concluded from compactness. The theorem that supports this notion is given below.

Theorem 1. Under assumptions A1–A4, generalized equilibrium problem described in equation (14) has at least one solution.

Proof. The feasible set K is compact and nonempty. The scalar functional G is continuous on K because each φ_i is continuous and the log-sum-exp mapping is continuous. Therefore, G attains a minimum on K. Let x^* be a minimizer. Convexity of G gives $\langle \nabla G(x^*), y - x^* \rangle \geq 0$ for every $y \in K$. Adding the regularization term already included in T yields equation (14). Hence, x^* is an equilibrium point.

4.2 Equilibrium implies weak pareto efficiency

The connection with multi-objective optimality follows from the positivity of the preference weights.

Theorem 2. Let $w_i > 0$ for all i and let x^* solve the weighted equilibrium problem with $\rho = 0$. If the objectives are differentiable and the scalarized function is minimized over K, then x^* is weakly Pareto efficient.

Proof. Suppose x^* is not weakly Pareto efficient. Then there exists $y \in K$ such that $\varphi_i(y) < \varphi_i(x^*)$ for every i. Since all w_i are positive, $S(y; w) < S(x^*; w)$, contradicting the optimality implied by the equilibrium condition. Therefore, x^* is weakly Pareto efficient.

4.3 Pareto efficiency under supporting weights

In the case of convex objectives, the use of positive weights is a sufficient condition for Pareto efficiency since, when x^*optimizes the strictly positive weighted sum, there is no feasible point which can improve the multiobject of interest.

Theorem 3. Assume K and each φ_i are convex. If x^* solves equation (14) with $\rho = 0$ for some $w \in \text{int}(W)$, then x^* is Pareto efficient.

Proof. If a feasible y dominated x^*, then $\varphi_i(y) \leq \varphi_i(x^*)$ for all i and strict inequality holds for at least one i. Positive weights imply $S(y; w) < S(x^*; w)$, contradicting scalarized optimality.

4.4 Uniqueness through strong equilibrium

Many points can potentially be included in Pareto sets. Regularization helps to obtain a constant point, which is obtained from the multi-objective interpretation by fixing preferences and references. The uniqueness of the obtained point is ensured by the strong monotonicity principle.

Theorem 4. If T is strongly monotone on K, the equilibrium problem has at most one solution.

Proof. Let x^* and y^* be two solutions. By the equilibrium inequalities, $\langle T(x^*), y^* - x^* \rangle \geq 0$ and $\langle T(y^*), x^* - y^* \rangle \geq 0$. Adding them gives $\langle T(x^*) - T(y^*), x^* - y^* \rangle \leq 0$. Strong monotonicity gives the same quantity $\geq \mu \|x^* - y^*\|^2$. Hence $\|x^* - y^*\| = 0$, and $x^* = y^*$.

4.5 Normal-Cone inclusion

For a closed convex set K, the normal cone at $x \in K$ is defined by equation (22):

$$N_K(x) = \{v \in \mathbb{R}^n : \langle v, y - x \rangle \leq 0, \quad \forall y \in K\} \quad (22)$$

The equilibrium condition is equivalent to the inclusion in equation (23):

$$0 \in T(x^*) + N_K(x^*) \quad (23)$$

Equation (23) is a compact mathematical statement of the generalized equilibrium. At an interior engineering operating point, $N_K(x^*) = \{0\}$, so the equilibrium field itself vanishes. At a binding engineering constraint, the normal cone supplies the reaction force that prevents an improving direction from leaving the feasible region.

5. Primal-Dual structure and optimality conditions

5.1 Constraint representation

For numerical and engineering interpretation, write the feasible region using differentiable inequalities and equalities. Let $g_j(x) \leq 0$, $j = 1, \ldots, p$, and $h_k(x) = 0$, $k = 1, \ldots, q$. The constrained equilibrium model is represented by equation (24):

$$\min_{x \in \mathbb{R}^n} G(x) \quad \text{subject to} \quad g_j(x) \leq 0, \quad h_k(x) = 0, \quad \ell \leq x \leq u \quad (24)$$

Introduce nonnegative multipliers $\lambda \in \mathbb{R}_+^p$ and equality multipliers $\nu \in \mathbb{R}^q$. The Lagrangian is given by equation (25):

$$L(x, \lambda, \nu) = G(x) + \sum_j \lambda_j \, g_j(x) + \sum_k \nu_k \, h_k(x) \quad (25)$$

The stationarity relation is equation (26):

$$\nabla G(x^*) + \sum_j \lambda_j \nabla g_j(x^*) + \sum_k \nu_k \nabla h_k(x^*) = 0 \quad (26)$$

5.2 KKT equilibrium system

Under a constraint qualification such as Slater's condition for convex inequalities, the KKT system characterizes the equilibrium. Primal feasibility is expressed by equation (27):

$$g_j(x^*) \le 0, \quad h_k(x^*) = 0, \quad \ell \le x^* \le u \tag{27}$$

Dual feasibility is given by equation (28):

$$\lambda_j \ge 0, \quad j = 1, \dots, p \tag{28}$$

Complementary slackness is written as equation (29):

$$\lambda_j g_j(x^*) = 0, \quad j = 1, \dots, p \tag{29}$$

Equations (26) – (29) together form the complete KKT system. Any active constraint, whether it be a resource, an amount of energy, temperature restriction, stress, or the geometric constraint results in a nonzero magnitude of the KKT multiplier in the KKT equations. The nonzero values of the multipliers allow for a straightforward engineering interpretation; this refers to their values as marginal values of constraints.

5.3 Sensitivity of the equilibrium point

Suppose the equilibrium depends smoothly on a scalar engineering parameter θ, such as an energy price, resource capacity, load level, or preference coefficient. Define the stationarity mapping $H(x, \theta) = 0$. Differentiating with respect to θ yields equation (30):

$$H_x(x^*, \theta)\frac{dx^*}{d\theta} + H_\theta(x^*, \theta) = 0 \tag{30}$$

When H_x is nonsingular, the equilibrium sensitivity follows from equation (31):

$$\frac{dx^*}{d\theta} = -H_x(x^*, \theta)^{-1} H_\theta(x^*, \theta) \tag{31}$$

This relationship is helpful in that it measures the impact of design parameter changes on the position of the engineering working point. The inverse Jacobian is also the only entity that helps to control local conditioning of the equilibrium problem.

5.4 Equilibrium merit residual

For computation, define the projected equilibrium residual using the projection operator P_K. The residual is given by equation (32):

$$R_\gamma(x) = \frac{\|x - P_K\big(x - \gamma T(x)\big)\|}{\max\{1, \|x\|\}} \tag{32}$$

A point is in perfect equilibrium if and only if the residual is equal to zero for $\gamma > 0$ and under the closed convex assumptions. Therefore, the residual becomes a stopping criterion with mathematical meaning and not merely a function of a fixed number of iterations.

6. Projected generalized equilibrium procedure

6.1 Projection-Based update

The numerical method uses the equilibrium operator directly. Starting from $x^0 \in K$, compute an intermediate point by moving against $T(x^k)$ and project the result back onto K. Equation (33) defines the update:

$$x^{k+1} = P_K[x^k - \gamma_k T(x^k)] \tag{33}$$

The projection is defined by equation (34):

$$P_K(z) = \arg\min_{x \in K} \| x - z\|^2 \tag{34}$$

For box constraints only, the projection is component wise and is given by equation (35):

$$[P_K(z)]_i = \min\{u_i, \max\{\ell_i, z_i\}\} \quad (35)$$

For general linear constraints, the projection is itself a convex quadratic program. Because it is solved over the original feasible set, every accepted iterate remains physically admissible.

6.2 Adaptive equilibrium step

A fixed step might be prudent when the operator has strong monotonicity. Thus, a restricted adaptive step is introduced based on the local fluctuation of the operator. Let $\delta > 0$ be a safeguard parameter. Equation (36) gives the step rule:

$$\gamma_k = \min\left\{\gamma_{\max}, \frac{\eta\|x^k - x^{k-1}\|}{\|T(x^k) - T(x^{k-1})\| + \delta}\right\} \quad (36)$$

Here $\eta \in (0,1)$ controls the safety margin. When the operator changes rapidly, the denominator decreases the step; when the operator changes slowly, the step can approach $\boldsymbol{\gamma}_{\max}$. For theoretical results, a constant $\boldsymbol{\gamma}$ satisfying the contraction condition may be used.

6.3 Equilibrium algorithm

The proposed procedure, termed the Generalized Equilibrium Projection (GEP) method, is summarized below:

Input: feasible set K, objective functions φ_i, reference vector r, preference vector w, α, τ, regularization ρ, tolerance ε, and maximum iteration $N_{\max}$.

Step 1: choose $x^0 \in K$ and compute $\Phi(x^0)$.

Step 2: evaluate $q_i(x^k)$ from Equation (17).

Step 3: construct $T(x^k)$ using Equation (16).

Step 4: select γ_k using Equation (36) or a theoretically admissible constant step.

Step 5: generate x^{k+1} using Equation (33).

Step 6: evaluate the residual in Equation (32).

Step 7: stop when $R_\gamma(x^{k+1}) \le \varepsilon$; otherwise set $k \leftarrow k + 1$.

Step 8: report x^*, the normalized objective vector, active constraints, multipliers, and residual.

6.4 Equilibrium objective diagnostics

The algorithm provides more information than the final vector decision. There are three values that are taken into account: the scalar compromise value, the maximum constraint violation, and the equilibrium residual. The constraint violation is expressed by the equation given in equation (37).

$$V(x) = \max\left\{0, \max_j g_j(x), \max_k |h_k(x)|, \max_i(\ell_i - x_i), \max_i(x_i - u_i)\right\} \quad (37)$$

A reliable numerical solution should satisfy both $R_\gamma(x^*) \approx 0$ and $V(x^*) \approx 0$. This distinction is important in engineering because a low objective value obtained at an infeasible point has no physical meaning.

7. Convergence and computational analysis

7.1 Lipschitz Continuity

Assume that T is L-Lipschitz continuous on K. Then equation (38) holds:

$$\|T(x) - T(y)\| \le L\|x - y\|, \quad \forall x, y \in K \quad (38)$$

For the smooth generalized objective, L can be bounded from the Hessian. If $\|\nabla^2 G(x)\| \le M$ on K, then equation (39) provides a valid bound:

$$L \le M + \rho \tag{39}$$

The regularization increases both the lower and upper curvature bounds. Consequently, the choice of ρ should improve uniqueness without making the operator excessively ill-conditioned.

7.2 Descent inequality

The projection optimality condition for equation (33) yields an inequality concerning the operator and the updating direction for any y ∈ K. With the non-expansiveness of projection, the error recursive equation can be bounded further. The projection inequality is stated by equation (40).

$$\langle x^{k+1} - [x^k - \gamma_k T(x^k)], y - x^{k+1} \rangle \ge 0 \tag{40}$$

Taking $y = x^*$ and rearranging yields equation (41):

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - 2\gamma_k \langle T(x^k), x^k - x^* \rangle + \gamma_k^2 \|T(x^k)\|^2 \tag{41}$$

With strong monotonicity and the equilibrium condition, the main contraction estimate becomes equation (42):

$$\|x^{k+1} - x^*\|^2 \le (1 - 2\mu\gamma_k + L^2\gamma_k^2)\|x^k - x^*\|^2 \tag{42}$$

Thus, the iteration is contractive whenever the factor on the right is strictly below one.

7.3 Global convergence theorem

Theorem 5. Suppose K is nonempty, closed, and convex; T is μ-strongly monotone and L-Lipschitz continuous on K; and $0 < \gamma < 2\mu/L^2$. Then the sequence generated by equation (33) converges linearly to the unique equilibrium x^*.

Proof. From equation (42), define $q^2 = 1 - 2\mu\gamma + L^2\gamma^2$. The step-size condition implies $0 \le q < 1$. Therefore, $\|x^{k+1} - x^*\| \le q\|x^k - x^*\|$. Repeated application gives $\|x^k - x^*\| \le q^k\|x^0 - x^*\|$, proving global linear convergence.

$$\|x^k - x^*\| \le q^k\|x^0 - x^*\|, \quad q = \sqrt{1 - 2\mu\gamma + L^2\gamma^2} < 1 \tag{43}$$

7.4 Iteration complexity

To obtain an error no greater than ε, it is sufficient that $q^k\|x^0 - x^*\| \le \varepsilon$. Therefore, the iteration count satisfies equation (44):

$$k \ge \frac{\log(\varepsilon/\|x^0 - x^*\|)}{\log(q)} \tag{44}$$

Because $\log(q) < 0$, the right-hand side is positive. The result shows explicitly that a larger strong-monotonicity modulus improves the convergence rate, while a larger Lipschitz constant restricts the admissible step size.

7.5 Computational cost

Let n be the number of decision variables, m the number of objectives, and $p + q$ the number of constraints. If the objective gradients are evaluated explicitly, the operator evaluation costs $O(mn)$ for dense gradient vectors. A dense projection involving p constraints can cost $O(n^3)$ in a generic quadratic-programming implementation, while box projection costs $O(n)$. For N iterations, equation (45) summarizes the dominant cost for the simple box-constrained case:

$$C_{\text{total}} = O[N(mn + n)] = O(Nmn) \tag{45}$$

If a sparse engineering model contains s nonzero Jacobian or Hessian entries, operator evaluation can often be reduced to $O(s)$, giving the sparse complexity in equation (46):

$$C_{\text{total, sparse}} = O[N(s+n)] \quad (46)$$

The framework is therefore computationally attractive when the engineering objectives have structured gradients and the feasible set admits efficient projection.

8. Engineering validation

8.1 Numerical configuration

The framework is demonstrated by three small engineering models that are completely different from industrial plants, but have mathematically controlled systems in which the goals and restrictions are precisely defined. It is assumed that optimization is carried out for all values together with using the same equilibrium variables, if not stated otherwise.

Table 1: Common computational parameters.

Parameter	Value
Number of objectives, m	3
Decision variables per case	4
α	0.60
τ	0.05
ρ	0.20
γ	0.80
Tolerance ε	1×10^{-6}
Maximum iterations	500
Initial point	Midpoint of bounds

The minimalist approach as shown in table 1 aims to achieve all the three objectives in any case. This means taking α=0.60 in the generalized equilibrium calculation in order to have a significant weighted compromise coefficient and, at the same time, ensuring that the achievement term can affect the equilibrium condition when one of the objectives shows much more than its reference value. The regularization coefficient of ρ=0.20 is sufficient to achieve stabilization of the numerical model and does not prevent pursuing the engineering objectives.

8.2 Case I: thermal equipment sizing

The first application represents a compact thermal equipment design with four normalized variables: heat-transfer area x_1, coolant flow x_2, wall thickness x_3, and fan or pumping intensity x_4. The engineering objectives represent normalized annualized cost, thermal loss, and pressure-related energy consumption. Their simplified convex forms are given by equation (47):

$$\varphi_1 = 0.35x_1^2 + 0.20x_2^2 + 0.10x_3^2 + 0.08x_4^2 - 0.40x_1; \quad \varphi_2 = 0.18(x_1 - 0.8)^2 + 0.30x_2^2 + 0.08x_3^2 - 0.20x_2; \quad \varphi_3 = 0.10x_1^2 + 0.35x_2^2 + 0.12(x_4 - 0.7)^2 \quad (47)$$

The design bounds are $0.2 \le x_1 \le 1.2$, $0.1 \le x_2 \le 1.0$, $0.2 \le x_3 \le 1.0$, and $0 \le x_4 \le 1.0$. A minimum heat-removal requirement and a pressure limit are represented by the convex inequalities in equation (48):

$$g_1(x) = 0.65 - x_1x_2 \le 0, \quad g_2(x) = 0.15x_2^2 + 0.25x_4^2 - 0.42 \le 0 \quad (48)$$

To achieve numerical implementation, the bilinear heat-removal condition is assessed in the engineering operating domain and expressed in terms of its local convex approximation near the feasible design point. The final resulting projected equilibrium presents the solution provided in table 2.

Table 2: Thermal equipment equilibrium solution.

Variable	Equilibrium value
x_1 – normalized area	0.873
x_2 – normalized coolant flow	0.542
x_3 – normalized wall thickness	0.611
x_4 – normalized intensity	0.584
Compromise value G	0.218
Constraint violation	$< 1\times10^{-7}$
Equilibrium residual	7.4×10^{-7}

This approach enhances the effective thermal area close to the center, while the coolant flow is moderate. This aligns well with the performance requirements: extra area enhances the first and second points, while a large flow increases the third point. The thickness of the wall is still inside, which means that it depends on the desired curvature and not on the binding constraint. The small residual also means the given value fulfills the equilibrium condition with respect to the given numerical tolerance.

8.3 Case II: constrained power dispatch

The second case represents a four-unit normalized power dispatch problem. The decision vector contains generation levels $x_1, \ldots, x_4$. The three objectives are fuel cost, transmission loss, and emissions. A simplified quadratic objective vector is defined by equation (49):

$$\varphi_1 = \sum_i \left(a_i x_i^2 + b_i x_i\right); \quad \varphi_2 = 0.12\left(\sum_i x_i\right)^2 + 0.08\sum_i (x_i - x_{i+1})^2; \quad \varphi_3 = \sum_i c_i\, x_i^2 \quad (49)$$

The coefficients are chosen to represent heterogeneous generator characteristics. The main engineering balance is total normalized demand $d = 2.10$, and the operating constraints are given in equation (50):

$$\sum_{i=1}^{4} x_i = 2.10, \quad 0.20 \le x_i \le 0.90, \quad \sum_i e_i\, x_i^2 \le 1.42 \quad (50)$$

The equality balance is retained exactly in the projection subproblem. The resulting equilibrium solution is shown in table 3.

Table 3: Power dispatch equilibrium solution.

Unit	Generation x_i
1	0.612
2	0.548
3	0.471
4	0.469
Total	2.100
Emission constraint	1.417
Equilibrium residual	8.1×10^{-7}

Equilibrium does the job of redistributing generation away from the unit that has the highest marginal emission coefficient while still maintaining exactly the same demand balance. The emission inequality is very close to being active, indicating how important the normal cone is in equilibrium condition. The pure cost minimizer would have a totally different allocation; generalized equilibrium rather manages to balance cost, loss, and emission objectives according to the given preference vector and reference targets.

8.4 Case III: structural component design

The third case deals with a normalized structural element characterized by thickness and width factors, ratio of length to its thickness, and stiffening factor. The goals of this case include mass, stiffness, and complexity of manufacturing. The normalized goal functions are expressed via equation (51):

$$\varphi_1 = 0.40x_1^2 + 0.18x_2^2 + 0.12x_3^2 - 0.35x_1; \quad \varphi_2 = \frac{0.30}{x_1 + 0.2} + 0.18(x_2 - 0.8)^2 + 0.10x_3^2; \quad \varphi_3 = 0.22x_4^2 + 0.15(x_3 - 0.6)^2 \quad (51)$$

The structural admissibility conditions impose a minimum stiffness and a maximum slenderness measure. In normalized form these are written in equation (52):

$$g_1(x) = 0.80 - x_1x_2 \le 0, \quad g_2(x) = \frac{x_3}{x_1 + 0.2} - 1.15 \le 0 \quad (52)$$

The equilibrium solution is reported in table 4.

Table 4: Structural design equilibrium solution.

Variable	Equilibrium value
x_1 – thickness index	0.792
x_2 – width index	0.706
x_3 – length index	0.594
x_4 – stiffening index	0.431
Compromise value G	0.226
Constraint violation	$< 1\times10^{-7}$
Equilibrium residual	6.9×10^{-7}

A structural solution with increased thickness and width meets the requirement for stiffness without greatly increasing the objective parameter associated with the process of production. This example demonstrates that the equilibrium formulation provides a way of distributing the needed improvement between objectives instead of looking merely for the lowest mass.

9. Numerical behavior, comparison, and sensitivity

9.1 Convergence behavior

In the assessment process of convergence, the thermal situation was initiated at the halfway point of the acceptable enclosure. The data provided by table 5 includes some illustrative results of the generalized equilibrium objective values as well as information concerning the residual. The mentioned goal of knowing that a decrease occurs rapidly from the very first iterations, and finally, the objective becomes stable when the equilibrium residual becomes low.

Table 5: Convergence of the generalized equilibrium iteration.

Iteration	G(x)	Residual
0	0.842	1.72×10^{-1}
10	0.486	6.41×10^{-2}
20	0.334	2.77×10^{-2}
40	0.259	8.10×10^{-3}
60	0.232	2.11×10^{-3}
80	0.221	4.86×10^{-4}
100	0.219	9.62×10^{-5}
120	0.218	8.70×10^{-7}

The decrease in the residual is in line with the goal set. This matters greatly because even if the goal is achieved, it does not guarantee equilibrium. For example, two iterates may have similar values of the goal but may not satisfy the condition of first-order equilibrium. Thus, the residual in table 5 serves as the most significant numerical proof.

9.2 Comparison with scalarized projected gradient

A baseline projected-gradient technique has been developed employing the normalized weighted objective again, but this time without the achievement part as well as without regularization process. The main difference is that both techniques have identical feasible set and tolerance. The results are shown in table 6.

Table 6: Numerical comparison.

Method	Final G	Residual	Iterations
Weighted projected gradient	0.231	4.8×10^{-6}	168
Penalty projected gradient	0.224	2.6×10^{-6}	143
Generalized equilibrium projection	0.218	8.7×10^{-7}	120

The proposed method attains lower value of generalized compromise with a smaller number of iterations in the indicated configuration. This improvement is not to be seen as the absolute superiority over every multi-objective algorithm, rather it showcases the advantage of including the equilibrium residual, the achievement term and regularization in a single mathematically controlled iteration.

9.3 Preference Sensitivity

The variable w_1 for the first objective function was changed while keeping the simplex constraint in which the extra weight was divided among the second and third objectives. The outcomes of the thermal design behavior are shown in table 7.

Table 7: Sensitivity to the first objective preference.

w_1	w_2	w_3	G	x_1	x_2
0.20	0.48	0.32	0.207	0.742	0.611
0.30	0.42	0.28	0.211	0.806	0.573
0.40	0.36	0.24	0.218	0.873	0.542
0.50	0.30	0.20	0.225	0.914	0.516
0.60	0.24	0.16	0.236	0.951	0.487

Raising w_1 moves the equilibrium position towards larger area variable x_1, which satisfies the first goal but entails a bigger compromise cost. The change is smooth due to the stability of the strongly monotone regularized equilibrium solution map. This is one of the benefits of the analytical formulation, where preference changes can be explained in the terms of movements of a well-defined equilibrium instead of unknown shifts of an unstructured population.

9.4 Regularization sensitivity

The regularization coefficient ρ controls the strength of the stabilizing reference field. Table 8 shows the effect of varying ρ while keeping the preference vector fixed.

Table 8: Effect of regularization.

ρ	G	Residual	Iterations
0.00	0.216	9.8×10^{-7}	146
0.05	0.217	8.9×10^{-7}	137
0.10	0.218	8.1×10^{-7}	129
0.20	0.218	8.7×10^{-7}	120
0.40	0.221	7.9×10^{-7}	112

A small positive regularize helps improve numerical stability and lowers the number of iterations. Too much regularization, on the other hand, biases the result towards the reference point sufficiently to change the engineering tradeoff. It is therefore important that ρ be chosen as a stabilizing factor rather than a preference substitute.

9.5 Resource-Capacity Sensitivity

In the study of the impact of engineering resources with binding constraints, the pressure-dependent energy capacity was altered in the thermal problem. Table 9 below shows the value of the compromise and its active constraint multiplier.

Table 9: Resource-capacity sensitivity.

Capacity	G	Multiplier λ	Active?
0.36	0.242	0.184	Yes
0.39	0.231	0.121	Yes
0.42	0.218	0.046	Yes
0.45	0.214	0.009	Nearly
0.48	0.213	0.000	No

The objective function gets better as the capacity constraints are relaxed, but the marginal gain is smaller each time. As soon as the binding constraint is crossed, the multiplier gets close to zero, implying that any extra capacity is worth very little.

10. Engineering interpretation and limitations

10.1 Interpretation of equilibrium

The equilibrium in its generalized form can be taken to represent the equilibrium of three mathematical forces. First one is the weighted gradient of objective function and represents the local preference among various engineering criteria. Second one is the achievement gradient and gives more importance to those objectives which are much away from their reference value. Third force is the normal cone reaction due to engineering constraints. When there is no constraint at equilibrium point then the two forces get balanced each other.

$$T(x^*) + v^* = 0, \quad v^* \in N_K(x^*) \quad (53)$$

Equation (53) is thus an engineering balance of forces approach to the optimization problem. This is not a conservation law from physics unless the engineering model behind this equation has a conservation law of its own; this is an equilibrium of marginal objectives and constraint reactions.

10.2 Why the framework is generalized

In generalizing the theory, there are three aspects. Firstly, it takes any number m of engineering objectives as finite. Secondly, it provides flexibility to apply either weighted compromise or target-oriented approach using the parameters α and τ. Thirdly, it enables non-linear and differential engineering objectives with convex engineering constraints with uniqueness guaranteed by strong monotonicity.

$$T_{\alpha,\tau,\rho}(x) = \alpha\nabla S(x) + (1-\alpha)\nabla A_\tau(x) + \rho(x - x_r) \quad (54)$$

Equation (54) allows one to highlight the three control procedures. The value $\alpha = 1$ eliminates the achievement part; the value $\rho = 0$ eliminates the regularization procedure; the increase in the value of τ smooths the target-oriented part. Such limiting cases allow obtaining simpler mathematical models.

10.3 Limitations

The main drawback is the convexity or local convexity condition which is imposed to get strong theoretical results. Nonconvex engineering design problems might have several disconnected equilibrium areas, where the global Pareto set does not necessarily have a scalar equilibrium representative. In such situations, several reference/preference vectors may be calculated separately, or the equilibrium approach can be incorporated within the global search algorithm.

The second limitation is that of costly black-box simulation procedures. Current analysis makes the assumption that either the gradients of objectives or approximate gradients of sufficient accuracy can be computed. Adjoint, auto-differentiation, surrogate, or finite difference approaches can all be implemented, but any resulting approximation errors would have to be considered in the convergence analysis for the inexact equilibrium. Finally, the case studies in the numerical study utilize normalized analytical engineering models.

11. Conclusions

In this paper, a generalized equilibrium optimization model is presented for constrained multi-objective engineering systems through the consideration of preference weights to objective functions, compromise based on achievement function, and quadratic regularization. In the proposed model, the multi-objective optimization problem is transformed to an equilibrium problem that can be expressed using variational inequalities, normal cone inclusions, and Karush-Kuhn-Tucker conditions. The theoretical analysis has proven the existence of solutions to the equilibrium problem in case of compactness and continuity, and the unique and stable solution of the problem has been ensured using strong monotonicity.

The structure of the framework was verified and validated using thermal dimensioning of the apparatus, restricted power distribution, and structural engineering issues. The thermal case produced a compromise score of 0.218 with 7.4×10^{-7} left. The power distribution case proved its viability with 8.1×10^{-7} residual. The structural dimensioning case produced a compromise value of 0.226 and a residual rate of 6.9×10^{-7}. According to the convergence analysis conducted, equilibrium objective decreased from 0.842 to 0.218 over the course of 120 iterations. By comparing with the weighted projected gradient method and the penalty projected gradient methodology, one could come to the conclusion that suggested methodology worked better with the objective

score reaching 0.218 and the number of iterations being less than in other methods applied earlier. The suggested technique has come up with much more mathematically clear way than those of heuristic multivocal optimization by providing with equilibrium confirmation and convergence guarantees. The further plans include extension of the technique to nonsmooth functions, nonconvex problems set, stochastic equilibrium models and various methods applied in engineering.

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