Minimax Theory and its Applications
Volume 11 (2026), No. 2s, 424-439

A unified variational and minimax optimization framework for robust engineering system design under uncertainty

V. Dhanalakshmi[1], Brijeshkumar M. Patel[2], Nagireddy Hanuman Reddy[3], Vaibhav Kaushik[4], Dr. Vinjamuri SNCH Dattu[5], Vimal Bibhu[6], Kiran Ingale[7]

[1]Assistant Professor, Department of Computer Science, Meenakshi College of Arts and Science, Meenakshi Academy of Higher Education and Research, Tamil Nadu, India. dhanalakshmiv@maher.ac.in

[2]Assistant Professor, Faculty of Engineering, Gokul Global University, Gujarat, India. bmpatel.me.hcet@gokuluniversity.ac.in, https://orcid.org/0009-0002-1292-7842

[3]Associate Professor, Department of Computer Science and Engineering, Vardhaman College of Engineering, Shamshabad, Hyderabad, India. hanuman@vardhaman.org

[4]Centre of Research Impact and Outcome, Chitkara University, Rajpura, Punjab, India. vaibhav.kaushik.orp@chitkara.edu.in, https://orcid.org/0009-0001-0234-3205

[5]Bonam Venkata Chalamayya Institute of Technology and Science, Amalapuram, Andhra Pradesh, India. dathuthermal@gmail.com

[6]Professor, Department of Computer Science & Engineering, Noida International University, Uttar Pradesh, India. vimal.bhibu@niu.edu.in

[7]Department of E&TC Engineering, Vishwakarma Institute of Technology, Pune, Maharashtra, India. kiran.ingale@vit.edu

Abstract

Engineering design concept under uncertain operating conditions requires contribution of mathematics that incorporates all the relevant criteria of competing performance objectives taking feasibility constraints into account and adverse realizations of uncertain parameters into consideration. The paper proposes a unified variational and minimax optimization approach that describes robust multi-objective engineering design problem as a scalarized worst-case functional and with the help of an equilibrium operator. The formulation considers normalized engineering criteria taking advantage of compact set of uncertainties and transforming semi-infinite robust constraints into expressions with the support function. The generalized equilibrium mapping is produced by performing certain operation with support function and normal cone of feasible design area. The existence of robust solution is established due to compactness and continuity of the model while uniqueness of the solution is determined by strong monotonicity property. The equivalent representations of primal, variational, saddle-point, and minimax formulations are derived in the paper, along with conditions of KKT and dual forms. In addition, a projected extragradient method is proposed for obtaining the numerical solution and the possibilities of this method are investigated from the standpoint of its descent property, residual contraction, convergence rate, and computational complexity. To demonstrate the proposed framework, three engineering design problems of energy distribution, thermal operation, and sizing of structural resources were chosen from the point of view of robustness. The numerical results indicate that the proposed framework is able to mitigate the effects of the adverse parameter realizations while ensuring that the performance of competing objectives is balanced. Sensitivity tests highlight the effect of the uncertainty radius and weights of the objectives on the robustness of the design. The framework allows for a mathematical bridge between the robust optimization and problems of variational inequalities and minimax analysis.

Keywords: Robust optimization, variational inequality, minimax optimization, uncertainty set, equilibrium problem, engineering design.

1. Introduction

The design challenges associated with engineering increasingly involve parameters that should not be treated as fixed values. Characteristics of materials, costs of energy, operating conditions, demand levels, climates, and rates of performance may all undergo variations during the lifespan of the system. Through robust optimization, uncertainty can be modeled deterministically, forcing the design to be feasible and efficient under the defined set of uncertainties (Bertsimas et al., 2021). Recent progress led to applications of the robust formulated approach to adjustable parameters, distribution uncertainties, multi-objective models, and uncertainty-dependent restrictions (Cai et al., 2021; Berthold et al., 2022). These improvements are relevant when a certain engineering system has to address multiple conditions rather than focusing on the cost only Kumar & Bawankar, (2025).

A key mathematical challenge here is that of the interaction between uncertainty and multi-objectivity (Irmanto et al., 2025). While an optimized design for nominal values can become infeasible or inefficient upon the realization of a bad scenario, minimization of a weighted objective can fail to capture the balance between conflicting engineering criteria (Duque et al., 2022). The focus in recent robust multi-objective optimization literature is, thus, on Pareto robustness, scalarization, and necessary and sufficient optimality conditions for uncertain data (Dineshkumar, 2026; Liu et al., 2022; Nie et al., 2023). Variational approaches to robust optimization problems prove especially powerful, as they characterize an optimal design in terms of inequalities in the set of admissible perturbations (Doan, 2022; Sedlmayer et al., 2023).

The minimax approach also offers a complementary formulation. In case the uncertainty set is convex and compact, the worst possible value of the engineering performance function can be formulated as a maximization over all admissible uncertainty realizations. In case appropriate convex-concave assumptions are made, the resulting robust optimization problem has a saddle point and minimax formulation (Shapiro et al., 2023). The most recent findings on minimax optimization, robust duality, and saddle point methods indicate that such formulations are applicable both theoretically and numerically (Detienne et al., 2024; Hung et al., 2024). Variational inequalities provide the necessary language of equilibria in order to relate the original robust design problem to the saddle points (Goberna et al., 2024).

The current study creates a mathematical framework based on the aforementioned link. In contrast to viewing robust optimization, variational inequality, and minimax theory as individual techniques, the suggested model puts them into one chain: multi-objective criteria for engineering are normalized and scalarized; uncertainty is modeled by a compact set; worst-case analysis yields a robust function; the robust function induces an equilibrium operator; and finally the generated variational inequality is proven to be equivalent to a constrained minimax problem under some conditions. The construction of the framework is done in the spirit of the mathematics of minimax theory, but its applications are confined to engineering problems.

This paper also attempts to contribute to the necessity of analytical properties explicitly. While numerical techniques are often utilized in engineering optimization literature without any proof of the point being a global robust solution or convergence of the sequence of iterations, the present framework begins by establishing the existence, uniqueness, monotonicity, saddle-point optimality, and convergence conditions before proceeding to numerical experiments (Chuong et al., 2024; Balter et al., 2024). Thus, the numerical approach is a natural extension of the theoretical model rather than an independent heuristic approach. Recent studies on robust optimization have stressed the importance of analytical approach to uncertainty sets, duality, and stability (Grecksch et al., 2024; Jiménez, 2024; Song et al., 2021).

Contribution of the work is as follows. First, a robust multi-objective engineering design model is formulated via normalized quadratic objective functions and an uncertainty set. Second, the worst-case functional is represented by means of support functions and a maximization operation. Third, variational equilibrium conditions are obtained via normal cone inclusion. Fourth, the primal-dual formulation of the problem and minimax formulation of the problem with sufficient conditions of strong duality are provided. Fifth, an iterative scheme called the projected extragradient method is studied via residual and contraction analysis. Sixth, computational complexity is obtained for uncertainty operators in dense and sparse representations. Finally, several engineering examples are provided to demonstrate robust energy, thermodynamic, and sizing designs.

The rest of the paper is structured as follows. Section 2 formulates the robust multi-objective design space and the uncertainty structure. Section 3 formulates the worst-case function and its properties. Section 4 formulates equilibrium and variational formulations. Section 5 derives saddle point, duality, and minimax formulations. Section 6 formulates the numerical algorithm. Section 7 proves convergence and computational results. Section 8 provides engineering design examples. Section 9 provides comparative analysis with other approaches. Section 10 analyzes uncertainty and weighting sensitivity. Section 11 discusses the engineering interpretation of the mathematical formulation, and Section 12 concludes the paper.

2. Robust multi-criteria design architecture

2.1 Design variables and feasible geometry

Let the engineering design be represented by a vector $\boldsymbol{x}$ in $\mathbb{R}^{\boldsymbol{n}}$. The components may denote equipment settings, resource allocations, geometric parameters, control intensities, or operating levels. The admissible design region is defined by convex engineering restrictions.

$$x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n \qquad (1)$$

Equation (1) defines the design vector used throughout the framework.

$$X = \{x \in \mathbb{R}^n : Ax \le b,\ Cx = d,\ \ell \le x \le u\} \quad (2)$$

The feasible set in equation (2) combines resource inequalities, balance equations, and engineering operating bounds.

The matrix A represents resource and capacity restrictions, C represents engineering balance relations, and ℓ and u provide lower and upper operating limits. The set X is assumed nonempty, closed, bounded, and convex. This assumption is important because the normal cone and projection operators used later are then well defined.

$$N_X(x) = \{v \in \mathbb{R}^n : v^T(y - x) \le 0,\ \forall y \in X\} \quad (3)$$

Equation (3) defines the normal cone that encodes active engineering restrictions.

2.2 Multi-Objective engineering criteria

Assume there are m criteria in engineering. These criteria may involve energy utilization, operating cost, thermal deviation, material consumption, reliability reduction, and environmental cost. As the units for these criteria are different, each one of them is normalized with respect to some reference scale.

$$f_i(x, \xi) = \frac{q_i(x, \xi) - q_i^{\text{ref}}}{s_i}, \quad i = 1, \dots, m \qquad (4)$$

Equation (4) converts heterogeneous engineering measures into dimensionless criteria.

$$F(x, \xi) = \sum_{i=1}^{m} w_i f_i(x, \xi), \quad w_i \ge 0, \quad \sum_{i=1}^{m} w_i = 1 \quad (5)$$

Equation (5) defines the scalarized multi-objective performance function.

Weights do not transform the multi-criteria nature of the model; instead, they identify the corresponding compromise solution from the set of reachable criteria values. Below is the robust reformulation which justifies this compromise under the given uncertainty set.

2.3 Uncertainty region and robust feasibility

Let $\xi \in U$ denote uncertain engineering parameters. A convenient uncertainty region is an ellipsoid centered at the nominal parameter $\bar{\xi}$. This choice permits a transparent analytical treatment while allowing correlated uncertainty.

$$U = \left\{\xi \in \mathbb{R}^p : (\xi - \bar{\xi})^T Q^{-1} (\xi - \bar{\xi}) \le \rho^2\right\}, \quad Q \succ 0 \qquad (6)$$

Equation (6) defines the compact ellipsoidal uncertainty set, with ρ controlling the uncertainty radius.

$$g_j(x,\xi) \le 0, \quad \forall \xi \in U, \quad j = 1, \dots, r \tag{7}$$

Equation (7) imposes robust feasibility for every admissible uncertainty realization.

The robust constraints in equation (7) are semi-infinite because each engineering restriction must hold for infinitely many values of ξ. For affine dependence on uncertainty, they can be reduced to finite-dimensional support-function constraints.

$$a_j(x) + c_j(x)^T \bar{\xi} + \rho\left[c_j(x)^T Q c_j(x)\right]^{1/2} \le 0 \tag{8}$$

Equation (8) gives the robust counterpart of an affine uncertain constraint under the ellipsoidal uncertainty model.

2.4 Standing analytical conditions

The following assumptions are used in the theoretical development. They are stated explicitly so that the

existence, uniqueness, duality, and convergence results can be interpreted without ambiguity.

Assumption 1. X is nonempty, compact, and convex. Assumption 2. U is nonempty, compact, and convex.

Assumption 3. Each $f_i(\cdot, \xi)$ is continuously differentiable and convex on X for every $\xi \in U$.

Assumption 4. The mapping $\nabla_x F(x,\xi)$ is continuous in (x,ξ) and Lipschitz continuous in x uniformly over U.

Assumption 5. The robust feasible region satisfies a Slater-type condition.

Assumption 6. The aggregate robust operator is strongly monotone with modulus $\mu > 0$ on X

3. Worst-Case scalarization and robust functional analysis

3.1 Adverse-Scenario functional

The robust value of a design is defined as the maximum scalarized engineering performance over the uncertainty set. This produces a deterministic worst-case functional while retaining explicit dependence on the uncertainty geometry.

$$\Phi(x) = \max_{\xi \in U} F(x,\xi) \tag{9}$$

Equation (9) is the central worst-case scalarization.

$$\Phi(x) = \sum_{i=1}^{m} w_i \max_{\xi \in U} f_i(x,\xi) \tag{10}$$

The expression in equation (10) represents a separable upper bound if the criteria are considered separately. However, when a unified approach is employed, equation (9) becomes more appropriate because all the criteria are equally affected by the uncertain realization. That is, the coupled worst-case approach does not allow incompatible realizations unless they are feasible physically.

$$x^* \in \arg\min_{x \in X} \Phi(x) \tag{11}$$

Equation (11) defines the robust engineering design problem.

3.2 Support-Function representation

Suppose the uncertainty-dependent component of the scalarized objective has the form $h(x)^T \xi$. The support function of U is defined as the maximum inner product over U.

$$\sigma_U(z) = \max_{\xi \in U} z^T \xi \tag{12}$$

Equation (12) defines the support function of the uncertainty set.

$$\sigma_U(z) = z^T\bar{\xi} + \rho(z^TQz)^{1/2} \qquad (13)$$

For the ellipsoid in equation (6), the support function has the closed form in equation (13).

$$\Phi(x) = \varphi_0(x) + \sigma_U(Hx + h) \qquad (14)$$

Equation (14) shows the functional robustness term as a combination of nominal term and uncertainty support term. The uncertainty support term functions like a robustness bonus in terms of mathematics, whose value depends on the uncertainty radius and sensitivity of the design to uncertainty.

3.3 Continuity and convexity

$$|\Phi(x) - \Phi(y)| \leq L_\Phi||x - y|| \qquad (15)$$

Equation (15) gives the Lipschitz estimate under the uniform Lipschitz condition.

Because F is continuous on the compact product set $X \times U$, the maximum in equation (9) is attained for every x. Convexity is inherited when F is convex in x for every ξ. Consequently, Φ is continuous and convex on X. If the nominal part is strongly convex or the resulting robust operator is strongly monotone, the robust minimizer becomes unique.

$$\Phi(\theta x + (1-\theta)y) \leq \theta\Phi(x) + (1-\theta)\Phi(y), \quad 0 \leq \theta \leq 1 \quad (16)$$

Equation (16) states the convexity property of the robust functional.

4. Equilibrium characterization of robust designs

4.1 Generalized robust equilibrium

The robust optimization problem can be converted into an equilibrium problem by considering the first-order response of the robust functional together with the geometry of X. Let $T(x)$ denote a generalized gradient or smooth gradient representation of Φ.

$$T(x) = \nabla\Phi(x) \qquad (17)$$

Equation (17) defines the robust equilibrium operator in the differentiable case.

$$0 \in T(x^*) + N_X(x^*) \qquad (18)$$

Equation (18) gives the normal-cone inclusion characterizing the robust optimum.

Equation (18) is central to the framework. It states that the negative robust gradient cannot point toward any feasible descent direction at x^*. When x^* lies in the interior of X, $N_X(x^*) = \{0\}$; when constraints are active, the normal cone balances the robust gradient.

4.2 Variational inequality form

$$T(x^*)^T(y - x^*) \geq 0, \quad \forall y \in X \qquad (19)$$

Equation (19) is the variational inequality associated with the robust design problem.

$$\text{Find } x^* \in X \text{ such that } \langle T(x^*), y - x^*\rangle \geq 0, \quad \forall y \in X \quad (20)$$

Equation (20) states the robust engineering variational inequality.

For a convex differentiable robust functional, equations (18) and (20) are equivalent. The equilibrium interpretation is particularly useful for engineering systems because y can be interpreted as any feasible alternative design. The inequality then states that the directional first-order change from x^* toward every feasible alternative is nonnegative.

4.3 Monotonicity and uniqueness

$$\langle T(x) - T(y), x - y \rangle \geq \mu ||x - y||^2 \quad (21)$$

Equation (21) defines strong monotonicity with modulus μ.

$$||T(x) - T(y)|| \leq L||x - y|| \quad (22)$$

Equation (22) defines Lipschitz continuity with constant L.

Strong monotonicity gives uniqueness of the equilibrium. Indeed, if x_1 and x_2 were two solutions, the variational inequalities applied reciprocally and added together would yield a zero lower bound for $\mu||x_1 - x_2||^2$, forcing $x_1 = x_2$.

$$\mu||x_1 - x_2||^2 \leq \langle T(x_1) - T(x_2), x_1 - x_2 \rangle \leq 0 \quad (23)$$

Equation (23) establishes uniqueness by contradiction.

4.4 Robust pareto interpretation

The scalarization weights define a robustly efficient compromise when all weights are positive. Let $f(x) = \big(f_1(x), \dots, f_m(x)\big)$ denote the vector of worst-case criteria. A robust Pareto design is one for which no feasible design improves every criterion simultaneously. Under convexity, a positive weighted scalarization provides a sufficient characterization of a weakly robust Pareto point.

$$f(x^*) \leq f(y) \text{ componentwise for no } y \in X \text{ with at least one strict inequality} \quad (24)$$

Equation (24) states the robust Pareto efficiency condition.

5. Saddle structure, dual variables, and minimax equivalence

5.1 Lagrangian construction

Let the robust finite-dimensional constraints be represented by $G(x) \leq 0$ and equality constraints by $Hx = d$. The Lagrangian introduces multipliers for the engineering restrictions.

$$\mathcal{L}(x, \lambda, v) = \Phi(x) + \lambda^T G(x) + v^T(Hx - d), \quad \lambda \geq 0 \quad (25)$$

Equation (25) defines the robust Lagrangian.

$$\nabla\Phi(x^*) + J_G(x^*)^T\lambda^* + H^T v^* = 0 \quad (26)$$

Equation (26) is the stationarity condition.

$$G(x^*) \leq 0, \quad Hx^* = d, \quad \lambda^* \geq 0 \quad (27)$$

Equation (27) states primal and dual feasibility.

5.2 Complementarity and KKT system

$$\lambda_j^* G_j(x^*) = 0, \quad j = 1, \dots, r \quad (28)$$

Equation (28) gives complementary slackness.

$$0 = \nabla\Phi(x^*) + J_G(x^*)^T\lambda^* + H^T v^*; \quad G(x^*) \leq 0; \quad \lambda^* \geq 0; \quad \lambda^* \circ G(x^*) = 0 \quad (29)$$

Equation (29) summarizes the KKT system.

Under the stated convexity and constraint qualification assumptions, the KKT system is sufficient for global robust optimality. The multiplier $\boldsymbol{\lambda_j^*}$ measures the first-order marginal value of relaxing the $\boldsymbol{j}$-th robust engineering constraint.

5.3 Primal-Dual equilibrium

$$z^* = (x^*, \lambda^*, v^*) \in \mathcal{Z}, \quad \mathcal{M}(z^*)^T(z - z^*) \geq 0, \quad \forall z \in \mathcal{Z} \quad (30)$$

Equation (30) expresses the coupled primal and dual equations in terms of equilibrium. The primal and dual formulations prove beneficial when the size of the uncertainty region and robust formulation is large. Instead of eliminating all multipliers, the algorithm updates design and dual variables simultaneously to form the equilibrium mapping.

5.4 Minimax representation

$$\min_{x \in X} \max_{\xi \in U} F(x, \xi) \quad (31)$$

Equation (31) is the explicit robust minimax formulation.

$$\min_{x \in X} \max_{\xi \in U} F(x, \xi) = \max_{\xi \in U} \min_{x \in X} F(x, \xi) \quad (32)$$

Under the convex-concave and compactness assumptions, equation (32) gives the minimax interchange.

The equality in equation (32) is the key unifying result. The engineering designer minimizes with respect to x while the uncertainty adversary maximizes the scalarized loss with respect to ξ. The saddle point therefore represents a robust equilibrium between design improvement and adverse uncertainty selection.

$$F(x^*, \xi) \leq F(x^*, \xi^*) \leq F(x, \xi^*), \quad \forall x \in X, \xi \in U \quad (33)$$

Equation (33) defines the saddle-point inequalities.

5.5 Dual robust value

$$D(\lambda, v) = \inf_{x \in X} \{\Phi(x) + \lambda^T G(x) + v^T (Hx - d)\} \quad (34)$$

Equation (34) defines the dual value function.

$$\max_{\lambda \geq 0, v} D(\lambda, v) \leq \min_{x \in X} \Phi(x) \quad (35)$$

Equation (35) gives weak duality.

In the case where Slater's condition is satisfied and the robust function is convex, strong duality can be guaranteed; hence, the inequality presented in equation (35) is an equality at the optimum point.

6. Projected extragradient solution mechanism

6.1 Predictor-Corrector iteration

A projected extragradient method is adopted because the equilibrium operator can contain both smooth robust-gradient components and constraint-induced monotone terms. Given $x_k \in X$ and step size α_k, the predictor is obtained first.

$$y_k = P_X[x_k - \alpha_k T(x_k)] \quad (36)$$

Equation (36) defines the predictor step.

$$x_{k+1} = P_X[x_k - \alpha_k T(y_k)] \quad (37)$$

Equation (37) defines the corrector step.

$$P_X(v) = \arg\min_{x \in X} | \, ||x - v||^2 \quad (38)$$

Equation (38) defines the Euclidean projection.

The two evaluations of T reduce the sensitivity of the iteration to the local rotation that can occur in monotone equilibrium mappings. Projection guarantees that every generated design remains feasible.

$$r_k = \frac{||x_k - P_X[x_k - T(x_k)]||}{1 + ||x_k||} \tag{39}$$

Equation (39) defines the normalized equilibrium residual used as a stopping criterion.

6.2 Step-Size condition

$$0 < \alpha_k < \frac{1}{L} \tag{40}$$

Equation (40) gives a sufficient constant-step restriction under L-Lipschitz continuity.

$$\alpha_k = \frac{\alpha_0}{(1 + \gamma k)^\tau}, \quad \alpha_0 > 0, \quad 0 \le \tau < 1 \tag{41}$$

Equation (41) gives an optional diminishing-step schedule.

In case of strong monotonicity, constant steps taken inside the feasible range results in linear contraction. If the monotonicity property holds but exact operator estimates are not available, one can use diminishing step size.

6.3 Algorithmic procedure

Input: X, U, weights w, uncertainty radius ρ, initial feasible point x_0, tolerance ε, and maximum iteration K.

Step 1: Construct the normalized criteria and robust functional Φ using equations (4) – (14).

Step 2: Initialize $x_0 \in X$ and choose α satisfying Equation (40).

Step 3: Evaluate $T(x_k)$ and compute the predictor y_k from Equation (36).

Step 4: Evaluate $T(y_k)$ and compute x_{k+1} from Equation (37).

Step 5: Evaluate the normalized residual r_{k+1} using Equation (39).

Step 6: Stop when $r_{k+1} \le \varepsilon$; otherwise set $k \leftarrow k + 1$ and repeat.

Step 7: Report the robust objective, active constraints, worst-case scenario, and engineering criteria.

7. Convergence guarantees and computational burden

7.1 Projection inequality

$$\langle x_{k+1} - v, z - x_{k+1} \rangle \ge 0, \quad \forall z \in X, \quad v = x_k - \alpha_k T(y_k) \tag{42}$$

Equation (42) is the fundamental projection inequality.

$$||x_{k+1} - x^*||^2 \le ||x_k - x^*||^2 - ||x_k - y_k||^2 - ||x_{k+1} - y_k||^2 + R_k \tag{43}$$

Equation (43) summarizes the extragradient descent relation, where R_k contains operator approximation terms.

Under exact operator evaluations, Lipschitz continuity bounds R_k by a multiple of $\alpha_k^2 ||x_k - y_k||^2$. Strong monotonicity then converts the descent relation into a contraction estimate.

7.2 Linear convergence

$$||x_{k+1} - x^*||^2 \le q ||x_k - x^*||^2, \quad 0 < q < 1 \tag{44}$$

Equation (44) gives the linear convergence estimate.

$$q = 1 - 2\alpha\mu + \alpha^2 L^2 \tag{45}$$

Equation (45) gives a representative contraction factor under the stated operator bounds.

$$k \geq \frac{\log(\varepsilon^2/||x_0 - x^*||2)}{\log(q)} \quad (46)$$

Equation (46) gives an iteration bound for achieving a prescribed squared-error tolerance.

This occurs because the condition q<1 is satisfied provided that the step size is chosen appropriately small compared to the strong monotonicity constant and the Lipschitz constant. Consequently, the method is guaranteed to converge and is not just converging heuristically.

7.3 Residual-Based stopping

$$r_k \leq (1 + L)||x_k - x^*|| \quad (47)$$

Equation (47) relates the equilibrium residual to the distance from the unique solution.

The residual is preferred over a specified number of iterations since it indicates nonconformity with the equilibrium criteria. The residual is also relevant even if the dimension of the engineering decision vector changes.

7.4 Complexity estimates

$$C_{\text{iter}} = O(n^2 + nr + np) \quad (48)$$

Equation (48) gives the dense per-iteration order for operator evaluation, robust constraints, and uncertainty terms.

$$C_{\text{total}} = O\big(K(n^2 + nr + np)\big) \quad (49)$$

Equation (49) gives the total cost for K iterations.

$$C_{\text{iter}}^{\text{sparse}} = O(\text{nnz}(T) + nr + np) \quad (50)$$

Equation (50) gives the sparse-operator cost.

The proposed framework is therefore tractable for moderate engineering dimensions and can exploit sparse matrices when network, structural, or resource interactions are localized.

8. Engineering design experiments under uncertainty

8.1 Common numerical configuration

In order to show various views of the decision vector, three cases of engineering problem solving are considered using the same mathematical approach. In all three cases, the same number of four normalized design variables, three criteria, an ellipsoid uncertainty set, and convex resource constraints are used. The problems are constructed in a mathematical engineering system form. Baseline parameters are sown in table 1.

$$x = [x_1, x_2, x_3, x_4]^T, \quad 0 \leq x_i \leq 1 \quad (51)$$

Equation (51) defines the normalized design vector used in the numerical cases.

$$U_\rho = \{\xi : ||Q^{-1/2}(\xi - \bar{\xi})||2 \leq \rho\} \quad (52)$$

Equation (52) specifies the uncertainty region used in the numerical experiments.

$$\Phi_\rho(x) = 0.35f_1(x, \xi) + 0.30f_2(x, \xi) + 0.20f_3(x, \xi) \text{ maximized over } \xi \in U_\rho \quad (53)$$

Equation (53) gives the baseline robust scalarization.

$$0.60x_1 + 0.45x_2 + 0.50x_3 + 0.30x_4 \leq 1.20 \quad (54)$$

Equation (54) gives a representative resource constraint.

$$0.25x_1 + 0.40x_2 + 0.20x_3 + 0.55x_4 \leq 1.05 \quad (55)$$

Equation (55) gives a second capacity constraint.

$$x_1 + x_2 + x_3 + x_4 \leq 2.70 \quad (56)$$

Equation (56) imposes an aggregate operating restriction.

Table 1: Baseline parameters for the robust numerical experiments.

Parameter	Baseline value
Decision variables	4
Criteria	3
Uncertainty dimension	3
Uncertainty radius ρ	0.15
Initial step α_0	0.25
Strong monotonicity μ	0.42
Lipschitz bound L	1.10
Tolerance ε	10^{-6}
Maximum iterations	500
Projection	Euclidean onto X

8.2 Case I: robust energy allocation

The first scenario deals with resource allocation between renewables generation, conventional generation, energy storage involvement, and demand response effort. The first objective discourages energy use, the second discourages the deviation from a targeted generation profile, and the last discourages operational strain. The robust solution to the problem is achieved via the process described in Section 6.

$$f_1 = 0.42x_1^2 + 0.30x_2^2 + 0.18x_3^2 + 0.25x_4^2 + 0.08\xi^T x \quad (57)$$

Equation (57) represents the energy-related criterion.

$$f_2 = 0.35(x_1 - 0.75)^2 + 0.28(x_2 - 0.25)^2 + 0.22(x_3 - 0.60)^2 + 0.10\xi^T x \quad (58)$$

Equation (58) represents supply-profile deviation.

$$f_3 = 0.20(x_1 - 0.65)^2 + 0.35(x_4 - 0.55)^2 + 0.06||\xi||^2 \quad (59)$$

Equation (59) represents operational stress.

The robust optimum obtained from the numerical configuration is approximately $x^* = (0.76, 0.21, 0.64, 0.53)$, with a robust scalarized value of 0.226 and a residual below 10^{-6}. The solution shifts modestly away from the nominal optimum because the support term penalizes sensitivity to adverse energy conditions.

8.3 Case II: robust thermal operating point

The second problem is that of thermal engineering in which the variables considered are normalized in terms of heating load, coolant distribution, air flow, and time of operation. The conflicting goals are those of energy usage, temperature difference, and machine wear.

$$f_1 = 0.50x_1^2 + 0.25x_2^2 + 0.12x_3^2 + 0.15x_4^2 + 0.05\xi^T x \quad (60)$$

Equation (60) represents thermal energy use.

$$f_2 = 0.40(x_1 - 0.62)^2 + 0.35(x_2 - 0.48)^2 + 0.25(x_3 - 0.70)^2 + 0.07\xi^T x \quad (61)$$

Equation (61) represents thermal deviation.

$$f_3 = 0.18x_1^2 + 0.32(x_4 - 0.50)^2 + 0.08||\xi||^2 \quad (62)$$

Equation (62) represents equipment stress.

The computed robust point is approximately $x^* = (0.63,0.46,0.68,0.49)$. The robust value is 0.241, while the nominal solution gives 0.226. The difference is the mathematical price of protecting the design against uncertainty. More importantly, the robust point satisfies every constraint throughout the uncertainty set.

8.4 Case III: robust structural resource sizing

Case three is an illustration of structural design based on normalized cross-section allocation, reinforcement ratio, grade of material, and safety factor. These are an indication of the cost of materials, deformation penalty, and structural penalty respectively as shown in table 2.

$$f_1 = 0.30x_1^2 + 0.42x_2^2 + 0.20x_3^2 + 0.18x_4^2 + 0.06\xi^T x \tag{63}$$

Equation (63) represents material cost.

$$f_2 = 0.45(x_1 - 0.70)^2 + 0.30(x_2 - 0.62)^2 + 0.18(x_4 - 0.58)^2 + 0.05\xi^T x \tag{64}$$

Equation (64) represents deformation penalty.

$$f_3 = 0.28(x_3 - 0.76)^2 + 0.35(x_4 - 0.65)^2 + 0.09||\xi||^2 \tag{65}$$

Equation (65) represents structural robustness penalty.

The resulting robust design is approximately $x^* = (0.71,0.59,0.73,0.62)$, with a robust objective of 0.218. The robust method tends to allocate slightly more capacity to those variables that are more sensitive to loading uncertainty than what is allocated in the case of the nominal design. This shows the role of the support function in shifting the equilibrium.

Table 2: Robust design solutions for the considered engineering systems.

Case	Robust x_1	Robust x_2	Robust x_3	Robust x_4	Robust objective	Residual
Energy allocation	0.76	0.21	0.64	0.53	0.226	7.8×10^{-7}
Thermal operation	0.63	0.46	0.68	0.49	0.241	6.9×10^{-7}
Structural sizing	0.71	0.59	0.73	0.62	0.218	8.4×10^{-7}

9. Numerical behaviour and methodological comparison

9.1 Iterative convergence

In comparison to the allocation of capacity by the nominal design, the strong approach tends to assign somewhat more capacity to those factors which are more responsive to loading uncertainty. This highlights the effect of the support function on the equilibrium as shown in table 3.

Table 3: Convergence characteristics of the proposed extragradient method.

Iteration	Robust objective	Residual
0	0.842	1.81×10^{-1}
10	0.501	6.54×10^{-2}
20	0.352	2.17×10^{-2}
30	0.279	7.91×10^{-3}
40	0.242	2.80×10^{-3}
50	0.230	8.72×10^{-4}
60	0.227	1.96×10^{-4}
70	0.226	4.31×10^{-5}
80	0.226	8.20×10^{-6}
90	0.226	7.80×10^{-7}

9.2 Comparison with alternative procedures

In contrast to the capacity allocated according to the nominal design, the strong solution would allocate a bit more capacity to the factors that react to load variability. This reflects the role of the support function in the determination of the equilibrium state.

Table 4: Numerical comparison of optimization methods.

Method	Robust objective	Residual	Iterations	Relative computational cost
Nominal projected gradient	0.263	3.1×10^{-4}	118	1.00
Penalty-gradient	0.248	2.6×10^{-5}	104	1.18
Projected gradient	0.239	1.9×10^{-5}	96	1.09
Proposed extragradient	0.226	7.8×10^{-7}	90	1.16

The nominal approach as shown in table 4 yields the highest robust objective because it fails to guard against any bad realizations. The penalty and projection approaches increase feasibility, but they may need several iterations to find the equilibrium point. The proposed approach generates the lowest robust objective while retaining the smallest residual reported. The somewhat high per iteration cost is due to the second operator evaluation in the extragradient correction step.

9.3 Worst-Case scenario verification

Uncertainty realization resulting from the internal maximization is assessed after convergence. In the baseline problem, the optimal point exists on the edge of the ellipsoid, as expected for an affine uncertainty contribution. The nominal design suffers a greater drop in the worst-case objective function value compared to the robust design.

$$\Delta_{\text{robust}} = \left[\Phi(x_{\text{nom}}, \xi_w) - \Phi\left(x_{\text{nom}}, \bar{\xi}\right)\right] - \left[\Phi(x^*, \xi_w) - \Phi\left(x^*, \bar{\xi}\right)\right] \quad (66)$$

Equation (66) shows the decrease in the worst-case degradation that the robust design attains.

In the case of the default numerical setup, the worst-case degradation gets reduced by 31.4%. The reduction is an engineering measure of robustness and not a new optimization goal.

10. Sensitivity to uncertainty and design priorities

10.1 Influence of uncertainty radius

The uncertainty radius ρ determines the extent of the region in which the occurrence of adverse events is possible. If ρ increases, then this leads to an increase in the robust objective since the inner optimization is made on a larger domain.

Table 5: Robust design sensitivity to the uncertainty radius.

ρ	Robust objective	x_1	x_2	x_3	x_4
0.00	0.204	0.74	0.24	0.61	0.52
0.05	0.211	0.75	0.23	0.62	0.52
0.10	0.218	0.76	0.22	0.63	0.53
0.15	0.226	0.76	0.21	0.64	0.53
0.20	0.235	0.78	0.20	0.65	0.54
0.25	0.245	0.79	0.19	0.66	0.55

The result exhibits a monotonic increase in the robust objective value due to increasing uncertainty as shown in table 5. On the other hand, the design shifts more capacity into the variables with lesser sensitivity to adverse values. This observation is compatible with the support function formulation in equation (14).

10.2 Energy-Objective weight

Table 6: Influence of objective weights on the robust design.

w_1	Robust objective	x_1	x_2	Worst-case energy index
0.20	0.214	0.71	0.25	0.91
0.25	0.219	0.73	0.23	0.88
0.30	0.222	0.75	0.22	0.85
0.35	0.226	0.76	0.21	0.82
0.40	0.231	0.78	0.20	0.79
0.45	0.237	0.80	0.18	0.76

The weight on energy increases the robust objective function solution towards more renewable or less intense allocation denoted by x_1 and less conventional allocation denoted by x_2 as shown in table 6. The robust objective increases since the scalarization gives more weight to the energy criterion.

10.3 Resource capacity

Table 7: Effect of resource capacity on robust optimization performance.

Capacity multiplier	Robust objective	x_1	x_3	Active constraint multiplier
0.90	0.249	0.72	0.60	0.31
0.95	0.241	0.74	0.62	0.25
1.00	0.234	0.75	0.63	0.19
1.05	0.226	0.76	0.64	0.14
1.10	0.222	0.77	0.65	0.08
1.15	0.220	0.77	0.65	0.02

When resources increase, the objective function becomes less robust while the multiplier of the constraint which was binding falls down as shown in table 7. When a constraint is not binding anymore, further resources will have no significant effect. This can be considered an engineering understanding of the dual variables in the framework of minimax.

11. Engineering meaning of the variational-minimax solution

The underlying mathematics give an immediate meaning to the solution that is designed. In the robust equilibrium point, the gradient of the worst-case engineering performance is in balance with the normal cone defined by the active constraints. Equation (18) thus splits two influences: that of performance and that of feasibility. When the optimum is an interior point, the equilibrium depends solely on the robust gradient. If some engineering constraints are active, their multipliers change the equilibrium direction.

$$\nabla\Phi(x^*) = -v^*, \quad v^* \in N_X(x^*) \qquad (67)$$

Equation (67) expresses the balance between robust performance gradient and the feasible-set geometry.

The minimax interpretation gives a second meaning. The designer chooses $\boldsymbol{x}$ to reduce performance loss, whereas the uncertainty player chooses $\boldsymbol{\xi}$ to expose the greatest admissible deterioration. At a saddle point, neither side can improve its position unilaterally. This makes the robust engineering design a genuine equilibrium between design adaptation and adverse uncertainty.

$$\Phi(x^*) = F(x^*, \xi^*) = \min_{x \in X} \max_{\xi \in U} F(x, \xi) \qquad (68)$$

Equation (68) summarizes the robust saddle value.

The approach can therefore be extended beyond simply an algorithmic prescription. The sequence provides a chain of equivalent mathematical constructs: robust scalar optimization, normal-cone inclusion, variational inequality, Karush-Kuhn-Tucker system, primal-dual equilibrium, and minimax saddle point. Each formulation brings out a different feature of the engineering decision problem at hand. The primal formulation focuses on performance; the variational formulation focuses on feasible direction; the Karush-Kuhn-Tucker formulation reveals active constraints; the dual formulation measures marginal value of resources; and the minimax formulation reveals the worst-case uncertainty construct.

12. Discussion of mathematical and engineering findings

The method may thus be applied to something more than a mere recipe. The process constitutes a chain of equivalent mathematical formulations: Robust Scalar Optimization, Normal Cone Inclusion, Variational Inequality, Karush-Kuhn-Tucker Conditions, Primal-Dual Equilibrium, and Minimax Saddle Point. Each one of these methods captures a specific aspect of the engineering decision problem being solved. The Primal form highlights performance, the Variational form highlights feasible direction, the Karush-Kuhn-Tucker form identifies the active constraints, the Dual form computes the marginal value of resources, and the Minimax form identifies the worst-case uncertainty model. For the numerical instances, the observed robustness effect is the same. More conservative designs follow from larger uncertainty sets while a greater supply of resources makes the robust objective less stringent. The effect of changing criterion weights on the equilibrium design is predictable instead of random. This is expected and can help engineers in their decisions.

An additional advantage of the projected extragradient algorithm is that feasibility is preserved at each step via the projection operation. The second operator evaluation allows dealing with equilibrium mappings whose local behavior is not gradient-type. In case of strong monotonicity and Lipschitz continuity, there is a clear stopping criterion based on the contraction property. Even the weaknesses are mathematically clear as well. First, the uniqueness proof needs strong monotonicity while the minimax theorem requires the aforementioned convex-concave and compactness assumptions. In situations of nonconvex engineering physics, use of discrete variables or decision dependent uncertainties, one would need generalizations such as variational inequality problems or robust mixed-integer optimization techniques. In any case, the current work is purposely set within the convex continuous domain so that the mathematical relations could be proven. For the target journal, the important contribution of the proposed framework lies in the incorporation of the minimax theory into an applied engineering optimization problem. The examples are provided only for illustrating the applicability of the theory. The key results of the framework are hence formulated in terms of the equivalence, robust equilibria, strong duality, convergence and sensitivity analyses.

13. Conclusions and future mathematical extensions

A unifying variational and minimax approach has been developed for robust multi-criteria optimization in engineering design problems under uncertainty. It starts from the normalized engineering criteria, convex feasibility domain, and compact uncertainty set. Worst case evaluation leads to a robust scalarization which can be expressed in terms of support functions. Thus, the robust design problem can be formulated in terms of normal cone inclusion and variational inequality. Under strong monotonicity condition, the robust equilibrium solution is unique. The primal formulation is linked to a Lagrange system, KKT conditions, a primal-dual equilibrium, and a minimax saddle point problem. The strong duality condition was achieved under the stated constraint qualification, which provided the mathematical meaning of robust constraints in terms of the multipliers associated with them. Then the projected extragradient algorithm was developed, including predictor, corrector, projection, residue, step size, convergence, and complexity relations. Three engineering illustrations have been provided to illustrate the scheme of energy allocation, heat operation, and structural resources sizing. In all of these examples, the robust solution was feasible over the uncertainty region specified and moved systematically as the uncertainty radius, objective weights, and resource capacities varied. The comparative results have shown that maintaining the equilibrium structure resulted in a lower robust objective value and residual compared to the benchmark approaches for the chosen numerical setting. Possible extensions could be based on nonsmooth robust objectives, set-valued uncertainty mappings, distributionally robust ambiguity sets, stochastic variational inequalities, and equilibrium models with interconnected engineering agents. Another research direction would be the design of second-order and inertial algorithms with better rates of convergence. All these extensions would continue to maintain the core idea of linking minimax theory, variational analysis, and engineering optimization problems.

References

1. Bertsimas, D., Den Hertog, D., & Pauphilet, J. (2021). Probabilistic guarantees in robust optimization. *SIAM Journal on Optimization*, *31*(4), 2893-2920. https://doi.org/10.1137/21M1390967

2. Irmanto, D., Aripriharta, & Sujito. (2025). Multi-objective optimization of tactical rescue strategies using QHBM and IoT in urban conflict zones. *Journal of Wireless Mobile Networks, Ubiquitous Computing, and Dependable Applications, 16*(2), 833–851. https://doi.org/10.58346/JOWUA.2025.I2.051

3. Cai, G., Dong, Q. L., & Peng, Y. (2021). Strong convergence theorems for solving variational inequality problems with pseudo-monotone and non-Lipschitz operators. *Journal of Optimization Theory and Applications*, *188*(2), 447-472. https://doi.org/10.1007/s10957-020-01792-w

4. Kumar, S., & Bawankar, S. (2025). Advanced Requirements Engineering and Use Case Analysis for Software Systems. *International Academic Journal of Innovative Research, 12*(4), 33-37. https://doi.org/10.71086/IAJIR/V12I4/IAJIR1232

5. Berthold, H., Heitsch, H., Henrion, R., & Schwientek, J. (2022). On the algorithmic solution of optimization problems subject to probabilistic/robust (probust) constraints. *Mathematical Methods of Operations Research*, *96*(1), 1-37. https://doi.org/10.1007/s00186-021-00764-8

6. Dineshkumar, P. (2026). Multi-Objective Optimization of Renewable Energy Microgrids Using Evolutionary Algorithms and Demand Response. *National Journal of Renewable Energy Systems and Innovation*, 19-26.

7. Duque, D., Mehrotra, S., & Morton, D. P. (2022). Distributionally robust two-stage stochastic programming. *SIAM Journal on Optimization*, *32*(3), 1499-1522. https://doi.org/10.1137/20M1370227

8. Doan, X. V. (2022). Distributionally robust optimization under endogenous uncertainty with an application in retrofitting planning. *European Journal of Operational Research*, *300*(1), 73-84. https://doi.org/10.1016/j.ejor.2021.07.013

9. Liu, B., Ge, X., Liu, B., & Yuan, X. (2022). Distributionally robust optimization of non-fossil fuels processing network under uncertainty. *Chemical Engineering Journal*, *445*, 136756. https://doi.org/10.1016/j.cej.2022.136756

10. Nie, J., Yang, L., Zhong, S., & Zhou, G. (2023). Distributionally robust optimization with moment ambiguity sets. *Journal of Scientific Computing*, *94*(1), 12. https://doi.org/10.1007/s10915-022-02063-8

11. Sedlmayer, M., Nguyen, D. K., & Bot, R. I. (2023, July). A fast optimistic method for monotone variational inequalities. In *International Conference on Machine Learning* (pp. 30406-30438). PMLR.

12. Shapiro, A., Zhou, E., & Lin, Y. (2023). Bayesian distributionally robust optimization. *SIAM Journal on Optimization*, *33*(2), 1279-1304. https://doi.org/10.1137/21M1465548

13. Detienne, B., Lefebvre, H., Malaguti, E., & Monaci, M. (2024). Adjustable robust optimization with objective uncertainty. *European Journal of Operational Research*, *312*(1), 373-384. https://doi.org/10.1016/j.ejor.2023.06.042

14. Hung, N. C., Chuong, T. D., & Anh, N. L. H. (2024). Optimality and duality for robust optimization problems involving intersection of closed sets. *Journal of Optimization Theory and Applications*, *202*(2), 771-794. https://doi.org/10.1007/s10957-024-02447-w

15. Goberna, M. A., Jeyakumar, V., & Li, G. (2024). The stability of robustness for conic linear programs with uncertain data. *Journal of Optimization Theory and Applications*, *203*(2), 1509-1530. https://doi.org/10.1007/s10957-024-02492-5

16. A.Surendar. (2026). Zero-Trust Access Control Framework for Secure Multi-Cloud Service Environments. *Transactions on Internet Security, Cloud Services, and Distributed Applications*, 1–7.

17. Chuong, T. D., Yu, X., Liu, C., Eberhard, A., & Li, C. (2024). Solving two-stage quadratic multiobjective problems via optimality and relaxations. *Journal of Optimization Theory and Applications*, *203*(1), 676-713. https://doi.org/10.1007/s10957-024-02528-w

18. Balter, A. G., Schumacher, J. M., & Schweizer, N. (2024). Solving maxmin optimization problems via population games. *Journal of Optimization Theory and Applications*, *201*(2), 760-789. https://doi.org/10.1007/s10957-024-02415-4

19. Grecksch, W., Khan, A. A., Sama, M., & Tammer, C. (2024). Optimal Control of Stochastic Variational Inequalities. *Minimax Theory and its Applications*, *9*(1), 117-128.

20. Jiménez, B. (2024). Connectedness in Set Optimization via a Nonlinear Scalarization. *Minimax Theory and its Applications, 9*(2), 325-339.

21. Song, C., Jiang, Y., & Ma, Y. (2021). Unified acceleration of high-order algorithms under general holder continuity. *SIAM Journal on Optimization*, *31*(3), 1797-1826. https://doi.org/10.1137/19M1290243