

On Quasi-Equilibrium Problems with Trifunctions

D. Inoan

*Technical University, Cluj-Napoca, Romania
daniela.inoan@math.utcluj.ro*

J. Kolumbán

Babeş-Bolyai University, Cluj-Napoca, Romania

Received: July 31, 2017

Accepted: January 12, 2018

We present an existence result for a quasi-equilibrium problem formulated with trifunctions. For this, we define the dual problem, which is a generalization of the dual for equilibrium problems given by bifunctions. From the main result follows also the existence of solutions for a hemivariational inequality problem governed by operators of two variables.

Keywords: Quasi-equilibrium problem, hemivariational inequality problem, pseudomonotone mapping.

2010 Mathematics Subject Classification: 47J20

1. Introduction

Let V be a real topological vector space, let C be a subset of V and let $f: C \times C \rightarrow \mathbb{R}$ be a given function. The equilibrium problem governed by the bifunction f is

$$\text{find } \bar{u} \in C \text{ such that } f(\bar{u}, v) \geq 0, \text{ for every } v \in C. \quad (1)$$

The problem

$$\text{find } \bar{u} \in C \text{ such that } f(v, \bar{u}) \leq 0, \text{ for every } v \in C \quad (2)$$

is called the *dual* of (1).

Equilibrium problems provide a common approach to several classes of problems, like variational or hemi-variational inequalities, complementarity problems, and have important applications in science (see for instance [1], [3], [12], [25], [26], [27], [28] and the references therein). Problem (1) was studied first by Ky Fan in [12]. The existence result was later extended in several ways, for instance by Brézis, Nirenberg and Stampacchia in [4] and, in another direction, in the papers [13], [14], where a quasi Ky Fan-like inequality is treated. We follow here the direction of [4], since this one can be applied also to variational inequalities governed by quasimonotone operators (see [26]).

The above way of defining the dual problem was already considered in [2] (see also [21]). Inspired by the particular case of variational inequalities governed by monotone operators, the existence of solutions is proved first for the dual problem (2) and then is shown that in certain conditions a solution of (2) is also a solution for (1).

This method can be applied also for certain variational inequalities, but not for variational inequalities that are governed by elliptic operators in divergence form (see for instance [22]), which are monotone only in those terms in the principal part, i.e. in the highest order terms (called quasimonotone in [26] and semi-monotone in [6]). In the paper [17] it was shown that such cases can be approached by using an equilibrium problem governed by functions of three variables.

A more general class of problems, studied in recent years, consists of quasi-equilibrium problems, in which the constraint set depends on the considered point (see [5], [9], [23] and the references therein). They cover several special cases, like quasivariational inequalities (see [24]) or Nash equilibrium problems. The constraints may be, for instance, *boundary constraints* (as in the Signorini problem of elasticity), *interior constraints* (as in elasticity theory to describe the equilibrium position of a membrane which is constrained below by an obstacle), *gradient constraints* (as in the elastic-plastic torsion problem in which the stress is pointwise bounded), etc. (see [26], p. 70-72).

In this paper we study the following quasi-equilibrium problem governed by a trifunction.

Let V be a real Hausdorff locally convex topological vector space, $C \subseteq V$ a nonempty convex set, $F: C \times C \times C \rightarrow \mathbb{R}$ a given function, and let $S_1, S_2: C \rightarrow 2^C$ be two set-valued mappings with nonempty values.

The *quasi-equilibrium problem* governed by F is:

$$(P) \quad \text{find } \bar{u} \in S_1(\bar{u}) \text{ such that } F(\bar{u}, v, \bar{u}) \geq 0, \text{ for every } v \in S_2(\bar{u}).$$

We define the dual of (P) as:

$$(DP) \quad \text{find } \bar{u} \in S_1(\bar{u}) \text{ such that } F(\bar{u}, v, v) \geq 0, \text{ for every } v \in S_2(\bar{u}).$$

For particular forms of the function F , we get various problems, as by the following examples:

Example 1.1. Let $F(u, v, w) = \sup_{f \in T(w, u)} \langle f, v - u \rangle + \varphi(v) - \varphi(u)$ and $S_1(u) =$

$S_2(u) = C$, where V is a real Hausdorff locally convex topological vector space, V^* its dual, $T: V \times V \rightarrow 2^{V^*}$ is a set-valued mapping, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V^* , and $\varphi: V \rightarrow \mathbb{R}$ is a convex and lower semi-continuous function.

In this case, the above problems become the following well-known variational inequalities of Stampacchia and Minty type, governed by set-valued operators of two variables (see [19]):

find $\bar{u} \in C$ such that $\sup_{f \in T(\bar{u}, \bar{u})} \langle f, v - \bar{u} \rangle + \varphi(v) - \varphi(\bar{u}) \geq 0$, for every $v \in C$,

respectively

find $\bar{u} \in C$ such that $\sup_{f \in T(v, \bar{u})} \langle f, v - \bar{u} \rangle + \varphi(v) - \varphi(\bar{u}) \geq 0$, for every $v \in C$.

Example 1.2. Let $F(u, v, w) = f(u, v) + g(w, v)$, where $f, g: C \times C \rightarrow \mathbb{R}$ and C is a convex subset of a topological vector space. Problem (P) is in this case a mixed equilibrium problem, where f is monotone and lower semi-continuous in the second argument, g is upper semi-continuous in the first argument, and both are convex in the second argument (see for instance [3], [18], [20] and the references therein).

Example 1.3. Let $F(u, v, w) = f(w, v) - f(w, u)$, where $f: C \times C \rightarrow \mathbb{R}$ is a given function with $f(u, u) = 0$ and $S_1(u) = S_2(u) = C$, for any $u \in C$. Problem (P) becomes problem (1) and (DP) becomes (2). So, the above defined duality concept for trifunctions generalizes the duality for bifunctions (see also [17]).

The general form of the problem and the presence of three arguments in the objective function allow to prove some existence results which can not be obtained using the classical theory of equilibrium points for bifunctions.

The paper is organized as follows. Section 2 contains the notions and auxiliary results used in the paper. In Section 3 we prove our main results. In Section 4 we apply the main theorem to a hemivariational inequality problem governed by operators of two variables, providing sufficient conditions for the existence of solutions.

2. Definitions and auxiliary results

Definition 2.1. (a) The function F is said to be *monotone* relative to (S_1, S_2) iff $F(u, v, v) \geq F(u, v, u)$, for every $u, v \in V$, with $u \in S_1(u)$ and $v \in S_2(u)$.

(b) F is said to be *A-pseudomonotone* relative to (S_1, S_2) iff, for every $u, v \in C$, with $u \in S_1(u)$ and $v \in S_2(u)$,

$$F(u, v, u) \geq 0 \text{ implies } F(u, v, v) \geq 0.$$

(c) F is said to be *B-pseudomonotone* relative to (S_1, S_2) iff, for every $u, v \in C$ with $v \in S_2(u)$, and for every net $(u_i)_{i \in I}$ on C with $u_i \rightarrow u$, $u_i \in S_1(u_i)$, $v \in S_2(u_i)$, for every $i \in I$, and

$$\liminf_{i \in I} F(u_i, u, u) \geq 0,$$

we have

$$\limsup_{i \in I} F(u_i, v, v) \leq F(u, v, v).$$

(d) F is said to be *C-pseudomonotone* relative to (S_1, S_2) iff for every $u, v \in C$ and for every net $(u_i)_{i \in I}$ on C with $u_i \rightarrow u$, $u_i \in S_1(u_i)$, $v \in S_2(u_i)$ and such

that $F(u_i, (1-t)u + tv, (1-t)u + tv) \geq 0$, for all $i \in I$ and $t \in [0, 1]$, we have $u \in S_1(u)$, $v \in S_2(u)$ and $F(u, v, v) \geq 0$.

Example 2.2. Let $C = S_1(u) = S_2(u) = \mathbb{R}$ for any $u \in \mathbb{R}$. The trifunction defined by $F(u, v, w) = w - u$ if $u \neq 0$ and $F(u, v, w) = 0$ if $u = 0$ is C-pseudomonotone but is not A-pseudomonotone nor B-pseudomonotone.

If $S_1(u) = S_2(u) = C$ for all $u \in C$, then the expression “relative to (S_1, S_2) ” will be omitted.

Remark 2.3. (i) In the case $S_1(u) = S_2(u) = C$ for all $u \in C$, if F is defined by $F(u, v, w) = f(w, v) - f(w, u)$, with $f(u, u) = 0$ for every $u \in C$, then F is monotone (A-pseudomonotone) if and only if f is monotone (algebraic pseudomonotone) as a bifunction, that is, $f(u, v) + f(v, u) \leq 0$ (from $f(u, v) \geq 0$ follows $f(v, u) \leq 0$), for all $u, v \in C$ (see [21], Definition 2.1).

(ii) The function F defined by $F(u, v, w) = f(u, v)$ is monotone in the sense of the previous definition, since $F(u, v, v) = F(u, v, u)$, for any $u, v \in C$.

(iii) If V is a normed space, the function F defined by $F(u, v, w) = \|w - u\|$ is also monotone, since $F(u, v, v) = \|v - u\| \geq 0 = F(u, v, u)$.

Remark 2.4. (i) If, for a given $f: C \times C \rightarrow \mathbb{R}$, we take $F(u, v, w) = f(u, v)$ then F is B-pseudomonotone iff f is topological pseudomonotone, which is pseudomonotonicity in the sense of Brézis, in the case of variational inequalities (see [18]).

(ii) If F has the form from (i), then it is C-pseudomonotone iff $-f$ satisfies condition (4) in [4], which is C-pseudomonotonicity in sense [16] (or 0-segmentary closedness in [11])

(iii) It is easy to see that B-pseudomonotonicity implies C-pseudomonotonicity, but the reverse implication is not true (see [16], Example 5).

In this paper we will use the following well-known notions (see [10], [18], [21]):

Definition 2.5. A function $f: C \rightarrow \mathbb{R}$ is said to be:

(a) *quasiconvex* iff, for each $u, v \in C$ and $\alpha \in [0, 1]$, we have

$$f(\alpha u + (1 - \alpha)v) \leq \max\{f(u), f(v)\},$$

(b) *explicitly quasiconvex* iff it is quasiconvex and, for each $u, v \in C$ such that $f(u) \neq f(v)$ and for each $\alpha \in]0, 1[$, we have

$$f(\alpha u + (1 - \alpha)v) < \max\{f(u), f(v)\},$$

(c) *0-explicitly quasiconvex* iff it is quasiconvex and, for each $u, v \in C$ such that $f(u) \neq f(v)$ and for each $\alpha \in]0, 1[$ such that $0 \leq f(\alpha u + (1 - \alpha)v)$, we have

$$f(\alpha u + (1 - \alpha)v) < \max\{f(u), f(v)\}.$$

Definition 2.6. Let $f: C \times C \rightarrow \mathbb{R}$, $S_1, S_2: C \rightarrow 2^C$ and $B \subseteq C$. We say that $(u, v) \mapsto f(u, v)$ is *0-transfer upper semicontinuous* relative to (S_1, S_2) on B , iff for every $u, v \in B$ with $u \in S_1(u)$, $v \in S_2(u)$, and $f(u, v) < 0$, there exists $v' \in B$ and a neighbourhood N of u such that $v' \in S_2(y)$ and $f(y, v') < 0$, for every $y \in N \cap B$.

Obviously, if f is upper semi-continuous relative to the first argument then f is 0-transfer upper semicontinuous relative to (S_1, S_2) on B , with $v' = v$. The converse is not true. For instance, let $C = B = \mathbb{R}$, $S_1(u) = \mathbb{R}$, $S_2(u) = [-u, u]$, for any $u \in \mathbb{R}$ and

$$f(u, v) = \begin{cases} -1, & \text{if } u = 0 \text{ and } v \neq 0, \\ 1, & \text{if } u \neq 0 \text{ and } v \neq 0, \\ -2, & \text{if } v = 0. \end{cases}$$

The function f is 0-transfer upper semicontinuous relative to (S_1, S_2) but is not upper semicontinuous in the first argument.

We denote by $\text{Fix } S_1$ the set of fixed points of S_1 and by $\text{co } A$ the convex hull of a set A .

Definition 2.7. A mapping $\Gamma: C \rightarrow 2^V$, is called

- (a) *KKM mapping* iff, for any $v_1, \dots, v_n \in C$, we have $\text{co}\{v_1, \dots, v_n\} \subseteq \cup_{i=1}^n \Gamma(v_i)$,
- (b) *transfer-closed valued* on $B \subseteq C$ iff, for any $u, v \in B$ with $u \notin \Gamma(v) \cap B$, there exists $v' \in B$ such that $u \notin \text{cl}(\Gamma(v') \cap B)$ (see [27]).

To prove the existence of solutions for problems (P) and (DP) we will use the following result, which is a generalization of the famous intersection lemma due to Brézis, Nirenberg and Stampacchia [4] (see [10], [16], [18]).

Lemma 2.8. Let V be a Hausdorff topological vector space, $C \subseteq V$ a nonempty convex set and let $\Gamma: C \rightarrow 2^V$ be such that:

- (i) Γ is a KKM mapping,
- (ii) for any finite subset A of C the mapping Γ is transfer-closed valued on $\text{co}A$,
- (iii) for any $u, v \in C$ it holds:

$$\text{cl}_C \left(\bigcap_{w \in [u, v]} \Gamma(w) \right) \cap [u, v] = \left(\bigcap_{w \in [u, v]} \Gamma(w) \right) \cap [u, v].$$

- (iv) $\text{cl}\Gamma(v_0)$ is compact for some $v_0 \in C$.

Then $\bigcap_{v \in C} \Gamma(v) \neq \emptyset$.

We end this section by recalling the following useful result which is needed in the following.

Lemma 2.9. ([15]) *Let U and V be topological spaces, $\Gamma: U \rightarrow 2^V$ a set-valued mapping and $\gamma: U \times V \rightarrow \mathbb{R}$. Denote by $h: U \rightarrow \mathbb{R}$, $h(u) = \sup_{v \in \Gamma(u)} \gamma(u, v)$ the marginal function. If the conditions:*

- (i) γ is upper semicontinuous on $U \times V$,
- (ii) $\Gamma(u_0)$ is compact for some $u_0 \in U$,
- (iii) Γ is upper semicontinuous at u_0 ,

are satisfied, then h is upper semicontinuous at u_0 .

3. Main result

In what follows, we consider the assumptions:

- (H1) F is A-pseudomonotone relative to (S_1, S_2) .
- (H2) (a) S_2 has convex values, $S_2(u) \subseteq S_1(u)$ for all $u \in C$,
(b) $u \in \text{Fix}S_1$ and $v \in S_2(u)$ imply that $]u, v] \subseteq S_2(u)$.
- (H3) The set of fixed points of S_1 is closed.
- (H4) For any $v \in C$ the set $S_2^{-1}(v)$ is open in C .
- (H5) $F(u, \cdot, w)$ is 0-explicitly quasiconvex for all $u, w \in C$.
- (H6) $F(u, u, u) = 0$ and $F(u, u, w) \leq 0$ for all $u, w \in C$.
- (H7) If $u, v \in C$ and $F(u, v, tv + (1-t)u) \geq 0$, for every $t \in]0, 1]$, then $F(u, v, u) \geq 0$.
- (H8) For any finite subset $A \subset C$, any $u, v \in \text{co}A$ with $v \in S_2(u)$ and $F(u, v, v) < 0$, there exists $v' \in \text{co}A$ such that, for any net $(u_i)_{i \in I}$ on $\text{co}A$ with $v' \notin S_2(u_i)$, or with $u_i \in S_1(u_i)$ and $F(u_i, v', v') \geq 0$, for every $i \in I$, there exists a neighborhood N of u and $i_0 \in I$ such that $u_i \notin N \cap \text{co}A$ for each $i > i_0$.
- (H9) For $u, v \in C$ and $t \in [0, 1]$, denote $v_t = (1-t)u + tv$. For any $u, v \in C$ with $v \in S_2(u)$ and $F(u, v, v) < 0$, and for any net $(u_i)_{i \in I}$ in C with $v_t \notin S_2(u_i)$, or $u_i \in S_1(u_i)$ and $F(u_i, v_t, v_t) \geq 0$, for every $i \in I$ and $t \in [0, 1]$, there exists a neighborhood N of u and a subnet $(u_{i_k})_{i_k \in I}$ such that $u_{i_k} \notin N$ for each $i_k \in I$.
- (H10) There exists a nonempty compact set $K \subseteq C$ and $v_0 \in C$ such that $F(u, v_0, v_0) < 0$ and $v_0 \in S_2(u)$ for all $u \in C \setminus K$.

Proposition 3.1. *If (H1) holds, then any solution of problem (P) is also a solution of problem (DP).*

Proof. Let \bar{u} be a solution of (P), that is $\bar{u} \in S_1(\bar{u})$ and $F(\bar{u}, v, \bar{u}) \geq 0$ for all $v \in S_2(\bar{u})$. Pseudomonotonicity of F relative to (S_1, S_2) implies $F(\bar{u}, v, v) \geq 0$ for all $v \in S_2(\bar{u})$, so \bar{u} is a solution of (DP). \square

Proposition 3.2. *If (H2), (H5), (H6) and (H7) hold, then any solution of problem (DP) is also a solution of problem (P).*

Proof. Let \bar{u} be a solution of (DP). Let $v \in S_2(\bar{u})$ be fixed. From (H2) we have

$$v_t = tv + (1 - t)\bar{u} \in S_2(\bar{u}) \quad \text{and} \quad F(\bar{u}, v_t, v_t) \geq 0, \text{ for every } t \in [0, 1].$$

From (H5) and (H6) we get $0 \leq F(\bar{u}, v_t, v_t) \leq F(\bar{u}, v, v_t)$.

This implies $0 \leq F(\bar{u}, v, v_t)$ for all $t \in]0, 1]$. Consequently, from (H7) follows

$$0 \leq F(\bar{u}, v, \bar{u}), \text{ for all } v \in S_2(\bar{u}),$$

so \bar{u} is a solution of (P). □

Theorem 3.3. *Suppose that $C \subseteq V$ is a nonempty closed convex subset of a real Hausdorff locally convex topological vector space V , $F: C \times C \times C \rightarrow \mathbb{R}$ is a given function and $S_1, S_2: C \rightarrow 2^C$ are two set-valued mappings. Suppose that the assumptions (H1)–(H10) are satisfied.*

Then problem (P) admits at least one solution.

Proof. For each $w \in C$, denote

$$\Gamma(w) = [C \setminus S_2^{-1}(w)] \cup \{u \in C \mid u \in S_1(u), F(u, w, w) \geq 0\}$$

It is easy to see that $\bigcap_{w \in C} \Gamma(w)$ is the set of solutions of (DP). Indeed, if $u \in \bigcap_{w \in C} \Gamma(w)$, let $v \in S_2(u)$ be fixed. This means $u \notin C \setminus S_2^{-1}(v)$, so since $u \in \Gamma(v)$, we must have $u \in S_1(u)$ and $F(u, v, v) \geq 0$ which gives that u is solution of (DP). Conversely, let u be a solution of (DP). Let $w \in C$ be arbitrary. If $w \notin S_2(u)$, then $u \in C \setminus S_2^{-1}(w)$ and so $u \in \Gamma(w)$. If $w \in S_2(u)$, then $F(u, w, w) \geq 0$ and $u \in S_1(u)$, so again $u \in \Gamma(w)$.

To prove that $\bigcap_{w \in C} \Gamma(w)$ is nonempty, we use Lemma 2.8. For each $w \in C$, we denote $\Gamma_1(w) = [C \setminus S_2^{-1}(w)] \cup \{u \in C \mid u \in S_1(u), F(u, w, u) \geq 0\}$. We prove first that Γ_1 is a KKM mapping. Let $w_1, \dots, w_n \in C$ and $u = \sum_{i=1}^n \lambda_i w_i$, with $\lambda_i \in [0, 1]$, $\sum_{i=1}^n \lambda_i = 1$. From (H5) and (H6) follows that

$$0 = F(u, u, u) = F(u, \sum_{i=1}^n \lambda_i w_i, u) \leq \max_i F(u, w_i, u)$$

which implies that there exists $j \in \{1, \dots, n\}$ such that $0 \leq F(u, w_j, u)$. If $w_i \in S_2(u)$, for every $i \in \{1, \dots, n\}$, then by the convexity of $S_2(u)$, we have $u \in S_2(u) \subseteq S_1(u)$ and so $u \in \Gamma_1(w_j)$. If, on the contrary, there exists $k \in \{1, \dots, n\}$ such that $w_k \notin S_2(u)$, then $u \in C \setminus S_2^{-1}(w_k)$ therefore $u \in \Gamma_1(w_k)$. This means

$$\text{co} \{w_1, \dots, w_n\} \subseteq \bigcup_{i=1}^n \Gamma_1(w_i).$$

Since (H1) ensures that $\Gamma_1(w) \subseteq \Gamma(w)$, for every $w \in C$, it follows that Γ is also a KKM mapping.

To verify hypothesis (ii) in Lemma 2.8 for Γ , let $A \subseteq C$ be a finite set and $u, v \in \text{co}A$ such that $u \notin \Gamma(v)$. This implies $v \in S_2(u)$ and at least one of the conditions $u \in S_1(u)$ or $F(u, v, v) \geq 0$ is not satisfied.

If $u \notin \text{Fix}S_1$, then we take $v' := v$. Since $\text{Fix}S_1$ is closed, there exists a neighborhood N_1 of u such that for all $y \in N_1$, $y \notin \text{Fix}S_1$. On the other hand, having $u \in S_2^{-1}(v)$ and $S_2^{-1}(v)$ open, there exists a neighborhood N_2 of u such that $y \in S_2^{-1}(v)$, for all $y \in N_2$. Taking now $N = N_1 \cap N_2$, we have $y \notin \Gamma(v)$ for all $y \in N \cap \text{co}A$.

On the other hand, in the case $F(u, v, v) < 0$, let $v' \in \text{co}A$ from (H8). If we have $u \in \text{cl}(\Gamma(v') \cap \text{co}A)$, then there exists a net $(u_i)_{i \in I}$ such that $u_i \rightarrow u$ and $u_i \in \Gamma(v') \cap \text{co}A$. By (H8) there exists a neighborhood N of u and $i_0 \in I$ such that $u_i \notin N \cap \text{co}A$ for each $i > i_0$, contradicting $u_i \rightarrow u$. Therefore $u \notin \text{cl}(\Gamma(v') \cap \text{co}A)$, so Γ is transfer closed valued on $\text{co}A$.

The next step is to prove the equality

$$\text{cl} \left(\bigcap_{w \in [u, v]} \Gamma(w) \right) \cap [u, v] = \left(\bigcap_{w \in [u, v]} \Gamma(w) \right) \cap [u, v], \tag{3}$$

for any $u, v \in C$.

Let $u \in \text{cl} \left(\bigcap_{w \in [u, v]} \Gamma(w) \right) \cap [u, v]$. Then there exists a net $u_i \in \bigcap_{w \in [u, v]} \Gamma(w)$ such that $u_i \rightarrow u$.

We have $u_i \in \Gamma(v_t)$, for every $i \in I$ and $t \in [0, 1]$ where $v_t = (1 - t)u + tv$. It follows that $u_i \in C \setminus S_2^{-1}(v_t)$, or $u_i \in S_1(u_i)$ and $F(u_i, v_t, v_t) \geq 0$.

We have to show that $u \in \Gamma(v_t)$ for all $t \in [0, 1]$. If we have $u \notin \Gamma(v)$, then $v \in S_2(u)$ and at least one of the conditions $u \in S_1(u)$ or $F(u, v, v) \geq 0$ is not satisfied. By (H3), $u_i \in S_1(u_i)$ and $u_i \rightarrow u$ imply $u \in S_1(u)$. So we get $v \in S_2(u)$ and $F(u, v, v) < 0$. Since $u_i \in \Gamma(v_t)$ for all $t \in [0, 1]$, by (H9) there exists a neighborhood N of u and $i_0 \in I$ such that $u_i \notin N$ for each $i > i_0$, contradicting $u_i \rightarrow u$. So equality (3) is proved.

From (H10) it follows that $\Gamma(v_0) \subseteq K$, so $\text{cl}\Gamma(v_0)$ is compact.

Since all the conditions of Lemma 2.8 are verified for Γ it follows that $\bigcap_{v \in C} \Gamma(v) \neq \emptyset$ and (DP) admits solutions. By Proposition 3.2, also (P) admits solutions. \square

Remark 3.4. (i) If property (H8) is verified for $v' = v$, then $\Gamma(w)$ is closed in $\text{co}A$, for every $w \in \text{co}A$ where $A \subset C$ is finite. Indeed, let w be fixed and let $(u_i)_{i \in I}$ be a net in $\Gamma(w)$ such that $u_i \rightarrow u$, $u \in C$. We have two possibilities: The first is that there exists a subnet $\{u_{i_j}\}$ such that $u_{i_j} \in C \setminus S_2^{-1}(w)$ for every j . By (H4), this gives $u \in C \setminus S_2^{-1}(w)$, so $u \in \Gamma(w)$.

The second case is where such a subnet does not exist. Then there exists a subnet $\{u_{i_j}\}$ such that $u_{i_j} \in S_1(u_{i_j})$ and $F(u_{i_j}, w, w) \geq 0$, for every $j \in I$. From (H3) we get $u \in S_1(u)$. Supposing that $F(u, w, w) < 0$ we get from (H8) that u_{i_j} can

not converge to u , which is a contradiction. This means $F(u, w, w) \geq 0$ and so $u \in \Gamma(w)$.

(ii) The closedness of $\Gamma(w)$ ensures that conditions (ii) and (iii) of Lemma 2.8 are verified.

(iii) Of course, (H8) is satisfied if the bifunction $(u, v) \mapsto F(u, v, v)$ is 0-transfer upper semicontinuous.

(iv) It is simple to see that, if F is C-pseudomonotone relative to (S_1, S_2) then assumption (H9) is verified.

Example 3.5. Let $a \leq 0 \leq b$, $C = [a, b]$, $S_1(u) = [u, b]$, for every $u \in C$, $S_2(u) =]u, b]$ for every $u \in [a, b[$ and $S_2(b) = \emptyset$,

$$F(u, v, w) = \begin{cases} v^2 - u^2 - (w - v)^2, & \text{if } u \neq 0, v, w \in C \\ 0, & \text{if } u = 0, v, w \in C. \end{cases}$$

Then

$$F(u, v, u) = \begin{cases} 2u(v - u), & \text{if } u \neq 0 \\ 0, & \text{if } u = 0, \end{cases} \quad \text{and} \quad F(u, v, v) = \begin{cases} v^2 - u^2, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0. \end{cases}$$

We have $F(u, v, v) \geq F(u, v, u)$ for all $u, v \in C$, so F is monotone and (H1) is fulfilled. The assumptions (H2), (H3), (H4) are verified, since $\text{Fix}S_1 = [a, b]$, $S_2^{-1}(v) = [a, v[$, open in C . The convexity of F in the second argument implies (H5). (H6) is obvious.

To check (H7), let $u, v \in C$ such that $F(u, v, tv + (1 - t)u) \geq 0$ for any $t \in]0, 1]$. This means $v^2 - u^2 - (1 - t)^2(u - v)^2 \geq 0$ for any $t \in]0, 1]$. Passing to the limit $t \rightarrow 0$ we get $F(u, v, u) \geq 0$.

(H8) is verified with $v' = v$.

If $u, v \in C$ are such that $F(u, v, v) < 0$ then $u \neq 0$ and $v^2 - u^2 < 0$. If $(u_i)_{i \in \mathbb{N}}$ is a sequence such that $F(u_i, v_t, v_t) \geq 0$ for any $i \in \mathbb{N}$ and $t \in [0, 1]$ we get $v^2 - u_i^2 \geq 0$, which makes the convergence $u_i \rightarrow u$ impossible and insures (H9).

Finally, the condition (H10) is obviously satisfied if we choose $K = [a, b]$.

All the assumptions of Theorem 3.3 are satisfied and the solution set is $[0, b]$.

4. Hemivariational inequalities governed by a semimonotone operator

In this section, let V be a real reflexive Banach space, V^* its dual, $C \subseteq V$ a nonempty closed convex set, $S_1, S_2: C \rightarrow 2^C$ with nonempty values, $T: V \times V \rightarrow 2^{V^*}$ and $\varphi: V \rightarrow \mathbb{R}$. Let $J: C \rightarrow \mathbb{R}$ be a locally lipschitz function and denote by J^0 its directional differential in the sense of Clarke [7],

$$J^0(\eta; \gamma) = \limsup_{\theta \rightarrow \eta, t \rightarrow 0_+} \frac{J(\theta + t\gamma) - J(\theta)}{t}.$$

We consider the following hemivariational inequality governed by the operator T : Find $\bar{u} \in \text{Fix}S_1$ such that

$$(HVI) \quad \sup_{f \in T(\bar{u}, \bar{u})} \langle f, v - \bar{u} \rangle + J^0(\bar{u}, v - \bar{u}) + \varphi(v) - \varphi(\bar{u}) \geq 0, \text{ for every } v \in S_2(\bar{u}).$$

Specific examples of hemivariational problems, that appear in engineering or economics are presented for instance by Naniewicz and Panagiotopoulos in [25]. We formulate the following assumptions, considering V endowed with the weak topology and V^* with the strong topology:

- (C1) For every $u, v \in C$, $\sup_{f \in T(v, u)} \langle f, v - u \rangle \geq \sup_{f \in T(u, u)} \langle f, v - u \rangle$.
- (C2) $\text{Fix}S_1 = \text{Fix}S_2$ and it is a closed set.
- (C3) $S_2(u) \subseteq S_1(u)$ for all $u \in C$.
- (C4) The set $S_2^{-1}(v)$ is open in C for every $v \in C$.
- (C5) φ is convex and lower semicontinuous.
- (C6) For each $u, v \in C$, the mapping $t \mapsto T(tv + (1 - t)u, v)$ is upper semicontinuous at $t = 0$.
- (C7) For each $u \in C$, the mapping $T(u, \cdot)$ is upper semicontinuous.
- (C8) For each $u, v \in C$ the set $T(u, v)$ is compact.
- (C9) The function f defined by $f(u, v) = J^0(u, v - u)$ for all $u, v \in C$ is B-pseudomonotone relative to (S_1, S_2) .
- (C10) There exists a nonempty compact set $K \subseteq C$ and $v_0 \in C$ such that $\sup_{f \in T(v_0, u)} \langle f, v_0 - u \rangle + J^0(u, v_0 - u) + \varphi(v_0) - \varphi(u) < 0$ for all $u \in C \setminus K$.

As an application of Theorem 3.3 we get the following:

Theorem 4.1. *If the assumptions (C1)–(C10) are satisfied, then problem (HVI) admits at least one solution.*

Proof. We apply Theorem 3.3 by taking

$$F(u, v, w) = \sup_{f \in T(w, u)} \langle f, v - u \rangle + J^0(u, v - u) + \varphi(v) - \varphi(u),$$

for all $u, v, w \in C$. Directly from (C1) it follows that F is monotone, so (H1) holds. The hypotheses (H2), (H3) and (H4), referring to S_1 and S_2 are obviously verified. Clearly $F(u, u, w) = 0$ for any $u, w \in C$, which ensures (H6).

It is known that $J^0(u, \cdot)$ is sublinear for any $u \in C$ (see for instance [7]). Together with the convexity of φ and the linearity of $\langle f, \cdot \rangle$, this gives (H5).

From conditions (C6) and (C8), using Lemma 2.9, it follows that the mapping $t \mapsto \sup_{f \in T(tv + (1-t)u, u)} \langle f, v - u \rangle$ is upper semicontinuous at $t = 0$, which ensures (H7).

We notice that $\varphi(v) - \varphi(\cdot)$ is upper semicontinuous (since φ is lower semicontinuous) and, by (C9), J^0 is B-pseudomonotone relative to (S_1, S_2) . The upper

semicontinuity of $\sup_{f \in T(v, \cdot)} \langle f, v - \cdot \rangle$ is obtained by using (C7), (C8) and Lemma 2.9. These, together with the fact that the sum of two B-pseudomonotone mappings is B-pseudomonotone, imply the assumptions (H8) and (H9).

Finally, (H10) is a direct consequence of (C10). \square

Remark 4.2. (i) If $T: V \times V \rightarrow V^*$ is single-valued, then condition (C7) can be weakened to: $T(u, \cdot)$ is locally bounded on finite dimensional subspaces and the mapping $T(u, (1-t)u + tv)$ is upper semicontinuous at $t = 0$. Indeed, in this case, by (C1), $T(u, \cdot)$ is B-pseudomonotone. In this case we obtain an analogue of Theorem 5.1 from [18].

(ii) If V is endowed with the strong topology, then J^0 is upper semicontinuous in both arguments (see [7]), so (C6) can be omitted.

(ii) For $\varphi \equiv 0$, instead of (C1) we can suppose the following pseudomonotonicity condition: for any $u, v \in C$,

$$\sup_{f \in T(u, u)} \langle f, v - u \rangle \geq 0 \text{ implies } \sup_{f \in T(v, u)} \langle f, v - u \rangle \geq 0.$$

In this case, Theorem 4.1 implies Theorem 3.1 from [19].

(iii) In the paper [8] a particular case of the problem (HVI) was considered, and a direct proof of an existence theorem was given.

References

- [1] J. P. Aubin: *Optima and Equilibria*, Springer-Verlag, Berlin et al. (1993).
- [2] V. Z. Belenkii, V. A. Volkonskii: *Iterative Methods in Game Theory and Programming (Russian)*, Nauka, Moscow (1974).
- [3] E. Blum, W. Oettli: *From optimization and variational inequalities to equilibrium problems*, Math. Student 63 (1994) 123–145.
- [4] H. Brézis, L. Nirenberg, G. Stampacchia: *A remark on Ky Fan's minimax principle*, Boll. Unione Mat. Ital. 6 (1972) 293–300.
- [5] M. Castellani, M. Giuli: *An existence result for quasiequilibrium problems in separable Banach spaces*, J. Math. Anal. Appl. 425 (2015) 85–95.
- [6] Y. Q. Chen: *On semi-monotone operator theory and applications*, J. Math. Anal. Appl. 231 (1999) 177–192.
- [7] F. H. Clarke: *Optimization and Nonsmooth Analysis*, John Wiley, New York (1983).
- [8] A. M. Croicu, J. Kolumbán: *On a generalized hemivariational inequality on Banach spaces*, (2000) 12 pages.
- [9] X. P. Ding: *Existence of solutions for quasi-equilibrium problems in noncompact topological spaces*, Comput. Math. Appl. 39 (2000) 13–21.
- [10] M. Fakhar, J. Zafarani: *Generalized vector equilibrium problems for pseudomonotone multivalued bifunctions*, J. Optim. Theory Appl. 126 (2005) 109–124.

- [11] M. Fakhar, J. Zafarani: *On generalized variational inequalities*, J. Global Optim. 43 (2009) 503–511.
- [12] K. Fan: *A minimax inequality and applications*, in: *Inequalities III*, O. Shisha (ed.), Academic Press, New York (1972) 103–113.
- [13] J. Y. Fu: *Generalized vector quasi-equilibrium problems*, Math. Methods Oper. Res. 52 (2000) 57–64.
- [14] S. H. Hou, X. H. Gong, X. M. Yang: *Existence and stability of solutions for generalized Ky Fan inequality problems with trifunctions*, J. Optim. Theory Appl. 146 (2010) 387–398.
- [15] S. H. Hu, N. S. Papageorgiu: *Handbook of Multivalued Analysis, Volume 1. Theory*, Kluwer-Springer, Berlin (1997).
- [16] D. Inoan, J. Kolumbán: *On pseudomonotone set-valued mappings*, Nonlinear Anal. Theory 68 (2008) 47–53.
- [17] D. Inoan, J. Kolumbán: *Existence theorems via duality for equilibrium problems with trifunctions*, submitted.
- [18] E. M. Kalmoun: *On Ky Fan's minimax inequalities, mixed equilibrium problems and hemivariational inequalities*, J. Inequ. Pure Appl. Math. 2(1) (2001) 1–31.
- [19] G. Kassay, J. Kolumbán: *Variational inequalities given by semi-pseudomonotone mappings*, Nonlinear Analysis Forum 5 (2000) 35–50.
- [20] G. Kassay, M. Miholca: *Existence results for vector equilibrium problems given by a sum of two functions*, J. Global Optim. 63 (2015) 195–211.
- [21] I. V. Konnov, S. Schaible: *Duality for equilibrium problems under generalized monotonicity*, J. Optim. Theory Appl. 104 (2000) 395–408.
- [22] J. Leray, J. L. Lions: *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1965) 97–107.
- [23] L.-J. Lin, S. Park: *On some generalized quasiequilibrium problems*, J. Math. Anal. Appl. 224 (1998) 167–181.
- [24] U. Mosco: *Implicit variational problems and quasi variational inequalities*, in: *Lecture Notes in Math.* 543, Springer-Verlag, Berlin (1976) 83–156.
- [25] Z. Naniewicz, P. D. Panagiotopoulos: *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York (1995).
- [26] R. E. Showalter: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surveys and Monographs 49, Amer. Math. Society, Providence (1997).
- [27] G. Tian: *Generalizations of the FKKM theorem and the Ky Fan minimax inequality with applications to maximal elements, price equilibrium, and complementarity*, J. Math. Anal. Appl. 170 (1992) 457–471.
- [28] E. Zeidler: *Nonlinear Functional Analysis and Its Applications, II/B: Nonlinear Monotone Operators*, Springer, New York (1990).