

Mean Field Control and Mean Field Game Models with Several Populations

Alain Bensoussan*

*International Center for Decision and Risk Analysis, Jindal School of Management,
University of Texas at Dallas, 800 W. Campbell Road, Richardson, TX 75080, U.S.A.
axb046100@utdallas.edu*

Tao Huang

*Dept. of Mathematics, Wayne State University, 4841 Cass Avenue, Detroit, MI 48201, U.S.A.
qq3481@wayne.edu*

Mathieu Laurière†

*Department of Operations Research and Financial Engineering, Princeton University,
Charlton Street, Princeton, NJ 08544, U.S.A.
lauriere@princeton.edu*

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We investigate the interaction of two populations with a large number of indistinguishable agents. The problem consists in two levels: the interaction between agents of a same population, and the interaction between the two populations. In the spirit of mean field type control (MFC) problems and mean field games (MFG), each population is approximated by a continuum of infinitesimal agents. We define four different problems in a general context and interpret them in the framework of MFC or MFG. By calculus of variations, we derive formally in each case the adjoint equations for the necessary conditions of optimality. Importantly, we find that in the case of a competition between two coalitions, one needs to rely on a system of master equations in order to describe the equilibrium. Examples are provided, in particular linear-quadratic models for which we obtain systems of ODEs that can be related to Riccati equations.

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1. Introduction

1.1. General introduction

The evolution of a large group of interacting agents, who are trying to realize a certain goal either from an individual or a social viewpoint, is an important

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and interesting question in mathematics, physics and many other fields. When the number of agents grows to infinity, it becomes extremely hard to keep track of all the agent-to-agent interactions and to study the resulting global behavior. However, by assuming that every agent has the same importance, the impact of each single agent's choice on the group decreases as the size of the population increases. So, in order to efficiently approximate the global evolution of the group, one can replace the influences of all the other players on a given agent by their average influence. This is called *mean field approach*, whose name is borrowed from statistical physics. This approach is valid under some assumptions (in particular the one asserting that all the players have indistinguishable roles among the population) and allows to replace the microscopic viewpoint by a macroscopic one. The main advantage of such an approximation is that the macroscopic description is more tractable and in particular amenable to numerical treatment. Moreover, the larger the number of individuals, the more accurate the approximation.

Based on the idea of mean field approach, two important theories have recently emerged: *mean field games* (MFG) and *mean field (type) control* problems (MFC). Both of them study the behavior of a typical agent *via* the evolution of her state and the cost induced by this evolution. One can characterize the optimal control of this representative player by two coupled equations: a forward one, describing her dynamics (or, equivalently, the dynamics of the population), and a backward one, describing the evolution of her value function. Depending on whether one uses analytical or stochastic methods, partial differential equations (PDEs) or stochastic differential equations (SDEs) are obtained respectively. For more details, the reader can refer to [8] for the general theory, [29] for the regularity theory of the PDE system, and [14, 15] for the stochastic analysis.

There are several important differences between these two theories that have to be emphasized.

(1) Mean field games correspond to the limit of differential games, in which one wants to find Nash equilibria, when the number of players tend to infinity. They describe the global behavior of the group resulting from the selfish choices that individuals are making so as to minimize a certain cost (or maximize a certain benefit). This cost depends on the state of the agent and also on the statistical distribution of all the agents' states, that is, the global state of the system. Mean field game models have been introduced on the one hand by Lasry and Lions [32, 45, 46, 47, 50] (see also [12] for a written account of Lions' lectures at Collège de France), and on the other hand by Caines, Huang and Malhamé [33, 34, 35, 36, 37, 39]. They have since then attracted a lot of interest. Some important applications can be found in finance [4, 18, 42, 47], economy [2, 19, 28, 31, 30, 43], systemic risk [17, 27], energy production [20, 32], or crowd motion [3, 44, 50].

(2) Another kind of asymptotic regime leads to mean field type control problems, which are stochastic optimal control problems where the cost function and the parameters of the dynamics depend on the law of the controlled stochastic

process. In general, such problems correspond to the control of a large number of agents by a global planner. The problem is to find an optimal feedback rule that she can provide to all the agents, who then implement it in a distributed fashion. This type of problem can also be regarded as the problem of a single player who tries to optimize a cost involving the law of her own state, which evolves with a Mc-Kean Vlasov (MKV) dynamics. Depending on which of the two viewpoints one chooses, this theory is referred to *mean field (type) control* (e.g., by Bensoussan, Frehse, and Yam) or *control of Mc-Kean Vlasov dynamics* (e.g., by Carmona and Delarue). It has found applications such as risk management, portfolio management, or cybersecurity [24, 25, 40, 51].

In this paper, we will investigate the interaction of two populations with a large number of indistinguishable agents, which is a natural extension of the aforementioned theories. In this type of problems, two (or more) large populations interact and the behavior of each group is approximated using a mean field approach. This idea has been broached by several authors since the original contribution of Huang, Caines and Malhamé [35]. In [26], Feleqi has derived an adjoint system of Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Plank for the ergodic mean field game theory of several populations, by letting the number of members of each group go to infinity. In [21, 22] and [7], existence and uniqueness results for this type of systems with Neumann boundary conditions have been proved, in the stationary and the dynamic case respectively. In [1], Achdou, Bardi and Cirant introduced a MFG model describing the interactions between two populations in urban settlements and residential choice. They showed the existence of solutions for both stationary and evolution cases with periodic boundary conditions, and provided some numerical simulations. For a synthetic presentation, we refer the reader to the monographs [8] (Chapter 8) and [14] (Chapter 7). To the best of our knowledge, mean field control problems with several populations have been considered only in [6] and [44], where the authors introduced models for crowd motion (with local and non-local interactions respectively). They studied optimality conditions and provided numerical results.

Despite the increasing research activities on this topic, a global viewpoint was still missing: multi-population MFC problems have been considered only on some examples, and even in the more studied multi-population MFG setting, the problems were of a relatively special type due to the form of the Hamiltonians. The goal of the present paper is to introduce a general framework to tackle multi-population mean field control problems and mean field games. To this end, we consider two types of interactions, cooperation or competition, between two populations and between agents of one population. At a heuristic level, we can summarize as follows the different cases.

(1) In the same population, if the agents cooperate with each other, one obtains a mean field control problem (or control of McKean-Vlasov dynamics), whereas if they compete, one obtains a mean field game. In this paper, we only consider the case that different groups have the same type of interaction between their own agents.

(2) For two populations, if the interaction is cooperation, there is only one single objective function to optimize and the equilibrium between the two groups should be of social type. If it is competition, each group optimizes its own objective function and we look for a Nash equilibrium between the two populations.

Therefore, we will study four different cases of two population interactions in this paper. For each of them, we will describe the control problems and derive the associated system of PDEs by calculus of variation. Moreover the adjoint equations can also be deduced from the so-called *master equation* [9, 10, 13, 48, 49, 52, 53]. We will also explain generalizations of this approach for two populations.

For the sake of clarity, we will consider the interactions between only two populations but the ideas could be generalized to a larger number of populations. To alleviate the notations, we will assume that each population represents the same proportion of the total population but more complex situations could be tackled in a similar way. Furthermore, we will focus on the mean field limit and will not discuss the models with a finite number of agents.

1.2. Mathematical framework

In the sequel, we will consider two populations with densities $(x, t) \mapsto m_i(x, t)$, $i = 1, 2$ and $x \in \mathbb{R}^n$. We note by m the vector $(m_1, m_2)^*$, where the superscript $*$ denotes the *transpose* of a vector or a matrix. We note by m_t the function $x \mapsto m(x, t)$. *Feedback controls* are functions $(x, m) \mapsto v_i(x, m) \in \mathbb{R}^d$ to be chosen by the two populations. To simplify notation, we omit to write explicitly a possible dependence on time. We note by v the vector $(v_1, v_2)^*$.

For simplicity, we will assume that $m_i(\cdot, t) \in L^2(\mathbb{R}^n)$. This allows us to use functional derivatives in the following sense. For a function $f : L^2(\mathbb{R}^n)^2 \rightarrow \mathbb{R}^n$, we note by $\partial_{m_i} f$ the *Gâteaux derivative* of f with respect to the i -th density, so that for $m \in L^2(\mathbb{R}^n)^2$, $\tilde{m} \in L^2(\mathbb{R}^n)^2$,

$$\frac{d}{d\theta} f(m_i + \theta \tilde{m}_i, m_{-i}) \Big|_{\theta=0} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial m_i}(m)(\xi) \tilde{m}_i(\xi) d\xi, \quad (1)$$

where $i = 1, 2$ and $-i = 2, 1$ respectively. This notion of differentiability will be sufficient for the purpose of this work. For a more rigorous treatment beyond the L^2 setting, one could rely on a more general notion of differentiability introduced by P.-L. Lions in his lectures at Collège de France [50] and called L -derivative by Carmona and Delarue (see Chapter 5 in [14]).

The paper is organized as follows. In Section 2, we consider the common mean field control problem with a single objective function. In some sense it is the simplest model to present because it does not involve any fixed point argument. By using calculus of variations we derive formally the adjoint equations and obtain a system of forward-backward PDEs characterizing the optimal solution. In Section 3, we consider two McKean-Vlasov populations with their own objective

functions. The Nash equilibrium cannot be described with PDEs in finite dimension and in this case one *needs* to rely on a system of master equations. In Section 4, we study the corresponding mean field game settings, and obtain the adjoint equations for common or separated objective functions, respectively. In Section 5, we provide two types of examples: we first revisit an example of crowd dynamics from [44, 6] and we then turn our attention to linear-quadratic models, for which we obtain systems of ODEs that can be related to Riccati equations.

2. Common Mean Field Type Control

2.1. Definition of the problem

In this section, we investigate the situation where a global planner seeks to control in a distributed fashion two interacting populations driven by McKean-Vlasov dynamics, and tries to minimize a global cost. This setting can also be construed as a kind of social optimum for two populations (or two players with MKV dynamics) cooperating in order to minimize a cost which aggregates their objectives. We call this problem *common mean field type control* (CMFC for short) and define it as follows.

Problem 2.1. (CMFC) *Find a feedback control $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$ minimizing the functional*

$$J^{CMFC}(v_1, v_2) = \sum_{i=1}^2 \left[\int_0^T \int_{\mathbb{R}^n} f_i(x, m_t^v, v_i(x, m_t^v)) m_i^v(x, t) dx dt + \int_{\mathbb{R}^n} h_i(x, m_T^v) m_i^v(x, T) dx \right], \tag{2}$$

where $m^v = (m_1^v, m_2^v)^*$ solves the following system of Fokker-Planck (FP) equations: for $i = 1, 2$,

$$\frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \operatorname{div}_x (g_i(x, m_t, v_i(x, m_t)) m_i(x, t)) = 0, \tag{3}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ with initial condition $m_i(x, 0) = \rho_{i,0}(x)$, $x \in \mathbb{R}^n$.

The functions

$$g_i: \mathbb{R}^n \times L^2(\mathbb{R}^n)^2 \times \mathbb{R}^d \rightarrow \mathbb{R}^n, \quad (x, m, v_i) \mapsto g_i(x, m, v_i), \tag{4}$$

$$f_i: \mathbb{R}^n \times L^2(\mathbb{R}^n)^2 \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad (x, m, v_i) \mapsto f_i(x, m, v_i), \tag{5}$$

$$h_i: \mathbb{R}^n \times L^2(\mathbb{R}^n)^2 \rightarrow \mathbb{R}, \quad (x, m) \mapsto h_i(x, m), \tag{6}$$

are assumed to be differentiable with respect to all independent variables. These functions as well as the control v may also depend on time but we omit it to alleviate the notations. By A_i^* we denote the formal dual operator of the differential operator

$$A_i \varphi(x) = - \sum_{\alpha, \beta=1}^n a_i^{\alpha\beta}(x) \frac{\partial^2 \varphi(x)}{\partial x_\alpha \partial x_\beta}.$$

We assume sufficient smoothness on the drift functions g_i as well as on the feedback, to perform differentiation as needed. We note that in (3) the coupling holds only through the vector m .

One could consider more general cost functions (which can not be decomposed as a sum) but here we restrict our attention to the form given in (2) in order to facilitate the comparison with a setting where each population has its own cost (see Section 3).

Although we are going to focus on PDE formulations in the rest of this work, let us mention that this model can also be motivated from a stochastic viewpoint as the optimal control of two McKean-Vlasov dynamics. Indeed, one can consider a stochastic process $X^v = (X_1^v, X_2^v)$ in \mathbb{R}^{2n} with the following McKean-Vlasov dynamics

$$dX_i^v(t) = g_i\left(X_i^v(t), (\mathcal{L}(X_1^v(t)), \mathcal{L}(X_2^v(t)))^*, v_i\left(X_i^v(t), (\mathcal{L}(X_1^v(t)), \mathcal{L}(X_2^v(t)))^*\right)\right) dt + \sigma_i(X_i^v(t)) dW_i(t)$$

for $i = 1, 2$, with $X^v(0)$ such that $X_i^v(0)$ has distribution $\rho_{i,0}$. Here, $W = (W_1, W_2)$ is a pair of independent \mathbb{R}^n -valued Brownian motions on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Moreover $\mathcal{L}(X_i^v(t))$ denotes the distribution of $X_i^v(t)$. If we assume that these distributions have densities with respect to Lebesgue measure which are in $L^2(\mathbb{R}^n)$, then these densities satisfy, at least formally, the FP equations (3) with $a_i(x) = \frac{1}{2}\sigma_i(x)\sigma_i^*(x)$. Moreover, the objective functional (2) can be written as a sum of expectations, which correspond to the expected cost of each player.

2.2. Necessary conditions of optimality

We shall assume the existence of optimal feedbacks $(x, m) \mapsto \hat{v}_i(x, m)$ and look for necessary conditions of optimality. We denote by $m^{\hat{v}} = (m_1^{\hat{v}}, m_2^{\hat{v}})^*$ the solutions of (3) controlled by $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$. We consider feedbacks $(x, m) \mapsto \hat{v}_i(x, m) + \theta \tilde{v}_i(x, m)$, $\theta \in \mathbb{R}$, and we call $m_{i,\theta}$ the corresponding solutions of the FP equations (3). Then $(m_{i,\theta}(x, t) - m_i^{\hat{v}}(x, t))/\theta \rightarrow \tilde{m}_i(x, t)$ pointwise as $\theta \rightarrow 0$, solution of

$$\begin{aligned} \frac{\partial \tilde{m}_i}{\partial t}(x, t) + A_i^* \tilde{m}_i(x, t) + \operatorname{div}_x \left(g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) \tilde{m}_i(x, t) + \right. \\ \left. + \left[\sum_{j=1}^2 \int_{\mathbb{R}^n} \left(\partial_{m_j} g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))(\xi) + \right. \right. \right. \\ \left. \left. + \frac{\partial g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \partial_{m_j} \hat{v}_i(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_j(\xi, t) d\xi + \right. \\ \left. \left. + \frac{\partial g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \tilde{v}_i(x, m_t^{\hat{v}}) \right] m_i^{\hat{v}}(x, t) \right) = 0, \end{aligned} \quad (7)$$

with the initial condition $\tilde{m}_i(x, 0) = 0$. Recall that $\partial_{m_i} \varphi(m)$ denotes the functional derivative of a functional $(m_1, m_2)^* = m \mapsto \varphi(m)$ with respect to m_i ,

as defined by (1). Note that \tilde{m} depends on \hat{v} and $m^{\hat{v}}$, but we omit it to save notation. We next compute the Gâteaux derivative of the functional J^{CMFC} .

$$\begin{aligned}
 & \frac{d}{d\theta} J^{CMFC}(\hat{v} + \theta\tilde{v})|_{\theta=0} = \tag{8} \\
 & = \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^n} f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) \tilde{m}_i(x, t) dx dt + \\
 & + \sum_{i,j=1}^2 \int_0^T \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left(\partial_{m_j} f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))(\xi) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \partial_{m_j} \hat{v}_i(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_j(\xi, t) d\xi \right] m_i^{\hat{v}}(x, t) dx dt + \\
 & + \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^n} \frac{\partial f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \tilde{v}_i(x, m_t^{\hat{v}}) m_i^{\hat{v}}(x, t) dx dt + \\
 & + \sum_{i=1}^2 \int_{\mathbb{R}^n} h_i(x, m_T^{\hat{v}}) \tilde{m}_i(x, T) dx + \\
 & + \sum_{i,j=1}^2 \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \partial_{m_j} h_i(x, m_T^{\hat{v}})(\xi) \tilde{m}_j(\xi, T) d\xi \right] m_i^{\hat{v}}(x, T) dx.
 \end{aligned}$$

We then introduce the functions $(x, t) \mapsto u_i^{\hat{v}}(x, t), i = 1, 2$, solutions of the backward equations

$$\begin{aligned}
 & - \frac{\partial u_i}{\partial t}(x, t) + A_i u_i(x, t) = \tag{9} \\
 & = f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) + Du_i(x, t) \cdot g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) + \\
 & + \sum_{j=1}^2 \int_{\mathbb{R}^n} \left[\partial_{m_i} f_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))(x) + \right. \\
 & \quad \left. + \frac{\partial f_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))}{\partial v_j} \partial_{m_i} \hat{v}_j(\xi, m_t^{\hat{v}})(x) \right] m_j^{\hat{v}}(\xi, t) d\xi + \\
 & + \sum_{j=1}^2 \int_{\mathbb{R}^n} Du_j(\xi, t) \left(\partial_{m_i} g_j(\xi, m_t^{\hat{v}}, \hat{v}_i(\xi, m_t^{\hat{v}}))(x) + \right. \\
 & \quad \left. + \frac{\partial g_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))}{\partial v_j} \partial_{m_i} \hat{v}_j(\xi, m_t^{\hat{v}})(x) \right) m_j^{\hat{v}}(\xi, t) d\xi
 \end{aligned}$$

with terminal condition

$$u_i(x, T) = h_i(x, m_T^{\hat{v}}) + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} h_j(\xi, m_T^{\hat{v}})(x) m_j^{\hat{v}}(\xi, T) d\xi.$$

Using (9) in (8) it follows

$$\begin{aligned}
& \frac{d}{d\theta} J^{CMFC} (\hat{v} + \theta \tilde{v})|_{\theta=0} = \\
& = \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^n} \tilde{m}_i(x, t) \left[-\frac{\partial u_i^{\hat{v}}}{\partial t}(x, t) + A_i u_i^{\hat{v}}(x, t) - Du_i^{\hat{v}}(x, t) g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) - \right. \\
& \quad - \sum_{j=1}^2 \int_{\mathbb{R}^n} Du_j^{\hat{v}}(\xi, t) (\partial_{m_i} g_j(\xi, m_t^{\hat{v}}, \hat{v}_i(\xi, m_t^{\hat{v}}))(x) + \\
& \quad \left. + \frac{\partial g_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))}{\partial v_j} \partial_{m_i} \hat{v}_j(\xi, m_t^{\hat{v}})(x) \right) m_j^{\hat{v}}(\xi, t) d\xi \Big] dx dt + \\
& + \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^n} \frac{\partial f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \tilde{v}_i(x, m_t^{\hat{v}}) m_i^{\hat{v}}(x, t) dx dt + \\
& + \sum_{i=1}^2 \int_{\mathbb{R}^n} u_i^{\hat{v}}(x, T) \tilde{m}_i(x, T) dx.
\end{aligned}$$

Integrating by parts and using (7) after rearrangements, we obtain

$$\begin{aligned}
\frac{d}{d\theta} J^{CMFC} (\hat{v} + \theta \tilde{v})|_{\theta=0} & = \sum_{i=1}^2 \int_0^T \int_{\mathbb{R}^n} \left[\frac{\partial f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} + \right. \\
& \left. + Du_i^{\hat{v}}(x, t) \cdot \frac{\partial g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} \right] \tilde{v}_i(x, m_t^{\hat{v}}) m_i^{\hat{v}}(x, t) dx dt.
\end{aligned}$$

Since $\tilde{v}_i(x, m_t^{\hat{v}})$ can be an arbitrary function of (x, t) and $m_i^{\hat{v}}(x, t) > 0$, assuming the matrix $a_i^{\alpha\beta}(x)$ uniformly positive definite, we necessarily have

$$\frac{\partial f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} + Du_i^{\hat{v}}(x, t) \frac{\partial g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}}))}{\partial v_i} = 0, \text{ a.e. } x, t. \quad (10)$$

Let us introduce, for $i = 1, 2$, the Hamiltonians

$$H_i(x, m, q_i) = \inf_{v_i} [f_i(x, m, v_i) + q_i g_i(x, m, v_i)], \quad (11)$$

and the functions $(x, m, q_i) \mapsto \hat{v}_i(x, m, q_i)$ which achieve the infima, that is

$$H_i(x, m, q_i) = f_i(x, m, \hat{v}_i(x, m, q_i)) + q_i g_i(x, m, \hat{v}_i(x, m, q_i)).$$

We see from (10) that

$$\hat{v}_i(x, m_t^{\hat{v}}) = \hat{v}_i(x, m_t^{\hat{v}}, Du_i^{\hat{v}}(x, t)). \quad (12)$$

Hence we have

$$H_i(x, m_t^{\hat{v}}, Du_i^{\hat{v}}(x, t)) = f_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})) + Du_i^{\hat{v}}(x, t) g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})), \quad (13)$$

and, by (10) and (12),

$$\frac{\partial H_i}{\partial q_i}(x, m_t^{\hat{v}}, Du_i^{\hat{v}}(x, t)) = g_i(x, m_t^{\hat{v}}, \hat{v}_i(x, m_t^{\hat{v}})). \tag{14}$$

Moreover,

$$\begin{aligned} & \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} H_j(\xi, m_t^{\hat{v}}, Du_j^{\hat{v}}(\xi, t))(x) m_j^{\hat{v}}(\xi, t) d\xi = \\ & = \sum_{j=1}^2 \int_{\mathbb{R}^n} \left[\partial_{m_i} f_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))(x) + \right. \\ & \quad \left. + \frac{\partial f_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))}{\partial v_j} \partial_{m_i} \hat{v}_j(\xi, m_t^{\hat{v}})(x) \right] m_j^{\hat{v}}(\xi, t) d\xi + \\ & \quad + \sum_{j=1}^2 \int_{\mathbb{R}^n} Du_j^{\hat{v}}(\xi, t) \left[\partial_{m_i} g_j(\xi, m_t^{\hat{v}}, \hat{v}_i(\xi, m_t^{\hat{v}}))(x) + \right. \\ & \quad \left. + \frac{\partial g_j(\xi, m_t^{\hat{v}}, \hat{v}_j(\xi, m_t^{\hat{v}}))}{\partial v_j} \partial_{m_i} \hat{v}_j(\xi, m_t^{\hat{v}})(x) \right] m_j^{\hat{v}}(\xi, t) d\xi. \end{aligned} \tag{15}$$

Therefore $(u_i^{\hat{v}}, m_i^{\hat{v}})_{i=1,2}$ solve the following system of HJB-FP equations

$$\begin{aligned} -\frac{\partial u_i}{\partial t}(x, t) + A_i u_i(x, t) &= H_i(x, m_t, Du_i(x, t)) + \\ &+ \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} H_j(\xi, m_t, Du_j(\xi, t))(x) m_j(\xi, t) d\xi \end{aligned} \tag{16}$$

$$\frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \operatorname{div}_x \left(\frac{\partial H_i}{\partial q_i}(x, m_t, Du_i(x, t)) m_i(x, t) \right) = 0 \tag{17}$$

with terminal and initial condition

$$u_i(x, T) = h_i(x, m_T) + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} h_j(\xi, m_T)(x) m_j(\xi, T) d\xi, \quad m_i(x, 0) = \rho_{i,0}(x),$$

where $m = (m_1, m_2)^*$ denotes the vector of solutions of (17).

The PDE system (16)–(17) extends to two populations the PDE system for mean field control with a single population (see e.g. [8], Chapter 4, page 18). In Section 3, we will present a different way to extend to two populations the mean field control framework.

Remark 2.2. It is important to notice that finding the solution $(u_i, m_i)_{i=1,2}$ of the above PDE system allows to compute the functions $(x, t) \mapsto \hat{v}_i(x, m_t)$ but not the feedbacks $(x, m) \mapsto \hat{v}_i(x, m)$. In other words, the optimal controls can be computed only along the optimal flows of distributions (that is, the solution $t \mapsto m_t$ obtained by solving the PDE system), but $\hat{v}_i(x, m)$ is not known for all possible m . This can be known only through the master equations, as we explain below.

2.3. Master and Bellman equations

The notion of master equation (for a single population) has been introduced by P.-L. Lions in the context of mean field games [50] and has been studied e.g. in [9, 10, 13]. For more details, the reader is referred to [15].

This section is devoted to the formal introduction of an analogous equation (or, rather, a system of analogous equations) for two-population CMFC. In this setting, the master equations are equations for functions $(x, m, t) \mapsto U_i(x, m, t)$ such that, in particular, for m solving (17), there holds

$$u_i(x, t) = U_i(x, m_t, t). \quad (18)$$

These equations are self-contained, whereas the HJB equations (16) for u_i are not since they need to be coupled with the functions m_i solutions of the FP equations (17). Assuming (18) holds, using the FP equation (17) and integration by parts, we obtain

$$\begin{aligned} \frac{\partial u_i}{\partial t}(x, t) &= \frac{\partial U_i}{\partial t}(x, m_t, t) + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} U_i(x, m_t, t)(\xi) \frac{\partial m_j}{\partial t}(\xi, t) d\xi = \\ &= \frac{\partial U_i}{\partial t}(x, m_t, t) + \\ &+ \sum_{j=1}^2 \int_{\mathbb{R}^n} \left[A_{j\xi} + \frac{\partial H_j}{\partial q_j}(x, m_t, DU_j(x, t)) D_\xi \right] \partial_{m_j} U_i(x, m_t, t)(\xi) m_j(\xi, t) d\xi. \end{aligned}$$

Here, $\partial_{m_j} U_i(x, m, t)$ denotes a derivative in the sense of (1); it is a function of the space variable and $D_\xi \partial_{m_j} U_i(x, m, t)(\xi)$ denotes its gradient. The notation $A_{i\xi}$ is to be understood in a similar way. Using the adjoint equations (16), we identify the master equations

$$\begin{aligned} - \frac{\partial U_i}{\partial t}(x, m, t) + A_{ix} U_i(x, m, t) &= \quad (19) \\ &= - \sum_{j=1}^2 \int_{\mathbb{R}^n} A_{j\xi} \partial_{m_j} U_i(x, m, t)(\xi) m_j(\xi) d\xi + \\ &+ H_i(x, m, DU_i(x, m, t)) + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} H_j(\xi, m_t, DU_j(\xi, m, t))(x) m_j(\xi, t) d\xi + \\ &+ \sum_{j=1}^2 \int_{\mathbb{R}^n} D_\xi \partial_{m_j} U_i(x, m, t)(\xi) \frac{\partial H_j}{\partial q_j}(\xi, m, DU_j(\xi, m, t)) m_j(\xi) d\xi, \end{aligned}$$

with terminal condition

$$U_i(x, m, T) = h_i(x, m) + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_i} h_j(\xi, m)(x) m_j(\xi) d\xi.$$

We then identify U_i as a functional derivative in m_i , namely

$$U_i(x, m, t) = \partial_{m_i} V(m, t)(x), \tag{20}$$

with $(m, t) \mapsto V(m, t)$ solution of the Bellman equation

$$\begin{aligned} -\frac{\partial V}{\partial t}(m, t) + \sum_{j=1}^2 \int_{\mathbb{R}^n} A_j \partial_{m_j} V(m, t)(x) m_j(x) dx &= \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} H_j(x, m, \partial_{m_j} V(m, t)(x)) m_j(x) dx, \end{aligned} \tag{21}$$

with terminal condition

$$V(m, T) = \sum_{j=1}^2 \int_{\mathbb{R}^n} h_j(x, m) m_j(x) dx.$$

Thanks to the functions U_i we can completely characterize the feedbacks by (recall that \hat{v}_i minimizes the Hamiltonian defined by 11)

$$\hat{v}_i(x, m) = \hat{v}_i(x, m, DU_i(x, m, t)).$$

Remark 2.3. From (20) we can assert that

$$\partial_{m_j} U_i(x, m, t)(\xi) = \partial_{m_i} U_j(\xi, m, t)(x) = \partial_{m_i m_j}^2 V(m, t)(x, \xi). \tag{22}$$

Remark 2.4. The Bellman equation (21) could also be obtained directly by a dynamic programming argument similarly to the case of single population, see e.g. [13, 9, 49].

3. Nash mean field type control problem

3.1. Definition of the problem

In this section, we consider a situation where the agents among each population cooperate, but the two populations compete and we try to find a Nash equilibrium between them. A key point is that, here again, each population chooses a feedback $(x, m) \mapsto v_i(x, m)$, so we incorporate both m_1 and m_2 in the feedback as in the case CMFC. The FP equations describing the evolution of $m_i(x, t)$ are still given by (3). However there is not a common cost functional. Instead, each population has its own functional. We call this problem *Nash mean field control* (NMFC for short) and define it as follows.

Problem 3.1. (NMFC) Find a Nash equilibrium $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$ for the cost functionals

$$\begin{aligned} J_i^{NMFC}(v_1, v_2) &= \\ &= \int_0^T \int_{\mathbb{R}^n} f_i(x, m_t^v, v_i(x, m_t^v)) m_i^v(x, t) dx dt + \int_{\mathbb{R}^n} h_i(x, m_T^v) m_i^v(x, T) dx \end{aligned} \tag{23}$$

where $m^v = (m_1^v, m_2^v)^*$ satisfies the PDEs (3) with initial conditions $m_i(\cdot, 0) = \rho_{i,0}$.

Here again, g_i , f_i and h_i are functions as in (4), (5) and (6) respectively, hence $J^{CMFC} = \sum_{i=1}^2 J_i^{NMFC}$.

Remark 3.2. This situation can also be viewed as a competition between two players, each having a dynamics of McKean-Vlasov type.

In the sequel $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$ represents a solution to this problem (assuming it exists). In other words, we have for all $v = (v_1, v_2)^*$

$$J_1^{NMFC}(\hat{v}_1, \hat{v}_2) \leq J_1^{NMFC}(v_1, \hat{v}_2) \quad \text{and} \quad J_2^{NMFC}(\hat{v}_1, \hat{v}_2) \leq J_2^{NMFC}(\hat{v}_1, v_2).$$

We shall write necessary conditions for the existence of such an equilibrium \hat{v} .

3.2. Problem of player 1

For player 1, \hat{v}_2 is a fixed function and \hat{v}_1 solves the following (one-population) MFC problem.

Problem 3.3. (NMFC: Problem of player 1) *Minimize*

$$\begin{aligned} \mathcal{J}_1^{NMFC}(v_1) &= J_1^{NMFC}(v_1, \hat{v}_2) = & (24) \\ &= \int_0^T \int_{\mathbb{R}^n} f_1(x, m_t^{v_1, \hat{v}_2}, v_1(x, m_t^{v_1, \hat{v}_2})) m_1^{v_1, \hat{v}_2}(x, t) dx dt + \int_{\mathbb{R}^n} h_1(x, m_T^{v_1, \hat{v}_2}) m_1^{v_1, \hat{v}_2}(x, T) dx \end{aligned}$$

where $m^{v_1, \hat{v}_2} = (m_1^{v_1, \hat{v}_2}, m_2^{v_1, \hat{v}_2})^*$ solves (3) controlled by (v_1, \hat{v}_2) , that is,

$$\frac{\partial m_1}{\partial t}(x, t) + A_1^* m_1(x, t) + \text{div}_x(g_1(x, m_t, v_1(x, m_t)) m_1(x, t)) = 0 \tag{25}$$

$$\frac{\partial m_2}{\partial t}(x, t) + A_2^* m_2(x, t) + \text{div}_x(g_2(x, m_t, \hat{v}_2(x, m_t)) m_2(x, t)) = 0 \tag{26}$$

with the initial conditions $m_i(\cdot, 0) = \rho_{i,0}$.

In this section, we denote by $m^{\hat{v}} = (m_1^{\hat{v}}, m_2^{\hat{v}})^*$ the solutions of (25)–(26) when player 1 chooses \hat{v}_1 as a control. Consider a feedback $(x, m) \mapsto \hat{v}_1(x, m) + \theta \tilde{v}_1(x, m)$ and denote by $m_{1,\theta}$, $m_{2,\theta}$ the corresponding solutions of the FP equations (25), (26) respectively. Then

$$\frac{m_{1,\theta} - m_1^{\hat{v}}}{\theta}(x, t) \xrightarrow{\theta \rightarrow 0} \tilde{m}_1(x, t), \quad \frac{m_{2,\theta} - m_2^{\hat{v}}}{\theta}(x, t) \xrightarrow{\theta \rightarrow 0} \tilde{m}_{-1}(x, t).$$

Notice that we have written $\tilde{m}_{-1}(x, t)$ and not $\tilde{m}_2(x, t)$ for the second limit. Indeed, we are considering the problem of player 1, and the second limit describes the impact of his choice on the second FP equation (26). So the index 1 characterizes the first player, and the sign \pm refers to his FP equation or the FP equation of the opponent (player 2). We obtain for \tilde{m}_1

$$\begin{aligned}
 & \frac{\partial \tilde{m}_1}{\partial t}(x, t) + A_1^* \tilde{m}_1(x, t) + \operatorname{div}_x \left\{ g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) \tilde{m}_1(x, t) + \right. \\
 & \quad + \left[\int_{\mathbb{R}^n} \left(\partial_{m_1} g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_1(\xi, t) d\xi + \right. \\
 & \quad + \int_{\mathbb{R}^n} \left(\partial_{m_2} g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \right. \\
 & \quad \left. + \frac{\partial g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_{-1}(\xi, t) d\xi + \\
 & \quad \left. + \frac{\partial g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \tilde{v}_1(x, m_t^{\hat{v}}) \right] m_1^{\hat{v}}(x, t) \left. \right\} = 0,
 \end{aligned} \tag{27}$$

and for \tilde{m}_{-1}

$$\begin{aligned}
 & \frac{\partial \tilde{m}_{-1}}{\partial t}(x, t) + A_2^* \tilde{m}_{-1}(x, t) + \operatorname{div}_x \left\{ g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}})) \tilde{m}_{-1}(x, t) + \right. \\
 & \quad + \left[\int_{\mathbb{R}^n} \left(\partial_{m_1} g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}}))(\xi) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_1} \hat{v}_2(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_1(\xi, t) d\xi + \right. \\
 & \quad + \int_{\mathbb{R}^n} \left(\partial_{m_2} g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}}))(\xi) + \right. \\
 & \quad \left. + \frac{\partial g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_2} \hat{v}_2(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_{-1}(\xi, t) d\xi \left. \right] m_2^{\hat{v}}(x, t) \left. \right\} = 0,
 \end{aligned} \tag{28}$$

with the initial conditions

$$\tilde{m}_1(x, 0) = 0, \quad \tilde{m}_{-1}(x, 0) = 0.$$

Similar to the CMFC case, we can compute

$$\begin{aligned}
 & \frac{d}{d\theta} \mathcal{J}_1^{NMFC}(\hat{v}_1 + \theta \tilde{v}_1)|_{\theta=0} = \\
 & = \int_0^T \int_{\mathbb{R}^n} f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) \tilde{m}_1(x, t) dx dt + \\
 & \quad + \int_0^T \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left(\partial_{m_1} f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(x, m_t^{\hat{v}})(\xi) \right) \tilde{m}_1(\xi, t) d\xi \right] m_1^{\hat{v}}(x, t) dx dt + \\
 & \quad + \int_0^T \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left(\partial_{m_2} f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \right. \right.
 \end{aligned} \tag{29}$$

$$\begin{aligned}
& + \frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(x, m_t^{\hat{v}})(\xi) \tilde{m}_{-1}(\xi, t) d\xi \Big] m_1^{\hat{v}}(x, t) dx dt + \\
& + \int_0^T \int_{\mathbb{R}^n} \frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \tilde{v}_1(x, m_t^{\hat{v}}) m_1^{\hat{v}}(x, t) dx dt + \\
& + \int_{\mathbb{R}^n} h_1(x, m_T^{\hat{v}}) \tilde{m}_1(x, T) dx + \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \left(\partial_{m_1} h_1(x, m_T^{\hat{v}})(\xi) \tilde{m}_1(\xi, T) + \right. \right. \\
& \quad \left. \left. + \partial_{m_2} h_1(x, m_T^{\hat{v}})(\xi) \tilde{m}_{-1}(\xi, T) \right) d\xi \right] m_1^{\hat{v}}(x, T) dx.
\end{aligned}$$

3.3. HJB-FP system for player 1

We introduce the functions $u_1^{\hat{v}}$, and $u_{-1}^{\hat{v}}$, solutions of the equations

$$\begin{aligned}
& - \frac{\partial u_1}{\partial t}(x, t) + A_1 u_1(x, t) + \tag{30} \\
& = f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) + Du_1(x, t) \cdot g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) + \\
& \quad + \int_{\mathbb{R}^n} \left[\partial_{m_1} f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \\
& \quad \quad \left. + \frac{\partial f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right] m_1^{\hat{v}}(\xi, t) d\xi + \\
& \quad + \int_{\mathbb{R}^n} \left[Du_1(\xi, t) \left(\partial_{m_1} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
& \quad \quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right] m_1^{\hat{v}}(\xi, t) d\xi + \\
& \quad + \int_{\mathbb{R}^n} \left[Du_{-1}(\xi, t) \left(\partial_{m_1} g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
& \quad \quad \left. \left. + \frac{\partial g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_1} \hat{v}_2(\xi, m_t^{\hat{v}})(x) \right) \right] m_2^{\hat{v}}(\xi, t) d\xi
\end{aligned}$$

with terminal condition

$$\begin{aligned}
& u_1(x, T) = h_1(x, m_T^{\hat{v}}) + \int_{\mathbb{R}^n} \partial_{m_1} h_1(\xi, m_T^{\hat{v}})(x) m_1^{\hat{v}}(\xi, T) d\xi, \quad \text{and} \\
& - \frac{\partial u_{-1}}{\partial t}(x, t) + A_2 u_{-1}(x, t) = \tag{31} \\
& = Du_{-1}(x, t) \cdot g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}})) + \\
& \quad + \int_{\mathbb{R}^n} \left[\partial_{m_2} f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \\
& \quad \quad \left. + \frac{\partial f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right] m_1^{\hat{v}}(\xi, t) d\xi +
\end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^n} \left[Du_1(\xi, t) \left(\partial_{m_2} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right] m_1^{\hat{v}}(\xi, t) d\xi + \\
 & + \int_{\mathbb{R}^n} \left[Du_{-1}(\xi, t) \left(\partial_{m_2} g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \left. \left. + \frac{\partial g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_2} \hat{v}_2(\xi, m_t^{\hat{v}})(x) \right) \right] m_2^{\hat{v}}(\xi, t) d\xi
 \end{aligned}$$

with terminal condition $u_{-1}(x, T) = \int_{\mathbb{R}^n} \partial_{m_2} h_1(\xi, m_T^{\hat{v}})(x) m_1^{\hat{v}}(\xi, T) d\xi$.

We now use (30), (31) in (29) to obtain

$$\begin{aligned}
 & \frac{d}{d\theta} \mathcal{J}_1^{NMFC}(\hat{v}_1 + \theta \tilde{v}_1)|_{\theta=0} = \\
 & = \int_0^T \int_{\mathbb{R}^n} \tilde{m}_1(x, t) \left[-\frac{\partial u_1^{\hat{v}}}{\partial t}(x, t) + A_1 u_1^{\hat{v}}(x, t) - Du_1^{\hat{v}}(x, t) \cdot g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) - \right. \\
 & \quad - \int_{\mathbb{R}^n} \left\{ Du_1^{\hat{v}}(\xi, t) \cdot \left(\partial_{m_1} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right\} m_1^{\hat{v}}(\xi, t) d\xi - \\
 & \quad - \int_{\mathbb{R}^n} \left\{ Du_{-1}^{\hat{v}}(\xi, t) \cdot \left(\partial_{m_1} g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \quad \left. \left. + \frac{\partial g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_1} \hat{v}_2(\xi, m_t^{\hat{v}})(x) \right) \right\} m_2^{\hat{v}}(\xi, t) d\xi \Big] dx dt + \\
 & + \int_0^T \int_{\mathbb{R}^n} \tilde{m}_{-1}(x, t) \left[-\frac{\partial u_{-1}^{\hat{v}}}{\partial t}(x, t) + A_2 u_{-1}^{\hat{v}}(x, t) - Du_{-1}^{\hat{v}}(x, t) \cdot g_2(x, m_t^{\hat{v}}, \hat{v}_2(x, m_t^{\hat{v}})) - \right. \\
 & \quad - \int_{\mathbb{R}^n} \left\{ Du_{-1}^{\hat{v}}(\xi, t) \cdot \left(\partial_{m_2} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right\} m_1^{\hat{v}}(\xi, t) d\xi - \\
 & \quad - \int_{\mathbb{R}^n} \left\{ Du_{-1}^{\hat{v}}(\xi, t) \cdot \left(\partial_{m_2} g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\
 & \quad \quad \left. \left. + \frac{\partial g_2(\xi, m_t^{\hat{v}}, \hat{v}_2(\xi, m_t^{\hat{v}}))}{\partial v_2} \partial_{m_2} \hat{v}_2(\xi, m_t^{\hat{v}})(x) \right) \right\} m_2^{\hat{v}}(\xi, t) d\xi \Big] dx dt + \\
 & + \int_0^T \int_{\mathbb{R}^n} \frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \tilde{v}_1(x, m_t^{\hat{v}}) m_1^{\hat{v}}(x, t) dx dt + \\
 & + \int_{\mathbb{R}^n} \tilde{m}_1(x, T) u_1^{\hat{v}}(x, T) dx + \int_{\mathbb{R}^n} \tilde{m}_{-1}(x, T) u_{-1}^{\hat{v}}(x, T) dx.
 \end{aligned}$$

Integrating by parts and using the equations of $\tilde{m}_1, \tilde{m}_{-1}$, see (27), (28), we obtain

$$\begin{aligned} \frac{d}{d\theta} \mathcal{J}_1^{NMFC}(\hat{v}_1 + \theta \tilde{v}_1)|_{\theta=0} &= \\ &= \int_0^T \int_{\mathbb{R}^n} \left[\frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} + \right. \\ &\quad \left. + Du_1^{\hat{v}}(x, t) \cdot \frac{\partial g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} \right] \tilde{v}_1(x, m_t^{\hat{v}}) m_1^{\hat{v}}(x, t) dx dt. \end{aligned} \quad (32)$$

Since this quantity must be 0 for any possible $\tilde{v}_1(x, m)$, we must have

$$\frac{\partial f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} + Du_1^{\hat{v}}(x, t) \cdot \frac{\partial g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))}{\partial v_1} = 0. \quad (33)$$

So, using the Hamiltonian notation introduced in (11), we have

$$f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) + Du_1^{\hat{v}}(x, t) \cdot g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) = H_1(x, m_t^{\hat{v}}, Du_1^{\hat{v}}(x, t)), \quad (34)$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left[\partial_{m_1} f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \\ &\quad \left. + \frac{\partial f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right] m_1^{\hat{v}}(\xi, t) d\xi + \\ &\quad + \int_{\mathbb{R}^n} \left[Du_1^{\hat{v}}(\xi, t) \left(\partial_{m_1} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\ &\quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_1} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right] m_1^{\hat{v}}(\xi, t) d\xi = \\ &= \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t^{\hat{v}}, Du_1^{\hat{v}}(\xi, t))(x) m_1^{\hat{v}}(\xi, t) d\xi, \quad \text{and} \end{aligned} \quad (35)$$

$$\begin{aligned} &\int_{\mathbb{R}^n} \left[\partial_{m_2} f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \\ &\quad \left. + \frac{\partial f_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right] m_1^{\hat{v}}(\xi, t) d\xi + \\ &\quad + \int_{\mathbb{R}^n} \left[Du_1^{\hat{v}}(\xi, t) \left(\partial_{m_2} g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))(x) + \right. \right. \\ &\quad \left. \left. + \frac{\partial g_1(\xi, m_t^{\hat{v}}, \hat{v}_1(\xi, m_t^{\hat{v}}))}{\partial v_1} \partial_{m_2} \hat{v}_1(\xi, m_t^{\hat{v}})(x) \right) \right] m_1^{\hat{v}}(\xi, t) d\xi = \\ &= \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m_t^{\hat{v}}, Du_1^{\hat{v}}(\xi, t))(x) m_1^{\hat{v}}(\xi, t) d\xi. \end{aligned} \quad (36)$$

We can write a similar necessary condition for player 2 by introducing u_2 and u_{-2} . Unfortunately, this is not sufficient to write a self-contained system for $u_1, u_{-1}, u_2, u_{-2}, m_1,$ and m_2 . Unlike the CMFC case, we need to know explicitly the feedback $(x, m) \mapsto \hat{v}_2(x, m)$ to express its functional derivatives $\partial_{m_1} \hat{v}_2(x, m)(\xi), \partial_{m_2} \hat{v}_2(x, m)(\xi)$. This is a major difference with respect to the CMFC case, in which these terms did not appear. For this reason, in the NMFC problem we cannot avoid the master equations in general.

Remark 3.4. In some special cases, it is not necessary to introduce the master equations; see section 5.1 below.

3.4. Master equations

We look for functions $(x, m, t) \mapsto U_1(x, m, t), U_{-1}(x, m, t)$ such that $u_1^{\hat{v}}(x, t) = U_1(x, m_t^{\hat{v}}, t), u_{-1}^{\hat{v}}(x, t) = U_{-1}(x, m_t^{\hat{v}}, t)$ when $m^{\hat{v}}$ solves (3) controlled by \hat{v} . From (33) we have, recalling the definition of $\hat{\mathbf{v}}_1$, see (11),

$$\hat{v}_1(x, m_t^{\hat{v}}) \hat{\mathbf{v}}_1(x, m_t^{\hat{v}}, Du_1^{\hat{v}}(x, t)) = \hat{\mathbf{v}}_1(x, m_t^{\hat{v}}, DU_1(x, m_t^{\hat{v}}, t)).$$

We then rewrite (34) as

$$\begin{aligned} f_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) + Du_1^{\hat{v}}(x, t) \cdot g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) &= \\ &= H_1(x, m_t^{\hat{v}}, DU_1(x, m_t^{\hat{v}}, t)), \end{aligned} \tag{37}$$

and the right-hand sides of (35) and (36) become respectively

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t^{\hat{v}}, Du_1^{\hat{v}}(\xi, t))(x) m_1^{\hat{v}}(\xi, t) d\xi &= \\ &= \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t^{\hat{v}}, DU_1(\xi, m_t^{\hat{v}}, t))(x) m_1^{\hat{v}}(\xi, t) d\xi, \end{aligned} \tag{38}$$

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m_t^{\hat{v}}, Du_1^{\hat{v}}(\xi, t))(x) m_1^{\hat{v}}(\xi, t) d\xi &= \\ &= \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m_t^{\hat{v}}, DU_1(\xi, m_t^{\hat{v}}, t))(x) m_1^{\hat{v}}(\xi, t) d\xi. \end{aligned} \tag{39}$$

In addition, using (33), we have the relation

$$g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) = \frac{\partial H_1}{\partial q_1}(x, m_t^{\hat{v}}, D_x U_1(x, m_t^{\hat{v}}, t)). \tag{40}$$

We can then take the functional derivatives of both sides, with respect to m_1 and m_2 to obtain

$$\begin{aligned} \partial_{m_1} g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \frac{\partial g_1}{\partial v_1}(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) \partial_{m_1} \hat{v}_1(x, m_t^{\hat{v}})(\xi) &= \\ = \partial_{m_1} \frac{\partial H_1}{\partial q_1}(x, m_t^{\hat{v}}, D_x U_1(x, m_t^{\hat{v}}, t))(\xi) + & \tag{41} \\ + \frac{\partial^2 H_1}{\partial q_1^2}(x, m_t^{\hat{v}}, D_x U_1(x, m_t^{\hat{v}}, t)) \partial_{m_1} D_x U_1(x, m_t^{\hat{v}}, t)(\xi), \text{ and} \end{aligned}$$

$$\begin{aligned}
& \partial_{m_2} g_1(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}}))(\xi) + \frac{\partial g_1}{\partial v_1}(x, m_t^{\hat{v}}, \hat{v}_1(x, m_t^{\hat{v}})) \partial_{m_2} \hat{v}_1(x, m_t^{\hat{v}})(\xi) = \\
& = \partial_{m_2} \frac{\partial H_1}{\partial q_1}(x, m_t^{\hat{v}}, D_x U_1(x, m_t^{\hat{v}}, t))(\xi) + \\
& \quad + \frac{\partial^2 H_1}{\partial q_1^2}(x, m_t^{\hat{v}}, D_x U_1(x, m_t^{\hat{v}}, t)) \partial_{m_2} D_x U_1(x, m_t^{\hat{v}}, t)(\xi). \tag{42}
\end{aligned}$$

In equations (30), (31) we do not need directly (41), (42) but their equivalent for the feedback $(x, m) \mapsto \hat{v}_2(x, m)$, which involve functions U_2, U_{-2} such that $u_2^{\hat{v}}(x, t) = U_2(x, m_t^{\hat{v}}, t)$, $u_{-2}^{\hat{v}}(x, t) = U_{-2}(x, m_t^{\hat{v}}, t)$. Finally, we can write (30), (31) as

$$\begin{aligned}
& - \frac{\partial u_1}{\partial t}(x, t) + A_1 u_1(x, t) = \tag{43} \\
& = H_1(x, m_t, D u_1(x, t)) + \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t, D u_1(\xi, t))(x) m_1(\xi, t) d\xi + \\
& \quad + \int_{\mathbb{R}^n} D u_{-1}(\xi, t) \cdot \left[\partial_{m_1} \frac{\partial H_2}{\partial q_2}(\xi, m_t, D_\xi u_2(\xi, t))(x) + \right. \\
& \quad \left. + \frac{\partial^2 H_2}{\partial q_2^2}(\xi, m_t, D_x u_2(\xi, t)) \partial_{m_1} D_\xi U_2(\xi, m_t, t)(x) \right] m_2(\xi, t) d\xi, \text{ and} \\
& - \frac{\partial u_{-1}}{\partial t}(x, t) + A_2 u_{-1}(x, t) = D_x u_{-1}(x, t) \cdot \frac{\partial H_2}{\partial q_2}(x, m_t, D_x u_2(x, t)) + \tag{44}
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m_t, D u_1(\xi, t))(x) m_1(\xi, t) d\xi + \\
& + \int_{\mathbb{R}^n} D u_{-1}(\xi, t) \left[\partial_{m_2} \frac{\partial H_2}{\partial q_2}(\xi, m_t, D_\xi u_2(\xi, t))(x) + \right. \\
& \quad \left. + \frac{\partial^2 H_2}{\partial q_2^2}(\xi, m_t, D_\xi u_2(\xi, t)) \partial_{m_2} D_\xi U_2(\xi, m_t, t)(x) \right] m_2(\xi, t) d\xi,
\end{aligned}$$

with terminal conditions

$$\begin{aligned}
u_1(x, T) &= h_1(x, m_T) + \int_{\mathbb{R}^n} \partial_{m_1} h_1(\xi, m_T)(x) m_1(\xi, T) d\xi, \\
u_{-1}(x, T) &= \int_{\mathbb{R}^n} \partial_{m_2} h_1(\xi, m_T)(x) m_1(\xi, T) d\xi.
\end{aligned}$$

Similarly, we associate equations for u_2 and u_{-2} as follows

$$\begin{aligned}
& - \frac{\partial u_2}{\partial t}(x, t) + A_2 u_2(x, t) = \tag{45} \\
& = H_2(x, m_t, D u_2(x, t)) + \int_{\mathbb{R}^n} \partial_{m_2} H_2(\xi, m_t, D u_2(\xi, t))(x) m_2(\xi, t) d\xi + \\
& \quad + \int_{\mathbb{R}^n} D u_{-2}(\xi, t) \left[\partial_{m_2} \frac{\partial H_1}{\partial q_1}(\xi, m_t, D_\xi u_1(\xi, t))(x) + \right. \\
& \quad \left. + \frac{\partial^2 H_1}{\partial q_1^2}(\xi, m_t, D_x u_1(\xi, t)) \partial_{m_2} D_\xi U_1(\xi, m_t, t)(x) \right] m_1(\xi, t) d\xi, \text{ and}
\end{aligned}$$

$$\begin{aligned}
 -\frac{\partial u_{-2}}{\partial t}(x, t) + A_1 u_{-2}(x, t) &= D_x u_{-2}(x, t) \frac{\partial H_1}{\partial q_1}(x, m_t, D_x u_1(x, t)) + \\
 &+ \int_{\mathbb{R}^n} \partial_{m_1} H_2(\xi, m_t, D u_2(\xi, t))(x) m_2(\xi, t) d\xi + \\
 &+ \int_{\mathbb{R}^n} D u_{-2}(\xi, t) \left[\partial_{m_1} \frac{\partial H_1}{\partial q_1}(\xi, m_t, D_\xi u_1(\xi, t))(x) + \right. \\
 &\quad \left. + \frac{\partial^2 H_1}{\partial q_1^2}(\xi, m_t, D_\xi u_1(\xi, t)) \partial_{m_1} D_\xi U_1(\xi, m_t, t)(x) \right] m_1(\xi, t) d\xi,
 \end{aligned} \tag{46}$$

with the terminal conditions

$$\begin{aligned}
 u_2(x, T) &= h_2(x, m_T) + \int_{\mathbb{R}^n} \partial_{m_2} h_2(\xi, m_T)(x) m_2(\xi, T) d\xi, \\
 u_{-2}(x, T) &= \int_{\mathbb{R}^n} \partial_{m_1} h_2(\xi, m_T)(x) m_2(\xi, T) d\xi,
 \end{aligned}$$

and the FP equations are

$$\frac{\partial m_1}{\partial t} + A_1^* m_1 + \operatorname{div}_x \left(\frac{\partial H_1}{\partial q_1}(x, m_t, D_x u_1(x, t)) m_1 \right) = 0 \tag{47}$$

$$\frac{\partial m_2}{\partial t} + A_2^* m_2 + \operatorname{div}_x \left(\frac{\partial H_2}{\partial q_2}(x, m_t, D_x u_2(x, t)) m_2 \right) = 0 \tag{48}$$

$$m_1(x, 0) = \rho_{10}(x), \quad m_2(x, 0) = \rho_{20}(x).$$

A crucial remark is that the system of six equations composed of (43), (44), (45), (46), (47), and (48), is *not self-contained* in general since we cannot express $\partial_{m_1} D_\xi U_1(\xi, m_t, t)(x)$, $\partial_{m_2} D_\xi U_1(\xi, m_t, t)(x)$, $\partial_{m_2} D_\xi U_2(\xi, m_t, t)(x)$, and finally $\partial_{m_1} D_\xi U_2(\xi, m_t, t)(x)$ in terms of u_1 , u_{-1} , u_2 , and u_{-2} . So we need to write the system of equations for U_1 , U_{-1} , U_2 , and U_{-2} . We obtain

$$\begin{aligned}
 -\frac{\partial U_1}{\partial t}(x, m, t) + A_1 U_1(x, m, t) &= \\
 &= -\sum_{j=1}^2 \int_{\mathbb{R}^n} A_{j\xi} \partial_{m_j} U_1(\xi, m, t)(x) m_j(\xi) d\xi + \\
 &+ \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_1(\xi, m, t)(x) \frac{\partial H_j}{\partial q_j}(\xi, m, D_\xi U_j(\xi, m, t)) m_j(\xi) d\xi + \\
 &+ H_1(x, m, D_x U_1(x, m, t)) + \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m, D U_1(\xi, m, t))(x) m_1(\xi) d\xi + \\
 &+ \int_{\mathbb{R}^n} D U_{-1}(\xi, m, t) \cdot \left[\partial_{m_1} \frac{\partial H_2}{\partial q_2}(\xi, m, D_\xi U_2(\xi, m, t))(x) + \right. \\
 &\quad \left. + \frac{\partial^2 H_2}{\partial q_2^2}(\xi, m, D_\xi U_2(\xi, m, t)) \partial_{m_1} D_\xi U_2(\xi, m, t)(x) \right] m_2(\xi) d\xi, \quad \text{and}
 \end{aligned} \tag{49}$$

$$\begin{aligned}
& -\frac{\partial U_{-1}}{\partial t}(x, m, t) + A_2 U_{-1}(x, m, t) = \tag{50} \\
& = -\sum_{j=1}^2 \int_{\mathbb{R}^n} A_{j\xi} \partial_{m_j} U_{-1}(\xi, m, t)(x) m_j(\xi) d\xi + \\
& \quad + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_{-1}(\xi, m, t)(x) \frac{\partial H_j}{\partial q_j}(\xi, m, D_\xi U_j(\xi, m, t)) m_j(\xi) d\xi + \\
& \quad + D_x U_{-1}(x, m, t) \cdot \frac{\partial H_2}{\partial q_2}(x, m, D_x U_2(x, m, t)) + \\
& \quad + \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m, D U_1(\xi, m, t))(x) m_1(\xi) d\xi + \\
& \quad + \int_{\mathbb{R}^n} D U_{-1}(\xi, m, t) \left[\partial_{m_2} \frac{\partial H_2}{\partial q_2}(\xi, m, D_\xi U_2(\xi, m, t))(x) + \right. \\
& \quad \quad \left. + \frac{\partial^2 H_2}{\partial q_2^2}(\xi, m, D_\xi U_2(\xi, m, t)) \partial_{m_2} D_\xi U_2(\xi, m, t)(x) \right] m_2(\xi) d\xi,
\end{aligned}$$

with terminal conditions

$$\begin{aligned}
U_1(x, m, T) &= h_1(x, m) + \int_{\mathbb{R}^n} \partial_{m_1} h_1(\xi, m)(x) m_1(\xi) d\xi \\
U_{-1}(x, m, T) &= \int_{\mathbb{R}^n} \partial_{m_2} h_1(\xi, m)(x) m_1(\xi) d\xi.
\end{aligned}$$

Moreover, we have two additional equations for U_2, U_{-2}

$$\begin{aligned}
& -\frac{\partial U_2}{\partial t}(x, m, t) + A_2 U_2(x, m, t) = \tag{51} \\
& = -\sum_{j=1}^2 \int_{\mathbb{R}^n} A_{j\xi} \partial_{m_j} U_2(\xi, m, t)(x) m_j(\xi) d\xi + \\
& \quad + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_2(\xi, m, t)(x) \frac{\partial H_j}{\partial q_j}(\xi, m, D_\xi U_j(\xi, m, t)) m_j(\xi) d\xi + \\
& \quad + H_2(x, m, D_x U_2(x, m, t)) + \int_{\mathbb{R}^n} \partial_{m_2} H_2(\xi, m, D_\xi U_2(\xi, m, t))(x) m_2(\xi, t) d\xi + \\
& \quad + \int_{\mathbb{R}^n} D U_{-2}(\xi, m, t) \cdot \left[\partial_{m_2} \frac{\partial H_1}{\partial q_1}(\xi, m, D_\xi U_1(\xi, m, t))(x) + \right. \\
& \quad \quad \left. + \frac{\partial^2 H_1}{\partial q_1^2}(\xi, m, D_\xi U_1(\xi, m, t)) \partial_{m_2} D_\xi U_1(\xi, m, t)(x) \right] m_1(\xi) d\xi, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
& -\frac{\partial U_{-2}}{\partial t}(x, m, t) + A_1 U_{-2}(x, m, t) = \tag{52} \\
& = -\sum_{j=1}^2 \int_{\mathbb{R}^n} A_{j\xi} \partial_{m_j} U_{-2}(\xi, m, t)(x) m_j(\xi) d\xi +
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_{-2}(\xi, m, t)(x) \frac{\partial H_j}{\partial q_j}(\xi, m, D_\xi U_j(\xi, m, t)) m_j(\xi) d\xi + \\
 & + D_x U_{-2}(x, m, t) \frac{\partial H_1}{\partial q_1}(x, m, D_x U_1(x, m, t)) + \\
 & + \int_{\mathbb{R}^n} \partial_{m_1} H_2(\xi, m, D U_2(\xi, m, t))(x) m_2(\xi) d\xi + \\
 & + \int_{\mathbb{R}^n} D U_{-2}(\xi, m, t) \left[\partial_{m_1} \frac{\partial H_1}{\partial q_1}(\xi, m, D_\xi U_1(\xi, m, t))(x) + \right. \\
 & \quad \left. + \frac{\partial^2 H_1}{\partial q_1^2}(\xi, m, D_\xi U_1(\xi, m, t)) \partial_{m_1} D_\xi U_1(\xi, m, t)(x) \right] m_1(\xi) d\xi,
 \end{aligned}$$

with terminal conditions

$$\begin{aligned}
 U_2(x, m, T) & = h_2(x, m) + \int_{\mathbb{R}^n} \partial_{m_2} h_2(\xi, m)(x) m_2(\xi) d\xi, \\
 U_{-2}(x, m, T) & = \int_{\mathbb{R}^n} \partial_{m_1} h_2(\xi, m)(x) m_2(\xi) d\xi.
 \end{aligned}$$

The system of four master equations (49), (50), (51), (52) for the functions U_1, U_{-1}, U_2, U_{-2} is self-contained.

Recall that in our notation, the subscript “−1” does not mean “2”. Here U_{-1} stems from the problem of player 1 and reflects the impact of the variation of the state of the opponent. This intuition is made more precise in the next subsection.

3.5. Bellman system

We can check that

$$U_1(x, m, t) = \partial_{m_1} V_1(m, t)(x), \quad U_{-1}(x, m, t) = \partial_{m_2} V_1(m, t)(x), \quad (53)$$

$$U_2(x, m, t) = \partial_{m_2} V_2(m, t)(x), \quad U_{-2}(x, m, t) = \partial_{m_1} V_2(m, t)(x), \quad (54)$$

where $(m, t) \mapsto V_1(m, t), V_2(m, t)$ solve of the Bellman system

$$\begin{aligned}
 0 & = \frac{\partial V_1}{\partial t}(m, t) - \int_{\mathbb{R}^n} A_{1x} \partial_{m_1} V_1(m, t)(x) m_1(x) dx - \\
 & - \int_{\mathbb{R}^n} A_{2x} \partial_{m_2} V_1(m, t)(x) m_2(x) dx + \\
 & + \int_{\mathbb{R}^n} H_1(x, m, D_x \partial_{m_1} V_1(m, t)(x)) m_1(x) dx + \\
 & + \int_{\mathbb{R}^n} D_x \partial_{m_2} V_1(m, t)(x) \cdot \frac{\partial H_2}{\partial q_2}(x, m, D_x \partial_{m_2} V_2(m, t)(x)) m_2(x) dx,
 \end{aligned} \quad (55)$$

with terminal condition $V_1(m, T) = \int_{\mathbb{R}^n} h_1(x, m) m_1(x) dx$, and

$$0 = \frac{\partial V_2}{\partial t}(m, t) - \int_{\mathbb{R}^n} A_{1x} \partial_{m_1} V_2(m, t)(x) m_1(x) dx - \quad (56)$$

$$\begin{aligned}
& - \int_{\mathbb{R}^n} A_{2x} \partial_{m_2} V_2(m, t)(x) m_2(x) dx + \\
& + \int_{\mathbb{R}^n} H_2(x, m, D_x \partial_{m_2} V_2(m, t)(x)) m_2(x) dx + \\
& + \int_{\mathbb{R}^n} D_x \partial_{m_1} V_2(m, t)(x) \cdot \frac{\partial H_1}{\partial q_1}(x, m, D_x \partial_{m_1} V_1(m, t)(x)) m_1(x) dx,
\end{aligned}$$

with terminal condition $V_2(m, T) = \int_{\mathbb{R}^n} h_2(x, m) m_2(x) dx$.

Remark 3.5. To recover the equations (49) to (52) by taking the functional derivatives of Bellman equations, one needs to use the following symmetry properties stemming from (53)–(54)

$$\partial_{m_2} U_1(x, m, t)(\xi) = \partial_{m_1} U_{-1}(\xi, m, t)(x), \quad \partial_{m_1} U_2(x, m, t)(\xi) = \partial_{m_2} U_{-1}(\xi, m, t)(x).$$

Remark 3.6. After completion of this work, it has been brought to our attention that similar problems have been studied recently under the term *mean field type games*, see e.g. [54] for the general setting and [23] for a collection of applications to engineering. However, to the best of our knowledge, our work is the first, at this level of generality, to provide a comprehensive framework and to focus on the necessary conditions of optimality formulated in terms of PDEs.

4. Mean field game problems

In this section, we consider the interaction of two populations when the agents of each population are rational (i.e., try to minimize an individual cost and anticipate the rationality of other the players) and we look for non-cooperative equilibrium using a mean field game approach. For a single population, the mean field game viewpoint focuses on Nash equilibria among the population and, as such, the problem is defined through a fixed point procedure: first, given the distribution of the other players' states, an infinitesimal player finds her best response (that is, her optimal control) in order to minimize her cost; second, the distribution driven by the optimal control found in the first step should correspond to the distribution of the population. The reader is referred to e.g. [8] (Chapter 2) and [16] for more details. This idea can be extended to the case of several populations. The PDE system is expressed e.g. in [8] (Chapter 8, page 68) and in [14] (Chapter 7, page 625) in the setting that has been the most usual in the literature so far. Analogously to the mean field control approach, we will distinguish between two types of problems.

4.1. Nash mean field game

Let us start with a setting where each player compete with all the other players. We call it *Nash mean field game* (NMFG for short). In this problem each infinitesimal agent considers that the distributions of both populations are fixed and tries to minimize her own individual cost. Fixing the state of all the other

players is translated, in the mean field limit, by the fact that the probabilities $m = (m_1, m_2)^*$ entering in the functions $f_i(x, m, v_i)$, $g_i(x, m, v_i)$, $h_i(x, m)$ are considered as fixed parameters. For this reason, we look for feedbacks depending on x only instead of (x, m) , that is, the controls are functions $x \mapsto v_i(x)$.

Problem 4.1. (NMFG) Find (\hat{m}, \hat{v}) satisfying the two conditions

(1) $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$ is a Nash equilibrium for the functionals

$$\begin{aligned} J_{i, \hat{m}}^{NMFG}(v_1, v_2) &= \\ &= \int_0^T \int_{\mathbb{R}^n} f_i(x, \hat{m}_t, v_i(x)) m_i^{v, \hat{m}}(x, t) dx dt + \int_{\mathbb{R}^n} h_i(x, \hat{m}_T) m_i^{v, \hat{m}}(x, T) dx \end{aligned}$$

where $m^{v, \hat{m}} = (m_1^{v, \hat{m}}, m_2^{v, \hat{m}})^*$ satisfies

$$\frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \operatorname{div}_x (g_i(x, \hat{m}_t, v_i(x)) m_i(x, t)) = 0,$$

with initial conditions $m_i(\cdot, 0) = \rho_{i,0}$.

(2) $\hat{m} = m^{\hat{v}} = (m_1^{\hat{v}}, m_2^{\hat{v}})^*$ is a solution to (62) controlled by \hat{v} .

The first condition means that, for a given \hat{m} , for any v ,

$$J_{1, \hat{m}}^{NMFG}(\hat{v}_1, \hat{v}_2) \leq J_{1, \hat{m}}^{NMFG}(v_1, \hat{v}_2) \quad \text{and} \quad J_{2, \hat{m}}^{NMFG}(\hat{v}_1, \hat{v}_2) \leq J_{2, \hat{m}}^{NMFG}(\hat{v}_1, v_2).$$

The problem of player 1 is the following.

Problem 4.2. (NMFG: Problem of player 1) Find \hat{v}_1 minimizing

$$\begin{aligned} \mathcal{J}_1^{NMFG}(v_1) &= J_{1, \hat{m}}^{NMFG}(v_1, \hat{v}_2) = \\ &= \int_0^T \int_{\mathbb{R}^n} f_1(x, \hat{m}_t, v_1(x)) m_1^{v_1, \hat{v}_2, \hat{m}}(x, t) dx dt + \int_{\mathbb{R}^n} h_1(x, \hat{m}_T) m_1^{v_1, \hat{v}_2, \hat{m}}(x, T) dx \end{aligned}$$

where $(m_1^{v_1, \hat{v}_2, \hat{m}}, m_2^{v_1, \hat{v}_2, \hat{m}})$ solves

$$\frac{\partial m_1}{\partial t}(x, t) + A_1^* m_1(x, t) + \operatorname{div}_x (g_1(x, \hat{m}_t, v_1(x)) m_1(x, t)) = 0, \quad (57)$$

$$\frac{\partial m_2}{\partial t}(x, t) + A_2^* m_2(x, t) + \operatorname{div}_x (g_2(x, \hat{m}_t, \hat{v}_2(x)) m_2(x, t)) = 0, \quad (58)$$

with initial conditions $m_i(\cdot, 0) = \rho_{i,0}$.

By comparing Problem 4.2 and Problem 3.3, we see that in order to derive the necessary optimality conditions, we can reuse the computations done in the NMFC setting but now with a fixed parameter $\hat{m} = (\hat{m}_1, \hat{m}_2)^*$, and impose a posteriori that $\hat{m} = (\hat{m}_1, \hat{m}_2)^*$ should solve the FP equations. We shall only provide the

equations and skip the proof to alleviate the presentation. This leads to the PDE system

$$-\frac{\partial u_i}{\partial t}(x, t) + A_i u_i(x, t) = H_i(x, m_t, Du_i(x, t)) \quad (59)$$

$$\frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \operatorname{div}_x \left(\frac{\partial H_i}{\partial q_i}(x, m_t, Du_i(x, t)) m_i(x, t) \right) = 0 \quad (60)$$

with terminal and initial conditions

$$u_i(x, T) = h_i(x, m_T), \quad m_i(x, 0) = \rho_{i,0}(x).$$

This corresponds to the PDE system obtained in [8] (Chapter 8, page 68), for which the mean field problem had not been written explicitly but which has been shown to provide an approximate Nash equilibrium for a finite player game where all the players compete. We refer the interested reader to [8] and [14] for more details.

4.2. Common mean field game

We now consider a different viewpoint where, in analogy with the CMFC setting, there is only one (common) cost functional. We introduce the following problem, that we call *common mean field game* (CMFG for short). Here again, the states of the populations are fixed so we look for feedbacks as functions of x only instead of (x, m) .

Problem 4.3. (CMFG) Find (\hat{m}, \hat{v}) satisfying the two conditions

(1) $\hat{v} = (\hat{v}_1, \hat{v}_2)^*$ minimizes

$$\begin{aligned} J_{\hat{m}}^{CMFG}(v_1, v_2) &= \\ &= \sum_{i=1}^2 \left[\int_0^T \int_{\mathbb{R}^n} f_i(x, \hat{m}_t, v_i(x)) m_i^{v, \hat{m}}(x, t) dx dt + \int_{\mathbb{R}^n} h_i(x, \hat{m}_T) m_i^{v, \hat{m}}(x, T) dx \right], \end{aligned} \quad (61)$$

where $m^{v, \hat{m}} = (m_1^{v, \hat{m}}, m_2^{v, \hat{m}})^*$ satisfies the following PDE system

$$\frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \operatorname{div}_x (g_i(x, \hat{m}_t, v_i(x)) m_i(x, t)) = 0, \quad (62)$$

with initial conditions $m_i(\cdot, 0) = \rho_{i,0}$.

(2) $\hat{m} = m^{\hat{v}, \hat{m}}$ is a solution to (62) controlled by \hat{v} .

The control problem appearing in the first point above is solved ignoring the parameters \hat{m}_1, \hat{m}_2 . Eventually the value of these parameters is defined a posteriori, by a fixed point argument, equaling these parameters to the solution of the FP equations (62). We can use this viewpoint to apply the CMFC considered in Section 2. Referring to the system (16) of HJB-FP equations, it turns out

that we recover the equations (59)–(60) with the same terminal and initial conditions. Hence the necessary optimality conditions of the two mean field games (Problems 4.1 and 4.3) have the same PDE system. In other words, we have two different interpretations for this PDE system. This similarity between the NMFG and the CMFG problems is not entirely surprising since, by looking at the definition of Problem 4.3, one realizes that the problem can be split into two sub-problems (one minimization problem for each component of v) which are independent because $\hat{m} = (\hat{m}_1, \hat{m}_2)^*$ is fixed.

Although, as described above, one can reuse the computations done in the CMFC setting to derive the PDE system of CMFG, the difference between the two PDE systems should be stressed: the HJB equation (59) does not involve derivatives of the Hamiltonians with respect to m_i , $i = 1, 2$, whereas in (16) such derivatives do appear.

Remark 4.4. Studying rigorously the corresponding N -player game is an important question which is beyond the scope of the present work. However, at least at a heuristic level, the CMFG problem described above can be viewed as the mean field limit of a game with a finite number of players in at least two ways. One could imagine that the state of each player has two components, each with its own dynamics, and the player chooses a control for each component in order to minimize her global cost (which depends on both components). One could also consider a game with two populations of equal size, say $(X_1^i)_{i=1,\dots,N}$ and $(X_2^i)_{i=1,\dots,N}$, where the players work in pairs: X_1^i and X_2^i help each other and compete with $((X_k^j)_{k=1,2})_{j \neq i}$. This would be a game representing competition between couples composed of one player from each population: the players collaborate among each pair but compete at a global level.

Remark 4.5. The adjoint equations (59) may also be deduced with an approach based on the master equation point of view. We omit the details.

5. Examples

5.1. Special cases

Let us start with some situations in which the system of master equations is not needed and one can work instead with a system of PDEs in finite dimension. For future reference (see also the examples below), we report here two cases of interest for many applications. To the best of our knowledge, the examples studied in the literature so far fall in one of these cases.

Special case 1: When deriving the necessary optimality conditions in the previous sections, we have looked for controls under the form of functions of both x and $m = (m_1, m_2)^*$. This corresponds to a situation where each player observes her individual state together with both distributions. If, instead, one considers a more restricted information structure according to which the agents do not have access to the distributions, then one should look for controls under the form of functions of x only, i.e. $x \mapsto v_i(x)$. In this case, the optimal controls, in partic-

ular, are not allowed to depend on m . Hence in (30)–(31) and in the analogous equations for u_2, u_{-2} , the terms $(\partial_{m_i} \hat{v}_j)_{i=1,2,j=1,2}$ vanish. We can thus write a self-contained PDE system for $u_1, u_{-1}, u_2, u_{-2}, m_1$, and m_2 in the NMFC setting. Indeed, the equations (43)–(44) for u_1, u_{-1} of NMFC become

$$\begin{aligned} & -\frac{\partial u_1}{\partial t}(x, t) + A_1 u_1(x, t) = \\ & = H_1(x, m_t, Du_1(x, t)) + \\ & \quad + \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t, Du_1(\xi, t))(x) m_1(\xi, t) d\xi \\ & \quad + \int_{\mathbb{R}^n} Du_{-1}(\xi, t) \partial_{m_1} \frac{\partial H_2}{\partial q_2}(\xi, m_t, D_\xi u_2(\xi, t))(x) m_2(\xi, t) d\xi, \end{aligned} \quad (63)$$

and

$$\begin{aligned} & -\frac{\partial u_{-1}}{\partial t}(x, t) + A_2 u_{-1}(x, t) = \\ & = D_x u_{-1}(x, t) \frac{\partial H_2}{\partial q_2}(x, m_t, D_x u_2(x, t)) + \\ & \quad + \int_{\mathbb{R}^n} \partial_{m_2} H_1(\xi, m_t, Du_1(\xi, t))(x) m_1(\xi, t) d\xi \\ & \quad + \int_{\mathbb{R}^n} Du_{-1}(\xi, t) \partial_{m_2} \frac{\partial H_2}{\partial q_2}(\xi, m_t, D_\xi u_2(\xi, t))(x) m_2(\xi, t) d\xi, \end{aligned} \quad (64)$$

with terminal conditions

$$\begin{aligned} u_1(x, T) &= h_1(x, m_T) + \int_{\mathbb{R}^n} \partial_{m_1} h_1(\xi, m_T)(x) m_1(\xi, T) d\xi \\ u_{-1}(x, T) &= \int_{\mathbb{R}^n} \partial_{m_2} h_1(\xi, m_T)(x) m_1(\xi, T) d\xi. \end{aligned}$$

For u_2 and u_{-2} similar equations hold, and the FP equations remain (47)–(48).

Special case 2: If the controls are functions of x only (as in the first special case above), and in addition g_1 does not depend upon m_2 and g_2 does not depend on m_1 , then the unknowns u_{-1} and u_{-2} become superfluous and we can write a self-contained PDE system for u_1, u_2, m_1, m_2 . Indeed, for NMFC, the equation (63) for u_1 simplifies further and we obtain

$$\begin{aligned} & -\frac{\partial u_1}{\partial t}(x, t) + A_1 u_1(x, t) \\ & = H_1(x, m_t, Du_1(x, t)) + \int_{\mathbb{R}^n} \partial_{m_1} H_1(\xi, m_t, Du_1(\xi, t))(x) m_1(\xi, t) d\xi, \end{aligned} \quad (65)$$

with terminal condition

$$u_1(x, T) = h_1(x, m_T) + \int_{\mathbb{R}^n} \partial_{m_1} h_1(\xi, m_T)(x) m_1(\xi, T) d\xi.$$

A similar equation holds for u_2 and the FP equations remain (47)–(48). Notice the difference between (65) and the corresponding equation for CMFC, namely (16).

5.2. Aversion in crowd motion

We briefly revisit examples of aversion in crowd motion, in the framework of this paper. For the FP equations, we let

$$A_i^* m^i = -\frac{\sigma^2}{2} \Delta m^i, \quad g_i(x, m, v_i) = v_i,$$

where $\sigma > 0$ is a constant. In other words, the drift is the control, and the diffusion is a standard diffusion with constant volatility σ . In particular, g_1 (resp. g_2) does not depend upon m_2 (resp. m_1).

Model of Lachapelle and Wolfram [44].

In [44], the authors considered costs of the form

$$f_i(x, m, v_i) = \chi(v_i) + \varphi_{i,\lambda}(m(x)), \quad h_i(x, m) = \psi_i(x),$$

$$\chi(v) = \frac{|v|^2}{2}, \quad \varphi_{1,\lambda}(\mu_1, \mu_2) = \mu_1 + \lambda\mu_2, \quad \varphi_{2,\lambda}(\mu_1, \mu_2) = \mu_2 + \lambda\mu_1,$$

where $\lambda > 0$ is a constant and $\psi_i: \mathbb{R}^n \rightarrow \mathbb{R}$. In f_i , the terms $\varphi_{i,\lambda}$ model aversion (of an agent towards its own population or the other population). Notice that these costs are *local* in m , in the sense that they depend only on the value of the density at the point x under consideration. The FP equation for m_i becomes:

$$\frac{\partial m_i}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta m_i(x, t) + \operatorname{div}_x (v_i(x, t) m_i(x, t)) = 0.$$

The Hamiltonians defined by (11) are

$$H_1(x, m, q_1) = -\frac{1}{2} |q_1|^2 + m_1(x) + \lambda m_2(x), \quad H_2(x, m, q_2) = -\frac{1}{2} |q_2|^2 + m_2(x) + \lambda m_1(x).$$

For the CMFC model, the adjoint equations are

$$-\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) = -\frac{|\nabla u_1(x)|^2}{2} + 2 [m_1(x) + \lambda m_2(x)]$$

$$-\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) = -\frac{|\nabla u_2(x)|^2}{2} + 2 [m_2(x) + \lambda m_1(x)],$$

whereas for the NMFC model, if the controls are allowed to depend only on x (as in [44, Proposition 4.1]; see also [41, Chapter 4, Proposition 4.2.1] and its proof), the adjoint equations are

$$-\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) = -\frac{|\nabla u_1(x)|^2}{2} + 2m_1(x) + \lambda m_2(x),$$

$$-\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) = -\frac{|\nabla u_2(x)|^2}{2} + 2m_2(x) + \lambda m_1(x).$$

We recover the equations found in [44, Proposition 4.1] and, as noticed by Lachapelle and Wolfram, the adjoint equations for CMFC and NMFC are equivalent up to multiplying λ by a constant.

For both CMFG and NMFG, the system of adjoint equations is given by

$$\begin{aligned} -\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) &= -\frac{|\nabla u_1(x)|^2}{2} + m_1(x) + \lambda m_2(x) \\ -\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) &= -\frac{|\nabla u_2(x)|^2}{2} + m_2(x) + \lambda m_1(x). \end{aligned}$$

If the controls are allowed to depend on both x and m , the necessary optimality conditions for NMFC are expressed in terms of master equations: the system (49)–(52) rewrites, in this setting:

$$\begin{aligned} &-\frac{\partial U_1}{\partial t}(x, m, t) - \frac{\sigma^2}{2} \Delta_x U_1(x, m, t) = \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} \frac{\sigma^2}{2} \Delta_\xi \partial_{m_j} U_1(\xi, m, t)(x) m_j(\xi) d\xi - \\ &\quad - \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_1(\xi, m, t)(x) \cdot D_\xi U_j(\xi, m, t) m_j(\xi) d\xi - \\ &\quad - \frac{1}{2} |D_x U_1(x, m, t)|^2 + 2m_1(x) + \lambda m_2(x) - \\ &\quad - \int_{\mathbb{R}^n} D U_{-1}(\xi, m, t) \partial_{m_1} D_\xi U_2(\xi, m, t)(x) m_2(\xi) d\xi, \end{aligned}$$

and

$$\begin{aligned} &-\frac{\partial U_{-1}}{\partial t}(x, m, t) - \frac{\sigma^2}{2} \Delta_x U_{-1}(x, m, t) = \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} \frac{\sigma^2}{2} \Delta_\xi \partial_{m_j} U_{-1}(\xi, m, t)(x) m_j(\xi) d\xi - \\ &\quad - \sum_{j=1}^2 \int_{\mathbb{R}^n} \partial_{m_j} D_\xi U_{-1}(\xi, m, t)(x) D_\xi U_j(\xi, m, t) m_j(\xi) d\xi - \\ &\quad - D_x U_{-1}(x, m, t) \cdot D_x U_2(x, m, t) + \lambda m_1(x) - \\ &\quad - \int_{\mathbb{R}^n} D U_{-1}(\xi, m, t) \partial_{m_2} D_\xi U_2(\xi, m, t)(x) m_2(\xi) d\xi, \end{aligned}$$

complemented with terminal conditions and analogous equations for U_2, U_{-2} .

Model of Aurell and Djehiche [6].

In [6], the authors considered a variant of the above model with *non-local* running cost of the form

$$f_i(x, m, v_i) = \chi(v_i) + \Phi_{i,\Lambda}(x, m),$$

$$\chi(v) = \frac{|v|^2}{2}, \quad \Phi_{i,\Lambda}(x, m) = \sum_{k=1}^2 \Lambda_{i,k} \phi * \mu_k(x),$$

where $\Lambda_{i,k} \geq 0$ are constants, $*$ denotes the convolution and $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function such as

$$\phi(x) = \gamma_\delta * \mathbb{I}_{B_r}(x), \quad \gamma_\delta(x) = \gamma(x/\delta)/\delta, \tag{66}$$

where $\delta > 0$, γ is a mollifier, and \mathbb{I}_{B_r} is the indicator function of the ball with radius r centered at 0 normalized by the volume of this ball. Here, $\Phi_{i,\Lambda}$ models aversion. For the sake of comparison, let us consider the case where the controls are functions of x and do not depend upon m (see [6], Assumption (C4) p.444). One can check that this example also falls in the second special case of Section 5.1.

For the CMFC model, the adjoint equations are

$$-\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) =$$

$$= -\frac{|\nabla u_1(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{1,k} \phi * m_{k,t}(x) + \sum_{j=1}^2 \Lambda_{j,1} \bar{\phi} * m_{j,t}(x)$$

$$-\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) =$$

$$= -\frac{|\nabla u_2(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{2,k} \phi * m_{k,t}(x) + \sum_{j=1}^2 \Lambda_{j,2} \bar{\phi} * m_{j,t}(x),$$

whereas for the NMFC model, the adjoint equations are

$$-\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) =$$

$$= -\frac{|\nabla u_1(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{1,k} \phi * m_{k,t}(x) + \Lambda_{1,1} \bar{\phi} * m_{1,t}(x)$$

$$-\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) =$$

$$= -\frac{|\nabla u_2(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{2,k} \phi * m_{k,t}(x) + \Lambda_{2,2} \bar{\phi} * m_{2,t}(x),$$

where $\bar{\phi}(x) = \phi(-x)$. Similarly to the example of [44] and as noticed in [6], the NMFC system can be seen as a CMFC system (up to changing the coefficients $\Lambda_{i,k}$ by a multiplicative constant) if one assumes that $\Lambda_{1,2} = \Lambda_{2,1}$ and ϕ is even (which is the case with (66) for example).

For both CMFG and NMFG, the system of adjoint equations is given by

$$\begin{aligned} -\frac{\partial u_1}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_1(x, t) &= -\frac{|\nabla u_1(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{1,k} \phi * m_{k,t}(x) \\ -\frac{\partial u_2}{\partial t}(x, t) - \frac{\sigma^2}{2} \Delta u_2(x, t) &= -\frac{|\nabla u_2(x)|^2}{2} + \sum_{k=1}^2 \Lambda_{2,k} \phi * m_{k,t}(x). \end{aligned}$$

Remark 5.1. Here again, if the controls were allowed to depend on both x and m , the necessary optimality conditions for NMFC would be expressed in terms of the master equations (49)–(52).

5.3. Linear-quadratic models

In this section, we consider the linear-quadratic setting. In the case of a single population, we refer the reader to, e.g., the papers [11, 37, 38], the monographs [8] (Chapter 6) and [14] (Section 3.5), as well as the references therein. In the case of several populations, a model with a finite number of agents has been studied in [5]. Here we focus on the limit mean field models, in the different cases of interactions introduced above.

In the sequel, for $m \in L^2(\mathbb{R}^n)^2$, we denote by $\bar{m}_k = \int x m_k(x) dx$ the first moment of m_k . We will denote by $\dot{\varphi}$ the time derivative of a function φ .

We consider, for $x \in \mathbb{R}^n$, $m = (m_1, m_2) \in L^2(\mathbb{R}^n)^2$, $v_i \in \mathbb{R}^d$, drift and cost functions of the form:

$$g_i(x, m, v_i) = \mathbf{A}^i x + \sum_{j=1}^2 \bar{\mathbf{A}}_j^i \bar{m}_j + \mathbf{B}^i v_i \quad (67)$$

$$f_i(x, m, v_i) = \frac{1}{2} \left[x^* \mathbf{Q}^i x + (v_i)^* \mathbf{R}^i v_i + \sum_{j=1}^2 (x - \mathbf{S}_j^i \bar{m}_j)^* \bar{\mathbf{Q}}_j^i (x - \mathbf{S}_j^i \bar{m}_j) \right] \quad (68)$$

$$h_i(x, m) = \frac{1}{2} \left[x^* \mathbf{Q}_T^i x + \sum_{j=1}^2 (x - \mathbf{S}_{T,j}^i \bar{m}_j)^* \bar{\mathbf{Q}}_{T,j}^i (x - \mathbf{S}_{T,j}^i \bar{m}_j) \right], \quad (69)$$

where \mathbf{A}^i , $\bar{\mathbf{A}}_j^i$, and \mathbf{B}^i are bounded deterministic matrix-valued functions in time

of suitable sizes, \mathbf{S}_j^i (respectively \mathbf{Q}^i , $\bar{\mathbf{Q}}_j^i$ and \mathbf{R}^i) are bounded deterministic (respectively, non-negative and positive definite) matrix-valued functions in time of suitable sizes. Except for $t = T$, we omit the dependence on time to alleviate the notations. Moreover M^* denotes the transpose of a matrix M .

In this setting the Fokker-Planck equation (3) rewrites

$$\begin{aligned} & \frac{\partial m_i}{\partial t}(x, t) + A_i^* m_i(x, t) + \\ & + \operatorname{div}_x \left(\left[\mathbf{A}^i x + \sum_{j=1}^2 \overline{\mathbf{A}}_j^i \overline{m}_{j,t} + \mathbf{B}^i v_i(x, t) \right] m_i(x, t) \right) = 0. \end{aligned} \tag{70}$$

Common mean field control problem. We first investigate the CMFC problem. We look for adjoint states of the form

$$u_i(x, t) = \frac{1}{2} x^* P_t^i x + x^* \nu_t^i + \tau_t^i. \tag{71}$$

We have $Du_i(x, t) = P_t^i x + \nu_t^i$.

The Hamiltonian (11) rewrites, for $x \in \mathbb{R}^n, m = (m_1, m_2)^* \in L^2(\mathbb{R}^n)^2, q_i \in \mathbb{R}^n$, as

$$\begin{aligned} H_i(x, m, q_i) &= q_i^* \left[\mathbf{A}^i x + \sum_{j=1}^2 \overline{\mathbf{A}}_j^i \overline{m}_j \right] - \frac{1}{2} q_i^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* q_i \\ &+ \frac{1}{2} \left[x^* \mathbf{Q}^i x + \sum_{j=1}^2 (x - \mathbf{S}_j^i \overline{m}_j)^* \overline{\mathbf{Q}}_j^i (x - \mathbf{S}_j^i \overline{m}_j) \right], \end{aligned} \tag{72}$$

since $\hat{v}_i(x, m, q_i) = -(\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* q_i$. Using (70) and taking into account the expression of \hat{v} , the first moments \overline{m}_i of m_i solve the system of ODEs

$$\dot{\overline{m}}_{i,t} = \mathbf{A}^i \overline{m}_{i,t} + \sum_{j=1}^2 \overline{\mathbf{A}}_j^i \overline{m}_{j,t} - \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* P_t^i \overline{m}_{i,t} - \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* \nu_t^i, \tag{73}$$

and the initial condition $\overline{m}_i(0) = \int x \rho_{i,0}(x) dx$. We see that the integral term in (16) rewrites

$$\begin{aligned} & \int \sum_{j=1}^2 \partial_{m_i} H_j(\xi, m, Du_j(\xi, t))(x) m_j(\xi, t) d\xi = \\ & = \sum_{j=1}^2 \left[(P_t^j \overline{m}_{j,t} + \nu_t^j)^* \overline{\mathbf{A}}_i^j - (\overline{m}_{j,t} - \mathbf{S}_i^j \overline{m}_{i,t})^* \overline{\mathbf{Q}}_i^j \mathbf{S}_i^j \right] x. \end{aligned} \tag{74}$$

Replacing u_i by its expression (71) in the adjoint equation (16), we obtain that $(P^1, P^2), (\nu^1, \nu^2)$ and (τ^1, τ^2) solve the following system of ODEs, which is coupled with the equations on the first moments (73),

$$\dot{P}_t^i + (P_t^i)^* \mathbf{A}^i + (\mathbf{A}^i)^* P_t^i - (P_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* P_t^i + \mathbf{Q}^i + \sum_{j=1}^2 \overline{\mathbf{Q}}_j^i = 0,$$

$$\begin{aligned}
-\dot{\nu}_t^i &= \left[(\mathbf{A}^i)^* - (P_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* \right] \nu_t^i + \\
&\quad + \frac{1}{2} \sum_{j=1}^2 \left[(P_t^i)^* \bar{\mathbf{A}}_j^i + (\bar{\mathbf{A}}_j^i)^* P_t^i - \bar{\mathbf{Q}}_j^i \mathbf{S}_j^i - (\mathbf{S}_j^i)^* \bar{\mathbf{Q}}_j^i \right] \bar{m}_{j,t} + \\
&\quad + \sum_{k=1}^2 \left[(\bar{\mathbf{A}}_i^k)^* \left(P_t^k \bar{m}_{k,t} + \nu_t^k \right) - (\mathbf{S}_i^k)^* \bar{\mathbf{Q}}_i^k \left(\bar{m}_{k,t} - \mathbf{S}_i^k \bar{m}_{i,t} \right) \right], \\
-\dot{\tau}_t^i &= \text{tr } a^i P_t^i + \sum_{j=1}^2 (\nu_t^i)^* \bar{\mathbf{A}}_j^i \bar{m}_{j,t} - \frac{1}{2} (\nu_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* \nu_t^i + \\
&\quad + \frac{1}{2} \sum_{j=1}^2 \left(\mathbf{S}_j^i \bar{m}_{j,t} \right)^* \bar{\mathbf{Q}}_j^i \left(\mathbf{S}_j^i \bar{m}_{j,t} \right),
\end{aligned}$$

with the terminal conditions

$$\begin{aligned}
P_T^i &= \mathbf{Q}_T^i + \sum_{j=1}^2 \bar{\mathbf{Q}}_{T,j}^i \\
\nu_T^i &= - \sum_{j=1}^2 \bar{\mathbf{Q}}_{T,j}^i \mathbf{S}_{T,j}^i \bar{m}_{j,T} - \sum_{k=1}^2 (\mathbf{S}_{T,i}^k)^* \bar{\mathbf{Q}}_{T,i}^k \left(\bar{m}_{k,T} - \mathbf{S}_{T,i}^k \bar{m}_{i,T} \right) \\
\tau_T^i &= \frac{1}{2} \sum_{j=1}^2 \left(\mathbf{S}_{T,j}^i \bar{m}_{j,T} \right)^* \bar{\mathbf{Q}}_{T,j}^i \mathbf{S}_{T,j}^i \bar{m}_{j,T}.
\end{aligned}$$

We can relate this system of ODEs to a Riccati equation as follows. Let us look for K^i such that

$$K_t^i \bar{m}_{i,t} = \int_{\mathbb{R}^n} Du_i(x, t) m_i(x, t) dx. \quad (75)$$

Taking the derivative (with respect to time) on both sides of the above equality, using that $\nu^i = (K^i - P^i) \bar{m}_i$, and using the equations for \dot{P}_t^i and $\dot{\nu}_t^i$ yields

$$\begin{aligned}
\dot{K}_t^i \bar{m}_{i,t} &= \left[K_t^i \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* K_t^i - K_t^i \left(\mathbf{A}^i + \bar{\mathbf{A}}_i^i \right) - \left(\mathbf{A}^i + \bar{\mathbf{A}}_i^i \right)^* K_t^i - \mathbf{Q}^i - \right. \\
&\quad \left. - \sum_{j=1}^2 \bar{\mathbf{Q}}_j^i + \bar{\mathbf{Q}}_i^i \mathbf{S}_i^i + (\mathbf{S}_i^i)^* \bar{\mathbf{Q}}_i^i - \sum_{j=1}^2 (\mathbf{S}_i^j)^* \bar{\mathbf{Q}}_i^j \mathbf{S}_i^j \right] \bar{m}_{i,t} + \\
&\quad + \left[-K_t^i \bar{\mathbf{A}}_{-i}^i - \bar{\mathbf{A}}_{-i}^i K_t^i + (\mathbf{S}_{-i}^i)^* \bar{\mathbf{Q}}_{-i}^i + \bar{\mathbf{Q}}_{-i}^i \mathbf{S}_{-i}^i \right] \bar{m}_{-i}(t).
\end{aligned}$$

This system of equations can be synthetically written under the following form, which turns out to be a *symmetric* Riccati equation

$$\dot{K}_t = K_t \mathbf{B} K_t - (\mathbf{A} + \bar{\mathbf{A}})^* K_t - K_t (\mathbf{A} + \bar{\mathbf{A}}) + \mathbf{G}, \quad K_T = \mathbf{G}_T,$$

where
$$K = \begin{pmatrix} K^1 & 0 \\ 0 & K^2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}^1(\mathbf{R}^1)^{-1}(\mathbf{B}^1)^* & 0 \\ 0 & \mathbf{B}^2(\mathbf{R}^2)^{-1}(\mathbf{B}^2)^* \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^1 & 0 \\ 0 & \mathbf{A}^2 \end{pmatrix}, \quad \bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{A}}_1^{-1} & \bar{\mathbf{A}}_2^{-1} \\ \bar{\mathbf{A}}_1^{-2} & \bar{\mathbf{A}}_2^{-2} \end{pmatrix}, \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} \mathbf{G}_1^1 & \mathbf{G}_1^2 \\ \mathbf{G}_2^1 & \mathbf{G}_2^2 \end{pmatrix} \quad (76)$$

and
$$\mathbf{G}_i^i = -\mathbf{Q}^i - \sum_{j=1}^2 \bar{\mathbf{Q}}_j^i + \bar{\mathbf{Q}}_i^i \mathbf{S}_i^i + (\mathbf{S}_i^i)^* \bar{\mathbf{Q}}_i^i - \sum_{j=1}^2 (\mathbf{S}_i^j)^* \bar{\mathbf{Q}}_i^j \mathbf{S}_i^j$$

$$\mathbf{G}_i^{-i} = (\mathbf{S}_i^{-i})^* \bar{\mathbf{Q}}_i^{-i} + \bar{\mathbf{Q}}_{-i}^i \mathbf{S}_{-i}^i.$$

and, for the final condition, \mathbf{G}_T is defined similarly.

Mean field game

We focus on the system (59)–(60). We look for u_i under the form

$$u_i(x, t) = \frac{1}{2} x^* P_t^i x + x^* \nu_t^i + \tau_t^i. \quad (77)$$

Following the same approach as above for the CMFC problem and noting that the integral term (74) does not appear in the adjoint equation (59) for CMFG, we obtain that \bar{m} satisfies (73) and P^i, ν^i, τ^i satisfy the system of ODEs

$$\begin{aligned} \dot{P}_t^i + (P_t^i)^* \mathbf{A}^i + (\mathbf{A}^i)^* P_t^i - (P_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* P_t^i + \mathbf{Q}^i + \sum_{j=1}^2 \bar{\mathbf{Q}}_j^i &= 0, \\ -\dot{\nu}_t^i &= [(\mathbf{A}^i)^* - (P_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^*] \nu_t^i + \\ &+ \frac{1}{2} \sum_{j=1}^2 [(P_t^i)^* \bar{\mathbf{A}}_j^i + (\bar{\mathbf{A}}_j^i)^* P_t^i - \bar{\mathbf{Q}}_j^i \mathbf{S}_j^i - (\mathbf{S}_j^i)^* \bar{\mathbf{Q}}_j^i] \bar{m}_{j,t}, \\ -\dot{\tau}_t^i &= \text{tr } a^i P_t^i + \sum_{j=1}^2 (\nu_t^i)^* \bar{\mathbf{A}}_j^i \bar{m}_{j,t} - \frac{1}{2} (\nu_t^i)^* \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* \nu_t^i + \\ &+ \frac{1}{2} \sum_{j=1}^2 (\mathbf{S}_j^i \bar{m}_{j,t})^* \bar{\mathbf{Q}}_j^i (\mathbf{S}_j^i \bar{m}_{j,t}), \end{aligned}$$

with the terminal conditions

$$P_T^i = \mathbf{Q}_T^i + \sum_{j=1}^2 \bar{\mathbf{Q}}_{T,j}^i, \quad \nu_T^i = - \sum_{j=1}^2 \bar{\mathbf{Q}}_{T,j}^i \mathbf{S}_{T,j}^i \bar{m}_T^j,$$

and
$$\tau_T^i = \frac{1}{2} \sum_{j=1}^2 (\mathbf{S}_{T,j}^i \bar{m}_T^j)^* \bar{\mathbf{Q}}_{T,j}^i \mathbf{S}_{T,j}^i \bar{m}_T^j.$$

Here again, we can relate this system of ODEs to a Riccati equation as follows. Let us look for K^i such that

$$K_t^i \bar{m}_{i,t} = \int_{\mathbb{R}^n} Du_i(x, t) m_i(x, t) dx.$$

Taking the derivative (with respect to time) on both sides of the above equality and using the equations for P_t^i and ν_t^i yields

$$\begin{aligned} \dot{K}_t^i \bar{m}_{i,t} = & \left[K_t^i \mathbf{B}^i (\mathbf{R}^i)^{-1} (\mathbf{B}^i)^* K_t^i - K_t^i \mathbf{A}^i - (\mathbf{A}^i)^* K_t^i - K_t^i \bar{\mathbf{A}}_i^i - \right. \\ & \left. - \mathbf{Q}^i - \sum_{j=1}^2 \bar{\mathbf{Q}}_j^i + \bar{\mathbf{Q}}_i^i \mathbf{S}_i^i \right] \bar{m}_{i,t} - \left[K_t^i \bar{\mathbf{A}}_{-i}^i + \bar{\mathbf{Q}}_{-i}^i \mathbf{S}_{-i}^i \right] \bar{m}_t^{-i}. \end{aligned}$$

This system of equations can be written under the following compact form which turns out to be a *non-symmetric* Riccati equation

$$\dot{K}_t = K_t \mathbf{B} K_t - \mathbf{A}^* K_t - K_t \mathbf{A} - K_t \bar{\mathbf{A}} + \tilde{\mathbf{G}}, \quad K_T = \tilde{\mathbf{G}}_T, \quad (78)$$

where \mathbf{B} , \mathbf{A} , and $\bar{\mathbf{A}}$ are defined by (76), and

$$\tilde{\mathbf{G}} = \begin{pmatrix} \tilde{\mathbf{G}}_1^1 & \tilde{\mathbf{G}}_1^2 \\ \tilde{\mathbf{G}}_2^1 & \tilde{\mathbf{G}}_2^2 \end{pmatrix}$$

with

$$\begin{aligned} \tilde{\mathbf{G}}_i^i &= -\mathbf{Q}^i - \sum_{j=1}^2 \bar{\mathbf{Q}}_j^i + \frac{1}{2} \left(\bar{\mathbf{Q}}_i^i \mathbf{S}_i^i + (\mathbf{S}_i^i)^* \bar{\mathbf{Q}}_i^i \right), \\ \tilde{\mathbf{G}}_i^{-i} &= \frac{1}{2} \left(\bar{\mathbf{Q}}_{-i}^i \mathbf{S}_{-i}^i + (\mathbf{S}_{-i}^i)^* \bar{\mathbf{Q}}_{-i}^i \right), \end{aligned}$$

and, for the final condition, $\tilde{\mathbf{G}}_T$ is defined similarly.

Remark 5.2. In particular, by comparing (75) and (78), one can see that linear-quadratic CMFC and CMFG are in general different. One can also check that the NMFC problem provides yet a different system of ODEs since this LQ model does not fall in the second special case of Section 5.1. Details are omitted here and a study of this LQ setting with a comparison of all the ODE systems will be done elsewhere.

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