

An Ergodic Problem for Mean Field Games: Qualitative Properties and Numerical Simulations

Simone Cacace

*Dip. di Matematica e Fisica, Università degli Studi Roma Tre,
Largo San Leonardo Murialdo 1, 00146 Roma, Italy
cacace@mat.uniroma3.it*

Fabio Camilli

*Dip. di Scienze di Base e Applicate per l'Ingegneria,
Università di Roma "Sapienza", Via Scarpa 16, 00161 Roma, Italy
camilli@dmmm.uniroma1.it*

Annalisa Cesaroni

*Dip. di Scienze Statistiche, Università di Padova,
Via Cesare Battisti 241-243, 35121 Padova, Italy
annalisa.cesaroni@unipd.it*

Claudio Marchi

*Dip. di Ingegneria dell'Informazione, Università di Padova,
Via Gradenigo 6/B, 35131 Padova, Italy
claudio.marchi@unipd.it*

Received: January 26, 2018

Accepted: July 20, 2018

This paper is devoted to some qualitative descriptions and some numerical results for ergodic Mean Field Games systems which arise, e.g., in the homogenization with a small noise limit. We shall consider either power type potentials or logarithmic type ones. In both cases, we shall establish some qualitative properties of the effective Hamiltonian \bar{H} and of the effective drift \bar{b} . In particular we shall provide two cases where the effective system keeps/looses the Mean Field Games structure, namely where $\nabla_P \bar{H}(P, \alpha)$ coincides or not with $\bar{b}(P, \alpha)$.

On the other hand, we shall provide some numerical tests validating the aforementioned qualitative properties of \bar{H} and \bar{b} . In particular, we provide a numerical estimate of the discrepancy $\nabla_P \bar{H}(P, \alpha) - \bar{b}(P, \alpha)$.

Keywords: Mean field games, periodic homogenization, small noise limit, ergodic problems, continuous dependence of solution on parameters, finite difference schemes.

2010 Mathematics Subject Classification: 35B27, 35B30, 35B40, 35K40, 35K59, 65M06, 91A13.

1. Introduction

This paper is devoted to some qualitative descriptions and some numerical results of ergodic problems arising in the homogenization of Mean Field Games (briefly,

MFG) in the small noise limit. Our starting point is the homogenization result stated in [13]. In that paper, the authors tackle the asymptotic behaviour as $\epsilon \rightarrow 0$ of the solution to

$$\begin{cases} -u_t^\epsilon - \epsilon \Delta u^\epsilon + \frac{1}{2} |\nabla u^\epsilon|^2 = V\left(\frac{x}{\epsilon}, m^\epsilon\right), & x \in \mathbb{R}^n, t \in (0, T) \\ m_t^\epsilon - \epsilon \Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla u^\epsilon) = 0, & x \in \mathbb{R}^n, t \in (0, T) \end{cases} \quad (1)$$

with initial/terminal condition $u^0(x, T) = u_0(x)$ and $m^0(x, 0) = m_0(x)$ where the potential $V(y, m)$ is 1-periodic in y , increasing in m , C^1 and *bounded*. It turns out that (u^ϵ, m^ϵ) converge (in a suitable sense) to the solution of the effective Cauchy problem

$$\begin{cases} -u_t^0 + \bar{H}(\nabla u^0, m^0) = 0, & x \in \mathbb{R}^n, t \in (0, T) \\ m_t^0 - \operatorname{div}(m^0 \bar{b}(\nabla u^0, m^0)) = 0, & x \in \mathbb{R}^n, t \in (0, T) \end{cases} \quad (2)$$

with the same initial/terminal condition. The effective operators $\bar{H}(P, \alpha)$ and $\bar{b}(P, \alpha)$ are obtained as follows: for every $P \in \mathbb{R}^n$ and $\alpha \geq 0$, the value $\bar{H}(P, \alpha)$ is the (unique) constant for which there exists a solution to the ergodic MFG system:

$$\begin{cases} \text{(i)} & -\Delta u + \frac{1}{2} |\nabla u + P|^2 - V(y, \alpha m) = \bar{H}(P, \alpha), & y \in \mathbb{T}^n \\ \text{(ii)} & -\Delta m - \operatorname{div}(m(\nabla u + P)) = 0, & y \in \mathbb{T}^n \\ \text{(iii)} & \int_{\mathbb{T}^n} u = 0 \quad \int_{\mathbb{T}^n} m = 1, \end{cases} \quad (3)$$

while \bar{b} is given by
$$\bar{b}(P, \alpha) = \int_{\mathbb{T}^n} (\nabla u + P) m dy, \quad (4)$$

where (u, m) is the solution to (3). Moreover the following relation holds true

$$\frac{\partial H}{\partial P_i}(P, \alpha) = \bar{b}_i(P, \alpha) - \alpha \int_{\mathbb{T}^n} V_m(y, \alpha m) \tilde{m}_i m dy \quad (5)$$

where $V_m = \partial V / \partial m$ and $(\tilde{u}_i, \tilde{m}_i, \frac{\partial H}{\partial P_i}(P, \alpha)) \in C^{2,\gamma} \times W^{1,p} \times \mathbb{R}$ is the solution of the ergodic problem

$$\begin{cases} \text{(i)} & -\Delta \tilde{u}_i + \nabla \tilde{u}_i \cdot (\nabla u + P) + (\nabla u + P) \cdot e_i - V_m(y, \alpha m) \alpha \tilde{m}_i = \frac{\partial H}{\partial P_i}(P, \alpha) \\ \text{(ii)} & -\Delta \tilde{m}_i - \operatorname{div}((P + \nabla u) \tilde{m}_i) = \operatorname{div}(m(\nabla \tilde{u}_i + e_i)) \\ \text{(iii)} & \int_{\mathbb{T}^n} \tilde{m}_i = \int_{\mathbb{T}^n} \tilde{u}_i = 0. \end{cases} \quad (6)$$

It is worth to observe that, for the homogenization of a MFG system with finite noise, the cell problem is *decoupled*, see [13, Sect.3.1.2]; hence, from a mathematical point of view, the most interesting case is the small noise one as in (1).

Let us recall that the ergodic system (3) is not related only to the homogenization problem; actually, it has its own interpretation as the MFG system when agents

pay an infinite horizon cost (see [3, 20, 21]) and it arises also in the study of the asymptotic behaviour as $t \rightarrow +\infty$ of evolutive MFG system (see [9, 10, 11]). For a general overview on MFG systems and on their applications, we refer the reader to the monographs [1, 4, 7, 8, 12, 15, 17, 19] and references therein.

The aim of this paper is to study the ergodic systems (3) and (6) under more general assumptions, especially we shall consider *unbounded* potential of the form

- (i) Power type nonlinearities: $V(y, m) = v(y) + m^q$ for some $q > 0$;
- (ii) Logarithmic nonlinearities: $V(y, m) = v(y) + \log m$

where $v: \mathbb{T}^n \rightarrow \mathbb{R}$ is Lipschitz continuous and 1-periodic. Without any loss of generality, we shall assume that $v \geq 0$. Let us stress that our methods apply to regular solutions to (3), that is $(u, m) \in C^{2,\gamma}(\mathbb{T}^n) \times W^{1,p}(\mathbb{T}^n)$, for all $\gamma \in (0, 1)$ and for all $p > 1$. As a matter of fact, by the regularity of the Hamiltonian, under our assumptions, the solution to (3) belongs to $C^{2,\gamma}(\mathbb{T}^n) \times C^{2,\gamma}(\mathbb{T}^n)$, for all $\gamma \in (0, 1)$.

An interesting feature is that the limit system (2) may lose the MFG structure, namely $\nabla_P \bar{H}(P, \alpha)$ may not coincide with $\bar{b}(P, \alpha)$. This feature was already pointed out in [13, Sect.6.1] for the potential $V(y, m) = v(y) + m$; in Corollary 2.3 we shall extend it to more general cases of power type nonlinearities. On the other hand, in Theorem 3.2 we shall show $\nabla_P \bar{H}(P, \alpha) = \bar{b}(P, \alpha)$ for MFG systems with logarithmic potentials. We will finally explore these results numerically, in particular by computing the discrepancy $\nabla_P \bar{H} - \bar{b}$.

2. Mean field games with power nonlinearities

In this section we consider potential of the form $V(y, m) = v(y) + m^q$ with $q > 0$. In this case, the ergodic problem (3) reads

$$\begin{cases} \text{(i)} & -\Delta u + \frac{|\nabla u + P|^2}{2} - v(y) - \alpha^q m^q = \bar{H}(P, \alpha), & y \in \mathbb{T}^n \\ \text{(ii)} & -\Delta m - \operatorname{div}(m(\nabla u + P)) = 0, & y \in \mathbb{T}^n \\ \text{(iii)} & \int_{\mathbb{T}^n} u = 0 \quad \int_{\mathbb{T}^n} m = 1. \end{cases} \tag{7}$$

Throughout this section we shall assume

$$q > 0 \quad \text{if } n \leq 4, \quad 0 < q \leq \frac{2}{n-2} \quad \text{if } n > 4.$$

We collect some qualitative properties of the effective operators.

Proposition 2.1. *There hold*

- (i) For every $P \in \mathbb{R}^n$, $\alpha \geq 0$ there exists a unique constant $\bar{H}(P, \alpha)$ such that (7) admits a solution (u, m) . Moreover this solution is unique, $(u, m) \in C^{2,\gamma}(\mathbb{T}^n) \times W^{1,p}(\mathbb{T}^n)$ for every $\gamma \in (0, 1)$ and $p > 1$ and $m > 0$.
- (ii) $\bar{H}(P, \alpha)$ is decreasing in α .

- (iii) $\bar{H}(P, \alpha)$ is coercive in P , with quadratic growth, in particular there exists a constant $R_q \geq 1$ depending only on q such that

$$\frac{|P|^2}{2} - (1 + R_q) \int_{\mathbb{T}^n} v(y) dy - R_q \alpha^q \leq \bar{H}(P, \alpha) \leq \frac{|P|^2}{2} - \alpha^q.$$

- (iv) There hold $\lim_{|P| \rightarrow +\infty} \frac{\bar{H}(P, \alpha)}{|P|^2} = \frac{1}{2}$ and $\lim_{|P| \rightarrow +\infty} \frac{|\bar{b}(P, \alpha) - P|}{|P|} = 0$,

locally uniformly for $\alpha \in [0, +\infty)$.

Proof. (i) This existence result can be found in [16, Thm 7.1], [14, Theorem 1.4], see also [22] and [12].

- (ii) Following the same argument as in [13, Proposition 3] we get for $\alpha_1, \alpha_2 \geq 0$,

$$(\bar{H}(P, \alpha_1) - \bar{H}(P, \alpha_2))(\alpha_1 - \alpha_2) = - \int_{\mathbb{T}^n} (m_1^q - m_2^q)(m_1 - m_2) dy$$

where m_1 and m_2 are respectively the solution to (7) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Since $q > 0$, we get the statement.

- (iii) We multiply equation (i) in (7) by m , we integrate and we get, recalling the periodicity assumptions and that m has mean 1,

$$\bar{H}(P, \alpha) = \int_{\mathbb{T}^n} -m \Delta u + \frac{1}{2} |\nabla u + P|^2 m - \alpha^q m^{q+1} - v(y) m dy.$$

We multiply equation (ii) in (7) by u , integrate, and subtract the previous to get

$$\bar{H}(P, \alpha) = \int_{\mathbb{T}^n} \frac{1}{2} (|P|^2 - |\nabla u|^2) m - \alpha^q m^{q+1} - v(y) m dy. \quad (8)$$

Therefore, recalling that $v \geq 0$, and that by Jensen inequality

$$\int_{\mathbb{T}^n} m^q dy \geq \left(\int_{\mathbb{T}^n} m dy \right)^q = 1,$$

we get $\bar{H}(P, \alpha) \leq \frac{|P|^2}{2} - \alpha^q$. Integrating the first equation in (7), we obtain

$$\begin{aligned} \bar{H}(P, \alpha) &= \int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2} - v(y) - \alpha^q m^q dy \\ &= \frac{|P|^2}{2} + \int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2} - v(y) - \alpha^q m^q dy. \end{aligned} \quad (9)$$

From this, using (8), recalling that m has mean 1, we get

$$\int_{\mathbb{T}^n} \frac{1}{2} |\nabla u|^2 (m + 1) + (m - 1)(\alpha^q m^q + v(y)) dy = 0. \quad (10)$$

By monotonicity of $m \mapsto \alpha^q m^q$ on $m \geq 0$, we observe that

$$\alpha^q m^q (m - 1) \geq \alpha^q (m - 1) \tag{11}$$

and then we conclude from (10) that

$$\int_{\mathbb{T}^n} \frac{1}{2} |\nabla u|^2 (m + 1) dy \leq - \int_{\mathbb{T}^n} (m - 1) v(y) dy \leq \int_{\mathbb{T}^n} v(y) dy. \tag{12}$$

Integrating (11), we get $\int_{\mathbb{T}^n} m^q dy \leq \int_{\mathbb{T}^n} m^{q+1} dy$. Hence, in (10), we obtain

$$0 \leq \int_{\mathbb{T}^n} \alpha^q m^{q+1} - \alpha^q m^q dy \leq \int_{\mathbb{T}^n} v(y) (1 - m) dy \leq \int_{\mathbb{T}^n} v(y) dy.$$

Therefore, by applying Hölder inequality, we obtain

$$\alpha^q \int_{\mathbb{T}^n} m^q dy \leq \alpha^q \int_{\mathbb{T}^n} m^{q+1} dy \leq \alpha^q \left(\int_{\mathbb{T}^n} m^{q+1} dy \right)^{\frac{q}{q+1}} + \int_{\mathbb{T}^n} v(y) dy.$$

Let us observe that this inequality can be written as $A \leq B + A^{\frac{q}{q+1}}$. So, if we choose $R_q > 1$ such that $R_q \geq 1 + R_q^{\frac{q}{q+1}}$, then the solutions to the inequality satisfies $A \leq R_q(B + 1)$. Hence, we conclude that

$$\alpha^q \int_{\mathbb{T}^n} m^q \leq \alpha^q \int_{\mathbb{T}^n} m^{q+1} \leq R_q \left(\int_{\mathbb{T}^n} v(y) dy + \alpha^q \right). \tag{13}$$

Replacing this inequality in (9), we obtain

$$\bar{H}(P, \alpha) \geq \frac{|P|^2}{2} - (1 + R_q) \int_{\mathbb{T}^n} v(y) dy - R_q \alpha^q.$$

(iv) The first limit is a direct consequence of item (iii). By definition (4), we get

$$\bar{b}(P, \alpha) - P = \int_{\mathbb{T}^n} \nabla u(y) m dy.$$

By Hölder's inequality and (12) we get

$$\int_{\mathbb{T}^n} |\nabla u(y)| m dy \leq \left(\int_{\mathbb{T}^n} |\nabla u|^2 m dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} m dy \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{T}^n} v(y) dy \right)^{\frac{1}{2}}.$$

This completes the proof. □

Using the properties stated in Proposition 2.1, we obtain regularity properties of the effective operators, and a relation between them.

Theorem 2.2. *The maps $(P, \alpha) \rightarrow \bar{H}(P, \alpha)$, $\bar{b}(P, \alpha)$ are locally Lipschitz continuous and admit partial derivatives everywhere. Moreover, there hold*

$$\nabla_P \bar{H}(P, \alpha) = \bar{b}(P, \alpha) - \frac{q}{q+1} \alpha^q \nabla_P \|m\|_{L^{q+1}}^{q+1} \quad (14)$$

$$\frac{\partial \bar{b}}{\partial P_i} \cdot e_i \geq 0. \quad (15)$$

Proof. The proof of the local Lipschitz continuity of \bar{H} follows the same arguments of [13, Proposition 5] by using the a priori bounds on $m, \nabla u$ obtained in the previous proof, in particular (12) and (13).

We fix $P \in \mathbb{R}^n$, $\alpha > 0$; let (u, m) be the solution to (7) associated to (P, α) . We introduce the following system for all $i = 1, \dots, n$

$$\begin{cases} \text{(i)} & -\Delta \tilde{u}_i + \nabla \tilde{u}_i \cdot (\nabla u + P) + (\nabla u + P) \cdot e_i - q\alpha^q m^{q-1} \tilde{m}_i = c_i(P, \alpha) \\ \text{(ii)} & -\Delta \tilde{m}_i - \operatorname{div}((P + \nabla u)\tilde{m}_i) = \operatorname{div}(m(\nabla \tilde{u}_i + e_i)) \\ \text{(iii)} & \int_{\mathbb{T}^n} \tilde{m}_i = \int_{\mathbb{T}^n} \tilde{u}_i = 0. \end{cases} \quad (16)$$

Arguing as in [13, Lemma 6], we get that for every i there exists a unique $c_i(P, \alpha)$ such that this system admits a solution $(\tilde{u}_i, \tilde{m}_i)$, which is regular and unique. Moreover, the compatibility condition for existence of solutions reads

$$c_i(P, \alpha) = \int_{\mathbb{T}^n} (\nabla u + P)_i m - \alpha^q q m^q \tilde{m}_i dy = \bar{b}_i(P, \alpha) - q\alpha^q \int_{\mathbb{T}^n} m^q \tilde{m}_i dy$$

where the latter equality is due to (4). We now fix $\delta \in \mathbb{R}$ and define u_i^δ, m_i^δ the solution to (7) associated to $(P + \delta e_i, \alpha)$. Then it is possible to prove arguing as in [13, Theorem 7] that $(u_i^\delta - u)/\delta, (m_i^\delta - m)/\delta$ converges as $\delta \rightarrow 0$ to a solution to (16) in a suitable sense (weakly in $W^{1,2}$ for $(u_i^\delta - u)/\delta$ and weakly in L^2 for $(m_i^\delta - m)/\delta$). This implies by uniqueness that, up to subsequences,

$$c_i(P, \alpha) = \lim_{\delta \rightarrow 0} \frac{\bar{H}(P + \delta e_i, \alpha) - \bar{H}(P, \alpha)}{\delta}.$$

So relation (14) is exactly the compatibility condition for existence of solutions to (16). Following the same arguments as in [13, Theorem 11], we get that \bar{b} is locally Lipschitz continuous and moreover

$$\frac{\partial \bar{b}}{\partial P_i} = e_i + \int_{\mathbb{T}^n} \tilde{m}_i \nabla u + m \nabla \tilde{u}_i dy.$$

We multiply (i) and (ii) in (16) respectively by \tilde{m}_i and by \tilde{u}_i , subtract and get

$$\int_{\mathbb{T}^n} \tilde{m}_i \nabla u \cdot e_i - q\alpha^q m^{q-1} \tilde{m}_i^2 - m(\nabla \tilde{u}_i + e_i) \cdot \nabla \tilde{u}_i dy = 0.$$

Therefore
$$\begin{aligned} \frac{\partial \bar{b}}{\partial P_i} \cdot e_i &= 1 + \int_{\mathbb{T}^n} \tilde{m}_i \nabla u \cdot e_i + m \nabla \tilde{u}_i \cdot e_i dy \\ &= 1 + \int_{\mathbb{T}^n} q \alpha^q m^{q-1} \tilde{m}_i^2 + m (\nabla \tilde{u}_i + e_i) \cdot \nabla \tilde{u}_i + m \nabla \tilde{u}_i \cdot e_i dy \\ &= 1 + \int_{\mathbb{T}^n} q \alpha^q m^{q-1} \tilde{m}_i^2 + m |\nabla \tilde{u}_i + e_i|^2 - m dy \\ &= \int_{\mathbb{T}^n} q \alpha^q m^{q-1} \tilde{m}_i^2 + m |\nabla \tilde{u}_i + e_i|^2 dy \end{aligned}$$

and relation (15) easily follows. □

Corollary 2.3. *Under the assumption of Theorem 2.2, for $n = 1$, there holds*

$$\nabla_P \bar{H}(P, \alpha) \neq \bar{b}(P, \alpha).$$

Proof. First of all we observe that by Jensen inequality $\|m\|_{L^{q+1}}^{q+1} > 1$.

Moreover, arguing as in [16, Proposition 3.4] or as in [13, Proposition 14] (see also [9, Lemma 2.5]) and still using (12), we have that \sqrt{m} is bounded in $W^{1,2}(\mathbb{T}^n)$, uniformly in P . In other words, there exists a constant $C > 0$ independent of P such that

$$\int_{\mathbb{T}^n} |\nabla \sqrt{m}|^2 dy \leq C \int_{\mathbb{T}^n} v(y) dy.$$

So, since $n = 1$, we conclude that $m \rightarrow \bar{m}$ strongly in $L^{q+1}(\mathbb{T}^1)$. On the other side, reasoning as in [13, Proposition 14] we can prove that m converges weakly to 1 in L^{q+1} . This implies that $\|m\|_{L^{q+1}}^{q+1} \rightarrow 1$ as $|P| \rightarrow +\infty$ and then in particular $\nabla_P \bar{H}(P, \alpha) \neq \bar{b}(P, \alpha)$ at some (P, α) . □

3. Mean field game with logarithmic nonlinearities

In this section we consider potentials of the form $V(y, m) = v(y) + \log m$. In this case, defining $\bar{H}(P)$ as

$$\bar{H}(P, \alpha) = \bar{H}(P) + \log \alpha \quad \forall \alpha > 0$$

the system (3) becomes

$$\begin{cases} \text{(i)} & -\Delta u + \frac{|\nabla u + P|^2}{2} - v(y) - \log m = \bar{H}(P), & y \in \mathbb{T}^n \\ \text{(ii)} & -\Delta m - \operatorname{div}(m(\nabla u + P)) = 0, & y \in \mathbb{T}^n \\ \text{(iii)} & \int_{\mathbb{T}^n} u = 0 \quad \int_{\mathbb{T}^n} m = 1. \end{cases} \tag{17}$$

We collect some qualitative properties of the effective operators.

Proposition 3.1.

- (i) For every $P \in \mathbb{R}^n$, there exists a unique constant $\bar{H}(P)$ such that (17) admits a solution (u, m) , which is unique, $(u, m) \in C^{2,\gamma}(\mathbb{T}^n) \times W^{1,p}(\mathbb{T}^n)$ for every $\gamma \in (0, 1)$ and $p > 1$ and $m > 0$.
- (ii) $\bar{H}(P)$ is coercive in P , with quadratic growth, in particular there exists $C > 0$ such that

$$\frac{|P|^2}{2} - C - 2 \int_{\mathbb{T}^n} v(y)dy \leq \bar{H}(P) \leq \frac{|P|^2}{2}.$$

- (iii) $\bar{H}(P)$ is strictly convex.

- (iv) There hold $\lim_{|P| \rightarrow +\infty} \frac{\bar{H}(P)}{|P|^2} = \frac{1}{2}$ and $\lim_{|P| \rightarrow +\infty} \frac{|\bar{b}(P) - P|}{|P|} = 0$.

Proof. Following the arguments in [18], we introduce the following minimization problem. For any ϕ smooth and periodic we define the energy

$$E_P(\phi) = \int_{\mathbb{T}^n} e^{-\Delta\phi + \frac{|\nabla\phi + P|^2}{2} - v(y)} dy$$

and consider the minimization problem

$$\inf\{E_P(\phi) \mid \phi \text{ smooth and periodic}\}.$$

Note that $\inf E_P(\phi) \leq E_P(0) \leq e^{\frac{|P|^2}{2}}$. According to [18, Theorem 1], there exists a unique (up to constants) smooth minimizer ϕ for this problem. We define

$$m_\phi(y) := \frac{e^{-\Delta\phi + \frac{|\nabla\phi + P|^2}{2} - v(y)}}{E_P(\phi)}.$$

- (i) By computing the Euler Lagrange equation associated to the minimization problem we get that $u = \phi$, $m = m_\phi$ are solution to (17) with \bar{H} given by

$$\bar{H}(P) = \log \inf\{E_P(\phi) \mid \phi \text{ smooth and periodic}\} \tag{18}$$

So, [18, Theorem 1] gives the existence and uniqueness of a regular solution to (17) (see also [16, Thm 7.1]).

- (ii) Using the representation formula (18) for $\bar{H}(P)$, we get that

$$\bar{H}(P) \leq \log E_P(0) \leq \frac{|P|^2}{2}.$$

We multiply the first equation in (17) by m , the second by u , integrate and subtract, and we obtain

$$\bar{H}(P) = \int_{\mathbb{T}^n} \frac{1}{2} (|P|^2 - |\nabla u|^2) m - m \log m - v(y) m dy.$$

Integrating the first equation in (17), we obtain

$$\begin{aligned} \bar{H}(P) &= \int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2} - \log m - v(y) \, dy \\ &= \frac{|P|^2}{2} + \int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2} - \log m - v(y) \, dy. \end{aligned} \tag{19}$$

The last two equalities give

$$\int_{\mathbb{T}^n} \frac{1}{2} |\nabla u|^2 (m + 1) + (m - 1)v(y) + (m \log m - \log m) \, dy = 0. \tag{20}$$

Note that $m \log m \geq \log m$, so we conclude that also in this case (12) holds. Moreover, we observe that there exists $C > 0$ such that $m \log m \geq 2 \log m - C$ and then

$$\int_{\mathbb{T}^n} (m - 1)v(y) + \log m \, dy - C \leq \int_{\mathbb{T}^n} (m - 1)v(y) + (m \log m - \log m) \, dy \leq 0.$$

This in turns gives
$$\int_{\mathbb{T}^n} \log m \, dy \leq C + \int_{\mathbb{T}^n} v(y) \, dy.$$

So, again by (19), we conclude $\bar{H}(P) \geq \frac{|P|^2}{2} - C - 2 \int_{\mathbb{T}^n} v(y) \, dy$.

(iii) The strict convexity is a consequence of the representation formula (18), see [18, Lemma1].

(iv) The proof is the same as in Proposition 2.1. □

As before, using the properties stated in Proposition 3.1, we obtain regularity properties of the effective operators, and a relation between them.

Theorem 3.2. *The maps $P \rightarrow \bar{H}(P), \bar{b}(P)$ are locally Lipschitz continuous and admit partial derivatives everywhere. Moreover, there hold*

$$\nabla_P \bar{H}(P) = \bar{b}(P). \tag{21}$$

Proof. The regularity comes from the properties of \bar{H} proved in Proposition 3.1. We fix $P \in \mathbb{R}^n$; let (u, m) be the solution to (17) associated to P . We introduce the following system for all $i = 1, \dots, n$

$$\begin{cases} \text{(i)} & -\Delta \tilde{u}_i + \nabla \tilde{u}_i \cdot (\nabla u + P) + (\nabla u + P) \cdot e_i - \frac{\tilde{m}_i}{m} = c_i(P) \\ \text{(ii)} & -\Delta \tilde{m}_i - \operatorname{div}((P + \nabla u)\tilde{m}_i) = \operatorname{div}(m(\nabla \tilde{u}_i + e_i)) \\ \text{(iii)} & \int_{\mathbb{T}^n} \tilde{m}_i = \int_{\mathbb{T}^n} \tilde{u}_i = 0. \end{cases} \tag{22}$$

Note that by [16, Theorem 5.1] $\frac{1}{m} \in L^\infty(\mathbb{T}^n)$. Arguing as in [13, Lemma 6], we get that for every i there exists a unique $c_i(P)$ for which this system admits a solution

$(\tilde{u}_i, \tilde{m}_i)$, which is regular and unique. Moreover, the compatibility condition for existence of solutions reads

$$c_i(P) = \int_{\mathbb{T}^n} (\nabla u + P)_i m - \tilde{m}_i dy = \int_{\mathbb{T}^n} (\nabla u + P)_i m dy = \bar{b}_i(P, \alpha).$$

Moreover, as in Theorem 3.2, we get that, up to subsequences,

$$c_i(P) = \lim_{\delta \rightarrow 0} \frac{\bar{H}(P + \delta e_i) - \bar{H}(P)}{\delta}.$$

So relation (21) is exactly the compatibility condition for the existence of solutions to (22). \square

4. Numerical Tests

This section is devoted to the numerical solution of the cell problems presented in the previous sections, in particular we provide a numerical validation of the properties involving the effective Hamiltonian \bar{H} and the effective drift \bar{b} . We recall that the computation of effective operators is an expensive task, since it requires the solution of a nonlinear problem for each pair (P, α) .

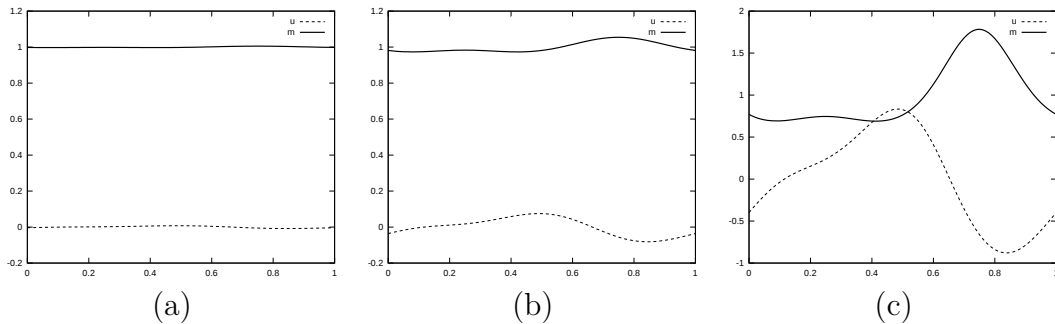


Figure 4.1: Correctors of the cell problem (3) depending on the amplitude A of the potential v : $A = 1$ (a), $A = 10$ (b), $A = 100$ (c).

To this end we apply the *direct method* we recently introduced in [5, 6], an efficient solver for ergodic problems associated to the homogenization of Hamilton-Jacobi equations. The new method avoids completely classical approximations such as the *small-delta* or the *large-time* methods (see [2, 23]). The ergodic constant \bar{H} appearing in the cell problem (3) is handled as an additional unknown, so that the corresponding nonlinear system is formally *overdetermined*, having more equations than unknowns. The solution is then meant in a *least-squares* sense and computed by a generalized Newton method for inconsistent systems. We refer the reader to [5, 6] for technical details and for the actual implementation of the method. Here we just remark that, after introducing a discretization of the torus \mathbb{T}^n with N nodes, collecting the approximations U_i, M_i, \bar{H} ($i = 1, \dots, N$) of the unknowns in a single vector $X = (U, M, \bar{H})$ of length $2N + 1$, and choosing a scheme which is consistent with the notion of viscosity solutions, we end up with

a nonlinear system of $2N + 2$ equations. The first $2N$ equations correspond to the partial differential equations in (3), whereas the last 2 equations correspond to the normalization conditions $\int_{\mathbb{T}^n} u = 0$ and $\int_{\mathbb{T}^n} m = 1$. Defining a suitable non linear map $F : \mathbb{R}^{2N+1} \rightarrow \mathbb{R}^{2N+2}$, the resolution of the cell problem is then reduced to

$$\text{Find } X \in \mathbb{R}^{2N+1} \text{ such that } F(X) = 0 \in \mathbb{R}^{2N+2}.$$

Note that, once a solution X is computed, we obtain the effective drift \bar{b} in (4) by a simple trapezoidal quadrature rule. Moreover, the system (6) can be discretized in a similar way and its solution easily computed in a single iteration, being an overdetermined but linear system. Hence we get the solution $(\tilde{U}, \tilde{M}, \frac{\partial \tilde{H}}{\partial P})$, from which we can readily compute the integral in (5), again by a trapezoidal quadrature rule. This gives, in particular, a practical strategy to validate numerically the relationship described by (5).

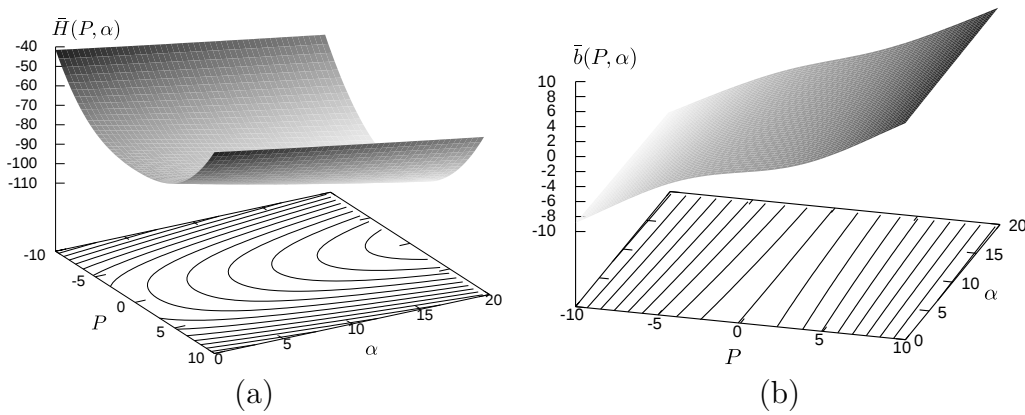


Figure 4.2: Surfaces and level sets of \bar{H} (a) and \bar{b} (b) for $q = 1$.

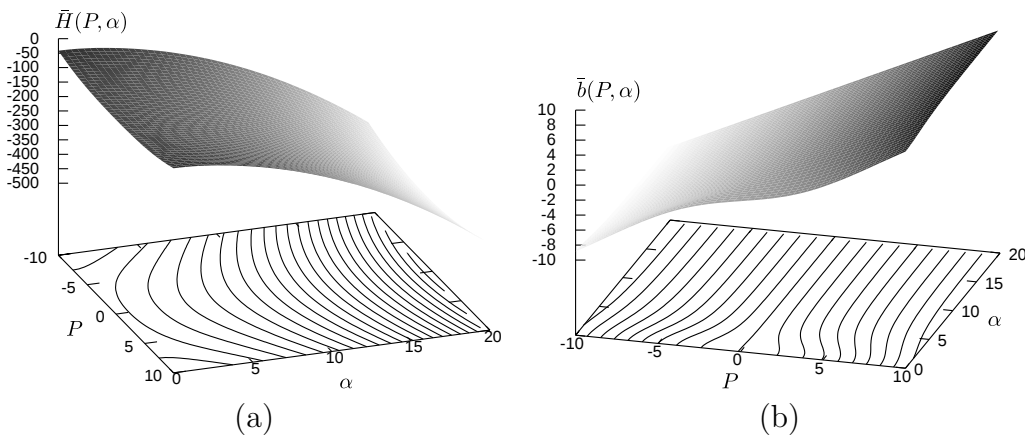


Figure 4.3: Surfaces and level sets of \bar{H} (a) and \bar{b} (b) for $q = 2$.

Here we consider for simplicity the one dimensional case $n = 1$, but the method can be easily implemented in any dimension. In all the tests we choose a nonnegative potential of the form $v(x) = A(1 + \frac{1}{2}(\sin(2\pi x) + \cos(4\pi x)))$ with $A > 0$. We observe that, for small values of the amplitude A , the corresponding correctors u

and m , i.e. solutions to (3), are quite close to 0 and 1 respectively, as shown in Figure 4.1 in the case $V(y, m) = v(y) + m$ with $P = 10$ and $\alpha = 2$. This considerably affects the shape of the effective Hamiltonian and effective drift, making difficult to explore numerically the features we are interested in. That’s why we enforce non trivial solutions by choosing $A = 100$ in what follows.

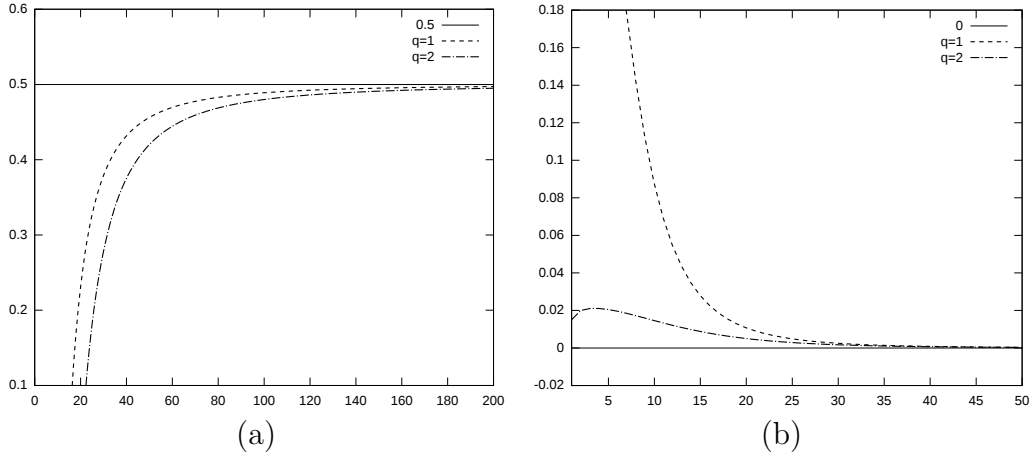


Figure 4.4: Asymptotic behavior in P of $\bar{H}(P, \alpha)/P^2$ (a) and $|\bar{b}(P, \alpha) - P|/P$ (b).

We start with the case of power nonlinearities, setting $V(y, m) = v(y) + m^q$, with $q = 1, 2$. We discretize the torus \mathbb{T}^1 with $N = 400$ nodes and we consider a uniform grid of 51×51 nodes for discretizing the space of parameters $[-10, 10] \times [0, 20] \ni (P, \alpha)$.

Figure 4.2 shows the surfaces and the level sets of the computed effective Hamiltonian $\bar{H}(P, \alpha)$ and the effective drift $\bar{b}(P, \alpha)$ as functions of (P, α) for $q = 1$, whereas Figure 4.3 shows the case $q = 2$.

In both cases $q = 1, 2$ we observe that $\bar{H}(P, \alpha)$ is “almost” quadratic in P and nonincreasing in α , while $\bar{b}(P, \alpha)$ exhibits a cubic-like behavior for P close to zero. Let us take a closer look to some slices of these effective surfaces, in order to analyze their behavior for P large. We choose $\alpha = 10$ and we plot in Figure 4.4 the graphs of $\frac{\bar{H}(P, \alpha)}{P^2}$ and $\frac{|\bar{b}(P, \alpha) - P|}{P}$ as functions of P for $q = 1, 2$. We clearly observe convergence to $\frac{1}{2}$ and 0 respectively, as expected by Proposition 2.1-(iv).

We proceed by examining the relationship between the effective operators described in the previous section. First of all we want to show that, in the case of power nonlinearities, the structure of mean field game is lost during the homogenization process, namely we have $\nabla_P \bar{H}(P, \alpha) \neq \bar{b}(P, \alpha)$. To this end, we consider the residual in (5) as a measure of this discrepancy:

$$\mathcal{R}(P, \alpha) = \left| \alpha \int_{\mathbb{T}^n} V_m(y, \alpha m) \tilde{m} m dy \right|,$$

where we recall that, for each (P, α) , the auxiliary \tilde{m} solves the linearized cell problem (6), in which the previously computed correctors (u, m) appear as given data.

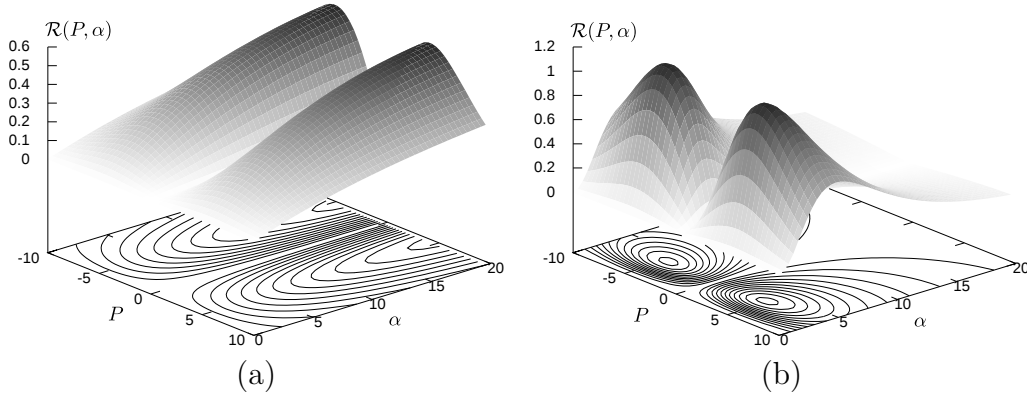


Figure 4.5: Surface and level sets of the residual \mathcal{R} for $q = 1$ (a) and $q = 2$ (b).

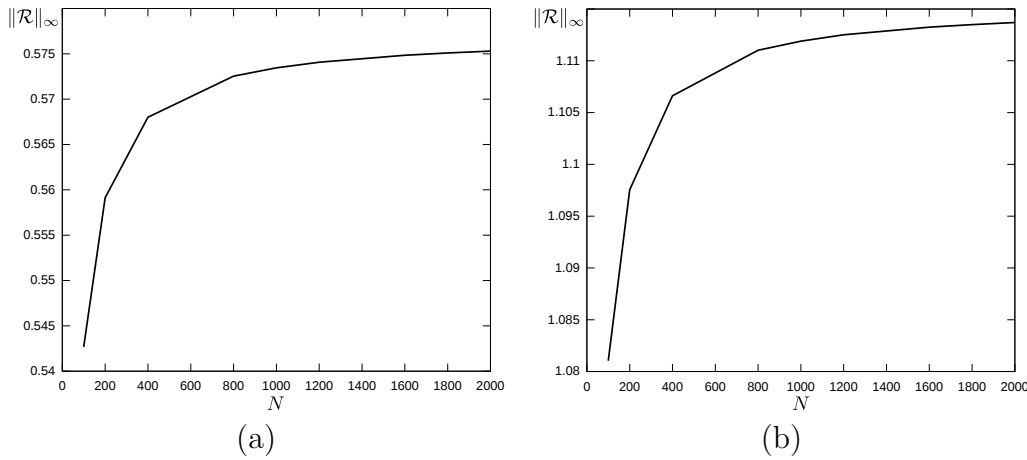


Figure 4.6: L^∞ norm of \mathcal{R} under grid refinement for $q = 1$ (a) and $q = 2$ (b).

Figure 4.5 shows the residuals respectively for $q = 1$ and $q = 2$ on the space of parameters $[-10, 10] \times [0, 20]$, while Figure 4.6 shows the behavior of the corresponding L^∞ norms under grid refinement. We clearly observe that, as N increases, the residual converges but not vanishes. Note that, in the case of power nonlinearities, the residual can be computed using also expression (14), in which appears the derivative in P of the $q + 1$ norm of the corrector m . Indeed, denoting by $m_{(P,\alpha)}$ the solution of (7), we can approximate $\frac{\partial}{\partial P} \|m_{(P,\alpha)}\|_{L^{q+1}}^{q+1}$ by finite differences of the form

$$\frac{\|m_{(P+\delta,\alpha)}\|_{L^{q+1}}^{q+1} - \|m_{(P,\alpha)}\|_{L^{q+1}}^{q+1}}{\delta} \quad \text{for some small } \delta > 0.$$

This approach leads to similar results, up to an additional error of order $\mathcal{O}(\delta)$.

We evaluate the numerical error in the relationship (5) by defining

$$\mathcal{E}_\infty = \left\| \frac{\partial H}{\partial P}(P, \alpha) - \bar{b}(P, \alpha) + \alpha \int_{\mathbb{T}^n} V_m(y, \alpha m) \tilde{m} m dy \right\|_\infty,$$

Figure 4.7 shows that, for both $q = 1$ and $q = 2$, the error \mathcal{E}_∞ has order of convergence 1, under grid refinement for a space step $h = 1/N$.

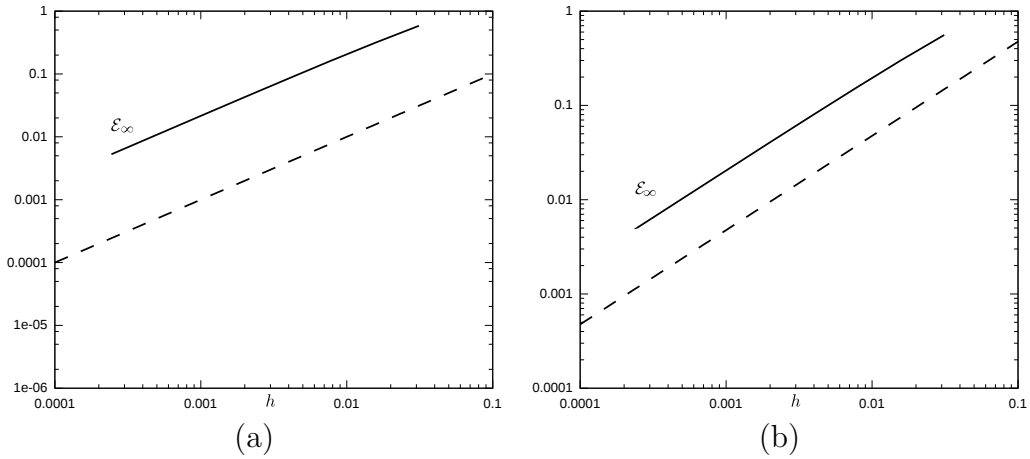


Figure 4.7: \mathcal{E}_∞ under grid refinement for $q = 1$ (a) and $q = 2$ (b) compared with straight (dashed) lines of slope 1.

We now consider the logarithmic case, setting $V(y, m) = v(y) + \log m$. We already know that in this case the dependency of \bar{H} on P and α is separated, however we solve the full cell problem (3) and not the reduced one in (17), in order to recover this feature numerically. Again, we discretize the torus \mathbb{T}^1 with $N = 400$ nodes and we consider a uniform grid of 51×51 nodes for discretizing the space of parameters $[-10, 10] \times [1, 20] \ni (P, \alpha)$ (here we avoid the singularity of the log function just starting with $\alpha > 0$).

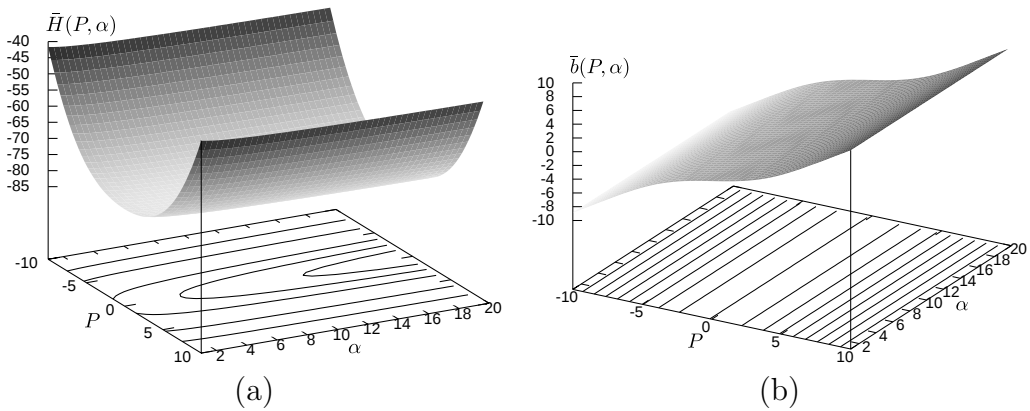


Figure 4.8: Surfaces and level sets of \bar{H} (a) and \bar{b} (b).

Figure 4.8 shows the surfaces and the level sets of the computed effective Hamiltonian $\bar{H}(P, \alpha)$ and the effective drift $\bar{b}(P, \alpha)$ as functions of (P, α) . We observe in particular the convexity of \bar{H} in P and the independence of \bar{b} on α . We finally recall that in this case the residual $\mathcal{R}(P, \alpha) = \left| \alpha \int_{\mathbb{T}^n} V_m(y, \alpha m) \tilde{m} m dy \right| = 0$, being $V_m = 1/m$ and \tilde{m} with zero mean.

Then, in Figure 4.9 we show the error $\mathcal{E}(P, \alpha) = |\nabla_P \bar{H}(P, \alpha) - \bar{b}(P, \alpha)|$ and the behavior of its L^∞ norm \mathcal{E}_∞ under grid refinement for a space step $h = 1/N$. We see that \mathcal{E}_∞ does not depend on α and it exhibits again an order of convergence

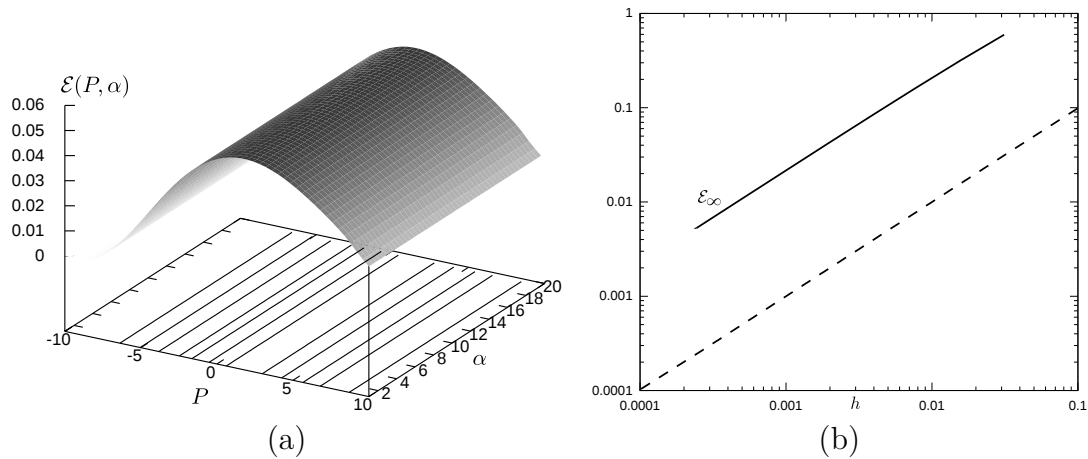


Figure 4.9: The error $\mathcal{E}(P, \alpha) = |\nabla_P \bar{H}(P, \alpha) - \bar{b}(P, \alpha)|$ (a), and its L^∞ norm under grid refinement compared with a straight (dashed) line of slope 1 (b).

1 as the space step goes to zero. This confirms the relation $\nabla_P \bar{H}(P, \alpha) = \bar{b}(P, \alpha)$ in the logarithmic case and also the separated dependency of \bar{H} on P and α .

References

- [1] Y. Achdou: *Finite difference methods for mean field games*, in: *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, P. Loreti and N. A. Tchou (eds.), Lecture Notes in Math. 2074, Springer, Heidelberg et al. (2013) 1–47.
- [2] Y. Achdou, I. Capuzzo Dolcetta: *Mean field games: numerical methods*, SIAM J. Numer. Anal. 48(3) (2010) 1136–1162.
- [3] M. Bardi, E. Feleqi: *Nonlinear elliptic systems and mean field games*, Nonlinear Diff. Equations Appl. 23(4) (2016) Art. No. 44.
- [4] A. Bensoussan, J. Frehse: *Mean Field Games and Mean Field Type Control Theory*, Springer Briefs in Mathematics, Springer, New York et al. (2013).
- [5] S. Cacace, F. Camilli: *A generalized Newton method for homogenization of Hamilton-Jacobi equations*, SIAM J. Sci. Computing 38(6) (2016) A3589–A3617.
- [6] S. Cacace, F. Camilli, C. Marchi: *A numerical method for mean field games on networks*, ESAIM: Math. Modelling Numer. Analysis 51(1) (2017) 63–88.
- [7] P. Cardaliaguet: *Note on Mean Field Games*, <https://www.ceremade.dauphine.fr/~cardalia/MFG20130420.pdf>.
- [8] P. Cardaliaguet, F. Delarue, J.-M. Lasry, P.-L. Lions: *The master equation and the convergence problem in mean field games*, preprint (2015), available at arxiv.org/abs/1509.02505.
- [9] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta: *Long time average of mean field games*, Netw. Heterog. Media 7(2) (2012) 279–301.
- [10] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta: *Long time average of mean field games with a nonlocal coupling*, SIAM J. Control Optim. 51(5) (2013) 3558–3591.

- [11] P. Cardaliaguet, A. Porretta: *Long time behavior of the master equation in mean-field game theory*, arxiv.org/abs/1709.04215.
- [12] A. Cesaroni, M. Cirant: *Introduction to Variational Methods for Viscous Ergodic Mean-Field Games with Local Coupling*, INDAM Lecture Notes, to appear.
- [13] A. Cesaroni, N. Dirr, C. Marchi: *Homogenization of a mean field game system in the small noise limit*, SIAM J. Math. Anal. 48(4) (2016) 2701–2729.
- [14] M. Cirant: *Stationary focusing mean field games*, Comm. Partial Diff. Equations 41(8) (2016) 1324–1346.
- [15] D. A. Gomes, L. Nurbekyan, E. Pimentel: *Economic Models and Mean-Field Games Theory*, IMPA Mathematical Publications, Rio de Janeiro (2015).
- [16] D. A. Gomes, S. Patrizi, V. Voskanyan: *On the existence of classical solutions for stationary extended mean field games*, Nonlinear Anal. 99 (2014) 49–79.
- [17] D. A. Gomes, E. A. Pimentel, V. Voskanyan: *Regularity Theory for Mean-Field Game Systems*, Springer Briefs in Mathematics, Springer, Berlin et al. (2016).
- [18] D. A. Gomes, H. Sánchez Morgado: *A stochastic Evans-Aronsson problem*, Trans. Amer. Math. Soc. 366(2) (2014) 903–929.
- [19] D. A. Gomes, J. Sáude: *Mean field games models: a brief survey*, Dyn. Games Appl. 4(2) (2014) 110–154.
- [20] J.-M. Lasry, P.-L. Lions: *Jeux à Champ Moyen. I: Le Cas Stationnaire*, C. R. Math. Acad. Sci. Paris 343 (2006) 619–625.
- [21] J.-M. Lasry, P.-L. Lions: *Mean field games*, Japan. J. Math. (N.S.) 2 (2007) 229–260.
- [22] E. Pimentel, V. Voskanyan: *Regularity theory for second order stationary mean-field games*, Indiana Univ. Math. J. 66(1) (2017) 1–22.
- [23] J. Qian: *Two approximations for effective Hamiltonians arising from homogenization of Hamilton-Jacobi equations*, UCLA, Dept. of Mathematics, preprint (2003).