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A Mathematical Framework for Balancing Learning Outcomes and Instructional Costs in Digital Education

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Abstract

Objective: Digital education service providers need to consider the cost of automation and instruction, but they should not assume that lower cost and better test results are the goals of optimization. This research proposes an interpretable model for the analysis of this tradeoff.Methodology: The institutional cost function includes two components: platform cost and cost of delivery. The response function is concave, indicating diminishing return on learning from further activities. There are two decision problems, namely maximizing the predicted gain under cost constraints and minimizing the cost under gain constraints. Marginal conditions show when a more costly activity should be included in the allocation.Findings: Example for illustration, automated practice costs 20 cost units per session while instructor support costs 80 cost units. Given a delivery budget of 200 cost units per learner, the best combination would be eight automated sessions and 0.50 instructor session per learner which will yield 29.61 gain in assessment scores. In the case where automated practice is done alone with 200 cost unit budget, it will yield a gain of 28.77 score. To achieve 30 gains in assessment score, the delivery budget must at least be 205.90 cost unit per learner.Value: The model connects educational objectives with resource needs and marginal gains without putting a dollar value on education. All numbers presented are for demonstration purposes only, not real-life outcomes of students. The application of this model would necessitate local impact assessment, cost evaluation, and validation against retention, accessibility, and teacher workload.

*Keywords:*Digital education, instructional costs, learning gains, constrained optimization, cost effectiveness.

1. The instructional spending decision

Digital learning modifies the delivery process, but it does not do away with the necessity of making resource choice(Janifa & Gokulakrishnan, 2026). Institutions incur costs related to platform construction, course materials, technical assistance, evaluation and teacher time. The learner also incurs time and may incur costs related to devices or connectivity. The programme could thus be inexpensive to deliver and yet require a lot of effort on the part of the learner or enhance the results of evaluation at a cost which the institution cannot afford. UNESO's study on the use of technology in learning highlights the need for relevance, inclusiveness,

and sustainability in evaluation of digital provision (UNESCO, 2023).

The planning problem is not just about the effectiveness of online teaching. The planning problem is about which activities to mix in a way that more spending leads to educational improvement(Al-Harbi & Al-Qahtani, 2024). Comparisons between online, hybrid, and in-class teaching should be analyzed together with the differences in teaching methods, conditions, and support provided(Means et al., 2013). The mere fact of delivery method cannot define the predictable learning impact.

Imagine a provider planning a short online course. It can provide either more practice sessions that are automated or more instructor-supported learning. The cost of automation is lower than the cost of the support. However, the benefit of automation decreases while a learner repeats similar tasks. The value of support is increasing when more automated practice sessions provide no improvements. The provider should have some criteria of when to implement it and how much to provide.

The following paper attempts to establish such a rule through an educational production function approach. The focus of the approach will be the expenditure vs. anticipated gain in education(Malhotra & Iyer, 2024). The significance of the research lies in its link between cost accounting, diminishing returns, and operational decisions regarding implementation of the best possible plan within budget limitations or the least costly plan that meets the set objective. It is a theoretical paper on planning with an illustrative example.

2. What existing evidence contributes

There are a couple of reasons from the research on tutoring to study both scalable activities and human interventions. The first is Bloom's analysis of the two-sigma problem that combined the advantages of tutoring with the problem of finding scalable solutions of group instruction. The second one is a comparison of human tutoring, intelligent tutoring, and other tutoring systems performed by VanLehn, (2011), who highlighted some differences between the types of interaction in the processes of instruction. Nevertheless, these studies raise resource allocation issues but do not provide universal prices and responses.

The benefits of intelligent tutoring systems have been reviewed by Kulik & Fletcher, (2016)and Ma et al., (2014), who showed educational gains from using this type of system. However, the results of these reviews are conditioned by specific comparisons, measures, and implementations. Therefore, the current framework does not include any pool effect size in the local course model. A score point response should be calculated using the same scale and population.

Even the quality of the task is a consideration. Mayer's research on multimedia learning argues for the need to pay attention to instructional design rather than the increase in digitized information(Mayer, 2002). Shute reviews formative feedback, giving insight into the reasons why effective feedback needs to provide relevant information about a learner's cognition or behavior(Shute, 2008). As such, in this model, instruction quantity is considered good only where the activity itself is educationally sound. It would not make sense for repetition of poor content to be given a good learning coefficient based on platform activity.

Cost-effectiveness analysis of educational interventions provides the accounting perspective. Levin and McEwan outline ways of identifying and measuring the resource requirements of the intervention(Levin & McEwan, 2001). Levin also outlines the difficulties in applying cost-effectiveness analysis to educational evaluations(Levin, 2001). The paper takes this perspective in its identification of fixed commitment, cost of delivery, and learning outcome. Financial return on investment is not calculated as assessment points do not have any monetary value.

The collected data could be helpful for future estimations. According to Koedinger et al., (2013), there are possibilities for data-driven development and optimization of the tutoring systems. Siemens considers the development and application of learning analytics (Siemens, 2013). Another example of modeling changes in the learner's knowledge is provided by knowledge tracing. However, both activity logs and prediction accuracy do not allow to estimate the causal learning gain of an extra lesson.

3. Accounting for the resources actually used

The analysis takes the provider's perspective over one course delivery period. Let N denote enrolment, F the fixed expenditure required to run the course, d the average number of automated practice sessions per learner, and h the average number of instructor-support sessions per learner. Let c_d and c_h be their respective delivery costs. Total institutional cost is

$$C(d,h;N) = F + N(c_d d + c_h h)$$

Fixed expenditures consist of course development and platform cost, which does not vary in the desired range of enrollment. Delivery costs will comprise of those elements that will vary according to the amount of teaching dose selected. Teacher preparation, session delivery, and after-class efforts need to be included if they vary according to added support. Those elements which have been covered in fixed costs should not be double counted in per session costs.

The corresponding average cost per learner is

$$\bar{C}(d,h;N) = \frac{F}{N} + c_d d + c_h h$$

This is important since a fixed and unchanging commitment will have no impact on how the already agreed delivery budget is allocated among the various activities. However, it will have an effect on the cost, as well as the affordability of the budget. Should the suggested service package have some requirements for license or staffing, the fixed cost would have to be included in the model and would possibly lead to another preferred strategy.

The framework adopts general cost units referred to as CU. These are accounting cost units that are being used in the example. In reality, the evaluation would need to include the year of price, the currency as well as the quantities of the resources, and also the treatment of gifts and shared resources (Levin & McEwan, 2001). Learners' and households' costs are not included in the present calculation of the cost incurred by the provider.

4. Turning educational aims into an allocation model

4.1. A response with diminishing returns

Let $G(d,h)$ be the expected improvement on a common assessment relative to a specified baseline course. A simple response function is

$$G(d,h) = a\ln(1+d) + b\ln(1+h), \qquad a > 0, \quad b > 0$$

Logarithms work on unitless number of sessions. Coefficients a and b have units of assessment points. Function is increasing for each additional activity, but the increment becomes smaller with each additional dose:

$$\frac{\partial G}{\partial d} = \frac{a}{1+d}, \qquad \frac{\partial G}{\partial h} = \frac{b}{1+h}.$$

This form has been selected due to its tractability and transparency. This form is not supposed to be an educational law that is established. The form considers the contributions of both practice and teaching without having the interaction between the two variables. It does not consider the concepts of fatigue, forgetting, and sequencing.

If the baseline assessment score is S_0, the predicted score is $S_0 + G(d,h)$. Predictions must remain within the assessment scale and the range of doses supported by data. The logarithmic function is unbounded and therefore should not be extrapolated indefinitely. The illustration uses budgets for which all predicted scores remain below the assumed maximum of 100.

4.2. The budget decision

Let B be the permitted delivery expenditure per learner after the fixed commitment has been accounted for. The provider solves

$$\max_{d,h \geq 0} G(d,h) \quad \text{subject to} \quad c_d d + c_h h \leq B$$

Since the gain function is strictly concave and the set of feasible budgets is convex and bounded, the optimal allocation will be unique. Such solution may be characterized by using standard optimization constraints(Boyd & Vandenberghe, 2004). In the absence of any limits on the level of activities, the full budget for delivery will be allocated since both marginal gains are positive.

If both activities are assigned a positive allocation, their marginal gains to cost are identical:

$$\frac{a}{c_d(1+d)} = \frac{b}{c_h(1+h)} = \mu$$

μ is the local measure of the gain in optimal gain per extra CU spent on delivering goods. In other words, this equality gives us the formula for allocation which states that we must spend money on the activity which brings a higher gain per CU until both activities bring equal gain.

4.3. The learning-target decision

An alternative question starts with a required gain T:

$$\min_{d,h\geq 0}(c_d d + c_h h) \quad \text{subject to} \quad G(d,h) \geq T$$

Since G is concave, its superlevel set is convex. This implies that the model can pinpoint the optimal feasible dosage that incurs minimal costs. In this particular case, the optimum perfectly matches the target since the response is increasing. A target outside the region justifiable based on data points should be considered unsupported instead of being adopted purely for mathematical solutions.

The two choices are aligned along the same efficient trade-off between costs and outcomes. Multiobjective optimization defines the efficient frontier, which can be considered as a Pareto frontier, as a curve where there is no way to increase learning without increasing cost or to decrease cost without decreasing learning. Highlighting several points on the curve will enable education managers to set a sensible target without pretending that mathematics solves everything.

5. A worked digital-course example

5.1. Assumptions and calculation

Assume a course with 100 learners, fixed expenditure of 1,000 CU, and a baseline assessment score of 40 points. An automated practice session costs 20 CU per learner; an instructor-support session costs 80 CU. The assumed response coefficients are $a = 12$ and $b = 8$. These values are constructed for the example and are not estimated from the cited studies.

The response and delivery cost are consequently

$$G(d,h) = 12\ln(1+d) + 8\ln(1+h), \qquad V(d,h) = 20d + 80h$$

At the initial dose, automated practice yields $12/20 = 0.60$ marginal assessment points per CU, whereas instructor support yields $8/80 = 0.10$. Automated practice therefore receives the first units of budget. When $h = 0$, instructor support becomes competitive once

$$\frac{12}{20(1+d)} = \frac{8}{80}$$

It happens at d=5, where the cost budget for delivery is 100 CU. Below that level, the cost-effective strategy includes only practice automation. Above it, both activities get resources. The resource allocation that follows is

$$d^* = \frac{B}{20},\ h^* = 0 \quad \text{when } 0 \leq B \leq 100$$

$$d^* = 0.03(B+100) - 1, \quad h^* = 0.005(B+100) - 1 \quad \text{when } B > 100$$

At $B = 200$, these expressions give $d^* = 8$ and $h^* = 0.50$. The model predicts a gain of 29.61 points and a final score of 69.61. Total institutional expenditure is 21,000 CU, comprising 1,000 fixed CU and 20,000 delivery CU. Instructor support totals 50 sessions across the cohort.

Fractional number of sessions is an average among cohorts and not a fraction of sessions for each participant. There must be a workable schedule that will implement the fractional number of sessions. Since the response is non-linear, gain obtained at an average dosage is different from average gain obtained by each individual; there

must thus be a workable way to model the dosage. In case the sessions are indivisible and each learner gets the same dose, integer constraints become necessary.

5.2. What the same budget purchases

Table 1: Model-calculated outcomes at the same delivery budget.

Allocation plan	Automated sessions per learner	Instructor sessions per learner	Predicted gain in points
Automated practice only	10.00	0.00	28.77
Instructor support only	0.00	2.50	10.02
Equal spending on both	5.00	1.25	27.99
Marginal-return allocation	8.00	0.50	29.61

Four different allocations using 200 CU of delivery expense per learner are given in table 1. The "equal-spending" allocation involves allocating funds in equal parts to all the activities involved; however, session numbers are not equal.

The optimized allocation does better than pure automation by 0.84 points, which is 2.90% of the value of the automation-only case. This is not much and not an empirical revolution; it is simply a proof that spending money on a more costly task can pay off if the marginal payoff of the less costly task decreases enough.

The poor performance of the tutor-only provision is due to our assumed coefficients and prices. But it does not mean that human tutoring is universally poor; the literature recommends looking at the instruction process, not claiming its inferiority based on the simple model (VanLehn, 2011; Kulik & Fletcher, 2016; Ma et al., 2014)

5.3. The cost–outcome boundary

Table 2 shows the optimal distribution for four levels of delivery budget. All the gains are computed using unrounded dosages but are presented up to two decimal places.

Table 2: Efficient budget choices under the assumed response model.

Delivery budget per learner in CU	Automated sessions	Instructor sessions	Predicted gain in points	Total cohort cost in CU
50	2.50	0.00	15.03	6,000
100	5.00	0.00	21.50	11,000
200	8.00	0.50	29.61	21,000
400	14.00	1.50	39.83	41,000

In figure 1, an increase in the delivery cost from 100 to 200 CU results in 8.11 more assessment points. An increase from 200 to 400 CU results in another 10.22 points but costs twice as much extra cost as before. In the first range, it costs 12.33 CU to earn an additional point. It is 19.58 CU for the second range. These numbers express the diminishing return relationship. They are not financial returns on investment.

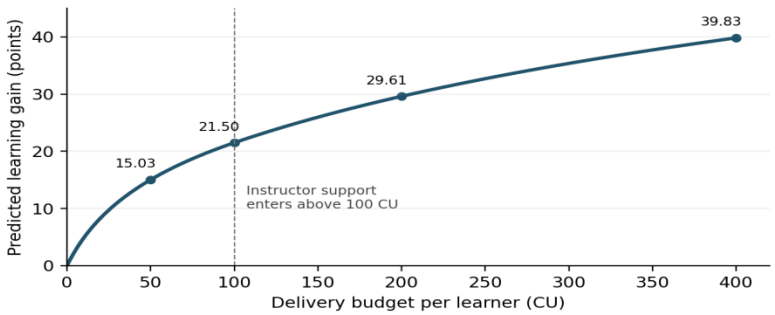


Figure 1: Calculated cost–outcome frontier for the illustrative model. The horizontal axis excludes the fixed platform expenditure. The curve represents assumed responses, not observed student data.

For the required improvement of 30 points, the inverse of the optimal response gives the delivery cost equal to 205.90 CU for each learner. Continuous allocation is 8.177 automated sessions and 0.5295 instructor sessions

per learner. The overall cost of the cohort, without rounding, is 21,590.11 CU. In practice, it is better to find the integer number of sessions using the integer allocation than rounding them and assuming the unchanged learning target.

The optimal allocation was confirmed with a constrained numerical optimizer. For the budget equal to 200 CU, the numerical result matched eight automated sessions and 0.50 instructor sessions.

5.4. Operational rules and sensitivity

First, marginal returns should be considered rather than total returns. There is no need for a high-performing component to get each new unit of the resource. The issue is about how much improvement the next unit brings at the current dose.

Second, it is important to differentiate between price and educational effect. In case when the cost of the instructor support increased to 160 CU from 80 CU and the coefficients stayed the same, then instructor support is included only after the delivery budget of 220 CU. With a delivery budget of 200 CU, the preferred mix is automation only with gain at 28.77 points. Thus, local staff price can change a mathematically preferred plan.

Third, there should be capacity constraints. In case the instructor support can be used only once per learner, the feasible optimum with the delivery budget of 400 CU will be 16 automated sessions and 1 instructor support session. The predicted gain will be 39.54 points versus 39.83 points without such a constraint. An advanced model can have total teacher hours, minimum support access, and learner workload constraints.

The fourth principle involves making scale assumption explicit. At a delivery budget of 200-CU, the average institutional cost at 50 learners, 100 learners, and 200 learners stands at 220 CU, 210 CU, and 205 CU, respectively, in the absence of any change in both the fixed cost commitment and the session costs. The above accounting effect does not necessarily mean the presence of real economy of scale, since more staffing and platform tiers, or quality deterioration can affect cost levels and learning responses.

Lastly, the frontier needs to be examined along with the minimum educational requirements. An aggregate target can ignore those learners who require special consideration and further assistance. The mandatory considerations must be placed prior to any allocation decision. It is important that technology assessment should continue to focus on inclusion and teachers' role (UNESCO, 2023).

5.5. Validation boundaries and an empirical next step

The framework has a solid math conclusion but lacks evidence. The numerical illustration of the framework involves hypothetical prices and response coefficients. There is no classroom experiment. Gains are predictions, not results. The illustration does not offer any significance testing, confidence intervals, or retention results.

The logarithmic response equation presupposes separability of benefits and gains at every dose level. Real-life teaching may show interactions, overload, absence of students, and absence of gains from teaching. Mastering of basics and subject matters influence the response equation as well. In addition, equality of averages of learners does not guarantee equal benefits. Individual support issues are beyond the two-variable framework.

The empirical illustration should start with the definition of common goal of learning, common scale of assessing performance, baseline, and period of observation. Resources needed for each activity should be accounted forKhamitdkhanovich, (2025). Randomization of doses of learning may help to estimate the effect of teaching as opposed to correlation between platform use and test scores. The fitted response should then be validated on new individuals or courses.

In this case, the comparison should be between the suggested resource allocation and other feasible allocations given the same amount of resources available. The immediate assessment, retained knowledge, the load on the learner, time on teachers, and the accessibility to resources should be considered simultaneously. Uncertainty in the cost and education effects would give a variety of possible frontiers. The decisions must be reviewed when the implementation changes the costs or education responses.

6. Concluding statement

It is only possible to balance learning and its cost through a model that clearly specifies both the learning goal and the budget limit. The model proposed by the author has the capacity to separate fixed and delivery costs, demonstrate the diminishing return concept and associate budget-maximizing learning with target-minimizing

costs. In the example given, the illustrative assignment provides 29.61 points of gains with 200 CU of delivery cost per learner while it requires 205.90 CU to attain a 30-point target. This will make it possible for marginal return to help in deciding on how much to spend and what the extra cost is.

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