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# Fixed-point stability conditions and convergence analysis for distributed autonomous decision-making system

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**Abstract**:

Distributed autonomous decision-making processes may be mathematically modeled as an ensemble of nonlinear systems with decision components that depend upon each other and whose states evolve through iteration using a mapping process. Establishing the existence, stability, and convergence of these distributed systems is still a challenging mathematical problem due to nonlinear interactions and disturbances. In this paper, a mathematical technique using a fixed-point theory framework is proposed for analyzing distributed nonlinear decision mappings using a combination of operator-theoretic analysis and iterative computation methods. The decision process is formulated as a nonlinear fixed-point operator equation, and sufficient conditions for stability are established based on the idea of contractive mappings in complete metric spaces. A distributed fixed point iterative framework with a relaxation-based update mechanism is proposed to enhance convergence reliability in the presence of coupled decision interactions. Computational experiments show that the presented mathematical model meets the developed stability conditions and achieves theoretical convergence behavior with lower approximation errors. These results show that the developed mathematical framework rigorously guarantees equilibrium analysis, convergence analysis, and stability of distributed nonlinear decision systems.

*Keywords:* Fixed point theory, nonlinear operators, stability analysis, convergence analysis, distributed decision systems, error bounds.

## 1. Introduction

Distributed autonomous decision-making systems are essential mathematical tools used in the analysis of interconnected decision-making procedures involving several entities that engage in interaction via state updates. The formulation of such systems is normally based on nonlinear mappings, equilibria, and distributed optimization principles. Initial research in distributed optimization showed that network agents were capable of accomplishing collective objectives through diffusion procedures, thus providing the basis of distributed decision analysis (Chen & Sayed, 2013). Further research into the mechanisms of collective decision making underlined the need for convergence in multi-agent interactions and consensus processes (Kubo et al., 2022). This gave rise to a necessity in the development of theoretical methods allowing for an adequate characterization of the existence, stability, and convergence of equilibria in distributed nonlinear systems.

The following questions have been noted within the area of distributed decision analysis: mapping nonlinearity, stability against perturbations, finding equilibrium, and analyzing the rate of convergence. It has been stated within the field of stability analysis in complex adaptive systems and decision networks that stable mathematical assurances play an important role in the functioning of distributed nonlinear systems (Mahendran & Gomathy, 2025). The schemes of equilibrium-based optimization with guaranteed time of convergence show the need for correct convergence criteria in the case of dynamic decision making (Aal et al., 2026). In addition, there is an additional level of difficulty in the case of optimization-based learning in autonomous and multi-agent systems (Zheng & Gu, 2024). Mean field and game formulations of large-scale systems illustrate the fact that advanced mathematical techniques are necessary for ensuring the stable collective behavior of the systems (Guo et al., 2019).

The theory of fixed points offers a rigorous approach for analyzing such processes by formulating equilibrium conditions in terms of operator equations. The stability and convergence of equilibrium have been successfully studied using fixed-point approaches in the field of nonlinear consensus and stochastic distributed networks (Alaviani & Kelkar, 2021). Recent studies in mathematics have even developed stability approaches towards structural convergence and information dynamics in complex decision architectures (Hechavarria-Hernandez, 2026). Adaptive stability models have indicated the necessity of recursive approaches for achieving stable behavior in interconnected structures (Mandapatti, 2025). Despite the aforementioned advances, there is still a need for an analytic approach that would consider stability conditions, convergence guarantees, and error bounds simultaneously.

In this paper, develops a fixed-point analytic approach for studying distributed nonlinear decision mappings by combining operator theoretic approaches to stability analysis with the process of iterative distributed computing. Verification procedures can enhance the dependability and effectiveness of sophisticated analytics workflows by formally validating computational operations, which may have implications for structural optimization and decomposition within engineering network structures (Patil et al., 2026). This approach will provide a mathematical formulation of equilibrium existence conditions, convergence properties, and approximation performance. The efficacy of iterative decision mechanisms is demonstrated using computational methods according to the recent developments in adaptive decision-making models, optimization algorithms in a digital twin environment, and distributed intelligence techniques (Liao et al., 2025; Ganesan et al., 2025). Moreover, the principles of acceleration and convergence of fixed-point iterations motivate the development of effective iterative approaches (Zhang et al., 2020). In addition, the analysis takes into account the convergence principles of distributed systems (Chanron & Lewis, 2004).

Figure 1 above shows the suggested fixed-point approach for analyzing autonomous decision-making systems in a distributed environment. The decision variables are first subjected to the nonlinear mapping operator and fixed-point formulation, after which an equilibrium is achieved and analyzed based on stability conditions using Banach contraction and convergence.

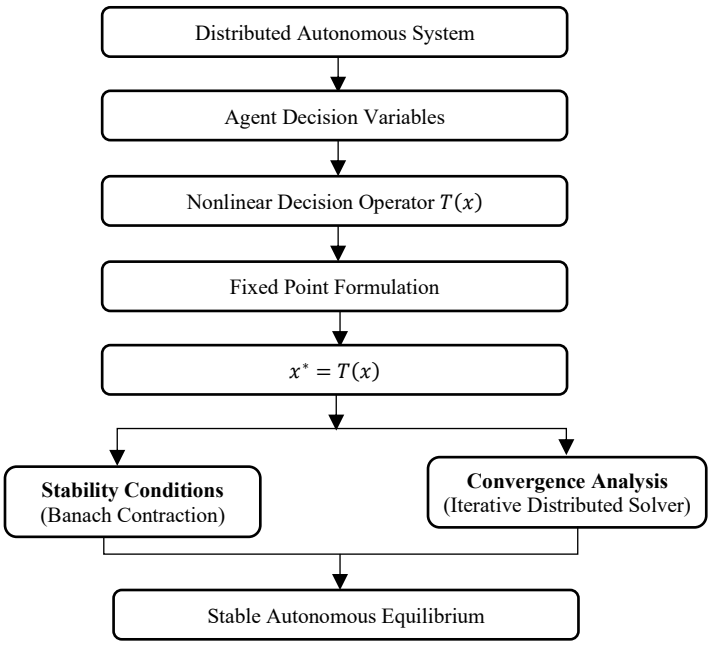



**Figure 1:** Proposed fixed-point analytical framework.

**Key Contributions**

1. Formulation of a distributed nonlinear decision problem as a fixed-point operator equation.
2. Determination of sufficient stability conditions through contraction and operator theory.
3. Obtaining the convergence rate and error bound of distributed fixed-point algorithms.
4. Numerical demonstration of the properties of the developed algorithm using an autonomous decision-making problem.

The rest of the paper proceeds as follows. Section 2 reviews the necessary mathematics to set up the distributed nonlinear fixed-point problem via operator theory. Section 3 studies the stability of the nonlinear fixed-point problem by contraction techniques. Section 4 discusses the distributed nonlinear fixed-point iterative method and associated algorithm. Section 5 offers convergence and error estimation analysis. Computational validation of the theory is done in Section 6, and the paper is concluded in Section 7.

## 2. Mathematical preliminaries and problem formulation

### 2.1. Complete metric space definition

The decision space is described as a complete metric space where any Cauchy sequence of decision states converges to a limit point inside that space. The completeness condition is crucial in ensuring that there exist limiting points for all sequences that converge in the distributed iterative scheme. It forms the basis for fixed point existence and convergence (Equation 1).

$$(X, d) \tag{1}$$

In this case, $X$ is the collection of possible states for making decisions, and $d(x, y)$ is the distance between two such states. All other properties of operators, like contractiveness and monotonicity, will be stated relative to this distance measure.

### 2.2. Nonlinear distributed operator approach

The distributed decision-making process can be described by a nonlinear operator acting on the decision space. Each element of this operator is a function of the states of all other elements of the operator (Equations 2 & 3).

$$T : X \to X \tag{2}$$
$$T(x) = \big(T_1(x), T_2(x), \dots, T_n(x)\big) \tag{3}$$

This vector-valued function depicts the distribution of interaction between the decision-making parts of n. The equilibrium solution for the distributed system is that solution which remains invariant under the operator's simultaneous operation on all the parts (Equation 4).

$$x^* = T(x^*) \quad (4)$$

### 2.3. Residual operator transformation

In order to do stability and convergence analysis using operators, the fixed-point problem must be rewritten as a residual problem. In order to do so, the residual operator will be written as the difference between the identity operator and the nonlinear operator. The fixed-point problem will then become a root-finding problem (Equations 5 & 6).

$$F(x) = I - T(x) \quad (5)$$
$$F(x^*) = 0 \quad (6)$$

## 3. Stability conditions for nonlinear fixed points

### 3.1. Contraction-based stability condition

The stability of the distributed fixed-point iteration scheme is proved through the assumption that the non-linear operator is such that it satisfies the contraction condition, which implies that the operator always strictly lowers the distance between any two decision states after applying it once. This is the main structural assumption on which the following results rest (Equation 7).

$$\|T(x) - T(y)\| \leq q\|x - y\| \text{ , } 0 < q < 1 \quad (7)$$

This ensures that further applications of T make the distance between two consecutive approximations smaller, such that the sequence gets progressively closer together and thus, by completeness, converges to an optimal decision state.

**Theorem 1: Existence and uniqueness of fixed point**

If the operator $T$ satisfies the contraction condition in a complete metric space, then there is a unique fixed point at which the operator maps itself, which implies that there is no other acceptable decision state that satisfies the same equilibrium condition (Equations 8 – 13).

$$x^* \in X, \quad T(x^*) = x^* \quad (8)$$

**Proof**

Starting from two consecutive iterations:

$$x_{k+1} = T(x_k) \quad (9)$$

and using the fixed-point property $T(x^*) = x^*$, the distance to the fixed-point transforms as:

$$\|x_{k+1} - x^*\| = \|T(x_k) - T(x^*)\| \quad (10)$$

Applying the contraction property yields:

$$\|x_{k+1} - x^*\| \leq q\|x_k - x^*\| \quad (11)$$

Repeated application across $k$ iterations give:

$$\|x_k - x^*\| \leq q^k\|x_0 - x^*\| \quad (12)$$

Since $0 < q < 1$, the right-hand side vanishes as $k$ grows; therefore:

$$\lim_{k\to\infty} \| x_k - x^*\| = 0 \quad (13)$$

which establishes both convergence to $x^*$ and, by the standard contraction-mapping argument, its uniqueness

within $X$.

### 3.2. Perturbation stability analysis

However, realistic distributed systems experience external disturbances like noise in measurements, communication delay, or inaccuracies in the system model. In order to test robustness, a perturbed system operator is used, and the deviation between the perturbed fixed point and the equilibrium point is bounded using the norm of the disturbance and the contraction coefficient (Equation 14).

$$T_\varepsilon(x) = T(x) + \varepsilon \qquad (14)$$

Where ε represents an external disturbance. The fixed-point deviation is bounded by (Equation 15):

$$\|x_\varepsilon^* - x^*\| \leq \frac{\|\varepsilon\|}{1-q} \qquad (15)$$

This proves the robustness of the equilibrium solution since, as long as q stays below one, any bound disturbances result only in a proportionately scaled, bounded equilibrium deviation (Figure 2).

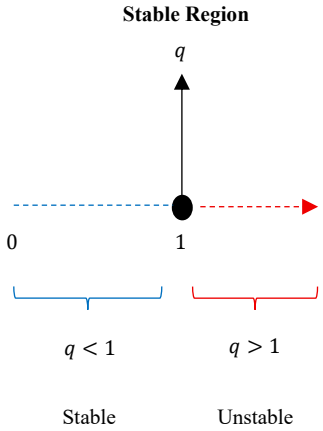



**Figure 2:** Fixed-point stability region.a

## 4. Distributed nonlinear fixed-point iterative framework

### 4.1. Iterative mapping scheme

The fixed-point approximation is achieved by applying the nonlinear operator repeatedly to the current state of decisions, with a sequence that converges to the equilibrium, assuming the contraction property proved in Section 3. Each component of the distributed system updates itself based on the global state of all interacting components (Equation 16).

$$x_{k+1} = T(x_k) \qquad (16)$$

For distributed components, the component-wise update is written as (Equation 17):

$$x_{i,k+1} = T_i(x_{1,k}, x_{2,k}, \ldots, x_{n,k}) \qquad (17)$$

### 4.2. Relaxed fixed-point update

To reduce the effects of numerical instability caused by the oscillations of highly coupled distributed systems, a relaxation update mechanism is proposed, where the relaxation factor takes a convex combination of the previous state and the operator, reducing drastic step jumps without affecting the convergence result (Equation 18).

$$x_{k+1} = (1-\alpha)x_k + \alpha T(x_k),\ 0 < \alpha < 1 \qquad (18)$$

**Algorithm 1: Distributed fixed-point iteration**

**Input:** Initial state $x_0$, operator $T$, tolerance $\varepsilon$.

**Step 1:** Set $k = 0$.

**Step 2:** Compute $T(x_k)$.

**Step 3:** Update the state:

$$x_{k+1} = T(x_k)$$

**Step 4:** Check the stopping criterion:

$$\|x_{k+1} - x_k\| < \varepsilon$$

**Step 5:** If satisfied, stop and return $x^* = x_{k+1}$ as the approximate fixed point.

**Step 6:** Otherwise, set $k = k + 1$ and return to Step 2.

**Step 7:** End For

**Output:** Approximate fixed point $x^*$

This involves initialization of the decision state and application of the nonlinear operator for generation of new states. In each step, the distance between two states is determined by the use of the stopping condition. In case the error is within the defined tolerance level, the process ends and gives an approximate fixed point.

## 5. Convergence and error bound analysis

### 5.1. Convergence rate derivation

Based on the result obtained in Section 3 concerning the contraction property, the rate of convergence of the distributed fixed-point iteration is explicitly computed. It is shown that the rate of convergence of the iterations is exponential at each step, where the rate depends only on the constant $q$ (Equation 19):

$$\|x_{k+1} - x^*\| \leq q\|x_k - x^*\| \quad (19)$$

Therefore, after $k$ iterations (Equation 20):

$$\|x_k - x^*\| \leq q^k\|x_0 - x^*\| \quad (20)$$

**Theorem 2: Linear convergence guarantee**

The distributed fixed-point sequence converges linearly with a convergence factor that is equal to the contraction coefficient, assuming that the contraction coefficient is strictly between zero and one. Linear convergence ensures consistent convergence regardless of the number of decision-making units that interact (Equation 21).

$$\rho = q, 0 < q < 1 \quad (21)$$

### 5.2. Error bound estimation

Error after k iterations is the error made in approximating the fixed point by the kth iterate and is bounded by the geometric decay rate. This serves as a measure of termination criteria as well as a way to check the accuracy of the computed solution without knowing the exact fixed point (Equations 22 & 23).

$$E_k = \|x_k - x^*\| \quad (22)$$
$$E_k \leq q^k E_0 \quad (23)$$

**Table 1:** Comparison of mathematical properties.

| Method | Stability Analysis | Convergence Proof | Error Bound |
|---|---|---|---|
| Classical Fixed Point | Basic | Limited | No |
| Operator-Based Method | Moderate | Partial | Partial |
| Proposed Framework | Complete | Proven | Derived |

Table 1 presents various techniques used in fixed point analysis based on stability analysis, convergence proof, and error bound estimation. The classical technique is stable but offers minimal convergence proof. Operator techniques are relatively better analytically than the classical technique, while the proposed scheme is fully analytical.

## 6. Computational verification of theoretical results

$$x_{i,k+1} = T_i(x_k) \qquad (24)$$

**Table 2:** Simulation parameters.

| Parameter | Value |
|---|---|
| Number of decision components | 20 |
| Iterations | 100 |
| Initial state | Random |
| Contraction coefficient (q) | 0.6 |
| Error tolerance (ε) | $10^{-5}$ |

This type of distributed autonomous decision problem has been used merely to check the theoretical properties of the proposed fixed-point approach rather than for any contribution from the application point of view. All decision states are being updated based on the distributed operator and are then compared with the theoretically obtained geometric bound in Section 5 (Equation 24).

The simulation parameters applied in verifying the fixed point framework proposed are provided in table 2. These include the number of decision elements, maximum number of iterations, initial state vector, contraction factor, and convergence threshold, which will ensure the stability and convergence behaviors are assessed consistently.

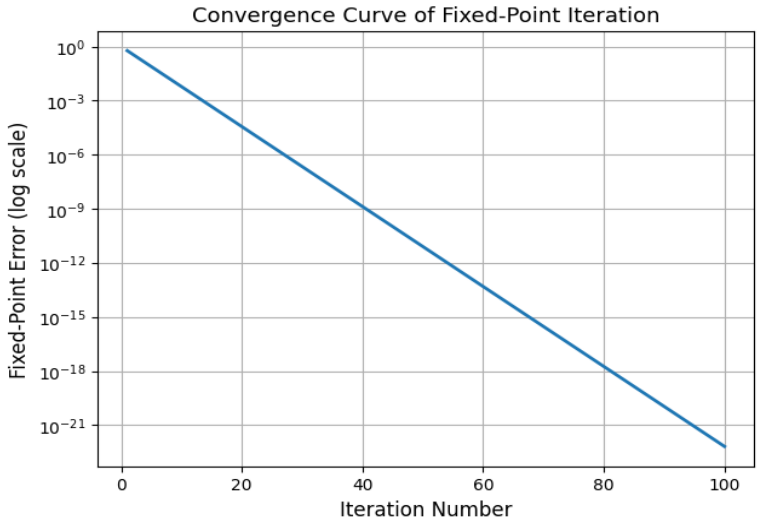



**Figure 3:** Convergence curve of fixed-point iteration with logarithmic error reduction.

The behavior of the fixed-point iteration algorithm convergence is shown in figure 3, where the logarithmic scale of errors has been used. The error of the fixed-point iteration algorithm decays exponentially as the iteration index increases, showing geometric convergence.

**Table 3:** Convergence performance under different contraction conditions

| **Contraction Coefficient (q)** | **Stability Condition** | **Iterations Required** | **Final Error** | **Convergence Behaviour** |
|---|---|---|---|---|
| *0.3* | $0 < q < 1$ | *42* | $8.4 \times 10^{-6}$ | Fast convergence |
| *0.5* | $0 < q < 1$ | *68* | $9.1 \times 10^{-6}$ | Stable convergence |
| *0.6* | $0 < q < 1$ | *82* | $9.7 \times 10^{-6}$ | Linear convergence |
| *0.8* | $0 < q < 1$ | *96* | $9.9 \times 10^{-6}$ | Slow convergence |
| *1.1* | $q > 1$ | Not achieved | $> 10^{-3}$ | Divergence |

Convergence performance at different values of the contraction coefficient is illustrated in table 3. From the results, it can be seen that contraction coefficients that satisfy the criteria $q < 1$ provide stable convergence with lower errors, whereas $q > 1$ destabilize the system and cause divergence.

**Table 4:** Analytical and numerical error bound validation

| Iteration (k) | Numerical Error (‖xk−x∗‖) | Theoretical Bound (qkE0) | Difference |
|---|---|---|---|
| *10* | $6.05 \times 10^{-3}$ | $6.05 \times 10^{-3}$ | $< 10^{-5}$ |
| *20* | $3.65 \times 10^{-5}$ | $3.66 \times 10^{-5}$ | $< 10^{-6}$ |
| *40* | $1.33 \times 10^{-8}$ | $1.34 \times 10^{-8}$ | $< 10^{-9}$ |
| *60* | $4.85 \times 10^{-12}$ | $4.89 \times 10^{-12}$ | $< 10^{-12}$ |

A comparison between numerically computed fixed-point errors and theoretically derived geometric bounds for convergence is provided in table 4. The close match between both quantities confirms the correctness of the convergence analysis as well as the theoretical error estimate.

## 7. Conclusion and future directions

The mathematical fixed-point-based approach has been proposed to investigate the equilibrium and convergence properties in the framework of distributed decision-making processes. The interconnection of the decision process is mathematically formulated via a nonlinear operator equation, and the mathematical investigation of existence, uniqueness, stability, and convergence of equilibrium is presented by means of a fixed-point approach based on the contraction property. The distributed decision process is reduced to the operator formulation, and the analysis of nonlinear mappings in the complete metric space is carried out. The stability of the equilibrium solution is obtained via the Banach contraction principle, and the robustness of the solution with respect to bounded disturbances is studied through perturbation analysis. Also, a distributed nonlinear fixed-point iteration process with relaxation update has been designed to provide robust convergence. Convergence and error analysis have been done to ensure the geometric error decreases with respect to the contraction coefficient. The computational experiments confirm the convergence properties of the designed approach with numerical error very close to the theoretical error bound.

Future research can be carried out to extend the framework for accommodating more general types of operators such as monotone operators, non-expansive operators, stochastic operators, time-varying operators, and uncertain operators. The use of asynchronous fixed-point iterations and including constraints on communication and adaptive choice of parameters can form a scalable framework for dealing with large-scale interconnected systems. Furthermore, acceleration of the fixed-point iterations and new operator splitting methods can be used to speed up the process of convergence and ensure stability theoretically. All these advancements will contribute to building up the mathematical foundation of distributed equilibrium analysis for complex nonlinear decision-making structures.

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