

On the Turnpike Property for Mean Field Games

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We consider the behavior of mean field games systems in the long horizon, under the assumption of monotonicity of the coupling term. Assuming that the Hamiltonian is globally Lipschitz and locally uniformly convex, we show that the time dependent solution is exponentially close to the ergodic stationary state in the long intermediate stages. This is evidence of the so-called exponential turnpike property for optimal control problems. Indeed, our proof follows a general approach which relies on the stabilization through the Riccati feedback of the associated linearized system.

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1. Introduction

The long time behavior of mean field games (MFG) systems has been extensively studied in the recent years (see e.g. [6]–[7] for finite states mean field games and [2], [3], [4] for the continuous case). Specifically, many papers have studied the behavior, as the horizon T tends to infinity, of the solutions (u^T, m^T) of the simplest form of MFG system:

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m(t)), & t \in (0, T) \\ m_t - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0, & t \in (0, T) \\ m(x, 0) = m_0(x), \quad u(x, T) = G(x, m(T)) \end{cases} \quad (1)$$

defined for x in the flat torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. The typical result which is proved is the stabilization towards the solution (\bar{u}, \bar{m}) of the stationary ergodic mean field game problem

$$\begin{cases} \bar{\lambda} - \Delta \bar{u} + H(x, D\bar{u}) = F(x, \bar{m}), & x \in \mathbb{T}^d \\ -\Delta \bar{m} - \operatorname{div}(\bar{m} H_p(x, D\bar{u})) = 0, & x \in \mathbb{T}^d \\ \int_{\mathbb{T}^d} \bar{u} dx = 0, \int_{\mathbb{T}^d} \bar{m} dx = 1. \end{cases} \quad (2)$$

in which the unknowns are now the ergodic constant $\bar{\lambda}$ and the functions \bar{u}, \bar{m} defined on \mathbb{T}^d .

This kind of convergence is quite natural from the viewpoint of stochastic control; however, compared to what happens for the long time behavior of a single Hamilton-Jacobi equation, the stabilization of MFG systems takes different forms and needs different tools in order to be proved. In particular, the dependence of the cost criterion from the distribution of the agents makes it impossible here to neglect the long time behavior of the optimal policy, namely the behavior of Du^T which drives the long time convergence of the Fokker-Planck equation. In addition, the forward-backward structure leads to a quite different statement of this convergence, since the proximity between time-dependent and stationary solutions is observed up to a possible boundary layer at initial and final time. One form of this statement is the following estimate which was proved, under different kind of assumptions, in [2], [3], [4]:

$$\|\tilde{u}^T - \bar{u}\| + \|m^T - \bar{m}\| \leq K(e^{-\omega t} + e^{-\omega(T-t)}) \quad \forall t \in [0, T] \quad (3)$$

where \tilde{u}^T is the u^T renormalized with zero average, and the norms may be weaker or stronger depending whether we are asking the coupling $F(x, m)$ to be a smoothing nonlocal operator or a purely local function. Notice that $\|\tilde{u}^T - \bar{u}\|$ could be replaced by $\|Du^T - D\bar{u}\|$ in some possibly different norm, meaning that we are giving here an estimate on the optimal feedbacks together with the distribution measures.

An estimate such as (3) is proved with different arguments in previous papers, either exploiting the strong monotonicity of the coupling ([2]) or using the linearized system around (\bar{u}, \bar{m}) ([3], [4]). In the recent paper [4], we also show that an exponential estimate as (3) is most important in order to handle the long time behavior of the master equation and a characterization of the convergence of $u^T - \bar{\lambda}(T-t)$ for any time scale t (whereas (3) gives significant information only for time scale $t \simeq \delta T$).

In the present paper, we show a different strategy for the proof of the exponential estimate (3), which is related to general methods for getting an exponential turnpike property of optimality systems. The turnpike property of optimal control problems is an old question, which was revisited in the recent literature since the papers [13], [14]. The terminology itself dates back to the Nobel laureate in economics Paul A. Samuelson, who wanted to stress that in a long horizon optimization, if the cost criteria are robust, optimal controls and trajectories are nearly stationary: *if we are planning long-run growth, no matter where we start, and where we desire to end up, it will pay in the intermediate stages to get into a steady growth phase* ([5]).

In this context, the *turnpike* is, by analogy, the stable stationary path which is convenient to take for a long intermediate time. This is a very clear description of what happens for the MFG system (1) under monotonicity assumption of the coupling F . Of course, this is not by chance: it is well known that the system (1) can be recast in the form of an optimality system for a convex functional (see e.g. [10]), so an estimate as (3) should be read as a manifestation of the turnpike property for the MFG system.

In order not to overlap with results proved before, we consider the case of Lipschitz Hamiltonians and local couplings. This is a simple but relevant situation, including the case that the set of controls is compact. Moreover, this setting does not seem to have been analyzed in previous results on the long time behavior, which were mostly considering nonlocal and smoothing couplings and coercive Hamiltonians.

In short terms, we prove in this paper that, if the Hamiltonian is Lipschitz continuous and locally uniformly convex, and if the nonlinearity F is locally Lipschitz continuous and nondecreasing, then the exponential turnpike property holds true, in the form of estimate (3). Some more technical regularity requirement upon F and H are required, and the precise statement is left to the next section.

The strategy that we adopt relies on the study of the linearized system around the stationary state (\bar{m}, \bar{u}) . In order to exploit the exponential stabilization of this linear system we borrow the approach developed in [13]. Thus, to describe it shortly, our proof of (3) will come out essentially through the following steps:

(a) We analyze the MFG system linearized around (\bar{u}, \bar{m}) , namely

$$\begin{cases} \mu_t^T - \Delta \mu^T - \operatorname{div}(\mu^T H_p(x, D\bar{u})) = \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv^T), & t \in (0, T) \\ -v_t^T - \Delta v^T + H_p(x, D\bar{u}) Dv^T = F'(\bar{m}) \mu^T, & t \in (0, T) \\ \mu^T(0) = \mu_0, \quad v^T(T) = 0. \end{cases}$$

We show that this system defines a feedback operator as

$$\mathcal{E}(T)\mu_0 := v^T(0) - \langle v^T(0) \rangle$$

and this operator $\mathcal{E}(T)$ is bounded and converges, as $T \rightarrow \infty$, to the corresponding infinite horizon operator $\hat{E}\mu_0 := \hat{v}(0) - \langle \hat{v}(0) \rangle$, where

$$\begin{cases} \hat{\mu}_t - \Delta \hat{\mu} - \operatorname{div}(\hat{\mu} H_p(x, D\bar{u})) = \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) D\hat{v}) & t \in (0, \infty) \\ -\hat{v}_t - \Delta \hat{v} + H_p(x, D\bar{u}) D\hat{v} = F'(\bar{m}) \hat{\mu} & t \in (0, \infty) \\ \hat{\mu}(0) = \mu_0. \end{cases} \quad (4)$$

Notice that $\mathcal{E}(T - t)$ and \hat{E} are nothing but the classical evolutionary and stationary Riccati feedback operators. Indeed, by time-shifting, we have $v^T(t) = \mathcal{E}(T - t)\mu^T(t)$, and $\hat{v}(t) = \hat{E}\hat{\mu}(t)$. The crucial point here is to prove the exponential estimate

$$\|\mathcal{E}(t) - \hat{E}\| \leq c e^{-\omega t}$$

as a consequence of the exponential decay of the infinite horizon problem. This is also where the exponential rate ω for the original MFG system comes from, namely ω is the decay rate of the semigroup generated by the evolution equation

$$\begin{aligned} \hat{\mu}^T - \Delta \hat{\mu} - \operatorname{div}(\hat{\mu} b(x)) &= \operatorname{div}(M(x) D\hat{E}\hat{\mu}), & t \in (0, \infty), \\ b(x) &:= H_p(x, D\bar{u}(x)), & M(x) &:= \bar{m}(x) H_{pp}(x, D\bar{u}(x)). \end{aligned}$$

(b) Given any f_1, f_2, g , we show that the unique solution to the nonhomogeneous system

$$\begin{cases} \mu_t^T - \Delta \mu^T - \operatorname{div}(\mu^T H_p(x, D\bar{u})) = \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv^T) + f_1 & t \in (0, T), \\ -v_t^T - \Delta v^T + H_p(x, D\bar{u}) Dv^T = F'(\bar{m}) \mu^T + f_2 & t \in (0, T), \\ \mu^T(0) = \mu_0, \quad v^T(T) = g \end{cases} \quad (5)$$

satisfies

$$\|v^T(t) - \mathcal{E}(T-t)\mu^T(t)\| \leq c e^{-\omega(T-t)} \|g\| + c \int_t^T e^{-\omega(s-t)} (\|f_1(s)\| + \|f_2(s)\|) ds.$$

(c) We look at the original system

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m(t)), & t \in (0, T) \\ m_t - \Delta m - \operatorname{div}(m H_p(x, Du)) = 0, & t \in (0, T) \\ m^T(0) = \bar{m} + \mu_0, \quad u^T(T) = \bar{u} + v_T. \end{cases}$$

as a perturbation of the linearized one, as in (5). Thanks to the estimate of the previous step, through a fixed point we construct a solution which satisfies the exponential estimate

$$\|\tilde{u}^T - \bar{u}\| + \|m^T - \bar{m}\| \leq K(e^{-\omega t} + e^{-\omega(T-t)}) \quad \forall t \in [0, T]$$

provided μ_0 and v^T are sufficiently small.

(d) By uniqueness and time-shifting, the solution (u^T, μ^T) of (1) satisfies a similar estimate provided, at *some* time t , $\tilde{u}^T(T-t)$ and $m^T(t)$ are close to \bar{u}, \bar{m} respectively. In this last step, we show that this is true as a consequence of the general stability of the MFG system, induced by monotone couplings.

Of course, the method we describe here can be applied to more general cases, and in particular whenever a uniform bound (global in time) for Du^T is known. Moreover, as developed in [4], one could stand on the exponential estimate in order to go further in the analysis of the long time behavior, specifically looking at the convergence of $u^T - \bar{\lambda}(T-t)$. However, we postpone this further step since it would require a much more technical part involving a deeper study of the associated master equation.

2. Notation, assumptions and statement of the result

Throughout the paper we work with periodic (in space) functions. Normalizing the period, we assume that all functions involved are \mathbb{Z}^d -periodic, we denote by $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ the usual flat torus and we work directly with functional spaces defined on the torus Ω . Namely, the spaces $L^2, H^1, H^2, W^{1,q}$ are the common Lebesgue or Sobolev spaces defined on Ω . The space $W^{1,\infty}(\Omega)$ coincides with the space of Lipschitz continuous functions on the torus. We will write $\|\cdot\|_2, \|\cdot\|_{H^1}$,

etc. for the usual norms in these spaces. For shortness, we will write $\|\cdot\|_p$ for the L^p and $\|\cdot\|_\infty$ for the L^∞ norm. We also denote by $\mathcal{P}(\Omega)$ the space of probability measures in Ω .

We use the notations: $\langle f \rangle = \frac{1}{|\Omega|} \int_\Omega f$, $\tilde{f} := f - \langle f \rangle$. $L_0^2(\Omega)$ denotes the subspace of L^2 functions with zero average in Ω and (f, g) the standard scalar product in $L^2(\Omega)$.

We now make precise the set of our assumptions. For the coupling function F , we assume that:

$$F \in C^1(\mathbb{T}^d \times \mathbb{R}) \text{ and } m \mapsto F(x, m) \text{ is non decreasing.} \tag{6}$$

We denote by F_x, F_m , etc.. the partial derivatives of the function F . We further assume that

$$\left\{ \begin{array}{l} \forall m \in \mathbb{R}, F_m(x, m) \text{ is differentiable with respect to } x, \text{ and } F_m, F_{mx} \text{ are} \\ \text{locally Lipschitz with respect to } m, \text{ uniformly in } x, \text{ i.e. } \forall K > 0, \exists L_K > 0: \\ |F_m(x, m) - F_m(x, m')| + |F_{mx}(x, m) - F_{mx}(x, m')| \leq L_K |m - m'| \\ \text{for every } x \in \mathbb{T}^d, m, m' \in [-K, K]. \end{array} \right. \tag{7}$$

As far as the Hamiltonian H is concerned, we assume that $H \in C^2(\mathbb{T}^d \times \mathbb{R})$ is globally Lipschitz:

$$\exists \beta > 0 : |H_p(x, p)| \leq \beta \quad \forall x \in \mathbb{T}^d, p \in \mathbb{R}^d \tag{8}$$

and locally uniformly convex in p :

$$\forall K > 0 \exists c_K > 0 : H_{pp}(x, p) \geq c_K Id, \quad \forall x \in \mathbb{T}^d, p \in \mathbb{R}^d \text{ with } |p| \leq K, \tag{9}$$

where again we denote by H_p, H_{pp} etc..the partial derivatives. We also need some more smoothness assumption upon H , namely:

$$\left\{ \begin{array}{l} \forall p \in \mathbb{R}^d, H_{pp}(x, p) \text{ is differentiable with respect to } x, \text{ and } H_{pp}, H_{ppx} \text{ are} \\ \text{locally Lipschitz with respect to } p, \text{ uniformly in } x, \text{ i.e. } \forall K > 0, \exists L_K > 0: \\ |H_{pp}(x, p) - H_{pp}(x, p')| + |H_{ppx}(x, p) - H_{ppx}(x, p')| \leq L_K |p - p'| \\ \text{for every } x \in \mathbb{T}^d, p, p' \in \mathbb{R}^d \text{ with } |p|, |p'| \leq K. \end{array} \right. \tag{10}$$

Finally, there is no loss of generality in assuming the normalization condition $H(x, 0) = 0$, since this can always be the case up to a modification of the function F .

Let us point out that, under assumptions (6) and (8), and assuming the convexity of $H(x, p)$ with respect to p , there exists a unique triple $(\bar{\lambda}, \bar{u}, \bar{m})$ solution of the stationary ergodic problem (2). The existence can be easily proved using standard fixed point theorems, e.g. by freezing m in the HJB equation and looking at a fixed point $m \mapsto u \mapsto m$. Since the Hamiltonian is globally Lipschitz, the drift in the Fokker-Planck equation is uniformly bounded (independently of u),

therefore the map $u \rightarrow m$ has range in a bounded, and actually compact, subset of continuous functions. The existence of a fixed point is trivial, in this setting. Uniqueness comes from the monotonicity of F and the convexity of H , as usual for MFG systems of this kind. For the interested reader, similar results for ergodic stationary problems in a more general context, even if with nonlocal smoothing functions F , can be found in [1].

Moreover, with bootstrap arguments, one can prove that the functions \bar{u}, \bar{m} are smooth (and $\bar{m} > 0$ in Ω) under the above smoothness conditions assumed upon F and H . In particular, (7) and (10) imply that $F_m(x, \bar{m}(x)), H_p(x, D\bar{u}(x)), H_{pp}(x, D\bar{u}(x)) \in W^{1,\infty}(\Omega)$; such a kind of regularity will be used in our study of the linearized problem.

As far as the initial-terminal data for (1) are concerned, it is enough to our purposes to assume that

$$G \in C^0(\mathbb{T}^d \times \mathbb{R}), \quad m_0 \in \mathcal{P}(\Omega). \quad (11)$$

Under this condition, the existence of a solution to the finite horizon problem (1) can be proved, again, by fixed point. Indeed, for any given probability measure m_0 , and any given u , the Fokker-Planck equation in (1) admits a (unique) weak solution which satisfies $\|m(t)\|_\infty \leq ct^{-d/2}$ by properties of the heat semigroup with bounded drift. This implies easily that a fixed point can be constructed for the system (1), and solutions will be smooth for $t \in (0, T)$ bootstrapping usual parabolic regularity (see also Lemma 6.1 later on).

In the following, by a solution to (1) we mean a solution of this kind. Precisely, we mean that u and m are classical solutions in $(0, T)$ and weak solutions in, respectively, $(0, T]$ and $[0, T)$, which gives sense to the initial-terminal data. Whenever G is nondecreasing and $m_0 \in L^\infty(\mathbb{T}^d)$ (beyond the convexity of H and monotonicity of F), a solution of this kind is also unique.

However, we stress that the long time behavior described below applies to *any* weak solution of (1), for *any* choice of G, m_0 satisfying (11). In particular, the monotonicity of G does not play any role. This is not surprising since this is exactly the spirit of the turnpike property, whereas initial and terminal conditions are irrelevant.

Eventually, the result that we prove in this paper is the following.

Theorem 2.1. *Assume that (6)–(11) hold true. Then there exists $\omega > 0$ and a constant K , independent of T , such that any solution of (1) satisfies for all $t \in [1, T - 1]$ the inequality*

$$\|\tilde{u}^T(t) - \bar{u}\|_{H^2 \cap W^{1,\infty}} + \|m^T(t) - \bar{m}\|_{H^1 \cap L^\infty} \leq K(e^{-\omega t} + e^{-\omega(T-t)}). \quad (12)$$

Throughout the proofs we denote by C possibly different constants depending on the data F, G, H , but not on T . Note that quantities depending on \bar{m} and \bar{u} can be considered as depending on F and H .

3. The linearized system

For shortness of notation, we set $F'(\bar{m}) = F'_m(x, \bar{m}(x))$. In this section we study the linearized system

$$\begin{cases} \mu_t^T - \Delta \mu^T - \operatorname{div}(\mu^T H_p(x, D\bar{u})) = -\operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv^T) & t \in (0, T) \\ -v_t^T - \Delta v^T + H_p(x, D\bar{u}) Dv^T = F'(\bar{m}) \mu^T & t \in (0, T) \\ \mu^T(0) = \mu_0, \quad v^T(T) = 0 \end{cases} \quad (13)$$

where we assume that $\mu_0 \in L^2_0(\Omega)$.

Existence of solutions to (13) can be proved, as it is usually done with MFG systems, through a fixed point argument. In particular, it can be deduced from the following a priori estimates. Let us point out that solutions of (13) exist and are unique in the energy space, i.e. if $\mu, v \in L^2([0, T]; H^1(\Omega))$ are weak solutions. In fact, as we will see later (see e.g. Proposition 4.3), solutions turn out to be smooth inside the interval $(0, T)$ because of bootstrap arguments.

Lemma 3.1. *Assume that $F'(\bar{m}) \in C^0(\bar{\Omega})$ and is non negative. Then, there exists a constant $C > 0$ (independent of T) such that the solution of (13) satisfies*

$$\|\mu^T(t)\|_2^2 + \|\tilde{v}^T(t)\|_{H^1}^2 + \int_0^T \|Dv^T(t)\|_2^2 dt \leq C \|\mu_0\|_2^2.$$

If in addition $F'(\bar{m}), H_p(x, D\bar{u}), H_{pp}(x, D\bar{u}) \in W^{1,\infty}(\Omega)$, then there exists a constant $C > 0$, independent of T , such that

$$\|Dv^T(0)\|_2 + \|D^2v^T(0)\|_2 \leq C \|\mu_0\|_2.$$

Proof. From the system (13), we have

$$(v^T(0), \mu_0) = \int_0^T \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) Dv^T Dv^T + \int_0^T \int_{\Omega} F'(\bar{m}) \mu^T \mu^T. \quad (14)$$

By Lemma 7.4 and the ellipticity of $H_{pp}(x, D\bar{u})$, we have that

$$\|\mu^T(t)\|_2^2 \leq c \int_0^T \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) Dv^T Dv^T + c \|\mu_0\|_2^2, \quad (15)$$

hence
$$\|\mu^T(t)\|_2^2 \leq c (v^T(0), \mu_0) + c \|\mu_0\|_2^2. \quad (16)$$

Similarly, using Lemma 7.2, we have

$$\|v^T(t) - \langle v^T(t) \rangle\|_2 \leq c \int_t^T \|F'(\bar{m}) \mu^T(s)\|_2 e^{-\sigma(s-t)} ds. \quad (17)$$

Applying this inequality with $t = 0$ implies

$$\|v^T(0) - \langle v^T(0) \rangle\|_2 \leq c \int_0^T \|\mu^T(s)\|_2 e^{-\sigma s} ds \leq c [|(v^T(0), \mu_0)| + \|\mu_0\|_2^2]^{\frac{1}{2}},$$

which yields, because μ_0 has zero average,

$$\|v^T(0) - \langle v^T(0) \rangle\|_2 \leq c \|\mu_0\|_2.$$

Coming back to (16), we deduce that $\|\mu^T(t)\|_2 \leq c \|\mu_0\|_2$ for all $t \geq 0$.

As a consequence of (14) and (17), we also deduce that

$$\|v^T(0) - \langle v^T(0) \rangle\|_2^2 + \int_0^T |Dv^T|^2 dxdt \leq c \|\mu_0\|_2^2, \quad \forall t \geq 0.$$

Now we use Lemma 7.3 to estimate Dv^T and D^2v^T . We know that, for all $t < t_0$,

$$(t_0 - t) \|Dv^T(t)\|_2^2 \leq c[(t_0 - t) + 1] \{ \|\tilde{v}^T(t_0)\|_2^2 + \|F'(\bar{m})\mu^T\|_{L^2((t,t_0) \times \Omega)} + \|\tilde{v}^T\|_{L^2((t,t_0) \times \Omega)} \}.$$

Applied with $t_0 = 2$, using (17) with $t = 2$, and the uniform L^2 bound for μ^T , this yields

$$\|Dv^T(t)\|_2^2 \leq c \|\mu_0\|_2^2 \quad \forall t \leq 1. \tag{18}$$

Moreover, since $H_{pp}(x, D\bar{u})Dv^T$ is bounded in $L^2((0, T) \times \Omega)$, standard energy estimates for the Fokker-Planck equation imply

$$\int_0^1 \int_{\Omega} |D\mu^T|^2 \leq c \|\mu_0\|_2^2. \tag{19}$$

Now we consider the derivative of the equation of v^T , namely $w := \partial_{\xi}v^T$ satisfies

$$-\partial_t w - \Delta w + Dw \cdot H_p(x, D\bar{u}) = -\partial_{\xi}(H_p(x, D\bar{u})) \cdot Dv^T + \partial_{\xi}(F'(\bar{m})\mu^T)$$

so that applying again Lemma 7.3 to w in $(0, 1)$ we get

$$\|Dw(t)\|_2^2 \leq c \left\{ \|\partial_{\xi}v^T(1)\|_2^2 + \|\partial_{\xi}(F'(\bar{m})\mu^T)\|_{L^2((0,1) \times \Omega)}^2 + \|Dv^T\|_{L^2((0,1) \times \Omega)}^2 \right\}.$$

Using (18)–(19) and the $W^{1,\infty}$ assumption on $F'(\bar{m})$, we conclude that

$$\|D^2v^T(0)\|_2^2 \leq c \|\mu_0\|_2^2. \quad \square$$

Associated to the linearized system (13), we define now the following operator $\mathcal{E}(T)$, defined on the space $L_0^2(\Omega)$ as

$$\mathcal{E}(T)\mu_0 := v^T(0) - \langle v^T(0) \rangle. \tag{20}$$

The crucial point of our strategy is to establish the long time behavior of the operator $\mathcal{E}(T)$. To this purpose, we introduce the infinite horizon problem

$$\begin{cases} \hat{\mu}_t - \Delta \hat{\mu} - \operatorname{div}(\hat{\mu}H_p(x, D\bar{u})) = \operatorname{div}(\bar{m}H_{pp}(x, D\bar{u})D\hat{v}) & t \in (0, \infty) \\ -\hat{v}_t - \Delta \hat{v} + H_p(x, D\bar{u})D\hat{v} = F'(\bar{m})\hat{\mu} & t \in (0, \infty) \\ \hat{\mu}(0) = \mu_0, \quad \langle \hat{v}(0) \rangle = 0, \\ D\hat{v} \in L^2((0, \infty) \times \Omega), \hat{\mu} \in L^\infty((0, \infty); L^2(\Omega)) \end{cases} \tag{21}$$

where last conditions are introduced so that the problem be well posed. Indeed, we will first prove that (21) admits a unique solution.

Lemma 3.2. *Problem (21) has a unique solution.*

Proof. We first prove that (21) admits a solution, obtained as limit of solutions of (13) as $T \rightarrow \infty$. For this purpose, we take any sequence $T_n \rightarrow \infty$ and set

$$v_n(t) := v^{T_n}(t) - \langle v^{T_n}(0) \rangle, \quad \mu_n(t) := \mu^{T_n}(t).$$

From the estimates of Lemma 3.1, $v_n(0)$ converges in $L^2(\Omega)$ up to a subsequence. In addition, integrating the equation of v^T we have

$$\langle v_n(t) \rangle = \int_0^t \int_{\Omega} H_p(x, D\bar{u}) Dv_n - \int_0^t \int_{\Omega} F'(\bar{m}) \mu_n$$

and the bounds obtained so far imply $|\langle v_n(t) \rangle| \leq c\sqrt{t}$. Therefore, $\langle v_n(t) \rangle$ is bounded for every t and so v_n is bounded in $L^2((0, r); H^1)$ for any $r > 0$. Then, by standard compactness arguments, there exists a further subsequence (eventually constructed through a diagonal process), which we do not relabel, and functions μ, v such that $v_n \rightarrow v$ and $\mu_n \rightarrow \mu$ strongly in $L^2((0, r); H^1) \cap C^0([0, r]; L^2)$ for every $r > 0$.

As a consequence of the estimates obtained before, we have:

$$Dv \in L^2((0, \infty) \times \Omega), \quad \mu \in L^\infty((0, \infty); L^2), \\ F'(\bar{m})\mu \in L^1((0, \infty) \times \Omega), \quad \langle v(0) \rangle = 0,$$

so that (μ, v) is a solution to (21).

Now we wish to prove that (μ, v) is unique satisfying the above properties. To this purpose, let $(\mu_1, v_1), (\mu_2, v_2)$ be two solutions of (21), and call $\tilde{\mu} = \mu_1 - \mu_2, \tilde{v} = v_1 - v_2$. Consider a function $\xi_R(t) := \xi(t/R)$, where ξ is a C^1 function such that $\xi \equiv 1$ in $(0, 1)$ and with support in $(-1, 2)$. Using $\psi = (\mu_1 - \mu_2)\xi_R$ as test function in the equation of $v_1 - v_2$, we get

$$\int_0^\infty \int_{\Omega} F'(\bar{m}) \tilde{\mu} \tilde{\mu} \xi_R + \int_0^\infty \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}^T D\tilde{v} \xi_R = \frac{1}{R} \int_R^{2R} \int_{\Omega} \tilde{v} \tilde{\mu} \xi' \left(\frac{t}{R} \right) dt. \\ \text{Since } \frac{1}{R} \int_R^{2R} \int_{\Omega} \tilde{v} \tilde{\mu} \xi' \left(\frac{t}{R} \right) dt = \frac{1}{R} \int_R^{2R} \int_{\Omega} (\tilde{v} - \langle \tilde{v} \rangle) \tilde{\mu} \xi' \left(\frac{t}{R} \right) dt \leq \\ \leq \frac{C}{R} \int_R^{2R} \|D\tilde{v}\|_2 \|\tilde{\mu}\|_2 dt \leq C \left(\frac{1}{R} \int_R^{2R} \|D\tilde{v}\|_2^2 dt \right)^{\frac{1}{2}} \xrightarrow{R \rightarrow \infty} 0$$

letting $R \rightarrow \infty$ we conclude that

$$\int_0^\infty \int_{\Omega} F'(\bar{m}) \tilde{\mu} \tilde{\mu} + \int_0^\infty \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) D\tilde{v}^T D\tilde{v} = 0$$

hence $Dv_1 = Dv_2$. Since $\mu_1(0) = \mu_2(0)$, using the equations of μ_1, μ_2 , this implies that $\mu_1 = \mu_2$ in $(0, \infty)$. Now we have that $v_1 - v_2$ is only time-dependent and

$(v_1 - v_2)_t = 0$ in weak sense. Since $\langle v_1(0) \rangle = \langle v_2(0) \rangle = 0$, this means that $v_1 = v_2$. Therefore, problem (21) has a unique solution $(\hat{\mu}, \hat{v})$.

We also stress that the above estimates imply that the solution of (21) satisfies

$$\int_0^\infty \int_\Omega |D\hat{v}|^2 + \sup_{t \in (0, \infty)} \|\hat{\mu}(t)\|_2^2 \leq C \|\mu_0\|_2^2 \tag{22}$$

for some constant C independent of μ_0 . □

The well-posedness result of Lemma 3.2 allows us to define a stationary operator in $L_0^2(\Omega)$ associated to the solutions of (21), by setting

$$\hat{E}\mu_0 := \hat{v}(0).$$

Our next goal, which is also the main result of this Section, is to prove that $\mathcal{E}(T)$ converges to \hat{E} as $T \rightarrow \infty$, in the topology of $\mathcal{L}(L_0^2(\Omega), L_0^2(\Omega))$, the space of linear bounded operators in $L_0^2(\Omega)$.

Henceforth, to shorten notations, we will define the following operators:

$$\begin{aligned} A(z) &:= -\Delta z - \operatorname{div}(zH_p(x, D\bar{u})), & A^*(z) &= -\Delta z + Dz \cdot H_p(x, D\bar{u}), \\ K(z) &:= -\operatorname{div}(\bar{m}H_{pp}(x, D\bar{u})Dz), & Q(z) &:= F'(\bar{m})z, \end{aligned}$$

which can all be considered as unbounded operators in $L_0^2(\Omega)$.

By time shifting and the uniqueness property of Lemma 3.2, we notice that $\hat{E}(\hat{\mu}(t)) = \hat{v}(t) - \langle \hat{v}(t) \rangle$. Hence, the equation of $\hat{\mu}$ can be rephrased in abstract notation as

$$\begin{cases} \hat{\mu}_t + (A + K\hat{E})\hat{\mu} = 0 & t \in (0, \infty) \\ \hat{\mu}(0) = \mu_0. \end{cases}$$

In terms of semigroups of unbounded operators, we will set $M := A + K\hat{E}$ and we will denote by $e^{-Mt}\mu_0$ the unique solution $\hat{\mu}(t)$ of (21). We are going to show now that this semigroup decays exponentially. Note however that we are not going to use any abstract result of semigroup theory but we are giving a direct self-contained proof of the decay.

Lemma 3.3. *Assume that $F'(\bar{m}) \in C^0(\bar{\Omega})$ and is non negative. Then we have:*

- (i) *the operator $\mathcal{E}(t)$ is bounded and converges to \hat{E} as $t \rightarrow \infty$.*
- (ii) *The operator $M := A + K\hat{E}$ is exponentially stable in $L_0^2(\Omega)$, i.e. there exists $\omega > 0$ such that*

$$\|e^{-Mt}\|_{\mathcal{L}(L_0^2(\Omega), L_0^2(\Omega))} \leq e^{-\omega t} \quad \forall t > 0.$$

- (iii) *We have $\|\mathcal{E}(t) - \hat{E}\|_{\mathcal{L}(L_0^2(\Omega), L_0^2(\Omega))} \leq ce^{-\omega t}$ for some $c > 0$ independent of t .*

Proof. (i) Since μ_0 has zero average, we have $(\mathcal{E}(T)(\mu_0), \mu_0) = (v^T(0), \mu_0)$, so

$$(\mathcal{E}(T)(\mu_0), \mu_0) = \int_0^T \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) Dv^T Dv^T + \int_0^T \int_{\Omega} F'(\bar{m}) \mu^T \mu^T.$$

In particular, $\mathcal{E}(T)$ is a nonnegative operator due to the convexity of H and monotonicity of F . Moreover, from Lemma 3.1 we estimate

$$|(v^T(0), \mu_0)| \leq c \|\mu_0\|_2^2, \quad \text{hence} \quad \|\mathcal{E}(T)(\mu_0)\|_2 \leq c \|\mu_0\|_2.$$

Here and below, we denote by c generic possibly different positive constants (independent of T). Now, let T_n be any sequence such that $T_n \rightarrow \infty$. By the bounds obtained so far, $\mathcal{E}(T_n)(\mu_0)$ would admit a subsequence converging in $L_0^2(\Omega)$, and, after the stability exploited in Lemma 3.8, its limit cannot be other than $\hat{v}(0)$, where $(\hat{\mu}, \hat{v})$ is the unique solution of (21). This implies that there exists the limit of $\mathcal{E}(T)(\mu_0)$ as $T \rightarrow \infty$ and we have, for every $\mu_0 \in L_0^2(\Omega)$

$$\mathcal{E}(T)(\mu_0) \rightarrow \hat{E}(\mu_0) \quad \text{as } T \rightarrow \infty,$$

since $\hat{E}(\mu_0) = \hat{v}(0)$ by definition.

(ii) Let us consider the function

$$(\hat{E}(\hat{\mu}(t)), \hat{\mu}(t)) = \int_{\Omega} \hat{v}(t) \hat{\mu}(t).$$

This function is decreasing in $(0, \infty)$, by integration of (21). We also know that the (unique) solution of (21) satisfies estimate (22). Therefore there exists a sequence $s_j \rightarrow \infty$ such that $\|D\hat{v}(s_j)\|_2^2 \leq \frac{1}{s_j} \rightarrow 0$ and so

$$\int_{\Omega} \hat{v}(s_j) \hat{\mu}(s_j) = \int_{\Omega} (\hat{v}(s_j) - \langle \hat{v}(s_j) \rangle) \hat{\mu}(s_j) \rightarrow 0$$

by the Poincaré-Wirtinger inequality and since $\hat{\mu}(t)$ is uniformly bounded in $L_0^2(\Omega)$. Since we have

$$\int_{\Omega} \hat{v}(t) \hat{\mu}(t) \geq \int_{\Omega} \hat{v}(s_j) \hat{\mu}(s_j) \quad \forall t < s_j,$$

letting $j \rightarrow \infty$ we deduce that $\int_{\Omega} \hat{v}(t) \hat{\mu}(t) \geq 0 \quad \forall t > 0$. (23)

Now we define
$$e(t) := \sup_{\|\mu_0\|_2 \leq 1} \int_{\Omega} \hat{v}(t) \hat{\mu}(t).$$

This function is decreasing in $(0, \infty)$, so there exists $e_{\infty} := \lim_{t \rightarrow \infty} e(t)$, and for every n there exist $t_n \rightarrow \infty$, $\{\mu_{0n}\}$ (with $\|\mu_{0n}\|_2 \leq 1$) and the corresponding solutions $(\hat{v}_n, \hat{\mu}_n)$ such that

$$e_{\infty} - \frac{1}{n} \leq \int_{\Omega} \hat{v}_n(t_n) \hat{\mu}_n(t_n) \leq e_{\infty} + \frac{1}{n}. \tag{24}$$

We set $w_n := \hat{v}_n(t+t_n) - \langle \hat{v}_n(t_n) \rangle$, and $\rho_n := \hat{\mu}_n(t+t_n)$, and since μ_{0n} is bounded in $L^2_0(\Omega)$, by (22) we deduce that ρ_n is uniformly bounded in $L^\infty((-t_n, +\infty); L^2_0(\Omega))$ and Dw_n is uniformly bounded in $L^2((-t_n, +\infty); L^2_0(\Omega))$. As in Lemma 3.2, a compactness argument implies that, up to subsequences, (w_n, ρ_n) converges to a solution (w, ρ) of the problem

$$\begin{cases} \rho_t - \Delta \rho - \operatorname{div}(\rho H_p(x, D\bar{u})) = \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dw), & t \in (-\infty, \infty) \\ -w_t - \Delta w + H_p(x, D\bar{u}) Dw = F'(\bar{m})\rho, & t \in (-\infty, \infty) \\ Dw \in L^2((-\infty, \infty) \times \Omega), \rho \in L^\infty((-\infty, \infty); L^2_0(\Omega)). \end{cases} \quad (25)$$

Passing to the limit in (24) we get

$$\int_{\Omega} w(0)\rho(0) = e_\infty$$

and since $\int_{\Omega} w_n(t)\rho_n(t) = \int_{\Omega} \hat{v}_n(t+t_n)\hat{\mu}_n(t+t_n) \leq e(t+t_n)$, we also have

$$\int_{\Omega} w(t)\rho(t) \leq e_\infty \quad \forall t \in (-\infty, +\infty).$$

Therefore, we get

$$\int_t^0 \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) Dw Dw + \int_t^0 \int_{\Omega} F'(\bar{m})\rho \rho = \int_{\Omega} w(t)\rho(t) - \int_{\Omega} w(0)\rho(0) \leq 0$$

hence
$$\int_t^0 \int_{\Omega} \bar{m} H_{pp}(x, D\bar{u}) Dw Dw + \int_t^0 \int_{\Omega} F'(\bar{m})\rho \rho = 0.$$

This implies that $Dw(t) = 0$ in $(t, 0)$, so $w(t) - \langle w(t) \rangle = 0$. Since $\langle w(0) \rangle = 0$, this implies $w(0) = 0$. In turn, we deduce

$$e_\infty = \int_{\Omega} w(0)\rho(0) = 0.$$

With a similar scaling argument we now show that

$$\sup_{\|\mu_0\|_2 \leq 1} \|\hat{\mu}(t)\|_2 \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (26)$$

Indeed, assume by contradiction that there exist $\delta > 0$ and sequences $t_n \rightarrow \infty$, μ_{0n} (with $\|\mu_{0n}\|_2 \leq 1$), and corresponding solutions $(\hat{v}_n, \hat{\mu}_n)$ such that

$$\|\hat{\mu}_n(t_n)\|_2 \geq \delta \quad \forall n.$$

As before, we define the rescaled functions w_n, ρ_n which, up to subsequences, converge to a solution (w, ρ) of (25). Now we know that, for all $t > -t_n$,

$$0 \leq \int_{\Omega} w_n(t)\rho_n(t) = \int_{\Omega} \hat{v}_n(t+t_n)\hat{\mu}_n(t+t_n) \leq e(t+t_n)$$

where we used (23) and the definition of $e(\cdot)$. As $n \rightarrow \infty$ we deduce that $\int_{\Omega} w(t)\rho(t) = 0$, for any $t \in (-\infty, +\infty)$. Since

$$\int_{t_1}^{t_2} \int_{\Omega} \bar{m}H_{pp}(x, D\bar{u})DwDw + \int_{t_1}^{t_2} \int_{\Omega} F'(\bar{m})\rho\rho = \int_{\Omega} w(t_1)\rho(t_1) - \int_{\Omega} w(t_2)\rho(t_2) = 0$$

this implies that $Dw = 0$. Using Lemma 7.1 we get

$$\|\rho(t)\|_2 \leq c e^{-\sigma(t-t_0)} \|\rho(t_0)\|_2 \quad \forall t > t_0$$

where t_0 is any real number. Since $\rho(t)$ is uniformly bounded in $L^2(\Omega)$, letting $t_0 \rightarrow -\infty$ we deduce that $\rho(t) = 0$. But, if $t = 0$, we have

$$\|\rho(0)\|_2 = \lim_{n \rightarrow \infty} \|\hat{\mu}(t_n)\|_2 \geq \delta$$

which gives a contradiction. Finally, we proved that (26) holds. The exponential decay now follows in a standard way; for example, one can fix t_0 such that

$$\sup_{\|\mu_0\|_2 \leq 1} \|\hat{\mu}(t_0)\|_2 \leq \frac{1}{2},$$

and, by time shifting and linearity, it will follow that $\|\hat{\mu}(nt_0)\|_2 \leq \frac{1}{2^n} \|\mu_0\|_2$.

In the end, this yields the existence of some $\omega > 0$ such that $\|\hat{\mu}(t)\|_2 \leq e^{-\omega t} \|\mu_0\|_2$.

Otherwise said, the operator $A + K\hat{E}$ is exponentially stable in $L^2_0(\Omega)$.

(iii) By integrating in $(0, T)$ the systems of (μ^T, v^T) and $(\hat{\mu}, \hat{v})$ we get

$$\begin{aligned} \int_0^T \int_{\Omega} F'(\bar{m})(\mu^T - \hat{\mu})(\mu^T - \hat{\mu}) + \int_0^T \int_{\Omega} \bar{m}H_{pp}(x, D\bar{u})D(v^T - \hat{v})D(v^T - \hat{v}) &= \\ &= \int_{\Omega} \hat{v}(T)(\mu^T(T) - \hat{\mu}(T)) \end{aligned}$$

hence, using the positivity of \bar{m} and $H_{pp}(x, D\bar{u})$, we have

$$\int_0^T \int_{\Omega} |D(v^T - \hat{v})|^2 \leq c \|\hat{v}(T) - \langle \hat{v}(T) \rangle\|_2 \|\mu^T(T) - \hat{\mu}(T)\|_2. \tag{27}$$

Now, by Lemma 7.4, we have

$$\|\mu^T(t) - \hat{\mu}(t)\|_2 \leq c \left(\int_0^t \int_{\Omega} |D(v^T - \hat{v})|^2 \right)^{\frac{1}{2}} \quad \forall t \in (0, T). \tag{28}$$

Using the above inequality with $t = T$ together with (27) we deduce

$$\|\mu^T(T) - \hat{\mu}(T)\|_2 \leq c \|\hat{v}(T) - \langle \hat{v}(T) \rangle\|_2 = c \|\hat{E}\hat{\mu}(T)\|_2 \leq c e^{-\omega T} \|\mu_0\|_2,$$

where we used the exponential decay of $\hat{\mu}$ and the boundedness of \hat{E} . From (27) in turn we deduce

$$\int_0^T \int_{\Omega} |D(v^T - \hat{v})|^2 \leq c e^{-2\omega T} \|\mu_0\|_2^2,$$

and using again (28) we conclude with

$$\|\mu^T(t) - \hat{\mu}(t)\|_2 \leq c e^{-\omega T} \|\mu_0\|_2 \quad \forall t \in (0, T). \tag{29}$$

Using Lemma 7.2, we now have:

$$\begin{aligned} \|v^T(t) - \hat{v}(t) - \langle v^T(t) - \hat{v}(t) \rangle\|_2 &\leq c e^{-\sigma(T-t)} \|\hat{v}(T) - \langle \hat{v}(T) \rangle\|_2 + \\ &+ c \int_t^T \|F'(\bar{m})(\mu^T(s) - \hat{\mu}(s))\|_2 e^{-\sigma(s-t)} ds \\ &\leq c e^{-\sigma(T-t)} e^{-\omega T} \|\mu_0\|_2 + c \int_t^T e^{-\omega T} \|\mu_0\|_2 e^{-\sigma(s-t)} ds \leq c e^{-\omega T} \|\mu_0\|_2. \end{aligned} \tag{30}$$

For $t = 0$ this reads as $\|\mathcal{E}(T)\mu_0 - \hat{E}\mu_0\|_2 \leq c e^{-\omega T} \|\mu_0\|_2$. □

Next, we need to exploit as well the regularizing effect of the operators \mathcal{E} , \hat{E} .

Lemma 3.4. *Assume that $F'(\bar{m}), H_p(x, D\bar{u}), H_{pp}(x, D\bar{u}) \in W^{1,\infty}(\Omega)$, then there exists a constant $C > 0$, independent of T , such that*

$$\|D\mathcal{E}(T)\mu_0\|_2 + \|D^2\mathcal{E}(T)\mu_0\|_2 \leq C \|\mu_0\|_2. \tag{31}$$

Moreover, the operators $K\mathcal{E}(T)$, as well as $K\hat{E}$, are bounded in $L_0^2(\Omega)$ and there exists a constant c independent of t such that

$$\|K(\mathcal{E}(t) - \hat{E})\|_{\mathcal{L}(L_0^2(\Omega), L_0^2(\Omega))} \leq c e^{-\omega t} \quad \forall t > 0. \tag{32}$$

Proof. Estimates (31) are just a consequence of Lemma 3.1 and the definition of $\mathcal{E}(T)$. As far as the operator $K\mathcal{E}(T)$ is concerned, by definition we have

$$K\mathcal{E}(T)\mu_0 = -\text{div}(\bar{m}H_{pp}(x, D\bar{u})Dv^T(0)).$$

Therefore, using the Lipschitz character of $\bar{m}, H_{pp}(x, D\bar{u})$ and the estimates of Lemma 3.1, we deduce that

$$\|K\mathcal{E}(T)\mu_0\|_2 \leq c \|\mu_0\|_2$$

so $K\mathcal{E}(T)$ is a bounded operator in $L_0^2(\Omega)$.

We can proceed to estimate the difference $v^T - \hat{v}$ in a similar way as we did in Lemma 3.1. Using now (29)–(30) with $t \leq 2$ and the equation of $v^T - \hat{v}$, we estimate first $D(v^T - \hat{v})(t)$ for $t \in (0, 1)$, and then we get

$$\|D^2(v^T - \hat{v})(0)\|_2^2 \leq c e^{-2\omega T} \|\mu_0\|_2^2$$

which implies that $K\hat{E}$ is also bounded in $L_0^2(\Omega)$ and (32) holds true. □

Remark 3.5. We observe that the proof of Lemma 3.3 only uses the fact that $F'(\bar{m}) := F_m(\cdot, \bar{m})$ defines a nonnegative bounded operator in $L^2_0(\Omega)$, while the smoothing effect stated in Lemma 3.4 requires that $F'(\bar{m})$ defines a bounded operator on $L^2((0, 1); H^1)$. In the case of possibly nonlocal functions F , those would be the requirements in order that the above statements hold.

Remark 3.6. We notice that the operator $\mathcal{E}(T)$ is self adjoint, and so is the operator $K\mathcal{E}(T)$. Since this latter one is bounded over all $L^2_0(\Omega)$, as a consequence of Lemma 3.4, we deduce that its adjoint $\mathcal{E}(T)K$ is also a bounded operator on its domain.

As a corollary of the previous results, we already deduce an exponential decay for the solution of (13).

Corollary 3.7. *Let (μ^T, v^T) be the solution of (13) and let ω be given by Lemma 3.3. Then we have*

$$\|\mu^T(t)\|_2 + \|\tilde{v}^T(t)\|_2 \leq c e^{-\omega t} \|\mu_0\|_2 \quad \forall t < T.$$

Proof. Recall that $\tilde{v}^T = \mathcal{E}(T - t)\mu^T$. Hence we recast the equation of μ^T as

$$\mu_t^T + (A + K\hat{E})\mu^T = -K(\mathcal{E}(T - t) - \hat{E})\mu^T.$$

By Duhamel’s formula, this implies

$$\|\mu^T(t)\|_2 \leq e^{-\omega t} \|\mu_0\|_2 + \int_0^t e^{-\omega(t-s)} \|K(\mathcal{E}(T - s) - \hat{E})\mu^T(s)\|_2 ds,$$

and then, using (32), we deduce

$$\|\mu^T(t)\|_2 \leq e^{-\omega t} \|\mu_0\|_2 + c \int_0^t e^{-\omega(t-s)} e^{-\omega(T-s)} \|\mu^T(s)\|_2 ds.$$

A standard Gronwall argument allows us to conclude that

$$\|\mu^T(t)\|_2 \leq c e^{-\omega t} \|\mu_0\|_2 \quad \forall t < T.$$

Since $\tilde{v}^T = \mathcal{E}(T - t)\mu^T$ and $\mathcal{E}(T - t)$ is bounded, we conclude for \tilde{v}^T as well. \square

4. The nonhomogeneous linearized system

We proceed by analyzing the nonhomogenous linearized system

$$\begin{cases} \mu_t^T - \Delta \mu^T - \operatorname{div}(\mu^T H_p(x, D\bar{u})) = \operatorname{div}(\bar{m} H_{pp}(x, D\bar{u}) Dv^T) + f_1, & t \in (0, T), \\ -v_t^T - \Delta v^T + H_p(x, D\bar{u}) Dv^T = F'(\bar{m})\mu^T + f_2, & t \in (0, T), \\ \mu^T(0) = \mu_0, \quad v^T(T) = g \end{cases} \quad (33)$$

where f_1, f_2 and g belong (at least) to $L^\infty((0, T); L^2(\Omega))$ and to $L^2(\Omega)$ respectively, with $\int_\Omega f_1(t) = 0$ for all t .

Now, we define h^T as the solution of the following problem

$$\begin{cases} -h_t^T + A^*h^T = -\mathcal{E}(T-t)Kh^T - \mathcal{E}(T-t)f_1 + f_2, & t \in (0, T) \\ h^T(T) = g. \end{cases} \quad (34)$$

We are going to see that h^T is in charge of the error between the homogeneous linearized system (13) and non homogeneous one (33).

Lemma 4.1. *Let (μ^T, v^T) be the solution of (33). Then we have*

$$v^T = \mathcal{E}(T-t)\mu^T + h^T \quad (35)$$

where h^T solves (34). In addition, we estimate

$$\|\tilde{h}^T(t)\|_2 \leq c e^{-\omega(T-t)}\|\tilde{g}\|_2 + c \int_t^T e^{-\omega(s-t)}[\|f_1(s)\|_2 + \|f_2(s)\|_2]ds \quad (36)$$

where $\tilde{h}^T(t) = h^T(t) - \langle h^T(t) \rangle$.

Proof. The representation (35) is just a consequence of duality pairings: indeed, we have

$$(v^T(t), \psi(t)) = (g, \psi(T)) - \int_t^T \langle \psi_t + A\psi, v^T \rangle + \int_t^T \int_{\Omega} F'(\bar{m})\mu^T \psi + \int_t^T (f_2, \psi) \quad (37)$$

for any $\psi \in L^2((0, T); H^1)$ such that $\psi_t + A\psi \in L^2((0, T); (H^1)')$. Let us now take (ψ, z) solution of the system

$$\begin{cases} \psi_\tau - \Delta\psi - \operatorname{div}(\psi H_p(x, D\bar{u})) = \operatorname{div}(\bar{m}H_{pp}(x, D\bar{u})Dz) & \tau \in (t, T), \\ -z_\tau - \Delta z + H_p(x, D\bar{u})Dz = F'(\bar{m})\psi & \tau \in (t, T), \\ \psi(t) = \xi, \quad z(T) = 0 \end{cases} \quad (38)$$

where $\xi \in L^2_0(\Omega)$. By definition of \mathcal{E} , we have $\tilde{z}(t) = \mathcal{E}(T-t)\xi$ and actually $\tilde{z}(\tau) = \mathcal{E}(T-\tau)\psi(\tau)$ for every $\tau \in [t, T)$. In particular, the equation of ψ can also be rephrased as

$$\psi_\tau + A\psi = -Kz = -K(\mathcal{E}(T-\tau)\psi).$$

Multiplying the second equation in (38) by μ^T we have

$$(\mu^T(t), z(t)) = \int_t^T \int_{\Omega} F'(\bar{m})\mu^T \psi - \int_t^T \langle Kv^T, z \rangle + \int_t^T (f_1, z).$$

However we recall that μ^T has zero average, so

$$(\mu^T(t), z(t)) = (\mu^T(t), \tilde{z}(t)) = (\mathcal{E}(T-t)\mu^T(t), \xi).$$

Similarly we write for the last term since $f_1(\tau) \in L^2_0(\Omega)$. Hence

$$(\mathcal{E}(T-t)\mu^T(t), \xi) = \int_t^T \int_\Omega F'(\bar{m})\mu^T \psi - \int_t^T \langle K v^T, z \rangle + \int_t^T (f_1, \mathcal{E}(T-\tau)\psi(\tau)). \quad (39)$$

Subtracting (39) from (37) and using symmetry of \mathcal{E} and K , we get

$$\begin{aligned} (v^T(t) - \mathcal{E}(T-t)\mu^T(t), \xi) &= (g, \psi(T)) + \int_t^T (f_2, \psi) - \int_t^T (\mathcal{E}(T-\tau)f_1, \psi(\tau)) d\tau \\ &= (h^T(t), \xi) \end{aligned} \quad (40)$$

where the last equality results after multiplying (34) by ψ and integrating in (t, T) . Therefore (35) is proved.

In order to estimate h^T , we use again (40). Applying to ψ Corollary 3.7, we have

$$\begin{aligned} \|\psi(s)\|_2 &\leq c e^{-\omega(s-t)} \|\xi\|_2 \quad \forall s \in (t, T], \quad \text{hence} \\ |(\tilde{h}^T(t), \xi)| &\leq c \|\xi\|_2 \left[e^{-\omega(T-t)} \|\tilde{g}\|_2 + \int_t^T e^{-\omega(s-t)} (\|\mathcal{E}(T-s)f_1\|_2 + \|f_2\|_2) ds \right]. \end{aligned}$$

Since $\mathcal{E}(T)$ is uniformly bounded, we deduce (36). □

Unfortunately, the L^2 estimate is not enough for h and we need to improve it up to the derivatives of h^T . We assume in the next corollary that f_1 and f_2 are well behaved.

Corollary 4.2. *Assume that $g \in H^2$, $f_2 \in L^\infty(0, T; H^1(\Omega))$ and there exists a constant M such that*

$$\|f_1(t)\|_2 + \|f_2(t)\|_2 + \|Df_2(t)\|_2 \leq M(e^{-2\omega t} + e^{-2\omega(T-t)}) \quad \forall t \in [0, T]. \quad (41)$$

Then we have

$$\|\tilde{h}^T(t)\|_{H^2} \leq c e^{-\omega(T-t)} \|\tilde{g}\|_{H^2} + c M(e^{-2\omega t} + e^{-\omega(T-t)}) \quad \forall t \in [0, T] \quad (42)$$

for some c independent of f_1, f_2, g, T .

Proof. From (36) we already deduced that

$$\begin{aligned} \|\tilde{h}^T(t)\|_2 &\leq c \left[e^{-\omega(T-t)} \|\tilde{g}\|_2 + M \int_t^T e^{-\omega(s-t)} [e^{-2\omega s} + e^{-2\omega(T-s)}] ds \right] \\ &\leq c [e^{-\omega(T-t)} \|\tilde{g}\|_2 + M(e^{-2\omega t} + e^{-\omega(T-t)})] \end{aligned}$$

so that (42) holds for $\|\tilde{h}^T(t)\|_2$. Now we apply Lemma 7.3 which yields

$$\begin{aligned} \|Dh^T(t)\|_2^2 &\leq c \left\{ \|\tilde{h}^T(t+1)\|_2 + \int_t^{t+1} \|\mathcal{E}(T-s)Kh^T(s)\|_2^2 ds + \right. \\ &\quad \left. + \int_t^{t+1} [\|f_1(s)\|_2^2 + \|f_2(s)\|_2^2 + \|\tilde{h}^T(s)\|_2^2] ds \right\}. \end{aligned}$$

Using the above estimate for \tilde{h}^T , and the fact that $\mathcal{E}(T-s)K$ is a bounded operator (see Remark 3.6), we conclude that estimate (42) also holds for $\|Dh^T(t)\|_2$ if $t < T - 1$. But the estimate holds in $(T - 1, T)$ as well, as a consequence of standard $L^\infty((T - 1, T); L^2(\Omega))$ estimates for Dh^T , because $g \in H^2$. So we deduce a complete estimate for $\|Dh^T(t)\|_2$.

In addition, the same Lemma 7.3 also gives, in the previous estimate, that

$$\int_t^{t+\frac{1}{2}} \|D^2h^T\|_2^2 ds \leq c [e^{-\omega(T-t)} \|Dg\|_2 + M(e^{-2\omega t} + e^{-\omega(T-t)})]^2. \tag{43}$$

Therefore, after taking one more derivative (say, ∂_ξ) in the equation and applying again Lemma 7.3 we obtain

$$\begin{aligned} \|Dh_\xi^T(t)\|_2^2 \leq c & \left\{ \|Dh^T(t + \frac{1}{2})\|_2^2 + \int_t^{t+\frac{1}{2}} \|\partial_\xi \mathcal{E}(t-s)Kh^T(s)\|_2^2 ds + \right. \\ & \left. + \int_t^{t+\frac{1}{2}} [\|\partial_\xi \mathcal{E}(t-s)f_1(s)\|_2^2 + \|\partial_\xi f_2(s)\|_2^2 + \int_t^{t+\frac{1}{2}} \|Dh^T(s)\|_2^2 ds] \right\} \end{aligned}$$

We recall that the operator $\partial_\xi \mathcal{E}(t)$ is a bounded operator in $L_0^2(\Omega)$, uniformly with respect to t , see (31). So, by arbitrariness of ξ , we deduce

$$\begin{aligned} \|D^2h^T(t)\|_2^2 \leq c & \left\{ \|Dh^T(t + \frac{1}{2})\|_2^2 + \int_t^{t+\frac{1}{2}} \|Kh^T(s)\|_2^2 ds + \right. \\ & \left. + \int_t^{t+\frac{1}{2}} [\|f_1(s)\|_2^2 + \|Df_2(s)\|_2^2 + \int_t^{t+\frac{1}{2}} \|Dh^T(s)\|_2^2 ds] \right\} \end{aligned}$$

Since $\|Kh^T\|_2 \leq c \|D^2h^T\|_2$, we can use (43) in the second integral, while for $\|Dh^T\|_2$ we can use the pointwise estimate (42) which was previously obtained for $\|\tilde{h}^T\|_{H^1}$. Finally, on account of (41) we get

$$\|D^2h^T(t)\|_2^2 \leq c [e^{-\omega(T-t)} \|\tilde{g}\|_{H^2} + M(e^{-2\omega t} + e^{-\omega(T-t)})]^2.$$

and (42) is completed for $t < T - \frac{1}{2}$. The estimate is then extended up to $t = T$ using the H^2 regularity of g . □

We conclude with the full consequence that assumption (41) yields for solutions of (33). Let us recall that, by what proved in Lemma 3.3, the operator $M := A + K\hat{E}$ generates a semigroup which is exponentially stable in $L_0^2(\Omega)$. In other words,

$$\begin{cases} \mu_t + M\mu = 0 \\ \mu(0) = \mu_0 \end{cases} \quad \Rightarrow \quad \|\mu(t)\|_2 \leq c e^{-\omega t} \|\mu_0\|_2$$

for every $\mu_0 \in L_0^2(\Omega)$.

Proposition 4.3. *Assume that $F'(\bar{m}), H_p(x, D\bar{u}), H_{pp}(x, D\bar{u}) \in W^{1,\infty}(\Omega)$, that f_1, f_2 satisfy (41) and that $g \in H^2$. Then the solution (μ^T, v^T) of (33) satisfies*

$$\begin{aligned} \|\mu^T(t)\|_{H^1} + \|\tilde{v}^T(t)\|_{H^2} &\leq \\ &\leq c(e^{-\omega(T-t)}\|\tilde{g}\|_{H^2} + e^{-\omega t}\|\mu_0\|_{H^1}) + cM(e^{-\omega(T-t)} + e^{-\omega t}) . \end{aligned} \tag{44}$$

Proof. On account of (35), we rephrase equation (33) as

$$\mu_t^T + A\mu^T = -Kv^T + f_1 = -K\mathcal{E}(T-t)\mu^T - Kh^T + f_1$$

and therefore

$$\mu_t^T + (A + K\hat{E})\mu^T = K(\hat{E} - \mathcal{E}(T-t))\mu^T - Kh^T + f_1 .$$

By the exponential stability of $A + K\hat{E}$ and Duhamel's formula, it follows

$$\begin{aligned} \|\mu^T(t)\|_2 &\leq e^{-\omega t}\|\mu_0\|_2 + \int_0^t \|K(\hat{E} - \mathcal{E}(T-s))\mu^T(s)\|_2 e^{-\omega(t-s)} ds + \\ &+ \int_0^t (\|Kh^T(s)\|_2 + \|f_1(s)\|_2) e^{-\omega(t-s)} ds . \end{aligned} \tag{45}$$

Since
$$\begin{aligned} Kh^T &= -\operatorname{div}(\bar{m}H_{pp}(x, D\bar{u})Dh^T) \\ &= -D[\bar{m}H_{pp}(x, D\bar{u})] \cdot Dh^T - \bar{m}\operatorname{Tr}(H_{pp}(x, D\bar{u})D^2h^T) \end{aligned}$$

using the regularity of $\bar{m}, H_{pp}(x, D\bar{u})$ and estimate (42), we have

$$\|Kh^T(t)\|_2 \leq c e^{-\omega(T-t)}\|\tilde{g}\|_{H^2} + cM(e^{-2\omega t} + e^{-\omega(T-t)}) .$$

Using also (32) and assumption (41), we estimate the right-hand side of (45) and we obtain

$$\begin{aligned} \|\mu^T(t)\|_2 &\leq e^{-\omega t}\|\mu_0\|_2 + c \int_0^t e^{-\omega(T-s)}\|\mu^T(s)\|_2 e^{-\omega(t-s)} ds \\ &+ c \int_0^t \{(\|\tilde{g}\|_{H^2} + M)e^{-\omega(T-s)} + Me^{-2\omega s}\} e^{-\omega(t-s)} ds . \end{aligned}$$

The above inequality implies

$$\|\mu^T(t)\|_2 \leq (e^{-\omega t}\|\mu_0\|_2 + e^{-\omega(T-t)}\|\tilde{g}\|_{H^2}) + cM[e^{-\omega t} + e^{-\omega(T-t)}] .$$

Since $\|D^2\mathcal{E}(T-t)\mu^T(t)\|_2 \leq c\|\mu^T(t)\|_2$ by (31), and using (42), from (35) we deduce that

$$\|D^2v^T(t)\|_2 \leq (e^{-\omega t}\|\mu_0\|_2 + e^{-\omega(T-t)}\|\tilde{g}\|_{H^2}) + cM[e^{-\omega t} + e^{-\omega(T-t)}] .$$

Finally, we observe that μ^T satisfies

$$\mu_t^T + A^*\mu^T = \mu^T \operatorname{div}(H_p(x, D\bar{u})) + \operatorname{div}(\bar{m}H_{pp}(x, D\bar{u})Dv^T)$$

hence applying Lemma 7.3 in $(t, t + 1)$ we get

$$\|D\mu^T(t)\|_2 \leq c(e^{-\omega t}\|\mu_0\|_2 + e^{-\omega(T-t)}\|\tilde{g}\|_{H^2}) + cM^2[e^{-\omega t} + e^{-\omega(T-t)}]$$

at least for $t > 1$. The inequality is then extended to $(0, 1)$ by using the H^1 character of μ_0 , so finally we complete the estimate (44). \square

5. Exponential stability around the stationary solution

In this section we apply the previous estimates to get the exponential stability for the system

$$\begin{cases} -u_t^T - \Delta u^T + H(x, Du^T) = F(x, m^T(t)), & t \in (0, T) \\ m_t^T - \Delta m^T - \operatorname{div}(m^T H_p(x, Du^T)) = 0, & t \in (0, T) \\ m^T(0) = m_0, \quad u^T(T) = u_T \end{cases} \quad (46)$$

when the initial-terminal data are close to the stationary solution.

Theorem 5.1. *Assume that (6)–(10) hold true and that $m_0, u_T \in C^1(\bar{\Omega})$, with $m_0 \in L^2_0(\Omega)$. Then there exists $\eta > 0$ such that whenever*

$$\|m_0 - \bar{m}\|_{H^1 \cap L^\infty} + \|\tilde{u}_T - \bar{u}\|_{H^2 \cap W^{1,\infty}} \leq \eta$$

then the solution of (46) satisfies

$$\|m^T(t) - \bar{m}\|_{H^1 \cap L^\infty} + \|\tilde{u}^T(t) - \bar{u}\|_{H^2 \cap W^{1,\infty}} \leq c[e^{-\omega t} + e^{-\omega(T-t)}] \quad \forall t \in [0, T]$$

where $c = c(\eta, F, H)$ and ω is the decay rate given by Lemma 3.3.

Proof. We apply a fixed point argument. Set

$$E := \{(z, w) \in C^0([0, T] \times \Omega)^2 : z(t), w(t) \in L^2_0(\Omega) \forall t \in [0, T]\}.$$

In E we consider the closed convex subset

$$X := \{(\mu, v) \in E : \|\mu(t)\|_{H^1 \cap L^\infty} + \|v(t)\|_{H^2 \cap W^{1,\infty}} \leq \gamma[e^{-\omega t} + e^{-\omega(T-t)}] \quad \forall t \in [0, T]\}$$

for a convenient γ to be fixed later. For $(\hat{\mu}, \hat{v}) \in X$, we define

$$\begin{aligned} f_1 &= \operatorname{div}(\bar{m} [H_p(x, D\bar{u} + D\hat{v}) - H_p(x, D\bar{u}) - H_{pp}(x, D\bar{u})D\hat{v}]) \\ &\quad + \operatorname{div}(\hat{\mu} [H_p(x, D\bar{u} + D\hat{v}) - H_p(x, D\bar{u})]), \\ f_2 &= -\{H(x, D\bar{u} + D\hat{v}) - H(x, D\bar{u}) - H_p(x, D\bar{u})D\hat{v}\} \\ &\quad + F(\bar{m} + \hat{\mu}) - F(\bar{m}) - F'(\bar{m})\hat{\mu}. \end{aligned} \quad (47)$$

and $g = u_T - \bar{u}$.

We observe that, on account of the assumptions on F and H , and specifically (7) and (10), if $(\hat{\mu}, \hat{v}) \in X$ then

$$\|f_1(t)\|_2 + \|f_2(t)\|_2 + \|Df_2(t)\|_2 \leq c\gamma^2(e^{-2\omega t} + e^{-2\omega(T-t)}) \quad \forall t \in [0, T]. \quad (48)$$

In X we define a map Φ such that $\Phi(\hat{\mu}, \hat{v}) = (\mu^T, \tilde{v}^T)$, where (μ^T, v^T) is a solution of the problem (33) with f_1, f_2, g defined above, and where $\tilde{v}^T(t, x) = v^T(t, x) - \langle v^T(t) \rangle$ as before. The reader may check that if $(\hat{\mu}, \hat{v})$ is a fixed point of the map Φ , then $m^T = \bar{m} + \hat{\mu}$ and $u^T = \bar{u} + \hat{v}$ give the unique solution of (46).

Henceforth, let $(\mu^T, \tilde{v}^T) = \Phi(\hat{\mu}, \hat{v})$; thanks to (48), and using Proposition 4.3, we have that

$$\begin{aligned} \|\mu^T(t)\|_{H^1} + \|\tilde{v}^T(t)\|_{H^2} &\leq \\ &\leq c(e^{-\omega(T-t)}\|\tilde{g}\|_{H^2} + e^{-\omega t}\|\mu_0\|_{H^1}) + c\gamma^2(e^{-\omega(T-t)} + e^{-\omega t}) \end{aligned} \quad (49)$$

where $g = u_T - \bar{u}$ and $\mu_0 = m_0 - \bar{m}$.

We now look for L^∞ bounds on $\mu^T(t)$ and $Dv^T(t)$. To this purpose, we observe first that, from the above definitions, being $(\hat{\mu}, \hat{v}) \in X$ we have

$$f_1(t) = \operatorname{div}(F_1(t)), \text{ where } \|F_1(t)\|_{L^\infty} \leq c\gamma^2(e^{-2\omega t} + e^{-2\omega(T-t)}) \quad \forall t \in [0, T]$$

and similarly $\|f_2(t)\|_{L^\infty} \leq c\gamma^2(e^{-2\omega t} + e^{-2\omega(T-t)}) \quad \forall t \in [0, T]$.

We now use the L^p -regularity theory for both equations with a bootstrap argument. Indeed, let us assume that, in the interval $(t-1, t+1)$, we know that μ^T is bounded in some $L^p((t-1, t+1) \times \Omega)$, $p > 2$; by regularity of the heat equation we deduce that Dv^T is bounded in $L^q((t-1, t+\frac{1}{2}) \times \Omega)$ for some $q > p$, with a corresponding estimate

$$\begin{aligned} \|Dv^T\|_{L^q((t-1, t+\frac{1}{2}) \times \Omega)} &\leq \\ &\leq c\{\|\mu^T\|_{L^p((t-1, t+1) \times \Omega)} + \|f_2\|_{L^p((t-1, t+1) \times \Omega)} + \|Dv^T\|_{L^2((t-1, t+1) \times \Omega)}\}. \end{aligned}$$

Then, we deduce from the equation of μ^T an improved summability property, say in $(t - \frac{1}{2}, t + \frac{1}{2})$:

$$\begin{aligned} \|\mu^T\|_{L^q((t-1, t+\frac{1}{2}) \times \Omega)} &\leq \\ c\{\|Dv^T\|_{L^q((t-1, t+\frac{1}{2}) \times \Omega)} + \|F_1\|_{L^q((t-1, t+\frac{1}{2}) \times \Omega)} + \|\mu^T\|_{L^2((t-1, t+1) \times \Omega)}\} & \end{aligned}$$

This is true for all $t \in (1, T-1)$, but we can replace local with global estimates in the intervals $(0, 2)$ or $(T-2, T)$ using the L^∞ regularity of Du_T and m_0 , respectively.

Since we can start with the estimate (49), a bootstrap argument in a finite number of steps leads us to a bound of $\mu^T(t)$ in L^∞ (as soon as q is sufficiently large, say $q > d+2$), and then a bound of $Dv^T(t)$ in L^∞ as well. Moreover we preserve the estimates obtained above in any interval $(t-1, t+1)$, so we can upgrade (49) including the L^∞ -norms of μ^T and Dv^T , namely:

$$\begin{aligned} \|\mu^T(t)\|_{H^1 \cap L^\infty} + \|\tilde{v}^T(t)\|_{H^2 \cap W^{1,\infty}} &\leq \\ &\leq c(e^{-\omega(T-t)}\|\tilde{g}\|_{H^2 \cap W^{1,\infty}} + e^{-\omega t}\|\mu_0\|_{H^1 \cap L^\infty}) + c\gamma^2(e^{-\omega(T-t)} + e^{-\omega t}). \end{aligned}$$

Eventually, if $\|\tilde{g}\|_{H^2 \cap W^{1,\infty}}$ and $\|\mu_0\|_{H^1 \cap L^\infty}$ are sufficiently small (e.g. if $\eta c < \frac{\gamma}{2}$), one can find some γ for which X is a convex invariant set for Φ .

Finally, by parabolic estimates we have that μ^T is bounded in $C^{0,\alpha}$ and v^T is bounded in $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, which means that the range of Φ is relatively compact in E . By Schauder's theorem, we conclude that Φ has a fixed point. \square

6. Turnpike property for general initial-terminal data

In this last section we show that the turnpike property holds for general initial-terminal data. We start with a regularity result on the solutions of the MFG system (1).

Lemma 6.1. *Let (u^T, m^T) be a solution of the MFG system (1). For every $\delta > 0$ there exists a constant C_δ , independent of the horizon T , such that we have*

$$\|m^T(t)\|_\infty + \|Du^T(t)\|_\infty + \frac{1}{T}\|u^T(t)\|_\infty \leq C_\delta \quad \forall t \in [\delta, T - \delta], \quad (50)$$

and C_δ is uniformly bounded for $\delta \geq 1$. Moreover, u^T and m^T are smooth and are classical solutions in $(0, T)$.

Proof. Since $H_p(x, Du^T)$ is uniformly bounded, by elliptic regularity we have

$$\|m^T(t + \delta)\|_\infty \leq C(\delta, \|m^T(t)\|_1)$$

for any possible $t > 0$. Since m^T is a probability density, we deduce that $\|m^T(t)\|_\infty$ is uniformly bounded as soon as $t > 0$, and with a universal constant (only depending on $d, \|H_p\|_\infty$) if $t \geq 1$. We deduce that $\|G(x, m(T))\|_\infty \leq C$, independently of T . By the maximum principle this implies that

$$\|u^T(t)\|_\infty \leq C_\delta T \quad \forall t \in [\delta, T],$$

where, here and below, C_δ is uniformly bounded for all $\delta \geq 1$. Now we notice that

$$H(x, Du^T) = V(t, x) \cdot Du^T, \quad V := \int_0^1 H_p(x, \lambda Du^T) d\lambda$$

where we used the normalization condition $H(x, 0) = 0$. Being H Lipschitz, we have that V is uniformly bounded. Therefore, we can apply Lemma 7.2 to u^T , and we get

$$\|\tilde{u}(t)\|_2 \leq C e^{-\sigma(T-t)} + C \int_t^T e^{-\sigma(s-t)} \|F(x, m^T(s))\|_2 ds \quad \forall t < T,$$

hence $\|\tilde{u}^T(t)\|_2 \leq C_\delta \quad \forall t \geq \delta > 0$.

Applying Lemma 7.3 we conclude a similar bound for $\|Du^T(t)\|_2$ at least for $t < T - \delta$. We consider now the equation satisfied by \tilde{u}^T , namely

$$-\partial_t \tilde{u}^T - \Delta \tilde{u}^T + H(x, D\tilde{u}^T) = F(x, m^T) + \int_\Omega [F(x, m^T) - H(x, Du^T)] dx.$$

The L^∞ bound on m^T and the L^2 bound on Du^T established before imply that the right-hand side is uniformly bounded. Therefore, in any $(t, t + 1)$ we get, by parabolic regularity

$$\|D\tilde{u}^T(t)\|_\infty \leq C + \sup_{[t,t+1]} \|\tilde{u}^T(s)\|_2 \leq C,$$

where, as before, C is uniformly bounded for $t \geq 1$ and, eventually, depends on δ if $t \in (\delta, 1)$ or $t \in (T - 1, T - \delta)$. This concludes the proof of (50).

Finally, once we know that m^T and Du^T are bounded, this means that $-u_t^T - \Delta u^T$ is bounded; this implies that Du^T actually belongs to $C^{0,\alpha}((0, T) \times \Omega)$ for some $\alpha \in (0, 1)$. Similarly we get that m^T belongs to $C^{0,\alpha}((0, T) \times \Omega)$ for some $\alpha \in (0, 1)$. This information can be used back in the equation of u^T and with a standard bootstrap argument we conclude that u^T and m^T are both smooth in $(0, T) \times \Omega$ and so the system is solved in a classical sense in $(0, T)$. \square

We will also use the following lemma which combines the decay estimates of the single equations, as stated in the Appendix.

Lemma 6.2. *Set $v := u^T - \bar{u}$ and $\mu := m^T - \bar{m}$. There exists a constant C such that, for all $t_0, t_1 \in [1, T - 1]$ we have, for every $t \in (t_0 + 1, t_1 - 1)$,*

$$\begin{aligned} & \|\tilde{v}(t)\|_2^2 + \|Dv(t)\|_2^2 + \|\mu(t)\|_2^2 \leq \\ & \leq C \left\{ e^{-\sigma(t_1-t)} \|\tilde{v}(t_1)\|_2^2 + e^{-\sigma(t-t_0)} \|\mu(t_0)\|_2^2 + \int_{t_0}^{t_1} \int_\Omega |Dv|^2 \right\}. \end{aligned}$$

Proof. As μ satisfies

$$\partial_t \mu - \Delta \mu - \operatorname{div}(\mu H_p(x, Du)) = -\operatorname{div}(\bar{m}(H_p(x, D\bar{u}) - H_p(x, Du))), \quad (51)$$

from Lemma 7.4 we get, for all $t \in (t_0, t_1)$,

$$\|\mu(t)\|_2^2 \leq e^{-\sigma(t-t_0)} \|\mu(t_0)\|_2^2 + c \int_{t_0}^t \int_\Omega |Dv|^2. \quad (52)$$

Now we turn to the equation satisfied by v , which is

$$-\partial_t v - \Delta v + Dv \cdot V = F(x, m) - F(x, \bar{m}), \quad V := \int_0^1 H_p(x, \lambda Du + (1 - \lambda) D\bar{u}) d\lambda \in L^\infty.$$

Since $t_0, t_1 \in [1, T - 1]$, μ is bounded in $L^\infty((t_0, t_1) \times \Omega)$ by Lemma 6.1, so F can be treated as a Lipschitz function. Applying Lemma 7.2 we deduce

$$\begin{aligned} \|\tilde{v}(t)\|_2^2 & \leq e^{-\sigma(t_1-t)} \|\tilde{v}(t_1)\|_2^2 + c \int_t^{t_1} e^{-\sigma(s-t)} \|\mu(s)\|_2^2 ds \\ & \leq e^{-\sigma(t_1-t)} \|\tilde{v}(t_1)\|_2^2 + c e^{-\sigma(t-t_0)} \|\mu(t_0)\|_2^2 + c \int_{t_0}^{t_1} \int_\Omega |Dv|^2. \end{aligned} \quad (53)$$

Similarly, using Lemma 7.3 in the interval $[t, t + \frac{1}{2}]$, with $t < t_1 - \frac{1}{2}$, we get

$$\begin{aligned} \|Dv(t)\|_2^2 &\leq c \|\tilde{v}(t + 1/2)\|_2^2 + \int_t^{t+\frac{1}{2}} [\|\tilde{v}(s)\|_2^2 + c \|\mu(s)\|_2^2] ds \\ &\leq e^{-\sigma(t_1-t)} \|\tilde{v}(t_1)\|_2^2 + c e^{-\sigma(t-t_0)} \|\mu(t_0)\|_2^2 + c \int_{t_0}^{t_1} \int_{\Omega} |Dv|^2, \end{aligned}$$

where we used both (52) and (53). □

We are ready now to conclude with the exponential estimate for general initial-terminal data. As in our previous papers [3], [4], the last step relies on the stability property of MFG systems which provides with a long time average convergence of (Du^T, m^T) towards $(D\bar{u}, \bar{m})$. This will allow us to find a suitable large interval $[\tau_0, \tau_1]$ where we can apply the exponential stability result of Theorem 5.1.

Proof of Theorem 2.1. We use (u^T, m^T) and $(\bar{u} + \bar{\lambda}(T - t), \bar{m})$ as a couple of solutions to the MFG system. The standard duality identity gives

$$\begin{aligned} -\frac{d}{dt} \int_{\Omega} (u^T - \bar{u})(m^T - \bar{m}) &= \int_{\Omega} \bar{m} \{H(x, Du^T) - H(x, D\bar{u}) - H_p(x, D\bar{u})D(u^T - \bar{u})\} \\ &\quad + \int_{\Omega} m^T \{H(x, D\bar{u}) - H(x, Du^T) - H_p(x, Du^T)D(\bar{u} - u^T)\} \\ &\quad + \int_{\Omega} [F(x, m^T) - F(x, \bar{m})][m^T - \bar{m}] \end{aligned}$$

where we dropped the term with $\bar{\lambda}$ since $m^T - \bar{m}$ has zero average. Using the monotone character of F and the uniform convexity of H on compact subsets together with the global bounds of Lemma 6.1, we get

$$-\frac{d}{dt} \int_{\Omega} (u^T - \bar{u})(m^T - \bar{m}) \geq c \int_{\Omega} (m^T + \bar{m}) |Du^T - D\bar{u}|^2 \quad \forall t \in [1, T - 1]. \quad (54)$$

Since \bar{m} is positive, by integration we deduce

$$C^{-1} \int_1^{T-1} \int_{\Omega} |Du^T - D\bar{u}|^2 \leq - \left[\int_{\Omega} (u^T - \bar{u})(m^T - \bar{m}) \right]_1^{T-1}.$$

Since the Poincaré-Wirtinger inequality and the estimates of Lemma 6.1 imply

$$\left| \int_{\Omega} (u^T(t) - \bar{u})(m^T(t) - \bar{m}) \right| \leq \|Du^T(t) - D\bar{u}\|_2 \|m^T(t) - \bar{m}\|_2 \leq C \quad \forall t \in [1, T - 1],$$

we conclude that
$$\int_1^{T-1} \int_{\Omega} |Du^T - D\bar{u}|^2 \leq C.$$

Let us now take some $\varepsilon < 1$. In consequence there exist points $t_0 \in (1, 1 + \frac{1}{\varepsilon})$ and $t_1 \in (T - (1 + \frac{1}{\varepsilon}), T - 1)$ such that

$$\int_{\Omega} |Du(t_0) - D\bar{u}|^2 \leq C \varepsilon, \quad \text{and} \quad \int_{\Omega} |Du^T(t_1) - D\bar{u}|^2 \leq C \varepsilon.$$

Using again (54) in the interval (t_0, t_1) we obtain

$$\int_{t_0}^{t_1} \int_{\Omega} |Du^T - D\bar{u}|^2 \leq C \{ \|Du^T(t_0) - D\bar{u}\|_2 \|m^T(t_0) - \bar{m}\|_2 + \|Du^T(t_1) - D\bar{u}\|_2 \|m^T(t_1) - \bar{m}\|_2 \} \leq C \varepsilon^{\frac{1}{2}}.$$

We now use Lemma 6.2 and we denote, as before, $\mu := m^T - \bar{m}$ and $v := u^T - \bar{u}$. So, for some constant C we have

$$\|\tilde{v}(t)\|_{H^1}^2 + \|\mu(t)\|_2^2 \leq C \left\{ e^{-\sigma(t_1-t)} \|\tilde{u}^T(t_1) - \bar{u}\|_2^2 + e^{-\sigma(t-t_0)} \|m^T(t_0) - \bar{m}\|_2^2 + \varepsilon^{\frac{1}{2}} \right\}.$$

If we take $t \in (1 + \frac{2}{\varepsilon}, T - 1 - \frac{2}{\varepsilon})$, then $t_1 - t$ and $t - t_0$ are bigger than $\frac{1}{\varepsilon}$, and we conclude that

$$\|\tilde{v}(t)\|_{H^1}^2 + \|\mu(t)\|_2^2 \leq C \varepsilon^{\frac{1}{2}} \quad \forall t \in \left(1 + \frac{2}{\varepsilon}, T - 1 - \frac{2}{\varepsilon} \right). \tag{55}$$

We recall that $Du^T, D\bar{u}$ are uniformly bounded, so for any $p > 2$ we have

$$\begin{aligned} \|\bar{m}(H_p(x, D\bar{u}) - H_p(x, Du^T(t)))\|_p &\leq C \|D\bar{u} - Du^T(t)\|_p \\ &\leq C \|D\bar{u} - Du^T(t)\|_2^{\frac{2}{p}} \leq C \varepsilon^{\frac{1}{2p}} \end{aligned}$$

whenever $t \in (1 + \frac{2}{\varepsilon}, T - 1 - \frac{2}{\varepsilon})$. By parabolic regularity applied to the equation of μ (see (51)), if $p > d$ we have

$$\|\mu(t+1)\|_{\infty} \leq C \sup_{[t, t+1]} [\|\mu(s)\|_2 + \|\bar{m}(H_p(x, D\bar{u}) - H_p(x, Du^T(s)))\|_p]$$

so we deduce that

$$\|\mu(t)\|_{\infty} \leq C [\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2p}}] \leq C \varepsilon^{\frac{1}{2p}} \quad \forall t \in \left(2 + \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon} \right). \tag{56}$$

Moreover, by a standard energy estimate for (51), where we use that H_p is uniformly bounded, in any interval $(t, t+1)$ we have

$$\int_t^{t+1} \int_{\Omega} |D\mu|^2 \leq C \|\mu(t)\|_2^2 + C \int_t^{t+1} \int_{\Omega} [\mu^2 + |Dv|^2]$$

hence (55) implies
$$\int_t^{t+1} \int_{\Omega} |D\mu|^2 \leq C \varepsilon^{\frac{1}{2}}$$

for any $t \in (1 + \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon})$. Using the mean value theorem from (6), and combining it with the inequalities (55) and (56), we conclude that there exists some $\tau_0 \in (2 + \frac{2}{\varepsilon}, 3 + \frac{2}{\varepsilon})$ such that

$$\|m^T(\tau_0) - \bar{m}\|_{H^1 \cap L^{\infty}} \leq C \varepsilon^{\frac{1}{2p}}. \tag{57}$$

Similarly we find some \tilde{v} , which satisfies the equation

$$\begin{aligned}
 -\partial_t \tilde{v} - \Delta \tilde{v} + D\tilde{v} \cdot V &= h, \quad V := \int_0^1 H_p(x, \lambda Du + (1 - \lambda)D\bar{u})d\lambda \\
 h &:= F(x, m) - F(x, \bar{m}) + \int_{\Omega} [F(x, m) - F(x, \bar{m}) - Dv \cdot V].
 \end{aligned}$$

Using (55) and (56), the function h is uniformly small in $L^\infty((t, t + 1) \times \Omega)$ for any $t \in (2 + \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon})$. By parabolic regularity we deduce that

$$\begin{aligned}
 \|Dv(t)\|_\infty &\leq C \sup_{[t, t+1]} [\|Dv(s)\|_2 + \|h\|_\infty] \\
 &\leq C \varepsilon^{\frac{1}{2p}} \quad \forall t \in \left(2 + \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon}\right).
 \end{aligned} \tag{58}$$

Moreover, if we consider the equation satisfied by $w := \partial_\xi v$, which is

$$-\partial_t w - \Delta w + \partial_\xi(Dv \cdot V) = \partial_\xi(F(x, m) - F(x, \bar{m})),$$

by energy estimate we obtain

$$\int_t^{t+1} \int_{\Omega} |Dw|^2 \leq C \|w(t+1)\|_2^2 + C \int_t^{t+1} \int_{\Omega} [|Dv|^2 + |\mu|^2] ds.$$

Recalling that $w = \partial_\xi v$, with arbitrary ξ , and using (55), we deduce

$$\int_t^{t+1} \int_{\Omega} |D^2v|^2 \leq C \varepsilon^{\frac{1}{2}} \tag{59}$$

for $t \in (2 + \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon})$.

Using the mean value theorem from (59), and combining it with (55) and (58), there exists some $\tau_1 \in (T - 3 - \frac{2}{\varepsilon}, T - 2 - \frac{2}{\varepsilon})$ such that

$$\|\tilde{u}^T(\tau_1) - \bar{u}\|_{H^2 \cap W^{1,\infty}} \leq C \varepsilon^{\frac{1}{2p}}. \tag{60}$$

Because of (57) and (60), if ε is sufficiently small, we can use Theorem 5.1 in the interval $[\tau_0, \tau_1]$. Hence, there exist $\delta, K > 0$ such that

$$\|\tilde{u}^T(t) - \bar{u}\|_{H^2 \cap W^{1,\infty}} + \|m^T(t) - \bar{m}\|_{H^1 \cap L^\infty} \leq K (e^{-\omega t} + e^{-\omega(T-t)}) \quad \forall t \in [\delta, T - \delta].$$

On the other hand, we know from Lemma 6.2 that u^T, m^T are smooth in $(0, T)$, so there exists C_δ such that

$$\|\tilde{u}^T(t) - \bar{u}\|_{H^2 \cap W^{1,\infty}} + \|m^T(t) - \bar{m}\|_{H^1 \cap L^\infty} \leq C_\delta \quad \forall t \in [1, \delta] \cup [T - \delta, T - 1],$$

and (12) is completed. □

7. Appendix

We recall here the decay estimates of viscous Hamilton-Jacobi and Fokker-Planck equations which have been systematically used in our analysis of MFG systems. Those are mainly classical results in parabolic theory. The proof of the statements below can be found in detail, among further results of the same kind, in the Appendix of [3].

Lemma 7.1. *Let $V \in L^\infty(\mathbb{R} \times \Omega)$ and $\rho_0 \in L^2_0(\Omega)$. There are constants $\sigma > 0$ and $C > 0$, depending only on $\|V\|_{L^\infty(\mathbb{R} \times \Omega)}$ such that, if ρ is the solution of*

$$\begin{cases} \rho_t - \Delta \rho - \operatorname{div}(\rho V) = 0 & \text{in } (0, \infty) \times \Omega, \\ \rho(0) = \rho_0 \end{cases} \tag{61}$$

then $\|\rho(t)\|_2 \leq C e^{-\sigma t} \|\rho_0\|_2 \quad \forall t > 0.$ (62)

By duality with Lemma 7.1, we get the following.

Lemma 7.2. *Let $V \in L^\infty(\mathbb{R} \times \Omega)$ and $v_0 \in L^2(\Omega)$. Let $\sigma > 0$ be as in Lemma 7.1. If v is the solution of*

$$\begin{cases} -v_t - \Delta v + Dv \cdot V = f & \text{in } (0, T) \times \Omega, \\ v(T) = v_0 \end{cases} \tag{63}$$

then $\tilde{v} := v - \langle v \rangle$ satisfies

$$\|\tilde{v}(t)\|_2 \leq C e^{-\sigma(T-t)} \|\tilde{v}_0\|_2 + C \int_t^T \|f(s)\|_2 e^{-\sigma(s-t)} ds \quad \forall t \leq T$$

where $C = C(\|V\|_\infty)$.

With standard energy methods, the derivatives of v can be bounded in L^2 as well.

Lemma 7.3. *Let v be the solution of (63). For every $0 < t < t_0$, we have*

$$(t_0 - t) \|Dv(t)\|_2^2 \leq \tag{64}$$

$$\leq C [(t_0 - t) + 1] \left\{ \|\tilde{v}(t_0)\|_2^2 + \|f\|_{L^2((t,t_0) \times \Omega)}^2 + \|\tilde{v}\|_{L^2((t,t_0) \times \Omega)}^2 \right\}, \tag{65}$$

and $\int_t^{t_0} \int_\Omega |Dv|^2 \leq C \|\tilde{v}(t_0)\|_2^2 + C \int_t^{t_0} \int_\Omega [|f|^2 + |\tilde{v}|^2]$ (66)

where C only depends on $\|V\|_\infty$.

Finally, we conclude with a Lemma which is often needed due to the coupling between the two equations appearing in MFG systems.

Lemma 7.4. *Assume $V \in L^\infty(\mathbb{R} \times \Omega)$, $F \in L^2_{loc}(0, T; L^2(\Omega))$ and take some function $f \in C^0([0, +\infty), L^2_0(\Omega))$. If $\rho \in L^1_{loc}([0, +\infty), L^1_0(\Omega))$ is the solution of*

$$\rho_t - \Delta \rho - \operatorname{div}(\rho V) = \operatorname{div}(F) + f \quad \text{in } (0, \infty) \times \Omega, \quad (67)$$

then we have, for every $t > t_0 \geq 0$:

$$\|\rho(t)\|_2 \leq C \left\{ e^{-\sigma(t-t_0)} \|\rho(t_0)\|_2 + \int_{t_0}^t e^{-\sigma(t-s)} \|f(s)\|_2 ds + \left(\int_{t_0}^t \int_{\Omega} |F|^2 \right)^{\frac{1}{2}} \right\} \quad (68)$$

for some $C = C(\|V\|_\infty)$, where σ is defined by Lemma 7.1.

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