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Time-Periodic Evans Approach to Weak KAM Theory

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We study the time-periodic version of Evans approach to weak KAM theory. Evans minimization problem is equivalent to a first order mean field game system. For the mechanical Hamiltonian we prove the existence of smooth solutions. We introduce the corresponding effective Lagrangian and Hamiltonian and prove that they are smooth. We also consider the limiting behavior of the effective Lagrangian and Hamiltonian, Mather measures and minimizers.

Keywords: Time-periodic weak KAM solution, effective Lagrangian and Hamiltonian, mean field game approximation.

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1. Introduction

We consider the extension, to time-periodic Hamiltonians $H\colon \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$, of Evans approach to weak KAM theory. For $k \in \mathbb{N}$, we address the problem of minimizing

$$I_k[u] := \int_{\mathbb{T}^{d+1}} e^{k(u_t + H(x,t,\nabla u))} \tag{1}$$

among functions $u\colon \mathbb{T}^{d+1} \to \mathbb{R}$ with $\int u = 0$. We assume that $H\colon \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ is a smooth function satisfying the hypotheses in Mather's seminal paper [7]:

Convexity: H_{pp} is positive definite.

Superlinearity: We have $\displaystyle\lim_{|p|\to\infty} \frac{H(z,p)}{|p|} = \infty$ uniformly on $z \in \mathbb{T}^{d+1}$.

Completeness: The Hamiltonian flow associated to H is complete.

The variational problem is equivalent to the first order mean field game system

$$\text{(MFG)} \qquad \begin{cases} u_t + H(x,t,\nabla u) = \frac{1}{k} \ln m + \bar{H}_k \\ m_t + \operatorname{div}(H_p m) = 0 \\ \int u = 0, \int m = 1 \end{cases}$$

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The minimization problem is also related through duality to an entropy penalized extension of Mather's problem (section 3). P. Cardaliaguet has also studied first order mean field games using variational principles in duality [2], [3] (with J. Graber), as well as other approaches [1]. Althought the mean field game we are considering is time dependent, the fact that we are searching for time-periodic solutions (u, m) together with the constant $\bar{H}_k$, makes more appropriate to consider it as an ergodic problem. The main difference with the ergodic problems that have been studied is that the new Hamiltonian $r + H(z, p)$ is neither coercive nor strictly convex in the variable (p, r).

According to [4] there exists a unique $\bar{H} \in \mathbb{R}$ such that the the Hamilton Jacobi equation

$$u_t + H(z, \nabla u) = \bar{H}, \quad z = (x, t) \tag{2}$$

has a Lipschitz viscosity solution $\phi \colon \mathbb{T}^{d+1} \to \mathbb{R}$. Moreover $\bar{H}$ is given by the min-max formula

$$\bar{H} = \inf_{u \in C^1(\mathbb{T}^{d+1})} \max_{z \in \mathbb{T}^{d+1}} u_t(z) + H(z, \nabla u(z)) \tag{3}$$

which motivates Evans problem of minimizing the functional I_k [5]. The convexity of the exponential and Hamiltonian functions imply that I_k is lower semicontinuous on $W^{1,q}(\mathbb{T}^{d+1})$ but it is not coercive due to the linear term u_t.

Using Jensen's inequality and $\int u_t = 0$, we have

$$e^{k \min H} \leq \exp \int_{\mathbb{T}^{d+1}} kH(z, \nabla u) \leq I_k[u],$$

Letting $e^{k\bar{H}_k} = \inf_u I_k[u]$, we have

$$e^{k \min H} \leq e^{k\bar{H}_k} \leq I_k[0] \leq e^{\max\{H(z,0): z \in \mathbb{T}^{d+1}\}}$$

and thus $$\min H \leq \bar{H}_k \leq \max_{z \in \mathbb{T}^{d+1}} H(z, 0).$$

A minimizer must satisfy the Euler Lagrange equation

$$\text{(EL)} \qquad (e^{k(u_t + H(z,\nabla u))})_t + \operatorname{div}(e^{k(u_t + H(z,\nabla u))} H_p(z, \nabla u)) = 0$$

which can be written as

$$u_{tt} + 2H_p \nabla u_t + \nabla^2 u(H_p, H_p) + \frac{1}{k} \operatorname{tr}(H_{pp} \nabla^2 u) + H_t + H_x \cdot H_p + \frac{1}{k} \operatorname{tr} H_{px} = 0 \tag{4}$$

where all derivatives of H are evaluated at $(z, \nabla u(z))$.

Letting $m = e^{k(u_t + H(z,\nabla u) - \bar{H}_k)}$, we transform (EL) together with the condition $\int u = 0$ and the definition of $\bar{H}_k$ into the mean field game system (MFG).

By the convexity of the exponential function

$$\int_{\mathbb{T}^{d+1}} e^{k(u_t + H(x,t,\nabla u))} \geq \int_{\mathbb{T}^d} e^{\int_0^1 kH(x,t,\nabla u) dt}.$$

If H does not depend on t, we define $\bar{u}(x) = \int_0^1 u(x,t)dt$ with the consequence of $\nabla\bar{u}(x) = \int_0^1 \nabla u(x,t)dt$. From the convexity of H we get

$$\int_0^1 kH(x, \nabla u(x,t))dt \geq kH(x, \nabla\bar{u}(x)), \tag{5}$$

$$\int_{\mathbb{T}^d} e^{\int_0^1 kH(x,\nabla u)dt} \geq \int_{\mathbb{T}^d} e^{kH(x,\nabla\bar{u})} \geq e^{k\bar{H}_{k,a}} \tag{6}$$

where $e^{k\bar{H}_{k,a}}$ is the minimum for the autonomous problem. If v is a minimizer for that problem, $v_t = 0$ and

$$\int_{\mathbb{T}^{d+1}} e^{kH(x,\nabla v)} = e^{k\bar{H}_{k,a}},$$

so v is also a minimizer for the time dependent problem and $\bar{H}_k = \bar{H}_{k,a}$.

2. Existence of classical solutions

Writing $z = (x,t)$, $q = (p,r)$, $Du(z) = (\nabla u(z), u_t(z))$, (4) can be written as

$$\operatorname{tr}(a_k(z, Du)D^2u) + b_k(z, Du) = 0, \tag{7}$$

where

$$a_k(z,q) = \begin{pmatrix} \frac{1}{k}H_{pp} + H_p \otimes H_p & H_p \\ H_p^T & 1 \end{pmatrix} = \sigma\sigma^T, \quad \sigma = \begin{pmatrix} (\frac{1}{k}H_{pp})^{\frac{1}{2}} & H_p \\ 0 & 1 \end{pmatrix}$$

$$b_k(z,q) = H_t + H_x \cdot H_p + \frac{1}{k}\operatorname{tr} H_{px}.$$

The main difficulty to establish the existence of classical solutions of (7) is that the maximal and minimal eigenvalues of $a_k(z,q)$ go as H_p^2 and $1/H_p^2$ when $|p|$ tends to ∞.

One can obtain apriori Lipschitz bounds for solutions in particular cases that include the most typical examples of Hamiltonians. Those are the cases when b_k is sublinear. More precisely,

Assumption 2.1. There exists a continuous function $\chi\colon [0,\infty) \to \mathbb{R}$ such that

$$\int^{\infty} \frac{ds}{\chi(s)+1} = \infty, \quad |b_k(z,q)| \leq \chi(|q|), (z,q) \in \mathbb{T}^{d+1} \times \mathbb{R}^{d+1}.$$

We observe that the assumption holds when

$$H(x,t,p) = \frac{1}{2}|p + \eta(t)|^2 + V(x,t). \tag{8}$$

In fact, $b_k(z,p) = (p+\eta(t)) \cdot \eta'(t) + V_t(x,t) + \nabla V(x,t)(p+\eta(t))$ does not depend on k, and we can take $\chi(s) = cs + d$ for some $c, d > 0$.

Lemma 2.2. *Under assumption* 2.1, *there exists* $K > 0$ *depending only on* χ *such that for a solution* $\phi \in C^2(\mathbb{T}^{d+1})$ *of* (7), *we have* $\|D\phi\|_\infty \leq K$.

Proof. For a suitable increasing concave function $\psi\colon [0,\infty)\to\mathbb{R}$ with $\psi(0)=0$, consider $h\colon \mathbb{R}^{d+1}\times\mathbb{R}^{d+1}\to\mathbb{R}$, where

$$h(z,w)=\phi(z)-\phi(w)-\psi(|z-w|).$$

Since ϕ is periodic and ψ is increasing, h achieves its global maximum in the cube $[0,1]^{d+1}\times[0,1]^{d+1}$. So we build ψ on $[0,2]$ and extend it by keeping it increasing and concave. Using Assumption 2.1, one can choose ψ to be a solution of

$$2\psi" = -\chi(\psi')-1 \text{ on } (0,2), \quad \psi(0)=0$$

with $\psi'>0$ on $[0,2]$. In fact, for K sufficiently large, the function

$$g(t)=\int_t^K \frac{2\,du}{\chi(u)+1}$$

is strictly decreasing and for some $a>0$, $[0,2]\subset g([a,K])$. Letting $\psi\colon[0,2]\to\mathbb{R}$ be the primitive of g^{-1} with $\psi(0)=0$, we have that ψ satisfies the requeriments. Observe that $\max\limits_{z,w} h(z,w)\geq 0$. We want to prove $\max h=0$ because in that case $\phi(z)-\phi(w)\leq\psi(|z-w|)\leq K|z-w|$, the last inequality being a consequence of the concavity of ψ. Assume by contradiction that $\max h$ is positive and is achieved at $(\bar z,\bar w)$. Then, for $q=(\bar z-\bar w)/|\bar z-\bar w|$,

$$(0,0)=Dh(\bar z,\bar w)=(D\phi(\bar z)-\psi'(|\bar z-\bar w|)q, -D\phi(\bar w)+\psi'(|\bar z-\bar w|)q),$$

and

$$0\geq D^2h(\bar z,\bar w)$$

$$=\begin{pmatrix} D^2\phi(\bar z) & 0\\ 0 & -D^2\phi(\bar w)\end{pmatrix}-\psi'(|\bar z-\bar w|)\begin{pmatrix} B & -B\\ -B & B\end{pmatrix}-\psi"(|\bar z-\bar w|)\begin{pmatrix} q\otimes q & -q\otimes q\\ -q\otimes q & q\otimes q\end{pmatrix},$$

where $B=(I-q\otimes q)/|\bar z-\bar w|$. Thus, $Bq=0$ and

$$q^T(D^2\phi(\bar z)-D^2\phi(\bar w))q=(q^T\ \ -q^T)\begin{pmatrix} D^2\phi(\bar z) & 0\\ 0 & -D^2\phi(\bar w)\end{pmatrix}\begin{pmatrix} q\\ -q\end{pmatrix}\leq 4\psi"(|\bar z-\bar w|).$$

Taking $v=\sigma^{-1}q$ we have

$$(\sigma v)^TD^2\phi(\bar z)\sigma v-(\sigma v)^TD^2\phi(\bar w)\sigma v\leq 4\psi"(|\bar z-\bar w|). \tag{9}$$

Similarly, for any $r\in\mathbb{R}^{d+1}$ we have

$$\begin{aligned}&(\sigma r)^TD^2\phi(\bar z)\sigma r-(\sigma r)^TD^2\phi(\bar w)\sigma r\\ &=((\sigma r)^T\ (\sigma r)^T)\begin{pmatrix} D^2\phi(\bar z) & 0\\ 0 & -D^2\phi(\bar w)\end{pmatrix}\begin{pmatrix} \sigma r\\ \sigma r\end{pmatrix}\leq 0.\end{aligned} \tag{10}$$

Inequalities (9) and (10) imply that

$$\operatorname{tr}\sigma^TD^2\phi(\bar z)\sigma-\operatorname{tr}\sigma^TD^2\phi(\bar w)\sigma\leq 4\psi"(|\bar z-\bar w|).$$

Since ϕ is a solution of (7), we have

$$\operatorname{tr}\sigma^T D^2\phi(\bar z)\sigma + b_k(\bar z, \psi'(|\bar z-\bar w|)q) = 0$$
$$\operatorname{tr}\sigma^T D^2\phi(\bar w)\sigma + b_k(\bar w, \psi'(|\bar z-\bar w|)q) = 0.$$

Thus

$$4\psi"(|\bar z-\bar w|) \geq b_k(\bar w, \psi'(|\bar z-\bar w|)q) - b_k(\bar z, \psi'(|\bar z-\bar w|)q) \geq -2\chi(\psi'(|\bar z-\bar w|)),$$

which leads to $-2 \geq 0$. □

Theorem 2.3. *For the Hamiltonian equation* (8) *the Euler-Lagrange equation* (EL) *has a unique smooth solution satisfying* $\int_{\mathbb{T}^{d+1}} u = 0$.

Proof. We use the continuation method. Consider the family of Hamiltonians

$$H_\lambda(x,t,p) = \frac{1}{2}|p+\lambda\eta(t)|^2 + \lambda V(x,t) \tag{11}$$

and the PDE

$$(e^{k(u_t+H_\lambda(x,t,\nabla u))})_t + \operatorname{div}(e^{k(u_t+H_\lambda(x,t,\nabla u))}D_pH_\lambda(x,t,\nabla u)) = 0 \tag{EL$_\lambda$}$$

for $u\colon \mathbb{T}^{d+1}\to\mathbb{R}$ with $\int_{\mathbb{T}^{d+1}} u = 0$. Define

$$\Lambda := \{\lambda \in [0,1] : (\mathrm{EL}_\lambda) \text{ has a smooth solution}\}.$$

It is clear that $0\in\Lambda$ and that constant functions are all the solutions of (EL_0).

We claim that Λ is closed. In fact, it is clear that Assumption 2.1 holds for H_λ with the same $\chi(s) = cs + d$ for all $\lambda \in [0,1]$. Therefore Lemma 2.2 implies that there is $K > 0$ such that for any $\lambda\in\Lambda$ the corresponding solution satisfies $\|Du_\lambda\|_\infty \leq K$. Elliptic regularity theory implies that we can bound uniformly in $\lambda\in\Lambda$ derivatives of u_λ of any order. Thus, any convergent sequence in Λ has a subsequence whose corresponding sequence of solutions converge uniformly, along with all derivatives.

We claim that Λ is open. Indeed, for $\lambda\in\Lambda$ the linearization of (EL_λ) about the solution u is given by

$$\mathcal{L}v := -(e^{k(u_t+H_\lambda(z,\nabla u))}(v_t + D_pH_\lambda(z,\nabla u)\cdot\nabla v))_t \tag{12}$$
$$-\operatorname{div}(e^{k(u_t+H_\lambda(z,\nabla u))}((v_t + D_pH_\lambda(z,\nabla u)\cdot\nabla v)D_pH_\lambda(z,\nabla u) + \frac{1}{k}\nabla v D_{pp}H_\lambda(z,\nabla u))$$

so that
$$\int_{\mathbb{T}^{d+1}} v\mathcal{L}v = \int_{\mathbb{T}^{d+1}} e^{k(u_t+H_\lambda(z,\nabla u))}Dv^T a_k(z,Du)Dv.$$

Then $\mathcal{L}$ is a symmetric, uniformly elliptic operator, whose null space consists of the constants. The Implicit Function Theorem yields that for any μ in a neighborhood of λ, equation (EL_μ) has a unique solution satisfying $\int_{\mathbb{T}^{d+1}} u = 0$.

Since Λ is nonempty, closed and open it coincides with $[0,1]$. Thus equation (EL) has a unique smooth solution satisfying $\int_{\mathbb{T}^{d+1}} u = 0$. □

3. Entropy penalized Mather theory

Given a Borel probability $\mu \in \mathcal{P}(\mathbb{T}^{d+1} \times \mathbb{R}^d)$ we consider its push forward $m_\mu \in \mathcal{P}(\mathbb{T}^{d+1})$ given by

$$\int_{\mathbb{T}^{d+1}} \varphi(z) dm_\mu(z) = \int_{\mathbb{T}^{d+1}\times\mathbb{R}^d} \varphi(z) d\mu(z,v). \tag{13}$$

In the set $\mathcal{A} \subset \mathcal{P}(\mathbb{T}^{d+1})$ of measures absolutely continuous with respect to Lebesgue measure, the mapping

$$m \mapsto S^*[m] := \int_{\mathbb{T}^{d+1}} \log m(z) dm(z)$$

is convex and lower semicontinuous. This mapping can be extended as a convex lower semicontinuous functional to $\mathcal{P}(\mathbb{T}^{d+1})$. The weak topology of $\mathcal{P}(\mathbb{T}^{d+1})$ is metrizable. Denoting by $B_r(n)$ the open ball of radius $r > 0$ centered at n we define the extension by

$$\bar{S}[n] = \liminf_{m\to n} S^*[m] = \sup_{r>0} \inf\{S^*[m] : m \in B_r(n) \cap \mathcal{A}\}.$$

Note that the map $\bar{S}$ is allowed to take the value $+\infty$, and since $y \ln y \geq -1/e$ we have $\bar{S} \geq -1/e$. Finally define $S[\mu] = \bar{S}[m_\mu]$.

Let $L\colon \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ be a C^2 function, strictly convex and superlinear in v

$$\frac{L(z,v)}{|v|} \to +\infty, \text{ as } |v| \to +\infty.$$

We consider the convex lower semicontinuous functional

$$A_{L,k}(\mu) = \int_{\mathbb{T}^d\times\mathbb{R}^d} L(z,v) d\mu + \frac{1}{k} S[\mu], \tag{14}$$

defined on the space $\mathcal{C}$ of measures $\mu \in \mathcal{P}(\mathbb{T}^{d+1} \times \mathbb{R}^d)$ such that

(a) $\displaystyle\int_{\mathbb{T}^{d+1}\times\mathbb{R}^d} |v| d\mu(z,v) < +\infty$

(b) For all $\varphi \in C^1(\mathbb{T}^{d+1})$, $\displaystyle\int_{\mathbb{T}^{d+1}\times\mathbb{R}^d} (\varphi_t + \nabla\varphi \cdot v) d\mu(x,t,v) = 0.$

From Mather's theory we know that for any $Q \in \mathbb{R}^d$ there is $\mu \in \mathcal{C}$ such that $Q = \int_{\mathbb{T}^{d+1}\times\mathbb{R}^d} v\, d\mu(z,v)$. We define the effective Lagrangian $\bar{L}_k\colon \mathbb{R}^d \to \mathbb{R}$ by

$$\bar{L}_k(Q) := \inf\{A_{L,k}(\mu) : \mu \in \mathcal{C}, Q = \int_{\mathbb{T}^{d+1}\times\mathbb{R}^d} v\, d\mu(z,v)\}. \tag{15}$$

Let $H\colon \mathbb{T}^{d+1} \times \mathbb{R}^d \to \mathbb{R}$ be Legendre transform of L.

Proposition 3.1.
$$\inf_{\mu\in\mathcal{C}} A_{L,k}(\mu) \geq -\frac{1}{k}\inf_{\varphi\in C^1}\log I_k[\varphi]. \tag{16}$$

Proof. Let $\mu\in\mathcal{C}$ and $\varphi\in C^1(\mathbb{T}^{d+1})$. Let m_μ be given by (13). We have

$$\int L(z,v)d\mu = \int (L(z,v)-\varphi_t-\nabla\varphi\cdot v)d\mu \geq -\int(\varphi_t+H(z,\nabla\varphi))dm_\mu,$$

since, for all $(z,v)\in\mathbb{T}^{d+1}\times\mathbb{R}^d$, $L(z,v)-\nabla\varphi(z)v\geq -H(z,\nabla\varphi(z))$, with equality only when $v=H_p(z,\nabla\varphi(z))$. Define $h_\varphi\colon\mathcal{P}(\mathbb{T}^{d+1})\to\mathbb{R}$ by

$$h_\varphi(m)=\int_{\mathbb{T}^{d+1}}(\varphi_t+H(z,\nabla\varphi))dm-\frac{1}{k}\bar{S}[m]$$

This implies $A_{L,k}(\mu)\geq -h_\varphi(m_\mu)$. Letting $m_\varphi(z)=e^{k(\varphi_t+H(z,\nabla\varphi))}/I_k[\varphi]$, we have $h_\varphi(m_\varphi)=\frac{1}{k}\log I_k[\varphi]$.

The convex function $t\mapsto t\log t$ has the Legendre transform $s\mapsto e^{s-1}$. In particular this implies that $t\log t+1\geq t$, and so, for any $m\in\mathcal{A}$ we obtain

$$h_\varphi(m)\leq h_\varphi(m_\varphi)+\int(\varphi_t+H(z,\nabla\varphi)-\frac{1}{k}\log m_\varphi-\frac{1}{k})(m(x)-m_\varphi(x))dx.$$

The convexity and an approximation argument show that in fact the previous inequality holds for any $m\in\mathcal{P}(\mathbb{T}^{d+1})$. From the definition of m_φ, and since m and m_φ are probability measures, the second term on the rhs vanishes and then

$$\frac{1}{k}\log I_k[\varphi]=\max_m h_\varphi(m).$$

Therefore, $A_{L,k}(\mu)\geq -\frac{1}{k}\log I_k[\varphi]$, and (16) follows. □

Proposition 3.2. *If* (EL) *has a smooth solution* u_k*,* $m_k=m_{u_k}$ *then*

$$\mu_k(z,v)=\delta(v-H_p(z,\nabla u_k(z)))m_k(z), \tag{17}$$

is a minimizer of (14) *and* u_k *is a minimizer of* (1).

Proof. Definition (17) means that for all continuous function $F\colon\mathbb{T}^{d+1}\times\mathbb{R}^d\to\mathbb{R}$ we have

$$\int_{\mathbb{T}^{d+1}\times\mathbb{R}^d}F(z,v)d\mu_k=\int_{\mathbb{T}^{d+1}}F(z,H_p(z,\nabla u))dm_k$$

and since (u_k,m_k) is a solution of (MFG), we get that $\mu_k\in\mathcal{C}$. Moreover,

$$A_{L,k}(\mu_k)=\int_{\mathbb{T}^{d+1}}[L(z,H_p(z,\nabla u_k))+\frac{1}{k}\log m_k]dm_k=-h_{u_k}(m_k)=-\frac{1}{k}\log I_k[u_k],$$

and therefore

$$A_{L,k}(\mu_k)=\inf_{\mu\in\mathcal{C}}A_{L,k}(\mu)=-\frac{1}{k}\inf_{\varphi\in C^1}\log I_k[\varphi]=-\frac{1}{k}\log I_k[u_k]. \qquad \square \tag{18}$$

Lemma 3.3. *The effective Lagrangian* $\bar{L}_k$ *is convex.*

Proof. Given $\varepsilon > 0$, let ν_1, ν_2 be such that $A_{L,k}(\nu_i) < \bar{L}_k(Q_i) + \varepsilon$ and $\int v\, d\nu_i = Q_i$. Note that, for $0 \le \lambda \le 1$,

$$\int v\, d(\lambda\nu_1 + (1-\lambda)\nu_2)) = \lambda \int v\, d\nu_1 + (1-\lambda) \int v\, d\nu_2 = \lambda Q_1 + (1-\lambda) Q_2,$$
$$m_{\lambda\nu_1+(1-\lambda)\nu_2} = \lambda m_{\nu_1} + (1-\lambda) m_{\nu_2}.$$

Since $\bar{S}$ is convex, we have that

$$\begin{aligned}\bar{L}_k(\lambda\nu_1 + (1-\lambda)\nu_2) &\le A_{L,k}(\lambda\nu_1 + (1-\lambda)\nu_2) \le \lambda A_{L,k}(\nu_1) + (1-\lambda) A_{L,k}(\nu_2) \\ &= \lambda \bar{L}_\varepsilon(Q_1) + (1-\lambda)\bar{L}_\varepsilon(Q_2) + \varepsilon.\end{aligned}$$ □

For $P \in \mathbb{R}^d$ we define $$\bar{H}_k(P) := \frac{1}{k} \inf_\varphi \int_{\mathbb{T}^{d+1}} e^{k(\varphi_t + H(x,t,P+\nabla\varphi))}. \tag{19}$$

Corollary 3.4. *The Legendre transform of* $\bar{L}_k$ *is* $\bar{H}_k$.

Proof. Since the Legendre transform of $L(z,v) - Pv$ is $H(z, P+p)$ and (16) is an equality, we have

$$\sup_Q PQ - \bar{L}_\varepsilon(Q) = -\inf_{\mu\in\mathcal{C}} A_{L-\langle P,\cdot\rangle,k}(\mu) = \frac{1}{k}\log\inf_\phi \int e^{k(\phi_t + H(z,P+\nabla\phi))} dx.$$ □

Lemma 3.5. *For the Hamiltonian* (8), $\bar{H}_k(P)$ *is strictly convex. Furthermore for each P,* (19) *admits at most one minimizer, up to the addition of constants.*

Proof. For $P \in \mathbb{R}^d$ the Hamiltonian $H(x,t,p+P) = \frac{1}{2}|p + P + \eta(t)|^2 + V(x,t)$ is of the same type. Suppose there are $P_0, P_1 \in \mathbb{R}^d$ and $0 < \lambda < 1$ such that

$$\bar{H}_k(\lambda P_0 + (1-\lambda)P_1) = \lambda \bar{H}_k(P_0) + (1-\lambda)\bar{H}_k(P_1).$$

Let $f_i \in C^2(\mathbb{T}^{d+1})$ be a solution of (EL) with $u = P_i x + f_i$, $i = 0, 1$ so that

$$\bar{H}_k(P_i) = \frac{1}{k}\log \int_{\mathbb{T}^{d+1}} e^{k(f_{it} + H(z,P_i+\nabla f_i))}.$$

For $\varphi = \lambda f_0 + (1-\lambda) f_1$ we have

$$\varphi_t = \lambda f_{1t} + (1-\lambda) f_{2t},$$
$$\nabla\varphi + \lambda P_0 + (1-\lambda)P_1 = \lambda(\nabla f_0 + P_0) + (1-\lambda)(\nabla f_1 + P_1),$$

and, by convexity of H,

$$H(z, \lambda P_0 + (1-\lambda)P_1 + \nabla\varphi) \le \lambda H(z, \nabla f_0 + P_0) + (1-\lambda)(H(z, \nabla f_1 + P_1)). \tag{20}$$

Convexity of the exponential function and Hölder inequality yield

$$\begin{aligned}
e^{k\bar{H}_k(\lambda P_0+(1-\lambda)P_1)} &\le \int_{\mathbb{T}^{d+1}} e^{k(\varphi_t+H(z,\lambda P_0+(1-\lambda)P_1+\nabla\varphi))} \\
&\le \int_{\mathbb{T}^{d+1}} e^{k(\lambda(f_{0t}+H(z,\nabla f_0+P_1))+(1-\lambda)(f_{1t}+H(z,\nabla f_1+P_1))} \\
&\le \left[\int_{\mathbb{T}^{d+1}} e^{k(f_{0t}+H(z,\nabla f_0+P_0))}\right]^{\lambda} \left[\int_{\mathbb{T}^{d+1}} e^{k(f_{1t}+H(z,\nabla f_1+P_1))}\right]^{(1-\lambda)} \\
&= e^{\lambda k\bar{H}_k(P_0)} e^{(1-\lambda)k\bar{H}_k(P_1)} = e^{k\bar{H}_k(\lambda P_0+(1-\lambda)P_1)}. \qquad (21)
\end{aligned}$$

Therefore all inequalities in (21) are equalities and so is (20). Since H is strictly convex $\nabla f_0 + P_0 = \nabla f_1 + P_1$ at all points. Hence $P_1 - P_0 = \nabla(f_0 - f_1)$ with $f_0 - f_1$ periodic, so $P_0 = P_1$. Since $P_0x + f_0$, $P_0x + f_1$ are solutions of (4) with $\nabla f_0 = \nabla f_1$ we have that $f_{0tt} = f_{1tt}$ and then $f_0 - f_1$ is constant. □

It follows from Theorem 2.3 and Proposition 3.2 that for the Hamiltonian given by (8) the functional I_k has a unique minimizer satisfying $\int_{\mathbb{T}^{d+1}} u = 0$ which is a solution of equation (EL).

Theorem 3.6. *For the Hamiltonian equation* (8) *the effective functions* $\bar{L}_k, \bar{H}_k$ *are smooth.*

Proof. For $P \in \mathbb{R}^d$ consider equation (EL) with $u = Px + \phi$ and define $F(P, \phi)$ as the l.h.s. of that equation. We have seen that for a solution $\phi = \phi(\cdot, P)$ of $F(P, \phi) = 0$, $\mathcal{L} = D_2F(P, \phi)$ is given by (12). The Implicit Function Theorem implies that $\phi(\cdot, P)$ is smooth in P and so $\bar{H}_k$ is smooth. Moreover $\bar{H}_k$ is stricltly convex so $D\bar{H}_k$ has a smooth inverse G_k and therefore $\bar{L}_k(Q) = QG_k(Q) - \bar{H}_k(G_k(Q))$ is smooth. □

4. Approximating weak KAM theory

In this section we assume the Hamiltonian is given by (8) so that b_k satisfies assumption (2.1) with χ independent of k. Let u_k be the minimizer of (1) with $\int u_k = 0$. From Lemma 2.2 there is K such that $\|Du_k\|_\infty \le K$ for any k, so passing to a subsequence, u_k converges uniformly to a Lipschitz $u\colon \mathbb{T}^{d+1} \to \mathbb{R}$ and $Du_k \rightharpoonup Du$ weakly in $L^q(\mathbb{T}^{d+1}, \mathbb{R}^{d+1})$, for any $1 \le q < \infty$.

As in the autonomous case we have the following Theorem with a similar proof.

Theorem 4.1. (i) $\lim_{k\to\infty} \bar{H}_k = \bar{H}$.

(ii) *The function u is a viscosity solution of*

$$u_{tt} + 2H_p\nabla u_t + \nabla^2 u(H_p, H_p) + H_t + H_x \cdot H_p = 0, \qquad (22)$$

which can be written as

$$(u_t + H(z, \nabla u))_t + \nabla(u_t + H(z, \nabla u)) \cdot H_p = 0,$$

looking like a differentiated form of the Hamilton-Jacobi equation (2).

(iii) *Moreover,* $u_t + H(z, \nabla u) \le \bar{H}$ *Lebesgue a.e. in* $\mathbb{T}^{d+1}$.

Proposition 4.2. *Let μ_k be the measure defined by* (17). *Passing to a subsequence such that $\mu_k \rightharpoonup \mu$, we have*

(a) *μ is a Mather measure; and* (b) $\displaystyle\lim_{k\to\infty}\frac{1}{k}S[\mu_k]=0.$

Proof. The function $\frac{1}{k}\log m_k = u_{kt}+H(z,\nabla u_k)$ is uniformly bounded. For $\lambda>0$,

$$\int_{\mathbb{T}^{d+1}}\frac{1}{k}\log m_k dm_k = \int_{\{\log m_k\geq -k\lambda\}}\frac{1}{k}\log m_k dm_k + \int_{\{\log m_k< -k\lambda\}}\frac{1}{k}\log m_k dm_k$$

$$\geq -\lambda\int_{\{\log m_k\geq -k\lambda\}} dm_k - \int_{\{\log m_k< -k\lambda\}} Ce^{-k\lambda}dz \geq -\lambda - Ce^{-k\lambda}.$$

Thus, $\displaystyle\liminf_{k\to\infty}\frac{1}{k}S[\mu_k]\geq -\lambda$. Since $\lambda>0$ is arbitrary

$$\liminf_{k\to\infty}\frac{1}{k}S[\mu_k]\geq 0. \tag{23}$$

We recall that $\displaystyle -\bar{H}=\min_{\mu\in\mathcal{C}}\int L\,d\mu$. From (23)

$$\int L\,d\mu = \lim_{k\to\infty}\int L\,d\mu_k = \lim_{k\to\infty} A_{L,k}(\mu_k)-\frac{1}{k}S[\mu_k] = -\lim_{k\to\infty}\bar{H}_k - \frac{1}{k}S[\mu_k]\leq -\bar{H}.$$

Hence, μ is a minimizing measure, the inequality is an equality and (b) holds. □

In the autonomous case and for dimension 1 we identified in [6] the Mather measures that are obtained as the limits in Proposition 4.2. It would be very interesting to obtain a similar result in the time-periodic case.

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