Minimax Theory and its Applications
Volume 11 (2026), No. 3s, 270 to 289

Certain fuzzy colorings on wheel related fuzzy graphs and an application tointerview scheduling optimization

Sreeja K S[1], T Kamaraj[2]

[1]Research scholar, Sathyabama Institute of Science and Technology, Chennai
[2]Assistant Professor,Sathyabama Institute of Science and Technology, Chennai

Abstract
The concept of a fuzzy graph is an extension of a graph where the degrees of membership of vertices and edges show ambiguous or imprecise interactions between them. The strength of the edge's incident to a vertex in a fuzzy graph determines the fuzzy color that can be applied to that vertex. In this research, we derive results on fuzzy coloring's chromatic numbers and fuzzy fractional coloring of several wheel-related fuzzy graphs. We then use these fuzzy coloring approaches to address a real-world interview schedule optimization problem.

Keywords: fuzzy coloring, fractional coloring, wheel graph, star graph,friendship graph,fan graph

1. Introduction

An illustration of the relationships inside a system or network is called a graph. A particular kind of vertex labeling called "graph coloring" use colors as labels so that no two neighboring vertices have the same color. The minimum quantity of colors needed to color a graph is known as the chromatic number. When Zadeh (1965) introduced fuzzy sets, recognizing the problem of ambiguity and vagueness, the scientific and technological community was revolutionized.

Fuzzy graphs were first presented by Rosenfeld (1975) as a diagrammatic description of fuzzy relations on fuzzy sets. The degree of fuzzy vertices, fuzzy paths, fuzzy trees, fuzzy-subgraphs, and complement of a fuzzy graph serve as the foundation for the numerous fuzzy graph ideas and definitions (Mordeson and Nair, 2000; Sunitha and Mathew, 2013). Fuzzy coloring of fuzzy graphs according to the intensity of an edge incident to a vertex was covered by Samanta et al. (2016). Tanmoy Mahapatra (2020) introduced fuzzy fractional coloring on fuzzy graphs as an extension of fuzzy coloring.

Wheel-related graphs offer a potential framework for cryptographic applications, secure communication networks, and scheduling improvements due to their distinctive structural and combinatorial characteristics (Gurjar, D. K. and Krishnaa, A. (2022)). For example, if any matching is eliminated from a friendship graph, they still stay connected. When building networks that are resistant to specific line failures, this feature is helpful. (Andrzej Proskurowski and Arthur M. Farley, 1982)

In this work, we address the limitations of these chromatic numbers based on the restriction on the location and number of weak edges. We also draw conclusions on the chromatic numbers of fuzzy coloring and fuzzy fractional coloring of several wheel related fuzzy graphs. We also use these fuzzy coloring approaches to tackle a real-world interview schedule optimization challenge.

1.1 Preliminaries

We review some fundamental graph definitions, fuzzy graphs, fuzzy coloring, fuzzy fractional coloring of graphs, and several kinds of wheel-related graphs.

Definition 1 Let G = (V, E) be a graph, where E ⊆ V×V is a set of edges and V is a non-empty set whose entries are termed vertices. Dots represent vertices, and lines connecting the pair of vertices (u, v) ∈ E represent edges. If there is an edge connecting two vertices, u and v, such that (u, v) ∈ E, then the two are said to be nearby vertices

Definition 2A relationship graph Constructed by merging n copies of cycle C3 (triangle) with a common vertex, F_n is a graph with 2n+1 vertices and 3n edges.

Definition 3A star graph Sn is a bipartite graph K_1, n, which contains n+1 vertices and n edges.

Definition 4A fan graph Where K1 is a complete graph on one vertex and Pn, n≥2 is the path graph on n vertices, F_n is the graph join K1+Pn.

Definition 5A wheel graph Wn is the graph join K1+Cn, where Cn is a cycle on n vertices (n≥3) created by joining all (rim) vertices of a cycle of length n to a single (apex) vertex.

Definition6[25] Zadeh (1965) The membership function is a mapping m : X → [0, 1] that characterizes a fuzzy set A on the universal set X. A = (X, m) represents a fuzzy set.

Definition7[5] The algebraic structure of a fuzzy graph ξ = (V, σ, μ) consists of a non-empty set V and two functions: σ: V → [0, 1] and μ: V × V → [0, 1]. such that μ (x, y) ≤ minimum of {σ(x), σ(y)}, and μ is a symmetric fuzzy relation on σ for every x, y ∈ V. In this case, the membership values of the vertex x and the edge (x, y) in ξ are denoted by σ(x) and μ(x, y), respectively. ξ* = (V, σ*, μ*) is the underlying crisp graph of ξ, where σ* = {u∈V/(σ(u)>0}) and μ* = {(u,v)∈VXV/(μ(u,v)>0}).

Definition 8(Samanta and others, 2016) Take a look at the fuzzy graph G = (V, α, β). If 1/2min {α (u), α (v)} ≤β (u, v), then two vertices u and v in V are strong nearby vertices, and the accompanying edge is a strong edge; if not, the vertices are called non-strong adjacent vertices, and the corresponding edge is a weak edge.

Definition 9 (Samanta et al. 2016) K(u, v) = β(u, v) / (min{α(u), α (v)} is the definition of the strength of an edge (u, v), represented by K(u, v).

Definition 10 According to Samanta et al. (2016), The collection of pairwise non-strong neighboring vertices is called an independent set in G = (V, α, β); the independence number of the largest independent set in G equals its size and is denoted by σ(G).

Definition 11 (Mũnoz and others, 2005) A proper n fuzzy coloring is a mapping that gives each network vertex one of the n colors, giving strong neighboring vertices a unique color.

Definition 12 (Samanta and others, 2016) Let C = {C_1, C_2,…..C_n} be the fundamental color set, with n ≥ 1. Let f be a mapping and let C → [0, 1]. The fuzzy color set {(C_i, f(C_i)) / C_i∈C} is thus present. If f(C_i)=1, then the fuzzy color of Ci is equal to the basic color Ci. The fuzzy chromatic number of fuzzy colorings is the bare minimum of basic colors needed to color a fuzzy graph.

Let us consider two fuzzy graphs whose underlying crisp graph is K4.

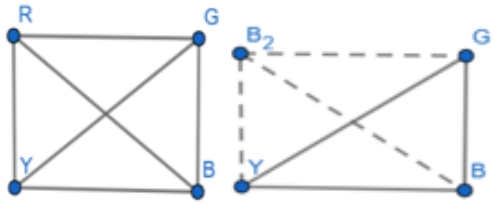


Figure 1. (a)Figure1(b)

A fuzzy graph with all of its edges being strong is depicted in Figure 1(a). The fuzzy graph has four colors, just like ordinary coloring of K_4. A fuzzy graph with edges incident to exactly one vertex of K_4 being weak (denoted by dotted lines) and the remaining edges being strong is depicted in Figure1(b). Here, the three pairwise mutually strong neighbor vertices are colored with three basic colors, say Y, B, and G. The vertex that is incident with three weak edges and whose other end vertices use three basic colors can be colored with any of the fuzzy colors of Y, B, or G. The membership value of the fuzzy color is determined by the strengths of these weak edges. The fuzzy chromatic number is three since there are three basic colors that are utilized.

Definition 13(Tanmoy et al.2020) Given a fuzzy graph G, let M(G) be the collection of all maximal independent sets I of vertices in G. Fuzzy fractional coloring of G is a mapping,

f_{frac}: M(G) $\rightarrow$ [0, 1] such that, $\sum_{v \in I \text{ and } I \in M(G)} f_{frac}(I) > 1, \quad \forall\, v \in V$. The weight of the fuzzy fractional coloring is $\sum_{I \in M(G)} f_{frac}(I)$. The fuzzy fractional chromatic number of G is the least weight that can be assigned to a fuzzy fractional coloring.

Now, we recall the fuzzy fractional coloring technique (Tanmoy et al.2020) applied to a fuzzy graph G.
Step 1: Remove all the weak edges i.e. the edges connecting the non-strong vertices.

Step 2: Remove the isolated vertices.

Step 3: Collect the maximal independent sets of vertices i.e.

$$M(G) = \{I \subseteq V / \, I \text{ is a maximal independent set}\}$$

Step 4**:** Choose one color C_I for each set $I \in M(G)$

Step 5: Allot a fraction (weight of the color), $f_{frac}(I)$ to each independent set I

(color C_I) , such $\sum_{v \in I \text{ and } I \in M(G)} f_{frac}(I) > 1 \quad \forall\, v \in V$.

Step 6: In Step 5, determine the lowest value of $\sum_{I \in M(G)} f_{frac}(I)$ that satisfies the inequality.

The fuzzy fractional chromatic number, represented by $\chi_{frac}(G)$, is this lowest value.

The following linear programming problem must be solved in order to complete step 6.

Minimize Z = Cx

Subject to: $Ax \geq \mathbf{1}$ and $x > \mathbf{0}$

where $x = [x_1, x_2, \ldots, x_n]^T$, where $j = 1, 2, \ldots n$, denotes the color weights for each x_j,,

$\mathbf{1} = [1, 1, \ldots, 1]^T = C^T$, and $0 = [0, 0, \ldots, 0]$. and the following is a technique to generate matrix A:

1. Columns correspond to independent sets $I \in M(G)$, while the rows correspond to vertices.

2. Every row i has entries equal to 1 on columns j if the vertex v_i is present in the independent

set I representing the column j correspond to the vertices in the independent set and 0 in

other cases, thus representing the characteristic function of the respective independent sets.

Step 7: Color each vertex v∈V with the multiple colors Ci's with the corresponding weights $f_{frac}(I)$whenever $v \in I \; and \; I \in M(G)$

Step 8: Color the other vertices (removed vertices in Step 2) in the original graph by the color of any one of the adjacent vertices.

Example 1.1

We consider a fuzzy wheel graph W_5 as shown in Figure 2, in order to explain the fuzzy fractional coloring technique.

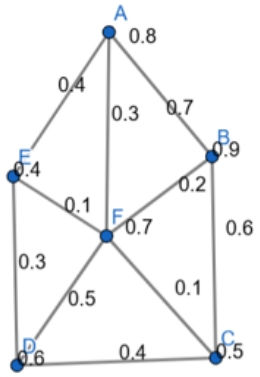


Figure 2

Here the weak edges are (a, f) , (b, f), (c, f) and (e, f). First, we remove all the weak edges from the graph. Then the Figure 2 becomes Figure 3

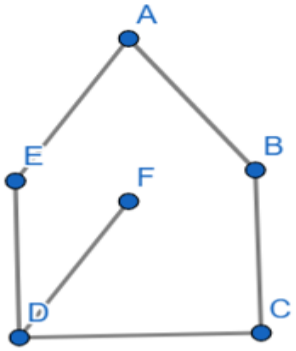


Figure 3

Here the independent sets are $S_1 = \{a, c, f\}$, $S_2 = \{a, d\}$, $S_3 = \{b, e, f\}$, $S_4 = \{b, d\}$. Assign a color to each independent set whose weights x_1, x_2, x_3, x_4, x_5 can be determined by solving the following linear programming problem

Minimize, $Z = x_1+x_2+x_3+x_4+x_5$

Subject to, $x_1+x_2 \geq 1$

$x_3+x_5 \geq 1$, $x_1+x_4 \geq 1$, $x_2+x_5 \geq 1$, $x_3+x_4 \geq 1$, $x_1+x_3+x_4 \geq 1$and

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Solving we get $x_1 = x_2 = x_3 = x_4 = x_5 = 0.5$

So, the fuzzy fractional chromatic number is $\sum x_i = 2.5$

The fuzzy fractional coloring of the above graph is shown in Figure 4

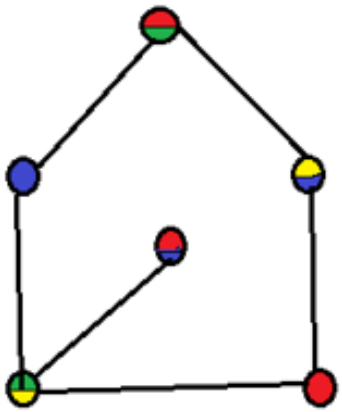


Figure 4

Hence the fuzzy fractional coloring of W_5 is shown in Figure 5

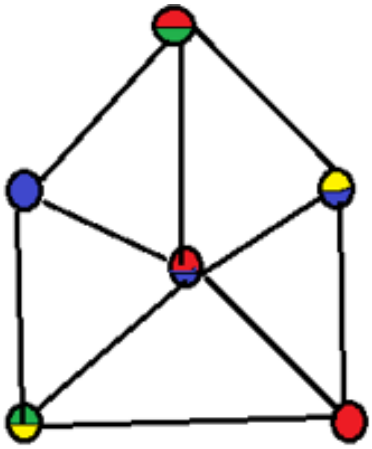


Figure 5

2. Fuzzy coloring and fuzzy fractional coloring of some wheel related fuzzy graphs

Consider a fuzzy friendship graph F_2 as in Figure 6, where two triangles are connected to the central vertex u. Let we assume that all the edges incident with u are strong and at least one of the edges not meeting with u are weak.

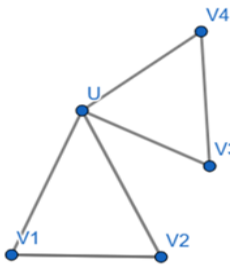


Figure 6

We discuss the fuzzy coloring and fuzzy fractional coloring of this graph in the following cases.

Case(i): One weak edge

In F_2, suppose (v_1, v_2) is a weak edge indicated by a dotted edge in Figure 7(a).

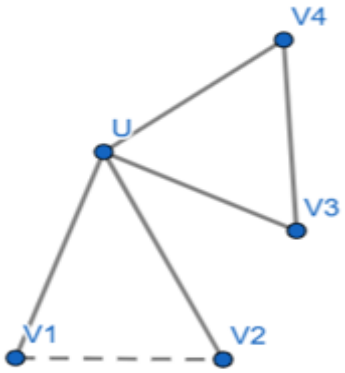


Figure 7(a)

For fuzzy coloring, we first consider the triangle in which all the edges are strong. Assign three basic colors for coloring the vertices of this triangle. The remaining vertices can be colored either by any of the three basic colors or a fuzzy color. Minimum number of basic colors used in a graph is its fuzzy chromatic number. Therefore, fuzzy chromatic number is 3.

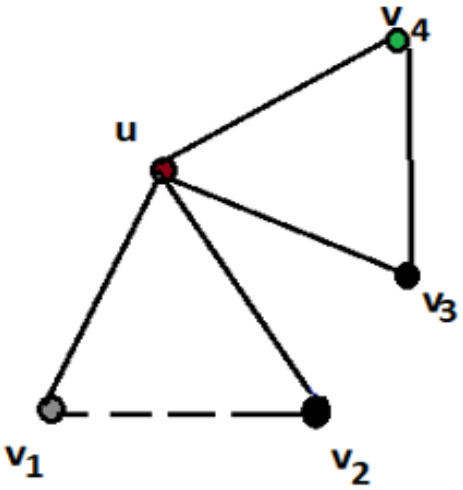


Figure 7(b)

In Figure7(b), three basic colors: red, green and black are used to color the vertices u, v4, v3 respectively and v1 is colored with a fuzzy black color.

For fuzzy fractional coloring, we delete the weak edge(v_1, v_2) as shown in Figure 7(c)

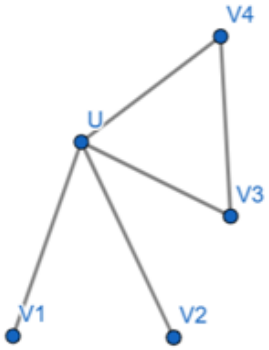


Figure7(c)

Now the independent sets are S_1= {v_1, v_2, v_3}, S_2= {v_1, v_2, v_4}, S_3={u}. Assign a color to each independent set whose weights y_1, y_2, y_3 can be determined by solving the following linear programming problem

Minimize Z =y_1+y_2+y_3

Subject to

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

i.e. $y_1+y_2 \geq 1$,

$y_1 \geq 1$

$y_2 \geq 1$

$y_3 \geq 1$

Solving we get $y_1 = y_2 = y_3 = 1$ and the fuzzy fractional chromatic number $\sum y_i = 3$. The fuzzy fractional coloring is given in Figure 7(d)

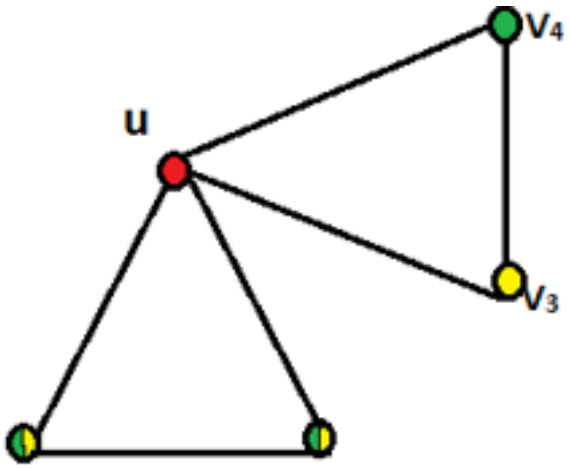


Figure 7(d)

Case (ii): Two weak edges

Suppose (v_1, v_2) and (v_3,v_4) are weak edges as in Figure 7(e).

For Fuzzy coloring, the central vertex u is assigned with a color say blue as in Figure 7(f). One of the adjacent vertices of u, joined by a strong edge, say v_2 is colored with another basic color say red. Color the vertex v_1 which is joined to v_2 by a weak edge, using a fuzzy red color. Here we use two basic colors. So, the fuzzy chromatic number is 2.

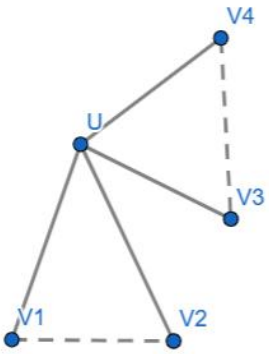


Figure 7(e)

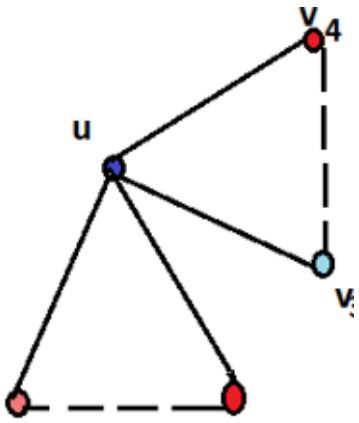


Figure 7(f)

For fuzzy fractional coloring, delete the weak edges as shown in Figure 7(g).

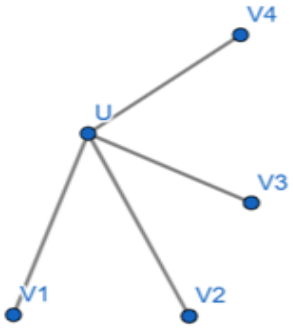


Figure 7(g)

Independent sets are $S_1 = \{v_1, v_2, v_3, v_4\}$, $S_2 = \{u\}$. Each independent set should be given a color whose weights y_1, y_2 can be found by solving the following linear programming problem.

Minimize, $Z = y_1 + y_2$

$$\text{Subject to, } \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and $y_1, y_2 \geq 0$

Solving we get $y_1 = y_2 = 1$ and the fuzzy fractional chromatic number is $\sum y_i = 2$. The fuzzy fractional coloring is shown in Figure 7(h)

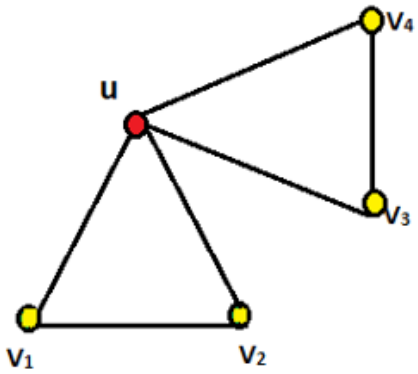


Fig.7(h)

Theorem 1.1:*Let G be a fuzzy friendship graph F_n with all the edges incident to its central vertex are strong edges. Let m denotes the number weak edges in G. Then the fuzzy chromatic number and fuzzy fractional chromatic number of G are given as*

$$\chi(G) = \chi_{frac}(G) = \begin{cases} 3, & \text{if } 1 \leq m \leq n-1 \\ 2, & \text{if } m = n \end{cases}$$

Proof:Consider a fuzzy friendship graph G = F_n, where all of the edges incident to the center vertex u are strong and there are n triangles connecting to the central vertex.

Case(i):There are m weak edges in G where $1 \leq m \leq n-1$

For Fuzzy coloring,

There exists at least one triangle of G with all edges strong. We know that if all the edges of a triangle are strong then three basic colors are needed to color its vertices. The remaining vertices of the graph G can be colored either by any of the three basic colors or a fuzzy colors. So, the fuzzy chromatic number is 3

Now we consider fuzzy fractional coloring.

Delete all the m weak edges say (v_i, v_{i+1}), where$i$=1, 3……2m-1

Now the independent sets are

S_1= {v_1, v_2, ……, v_{2m-1}, v_{2m}, v_{2m+1}, v_{2m+3}, v_{2m+5}, …., v_{2n-1}}, S_2={v_1, v_2, ……, v_{2m-1}, v_{2m}, v_{2m+2}, v_{2m+4}, v_{2m+6}, …., v_{2n}} and S_3={u}

Assign a color to each independent set whose weights x_1, x_2, x_3 can be determined by solving the following linear programming problem

Minimize Z= x_1+ x_2+ x_3

Subject to, AX ≥ 1

i.e.

$$\begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_{2m-1} \\ v_{2m} \\ v_{2m+1} \\ v_{2m+2} \\ v_{2m+3} \\ \vdots \\ v_{2n-1} \\ v_{2n} \\ u \end{matrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(2n+1)\times 3} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ \vdots \\ . \\ 1 \end{bmatrix}_{(2n+1)\times 1}$$

i.e. $x_1 + x_2 \geq 1$

$x_1 \geq 1$

$x_2 \geq 1$

$x_3 \geq 1$ and $x_1, x_2, x_3 \geq 0$

Solving we get $x_1 = x_2 = x_3 = 1$ and $\chi_{frac}(G) = \sum x_i = 3$

Case(ii): Number of weak edges $m = n$

For fuzzy coloring:

The end vertices of weak edges are colored with a basic color or its fuzzy color. The central vertex u is assigned with another basic color. So, the fuzzy chromatic number is 2

For Fuzzy fractional coloring:

Delete all the weak edgessay (v_i, v_{i+1}), i=1, 3, ……, 2n-1. Now the independent sets are

$S_1 = \{v_1, v_2 \ldots\ldots\ldots\ldots v_i, v_{i+1}, \ldots., v_{2n}\}$, $S_2 = \{u\}$

Give each independent set a color whose weights y_1, y_2 can be found by solving the LPP.

Min $Z = y_1 + y_2$

Subject to $y_1 \geq 1$, $y_2 \geq 1$ and

$y_1, y_2 \geq 0$

Solving we get $y_1 = y_2 = 1$ and $\chi_{frac}(G) = \sum y_i = 2$

Theorem 1.2:*If every triangle in a fuzzy friendship graph F_n is having exactly one weak edge incident with central vertex then the fuzzy chromatic number and fuzzy fractional chromatic number of $F_n$is 2.*

Proof:First consider fuzzy coloring. We prove the result using induction method. Consider n=2; that is F_2 contains exactly one weak edge in each of the two triangles which is incident to the central vertex as shown in Figure 8(a).

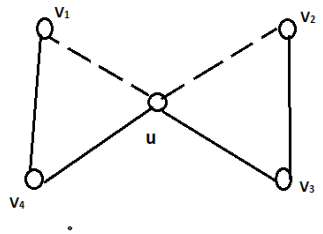


Figure 8(a)

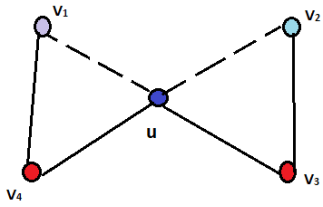


Figure 8(b)

In fuzzy coloring, first color the vertex u by a basic color blue. Since (u, v_4) and (u, v_3) are strong edges, v_3 and v_4 are colored with a color other than blue say red. Here (v_1, u) and (v_2, u) are weak edges. So v_1 and v_2 are colored with fuzzy blue color. Two basic colors are needed for coloring this graph as shown in Figure 8(b). Therefore, fuzzy chromatic number is 2.

Assume the result is true for n-1. Now consider F_n It consist of F_{n-1} together one triangle in which one edge meeting with u is a weak edge. Fuzzy chromatic number of F_{n-1} is 2 and we need only 2 colors to color the remaining one triangle. Hence by induction, fuzzy chromatic number of F_n is 2.

Consider the fuzzy fractional coloring of F_2 having exactly one weak edge in each of the two triangles which is incident to the central vertex

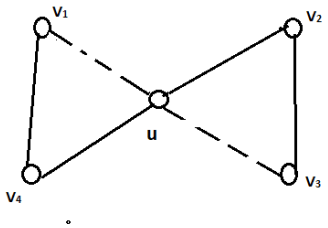


Figure 8(c)

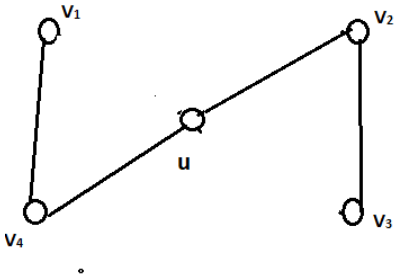


Figure 8(d)

In Figure 8(c), (v_1, u) and (v_2, u) are weak edges. First, we delete the weak edges as in Figure 8(d). Here the independent sets are S_1= {v_1, u, v_3}, S_2= {v_2, v_4}, S_3={v_1, v_2}, S_4={v_3, v_4}.Assign a color to each independent sets whose weight can be determined by

Min Z =x_1+x_2+ x_3+ x_4

x_1+ x_3 ≥ 1

x_2+ x_4 ≥ 1

x_2+ x_3 ≥ 1

x_1+ x_4 ≥ 1

x_1, x_2, x_3, x_4 ≥0

Solving we get x_1=1, x_2=1, x_3= x_4=0

Hence the fuzzy fractional chromatic number = 2. By induction, we can prove that for Fuzzy friendship graph Fn with n triangles each containing exactly one weak edge meeting with central vertex, the fuzzy fractional chromatic number is 2.

Theorem 1.3: *The fuzzy fractional chromatic number and fuzzy chromatic number of a fuzzy star graph S_n containing m weak edges, where 1 ≤ m < n-1, is always 2.*

Proof: Consider the fuzzy graph G as a star graph S_n(n≥3), having the m weak edges where

1≤ m < n-1

First, we discuss fuzzy fractional coloring of G

Suppose there are m weak edges are present in G where $1 \leq m < n-1$. Then delete all the m weak edges say (u, v_i), i=1, 2, ……, m

Now the independent sets are

S_1= {v_{m+1}, v_{m+2}, v_{m+3}, …., v_{n-1}}, S_2= {u}

Assign a color to each independent set whose weights x_1, x_2 can be determined by solving the following linear programming problem

Minimize Z= x_1+ x_2

Subject to:

$x_1 \geq 1$

$x_2 \geq 1$

and x_1, $x_2 \geq 0$

Solving we get $x_1 = x_2 = 1$

Therefore, the fuzzy fractional chromatic number is 2

Now, we consider fuzzy coloring of G, when $1 \leq m < n - 1$

Assign a color c_1 to the central vertex. Vertices which are strongly adjacent can be colored using a color other than c_1. Vertices which are weakly adjacent are assigned a fuzzy color of the central vertex. Here we use two basic colors to color the graph. Hence fuzzy chromatic number is 2.

Theorem 1.4:*The fuzzy fractional chromatic number of a fan graph $F_{1,n} = K_1 + P_n$ having all the edges of P_n as weak edges and other edges as strong edges is 2.*

Proof. Consider $G = F_{1,n} = K_1 + P_n$. In this case, $V(G) = \{u, v_1, v_2, v_3, \ldots\ldots\ldots\ldots v_n\}$ and $E(G) = E_1 \cup E_2$, where $E_1 = E(P_n) = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$ and $E_2 = \{u v_i / 1 \leq i \leq n\}$. Suppose that all of the edges in E_1 are weak, and all of the edges in E_2 are strong. Two independent sets, $S_1 = \{v_1, v_2, v_3, \ldots\ldots\ldots\ldots v_n\}$ and $S_2 = \{u\}$, are formed. The linear programming problem is

Minimize $Z = y_1 + y_2$

Subject to $y_1 \geq 1$

$y_2 \geq 1$

and $y_1, y_2 \geq 0$

Solving we get $y_1 = y_2 = 1$. Hence, fuzzy fractional chromatic number = 2.

Theorem 1.5:*The maximum value of fuzzy chromatic number of a fan graph $F_{1,n}$ having all the edges incident with central vertex as strong edges is 3.*

Proof:Consider fuzzy fan graph $F_{1,n} = K_1 + P_n$. A fuzzy path P_n has a chromatic number of less than or equal to 2. The end vertices of an edge that has at least one strong edge on the fuzzy path P_n are colored by two different basic colors. Since all the edges connecting to the central vertex are strong, the central vertex has to colored using a color other than 2 colors used. So, the chromatic number is 3. If the fuzzy path P_n has all weak edges, then the central vertex is colored with one basic color A and other end vertices are colored with another basic color B or fuzzy color a B. Hence maximum fuzzy chromatic number The chromatic number of a fan graph $F_{1,n}$having the edges incident with central vertex as strong edges is 3.

Theorem 1.6: *Let G be a fuzzy Wheel W_n with all the edges incident to its apex vertex are strong edges. Let m denotes the number weak edges in the cycle C_n of G. Then the fuzzy chromatic number of G is given as,*

$$\chi(G) = \begin{cases} 3, & \text{if } 1 \leq m \leq n-1 \\ 2, & \text{if } m = n \end{cases}$$

Proof: Consider a fuzzy wheel graph $G = W_n$, with all the edges incident to the apex vertex u be strong. Let m denotes the number weak edges in the cycle C_n of G.

Case(i):$1 \leq m \leq n - 1$

Let we assume that at most n-1 edges of the cycle C_n of G are weak edges. Then there exists at least one edge of the cycle C_n say (a, b) as strong edge. Since all the edges incident to the apex vertex u are strong edges, the triangle formed by the vertices: a, b, u has all strong edges. So three basic colors are needed to color its vertices. The remaining vertices of the graph G can be colored either by any of the three basic colors or its fuzzy colors. So, the fuzzy chromatic number is 3

Case(ii): m = n

Let we assume that all the n edges in the cycle C_n of G be weak edges. The end vertices of weak edges are colored with a basic color or its fuzzy color. The apex vertex is assigned another basic color. So, the fuzzy chromatic number is 2.

Theorem 1.7: *If G is a fuzzy Wheel graph W_n with all the edges incident to its apex vertex are strong and m consecutive edges of cycle $C_n$are weak edgesthen the fuzzy fractional chromatic number of G is given as,*

$$\chi_{frac}(G) = \begin{cases} 3, & \text{if } 1 \leq m \leq n-1 \\ 2, & \text{if } m = n \end{cases}$$

Proof: Let G be a fuzzy Wheel W_n with all the edges incident to its apex vertex are strong edges. Suppose weak edges are present only in the cycle C_n of G, as m consecutive edges.

Case(i):$1 \leq m \leq n-1$

Suppose there are only m consecutive weak edges are present in the cycle C_n of G where

$1 \leq m \leq n-1$. After deleting those weak edges say (v_i, v_{i+1}), i=1, 2,……,m, the independent sets are $S_{1=}$ {v_1, v_2, ……, v_m, v_{m+1}, v_{m+3}, v_{m+5}, …., v_{n-1}, v_1}, $S_{2=}$ { v_1, v_2, ……, v_m, v_{m+2}, v_{m+4}, v_{m+6}, …., v_n}, $S_{3=}${u}

Each independent set should be given a color whose weights y_1, y_2, and y_3 can be found by resolving the following linear programming problem

Minimize Z= y_1+ y_2+ y_3

Subject to,

i.e. $y_1+ y_2 + y_3 \geq 1$

$y_1 \geq 1$

$y_2 \geq 1$

$y_3 \geq 1$ and

$y_1, y_2, y_3 \geq 0$

Solving we get $y_1= y_2= y_3 =1$and $\chi_{frac}(G) = \sum y_i = 3$

Case(ii): m=n

Suppose there are n consecutive weak edges say (v_i, v_{i+1}), i=1, 2, ……, n

Delete all weak edges. Now the independent sets are

$S_{1=}$ {$v_{1,}$ v_2…… v_i, v_{i+1}, …., v_n}, $S_{2=}${u}

Give each independent set a color whose weights y_1, y_2 can be found by solving the LPP.

Min Z= y_1+ y_2

Subject to $y_1 \geq 1$

$y_2 \geq 1$

$y_{1,}$ $y_2 \geq 0$

Solving we get y_1= y_2 =1and $\chi_{frac}(G) = \sum y_i = 2$

Theorem 1.8: *In a fuzzy wheel graph W_n, if weak edges are not present on the cycle C_n then fuzzy fractional chromatic number need not be an integer*

Proof: In Example 1.1, the fuzzy graph W_5 has 4 weak edges which are not present in the cycle C_5. In that Example, we have shown that the fuzzy fractional chromatic number of that graph is 2.5. Hence the proof the theorem completes.

3. An application of fuzzy coloring and fuzzy fractional coloring

Recruitment is an essential and pivotal activity in any sector, exerting a substantial influence on the success and expansion of organizations. Throughout this process, the recruiters scrutinize and evaluate specific expatiation of the applicants, classify them into distinct categories based on their skills, and subsequently make their judgements based on multiple levels of interviews. The primary challenge encountered by recruiters is the significant amount of time and effort put into identifying the most suitable candidates from a large pool of applicants. To address this issue, we will examine the following scenario in an IT organization.

The applicants have myriad skills. The following are those skills

1.Python2.Java3.HTML4.PHP5.Android6.CSS7.R

8.Flask9.Django10.MATLAB11.SQL12.React JS13.MySQL

Here, each candidate must have three skills for applying for interview. The applicant should attend the interview for showcasing his one of the three skills on a particular day. It is expected that the recruitment schedule will specify the minimum number of days needed to finish the entire interview process. Let S_k denote the set of candidates appearing for the interview testing the skill p. Consider the fuzzy graph with 13 vertices a, b, c, d, e, f, g. h. i, j, k, l, m representing the skills namely Python, Java, .HTML, PHP, Android, CSS, R, Flask, Django, MATLAB, SQL, React JS, MySQL respectively. There is an edge between the a and b if Sa ∩ Sb ≠ Ø. The membership value of the vertices is determined by the demand of the candidate skills required for the particular job interview. If the demand of a vertex (skill) is 95% or higher among the total number of candidates, we consider its membership value to be "1." The membership value to be "0" if the demand of the skill is less than 5% among the total number of candidates. Each candidate's ability to demonstrate both talents is taken into consideration when determining the membership values of the edges. The corresponding fuzzy graph is shown in Fig.9

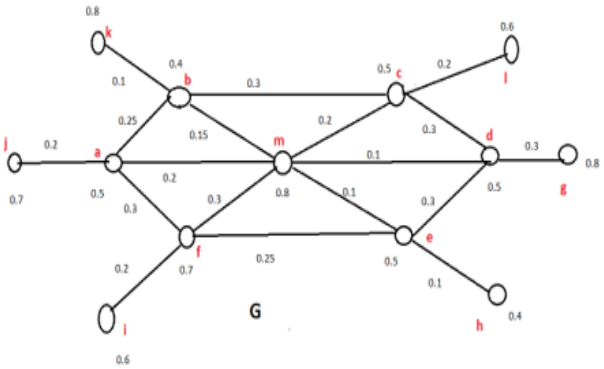


Figure 9

Our primary goal is to reduce the overall number of days needed to finish the recruitment. As a result, this problem is modeled as a fuzzy graph coloring problem where the objective is to determine the fuzzy graph's least chromatic number.

Figure 10 displays the fuzzy coloring of the fuzzy graph in Figure 9, by using the strongly adjacent vertices: a, b, c, d, e, f and g. Here we use three basic colours: green, red and blue and their fuzzy colors to color all the vertices of Figure 9. The fuzzy chromatic number is 3. Therefore, a total of 3 days is required to complete the entire recruitment process.

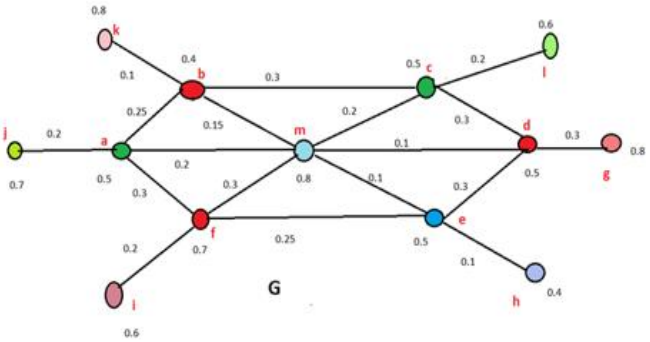


Figure10

Now we calculate the fuzzy fractional chromatic number for the graph in Figure 9. After removing the weak edges (i.e. edges connecting the non-strongly vertices), the independent sets are S_1={a, c, e ,g}, S_2={b, d, f}, S_3={b, f, g}, S_4={c, f, g}, S_5={b, e, g}, S_6={a, d}.Now assign a color to each independent set whose weights y_1, y_2, y_3, y_4, y_5, y_6 can be determined by solving the LPP

Minimize, $Z = y_1 + y_2 + y_3 + y_4 + y_5 + y_6$

Subject to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

i.e. $y_1 + y_6 \geq 1$

$y_2 + y_3 + y_5 \geq 1$

$y_1 + y_4 \geq 1$

$y_2 + y_6 \geq 1$

$y_1 + y_5 \geq 1$

$y_2 + y_3 + y_4 \geq 1$

$y_1 + y_3 + y_4 + y_5 \geq 1$ and $y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$

and solving we get the fuzzy fractional chromatic number as 2.5 and the corresponding fuzzy fractional coloring is depicted in Figure 11.It takes the same amount of days to complete the entire hiring procedure using the fuzzy coloring strategy and the fuzzy fractional coloring method. The fuzzy fractional chromatic number, on the other hand, is less than the fuzzy chromatic number of the fuzzy graph if the chromatic number accounts for the hours and the day.

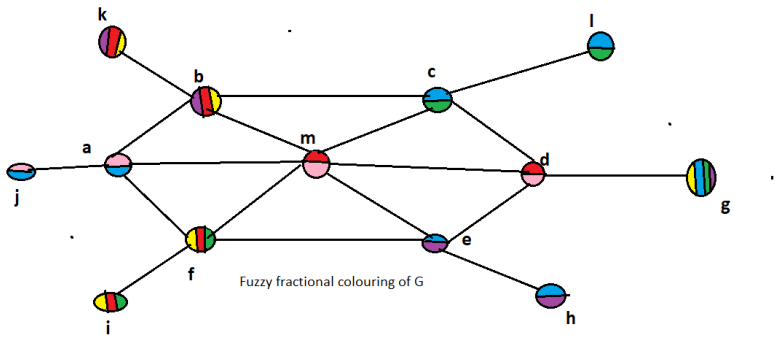


Figure 11

4. Conclusions

The fuzzy chromatic number and fuzzy fractional chromatic number of a few wheel-related graphs have been determined in this study. The bounds on the chromatic numbers based on the position and number of weak edges arediscussed. We can extend this research on some graph operations like product graphs, join of the graphs and explore their real-life applications in inter connected networks.

References

1. Akram M, Adeel A ,m-polar fuzzy graphs and m-polar fuzzy line graphs, J Discrete Math SciCryptogr ,pp. 1597–1617, 2017.

2. Akram M, Wassem N, DudekWA, Certain types of edge m-polar fuzzy graph,Iran J Fuzzy Syst 14(4), pp.27–50, 2016.

3. Ananthanarayanan M, Lavanya S, Fuzzy graph coloring using α-cut,Int J EngAppl Sci 4(10), pp.23–28B, 2014.

4. Anjali N, Mathew S, On blocks and stars in fuzzy graphs, J Intell Fuzzy Syst, pp.1659–1665,2015.

5. Farley, Arthur M.; Proskurowski, Andrzej (1982), "Networks immune to isolated line failures", *Networks*, **12** (4): 393–403, doi:10.1002/net.3230120404

6. A. Rosenfeld, Fuzzy graphs, in: L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and Their Applications, Academic Press, New York, pp.77-95, 1975.

7. Bhutani KR, Battou A, On M-strong fuzzy graphs, Inf Sci pp.103–109 2003.

8. Bhutani KR, Rosenfeld A, Strong arcs in fuzzy graph, Inf Sci pp.319–322,2003.

9. Chen J, Li S, Ma S, Wang X, m-polar fuzzy sets: an extension of bipolar fuzzy sets, Hindwai Publishing Corporation, 2014.

10. Eslahchi C, OnaghNB, Vertex strength of fuzzy graphs, Int J Math MathSci 2006:1–9, Hindawi Publishing Corporation, 2006.

11. Ghorai G, Pal M, On some operations and density of m-polar fuzzy graphs, Pac Sci Rev A Natural Sci Eng 17(1), pp.14–22, 2015.

12. Ghorai G, Pal M, some properties of m-polar fuzzy graphs, Pac Sci Rev a Natural Sci Eng 18, pp.38–46, 2016.

13. Ghorai G, Pal M, A study on m-polar fuzzy planar graphs, Int J ComputSci Math 7(3), pp.283–292, 2016.

14. Ghorai G, Pal M, Faces and dual of m-polar fuzzy planner graphs,JIntell Fuzzy Syst 31,pp.2043–2049, 2016.

15. Ghorai G, Pal M, Planarity in vague graphs with application, Acta Math AcadPaedagogiaceNyregyhziensis 33(2), pp.1–21, 2017.

16. Gurjar, D. K., &Krishnaa, A, Various antimagic labeled graphs from graph theory for cryptography applications. International Journal of Scientific Research in Mathematical and Statistical Sciences, 9(3), pp 11-18, 2022

17. Kaufman A, Introduction a la Theorie des Sous-emsemblesFlous, Mansson et Cie 1,1973.

18. Mandal S, Sahoo S, Ghorai G, Pal M, Genus value of m-polar fuzzy graphs, J Intell Fuzzy Syst ,34(3):1947–1957.

19. Mathew S, Sunitha MS, Fuzzy graphs: basics, concepts and applications, Lap Lambert Academic Publishing, Berlin,2012.

20. Mordeson JN, Nair PS, Fuzzy graph and fuzzy hypergraphs. Physica-Verlag, Heidelberg, 2013

21. Samanta S, Pal M, Fuzzy planar graph, IEEE Trans Fuzzy Syst23, pp.1936–1942, 2015.

22. Samanta S, Pramanik T, Pal M, Fuzzycolouring of fuzzy graphs, Afrika Math 27:37–50, 2016.

23. Sunitha MS, Mathew S, Fuzzy graph theory: a survey. Ann Pure ApplMath 4:92–110, 2013

24. TanmoyMahapatra, Ganesh Ghoral,MadhumangalPal, Fuzzy fractional coloring of fuzzy graph with its application, Springer-Verlag GmbH Germany, part of Springer Nature, 2020.

25. Wei Gao, Weifan Wang, Overview on fuzzy fractional coloring, International Journal of Cognitive Computing in Engineering, pp.196-201, 2021.