Minimax Theory and its Applications

Volume 11 (2026), No. 3s, 325 to 335

Constrained Optimal Energy Allocation in Wireless Sensor Networks Under Stochastic Channel Conditions

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Abstract

Wireless sensor networks (WSNs) are networks that often have very limited energy resources and operate with random wireless channel conditions. Thus, allocation of energy needs to be done so as to balance the transmission utility, the individual energy limitations, and the reliability of the communication. The current research on energy allocation has focused on maximizing the lifetime or fairness, coordinating sensing and transmission, adapting the channel, and leveraging energy harvesting; however, many energy-allocation formulations have assumed known or nominal channel distributions, or reliability requirements do not account for distributional uncertainty. To that end, this paper proposes a constrained stochastic energy-allocation setting where (unknown) channel statistics are modelled with a Wasserstein ambiguity set around an empirical channel distribution, and reliability is ensured via a distributionally robust tail-risk constraint. That Wasserstein distributionally robust optimization and conditional value-at-risk are not viewable as two independent corner results is not a concern, since the proposed analysis is used to extract the structural facts of the optimal allocation policy, which are standard results in Wasserstein distributionally robust optimization and conditional value-at-risk studies. An ambitious MV operator is added that can measure the worst-case MV of more energy, under distributional ambiguity. The monotonicity results in the scalar characterisation of the optimal allocation. The reliability requirement further creates a minimum energy threshold for every sensor, which is endogenously determined. This helps to arrive at an operating regime characterized by three different types of allocation policy, namely deactivation, marginal allocation and reliability-dominating allocation. The analysis proves the existence and uniqueness of the optimal allocation, monotonicity with respect to the total energy budget, sensitivity to distributional ambiguity and global convergence of a scalar bisection solution procedure. The numerical evaluation is structured by different channel and reliability conditions and illustrates the resulting allocation structure and how it differs from equal allocation, nominal optimization, and conventional water-filling approaches. The protocol offers a mathematically tractable solution towards achieving robust energy management for WSNs that are designed to have simultaneous stochastic-channel uncertainty as well as reliability requirements.

Keywords—wireless sensor networks, energy allocation, stochastic channels, distributionally robust optimization, Wasserstein ambiguity, CVaR, reliability constraint, marginal value, energy threshold, robust optimization.

1. Introduction

Wireless sensor networks (WSNs) gather, process and communicate information from a set of sensor nodes via wireless connections. However, as sensor nodes are battery-constrained, the allocation of energy becomes a fundamental part of the operation, quality and reliability of the communication. Research focuses on energy allocation via lifetime maximization (Cerulli *et al.*, 2014), fairness (Kumar & Kumar, 2016), routing, sensing and transmission coordination (Tang & Li, 2006).

Wireless channels are stochastic with fading and time-varying propagation such that the energy-allocation utility will vary. Goldsmith and Varaiya (1997) showed the necessity of CSI in achieving capacity optimisation over fading channels, and Wang *et al.* (2010) investigated channel-adaptive energy-efficient channel allocation on WSNs over fading TDMA channels. Other works investigated the cases of utility-based allocation (Liao *et al.*, 2008), distributed estimation (Li & Al-Regib, 2009), energy harvesting (Edalat & Motani, 2016; Mao *et al.*, 2014; Wu *et al.*, 2017), and energy balancing (Hasan *et al.*, 2017; Shah, 2020).

Uncertainty of channel distribution is still a factor. Limited observations are likely to result in an inaccurate empirical distribution, and conventional stochastic optimization may be based on an assumed or estimated distribution. This is an issue that is addressed by distributionally robust optimization (DRO), where multiple sets of probability distributions are considered. A moment-based DRO model was proposed by Delage and Ye (2010), while tractable approximations of DRO were studied by Goh and Sim (2010). Wasserstein ambiguity sets are a data-driven uncertainty model for empirical distributions, for which Esfahani and Kuhn (2018) obtained performance guarantees and tractable reformulations, and Gao (2023) obtained finite-sample guarantees for Wasserstein DRO.

Reliability gives weight to the fact that a high expected utility might result in an intolerable outcome if the conditions on the channel are bad. Following Rockafellar and Uryasev (2002), CVaR is a tail risk measure for unpleasant experiences.

This paper considers the analysis of optimal energy allocation under both simultaneous distributional uncertainty and reliability constraints. A strong marginal-value operator is a measure of the minimum energy threshold that arises endogenously, and also of the worst-case marginal benefit of added energy, the reliability. Compactness, continuity and concavity conditions make it monotone, so it is scalar-characterizable: there is a common energy shadow price. The strong reliability-threshold allocation consists of three regimes: deactivation, marginal allocation and reliability domination (Figure 3). The paper investigates the properties of existence and uniqueness, energy-budget monotonicity, sensitivity to uncertainties and convergence in a global sense. The model, analysis, solution procedure, numerical evaluation, discussion and conclusion are included in sections 2-6.

2. System Model and Problem Formulation

Assume a WSN containing.N sensor nodes indexed by

$i = 1,2, \ldots N$

Each sensor is connected to a fusion centre or a base station via a wireless link. In the paper, a network architecture is taken into account, as shown in Figure 1. The energy allocation of each sensor, which is represented by e_i is the amount of energy allocated for its transmission operation in the sensor allocation interval under consideration.

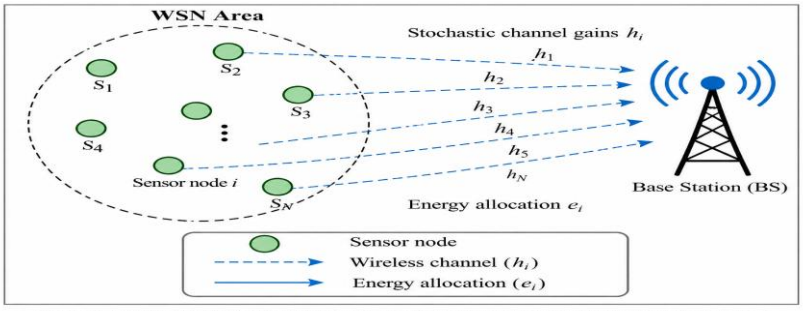


Figure 1. System Model for Constrained Energy Allocation Under Stochastic Channel Conditions.

Figure 1. System Model for Constrained Energy Allocation Under Stochastic Channel Conditions

$$0 \leq e_i \leq E_i^{\max}$$

The maximum energy that may be attributed to a sensor i is denoted $E_i^{\max}$. The amount of energy available on the network is E and therefore,

$$\sum_{i=1}^{N} e_i \leq E.$$

Definitions and notations that are used throughout the paper are listed in Table 1. The model is designed to show the rudimentary energy constraint faced by WSNs. However, previous works have proven that when allocating energy, network lifetime and communication requirements should be taken into consideration (Cerulli *et al.*, 2014; Kumar & Kumar, 2016). The utility-based allocation has also been studied in the context of WSNs in particular (Liao *et al.*, 2008).

Table 1. Model Parameters and Notation

Symbol	Description
N	Number of sensor nodes
e_{i}	Energy allocated to sensor i
$E_{\mathrm{i}}^{\max}$	Maximum allowable energy for a sensor i
E	Total network energy budget
h_{i}	Random channel state/gain of sensor i
$R_{\mathrm{i}}(e_{\mathrm{i}}, h_{\mathrm{i}})$	Transmission utility/rate
$R_{\mathrm{i}}^{\mathrm{re}\varphi}$	Required reliability-related rate level
$L_{\mathrm{i}}(e_{\mathrm{i}}, h_{\mathrm{i}})$	Rate-deficit loss
$P_{\mathrm{n}}^{\wedge}$	Empirical channel distribution
$\mathcal{P}\rho$	Wasserstein ambiguity set
ρ	Wasserstein ambiguity radius
α	Tail probability level
ε_{i}	Reliability-loss tolerance
λ	Common energy shadow price

Let $h_i \in \mathcal{H}_i$ be the random channel condition sensor i is associated with. The channel state may correspond to fading gain, effective channel quality or some other random communication parameter. The transmission utility is denoted by $R_i(e_i, h_i)$, and assumed to be an increasing function ofe_i for a given realization of the channel h_i,

$$R_i(e_i, h_i) = B_i \log\left(1 + \frac{h_i e_i}{N_0}\right)$$

where B_i are effective bandwidths and N_0 a parameter related to noise. It is nice to note that this very logarithmic form is not essential for the analysis;it only needs some monotonicity conditions and some concavity conditions. However, channel-adaptive allocation has been studied before, for example under fair energy-efficient allocation over TDMA channels (Wang *et al.*, 2010) and under distributed estimation over Rayleigh fading channels (Wardihani *et al.*, 2015).

Assume the actual distribution of the channel state is not known. An empirical distribution is created from M observations of channel samples as,

$$\hat{P}_M = \frac{1}{M} \sum_{m=1}^{M} \delta_{\hat{h}^m}$$

Rather than setting it equal to this empirical distribution as the actual channel distribution, let us define the Wasserstein ambiguity set:

$$\mathcal{P}_\rho = \left\{P : W\left(P, \hat{P}_M\right) \leq \rho\right\}$$

where $W(\cdot,\cdot)$is the Wasserstein measure and $\rho$is an ambiguity radius, $\rho \geq 0$. The purpose of the Wasserstein set is to account for the uncertainty in the statistical nature of the observed channel distribution. Such a realization

was based on the DRO approach developed by Esfahani and Kuhn (2018) and is data-driven. The finite-sample understanding of the ambiguity radius is related to statistical guarantees proposed by Gao (2023). However, it is important to note that the Wasserstein ambiguity set is not presented as a new contribution of this paper. It is meant to produce the stochastic uncertainty setting in which the brand-new allocation structure is drawn.

To incorporate reliability, define a rate-deficit loss,

$$L_i(e_i, h_i) = \left[R_i^{\mathrm{req}} - R_i(e_i, h_i)\right]_+$$

where $[x]_+ = \max\{x, 0\}$. The higher the value of L_i, the lower the transmission performance. A tail-risk measure is used to take care of the risk of such adverse events. Let the worst-case CVaR be defined, for a confidence level $\alpha \in (0,1)$, as

$$\sup_{P \in \mathcal{P}_\rho} \mathrm{CVaR}_\alpha^P(L_i(e_i, h_i)) \le \varepsilon_i)$$

CVaR is employed because it penalizes the tail value of the loss distribution as well as the expectation. CVaR's optimization interpretation is based on the existing one of Rockafellar and Uryasev (2002). The resulting formulation is a mixture of distributional uncertainty and reliability control. This is of significant relevance for the WSN allocation, as the policy with only expected utility might allocate too little energy to sensors operating under unfavourable channel states.

Let $U_i(e_i, h_i)$ denote the utility obtained from sensor i. The comprehensive network convenience program may be boiled down to networking components.

$$\Phi(\boldsymbol{e}) = \inf_{P \in \mathcal{P}_\rho} \mathbb{E}^P \left[\sum_{i=1}^{N} U_i\,(e_i, h_i)\right]$$

where $\boldsymbol{e} = (e_1, \ldots, e_N)$. The constrained allocation problem is therefore

$$\begin{aligned} \max_{\boldsymbol{e}} \quad & \inf_{P \in \mathcal{P}_\rho} \mathbb{E}^P \left[\sum_{i=1}^{N} U_i\,(e_i, h_i)\right] \\ \text{s.t.} \quad & \sum_{i=1}^{N} e_i \le E, \quad 0 \le e_i \le E_i^{\max}, \\ & \sup_{P \in \mathcal{P}_\rho} \mathrm{CVaR}_\alpha^P\big(L_i(e_i, h_i)\big) \le \varepsilon_i, \quad i = 1, \ldots, N. \quad (1)\big) \end{aligned}$$

Equation (1) is the central optimization problem studied in this paper.

3. Mathematical Analysis and Optimal Allocation

The robust marginal-value operator is the main mathematical object.

$$M_i(e_i) = \inf_{P \in \mathcal{P}_\rho} \mathbb{E}^P \left[\frac{\partial U_i(e_i, h_i)}{\partial e_i}\right] \quad (2)$$

The operator measures what they believe is the maximum additional output that the operator can extract from a particular unit of energy assigned to a sensor i. Marginal utility allocation decisions are often values that can be treated directly for a deterministic nominal model. However, if there is distributional uncertainty, the marginal value has to be computed under the worst distribution in the ambiguity set. This separation is between the proposed formulation and classical nominal allocation methods. Although conventional energy allocation has been extensively studied in WSNs (Wang *et al.*, 2010), energy-aware task allocation and harvesting-based schemes have been studied to cater to the additional resource constraints (Edalat & Motani, 2016; Wu *et al.*, 2017). The current formulation, however, deals with distributional ambiguity and reliability constraints' structure of the allocation.

Lemma 1. Assume that for each i, $U_i(e_i, h_i)$ is continuous and differentiable in e_i and strictly concave in e_i for almost all h_i. Then the strong MV operator $M_i(e_i)$ is clearly nonincreasing in e_i and under strong concavity it is strictly decreasing in the interior of the allocation interval.

Proof. Strict concavity implies for each fixed channel realization h_i that,

$$\frac{\partial^2 U_i(e_i, h_i)}{\partial e_i^2} < 0$$

Hence, $\frac{\partial U_i}{\partial e_i}$ has to be strictly decreasing in e_i. This monotonicity is preserved with taking the expectation with respect to any $P \in \mathcal{P}_\rho$. The infimum of decreasing functions is decreasing over their common domain of ambiguity. Hence, $e_i^1 < e_i^2$ implies $M_i(e_i^1) > M_i(e_i^2)$. Thus function $M_i(\cdot)$ is strictly decreasing.

The fact that this result is monotonic is the crucial property that enables such a multidimensional allocation problem to be represented in terms of a common shadow price.

$$\mathcal{E} = \left\{ e : 0 \le e_i \le E_i^{\max}, \sum_i e_i \le E \right\}$$

The individual allocation intervals are compact, and the total-energy constraint is closed; hence $\mathcal{E}$ is compact. Suppose that the Wasserstein ambiguity set $\mathcal{P}_\rho$ is weakly compact, and that the utility functions meet the regularity requirements for lower semicontinuity and worst-case expectation attainment.

Theorem 1. If the feasible reliability-constrained allocation set is nonempty, $\mathcal{P}_\rho$is weakly compact and the robust utility is upper semicontinuous on the feasible allocation set, then there is an optimum. If the strong goal is strictly concave, then there is only one optimal allocation.

*Proof.*The feasible set is closed. The robust objective is upper semicontinuous, which assures it reaches its maximum on this set. Hence, there is such an optimal allocation. Two sufficiently different feasible allocations which find the same value of the maximum, in this case, would yield a strictly larger value of the objective for their strict convex combination, and hence would not be optimal. Therefore, the Optimizer is unique.

$$\mathcal{R}_i(e_i) = \sup_{P \in \mathcal{P}_\rho} \mathrm{CVaR}_\alpha^P\big(L_i(e_i, h_i)\big) \quad (3)\big)$$

The rate deficits $R_i(e_i)$ are assumed to be nonincreasing as more energy is added, which increases the rate. The minimum reliability-feasible energy is defined as

$$\underline{e_i} = \inf\{e_i : \mathcal{R}_i(e_i) \le \varepsilon_i\} \quad (4)$$

An endogenous reliability threshold is added to equation (4). It is endogenous since it comes from the reliability constraint, the uncertain channel model and not from an arbitrary fixed lower bound. The whole network allocation problem is infeasible if there is no feasible e_isatisfies the reliability constraint.

Theorem 2. Suppose the robust utility function is strictly concave and differentiable; the robust marginal-value operator $M_i(e_i)$ is strictly decreasing, and the reliability function is nonincreasing. Second, if not all sensors are operated fully saturated at their maximum energy levels, then for all the others $u\big(x_i(\cdot)\big) = \lambda$ at their optimal allocations, where $u\big(x_i(\cdot)\big)$ is the marginal utility of sensor i. Assuming that the reliability constraint is active, the optimal allocation is, $e_i^* = \bar{e}_i$. Consequently.

$$e_i^* = \max\big\{\underline{e_i}, M_i^{-1}(\lambda)\big\} \quad (5)$$

subject to the upper bound$E_i^{\max}$.

Proof. Consider the Lagrangian associated with the total-energy constraint. Suppose the $\lambda \ge 0$ is the multiplier corresponding to $\sum_i e_i \le E$. There is an interior sensor, with an inactive part of the reliability constraint and an inactive part of the upper bound; for this sensor, first-order optimality conditions imply that $M_i(e_i^*) - \lambda = 0$ and so that $M_i(e_i^*) = \lambda$. Since M_i is strictly decreasing, it possesses an inverse over its effective range. Hence, $e_i^* = M_i^{-1}(\lambda)$. The marginal solution is typically not feasible if it is below the reliability threshold, so the allocation must be increased to $\bar{e}_i$.

Adding the two cases together gives Equation (5). The former energy constraint provides the respective projection onto $[0, E_i^{\max}]$.

The allocation structure of Theorem 2 can be split up into three regimes. In Regime I (Deactivation) if $M_i(0) \leq \lambda$, in case of no guaranteed reliability energy $E_i^{\min}$ is required, then the sensor i receives $e_i^* = 0$. In Regime II (Marginal Allocation), if $\bar{e}_i < M_i^{-1}(\lambda) < E_i^{\max}$, then $e_i^* = M_i^{-1}(\lambda)$. If $M_i^{-1}(\lambda) < \bar{e}_i$, then $e_i^* = \bar{e}_i$ (Regime III - Reliability Domination). An overview of these regimes is provided in Figure 2.

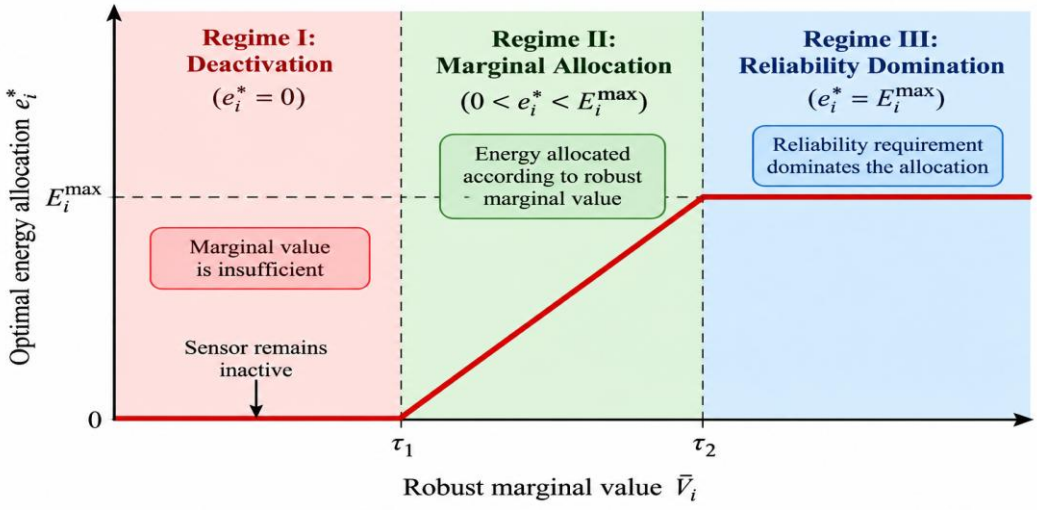


Figure 2. Three-Regime Structure of the Optimal Energy Allocation

The allocation is not only based on a marginal channel-quality condition as is done with conventional water-filling. If the reliability threshold is low, it may be necessary to run energy to a sensor when its value of energy would otherwise be less than the average shadow-value.

Figure 3 is a summary of the complete math framework. Images obtained by the channel are first used to define the empirical distribution. The Wasserstein ambiguity sets are then potential perturbations of this empirical distribution. The robustness margin utility and reliability functions are then calculated, and the robust marginal-value operator is finally calculated and the constrained allocation performed. The interpretation of distributional ambiguity is consistent with Esfahani and Kuhn (2018), and the finite-sample interpretation is done along the lines of Gao (2023). CVaR is estimating the tail risk part based on Rockafellar and Uryasev (2002).

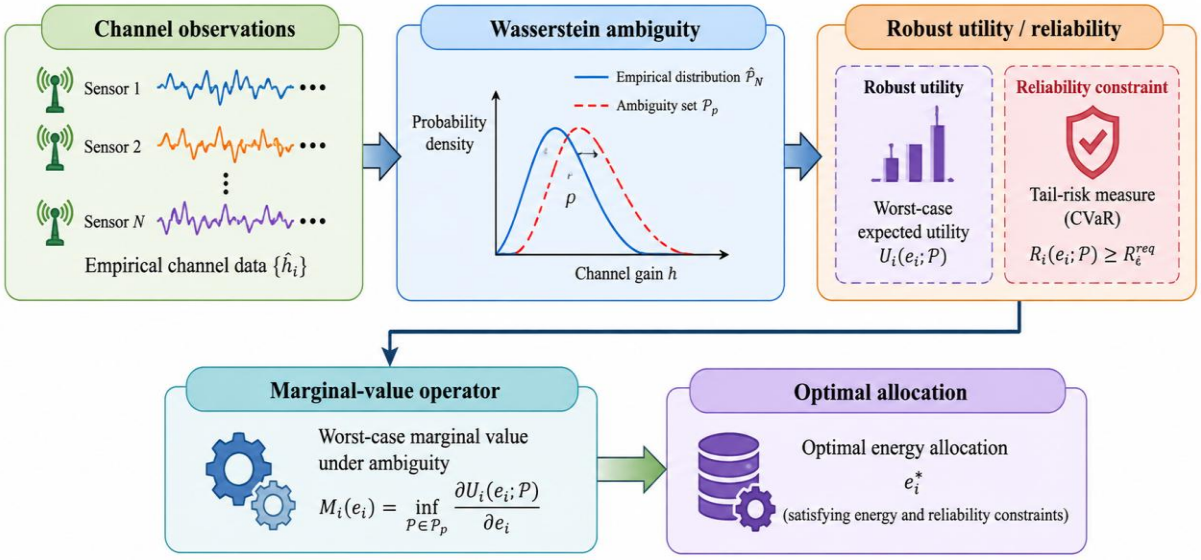


Figure 2. Robust Reliability-Threshold Energy Allocation Framework.

Figure 3. Robust Reliability-Threshold Energy Allocation Framework

Assume that the active-set structure does not change on this interval and let $E_1 < E_2$ on this interval.

Theorem 3. Monotonicity of M_i is strict, meaning a weak increase of the total energy budget E_1 to E_2 results in a corresponding weak increase of the allocation of each active sensor.

Proof: Under a given active set, an optimal allocation is obtained when $M_i(e_i^*) = \lambda$ and $\sum_i e_i^* = E$. Because $M_i^{-1}(\lambda)$ is decreasing in λ, the larger the value of E the smaller the common shadow price λ. Consequently, $M_i^{-1}\big(\lambda(E_2)\big) \geq M_i^{-1}\big(\lambda(E_1)\big)$. This means the individual allotments increase just a little bit as more energy is allocated.

This result is in line with the basic observation about energy availability in lifetime-oriented optimization of WSNs (Hasan *et al.*, 2017; Shah, 2020), but here it is an immediate consequence of the strong marginal-value structure.

$$\mathcal{P}_{\rho_1} \subseteq \mathcal{P}_{\rho_2}, \quad \rho_1 < \rho_2$$

Theorem 4. Under a single-crossing condition on the marginal utility, increasing the ambiguity radius weakly decreases the robust marginal-value operator:

$$M_i^{(\rho_2)}(e_i) \leq M_i^{(\rho_1)}(e_i) \quad (6)$$

Proof. The infimum for the robust marginal value of ρ_2 will be over a larger set because $\mathcal{P}_{\rho_1} \subseteq \mathcal{P}_{\rho_2}$. Consequently, the value of the strong margin is not allowed to augment. Hence Equation (6) follows. Mathematically explains the margins of response and why variance in the distribution is likely to create a more conservative marginal response.

The principle structural contribution is given by the mixed of Theorems 2–4. The above optimal allocation is not necessarily $e_i^* = M_i^{-1}(\lambda)$due to its possible violation of reliability. Rather, there is an endogenous lower bound on the allocation of:

$$e_i^* = \max\{\underline{e_i}, M_i^{-1}(\lambda)\}$$

Consequently, there are two different mechanisms in the optimal policy. First, the allocation of energy that is done at the margins incorporates a strong notion of marginal value. Second, reliability protection ensures a protection of a minimum level of energy for a sensor to be placed in energy reduction, which corresponds to the required worst-case tail performance. The distinction is very critical, especially in cases where there is uncertainty about the channel distribution. Even if a sensor is nominal, its channel can be expected to be poor, affecting its performance. However, if the same sensor is part of the robust formulation, it could get a minimum reliability allocation, as if too little energy is allocated, then the network would be exposed to too much tail loss. Such a policy can thus be regarded as a strong threshold allocation policy with a common energy shadow price.

4. Solution Method

The structural characterization reduces the unique N-dimensional optimization issue that is reduced to a scalar search of the common multiplier λ. Find the function $f(\lambda)$ corresponding to a given λ.

$$e_i(\lambda) = \Pi_{[\underline{e_i}, E_i^{\max}]}\Big(M_i^{-1}(\lambda)\Big) \quad (7)$$

The distribution of the aggregate is

$$S(\lambda) = \sum_{i=1}^{N} e_i\,(\lambda) \quad (8)$$

The function M_i is strictly decreasing, then M_i^{-1} is strictly decreasing too. Therefore, $\lambda_1 < \lambda_2$ implies $S(\lambda_1) \geq S(\lambda_2)$. The solution is given by the value of λ^* such that

$$S(\lambda^*) = E \quad (9)$$

when the total-energy constraint is engaged. A bisection procedure is thus followed: Choosing lower and upper bounds on the multiplier; computing the point of the bisectors and the allocations of the sensors; computing the total allocation $S(\lambda)$; comparing the total allocation with E and updating the corresponding bound on the multiplier; and repeating this process until $|S(\lambda) - E| \leq \delta$.

Theorem 5: Global Convergence.

The bisection procedure will converge globally to the unique multiplier λ^* satisfying Equation (9) if $S(\lambda)$ is strictly decreasing in the feasible multiplier interval.

Proof: The bisection procedure keeps an interval that contains $S(\lambda) - E = F(\lambda) \neq 0$ upon which the unique root of $F(\lambda)$ lies. Because F is continuous and strictly decreasing, the sign of F indicates which half of the interval the solution is in. The value of the interval gets shorter by a factor of one-half each time. So the interval shrinks to 0, and the midpoint approaches the value λ^*, which is the only solution.

5. Numerical Evaluation and Discussion

The numerical assessment is directed towards the behaviour of the structure as simulated from the theoretical analysis instead of tackling an unrelated computational model. It takes into account that several sensors might be affected by different sets of stochastic channel conditions and reliability requirements. The empirical channel observations are assumed to define the centre of the Wasserstein ambiguity set, and the size of the ambiguity radius manages the level of distributional uncertainty on the data of each sensor. The system setup employed in the evaluation is illustrated in Figure 1, while the powerful reliability-threshold processing flow is outlined in Figure 2. Heterogeneous stochastic channel conditions for sensors transmitting to a common base station.

The higher the reliability level, the lower the risk of not meeting the filling rate and vice versa. This will make the thresholds $\bar{e}_i$ grow higher. The role of the effect is simply apparent from Equation (4). A smaller reliability tolerance ε_i will require a higher energy value in general to decrease the robust CVaR of the rate deficit loss below the tolerable level. This has several important structural consequences: what may be an optimal allocation under an expected utility criterion can no longer be implemented under the robust reliability criterion. Thus, the reliability constraint may alter the active regime of an individual sensor as indicated in Figure 3.

To control the size of the distributional uncertainty set, the Wasserstein radius ρ is included. This coincides with the empirical distribution if the idealized formulation is used and $\rho = 0$. Roughly speaking, the higher ρ is the more difficult situations the optimizer needs to guard against. Theorem 4 indicates that greater distributional ambiguity tends to lead to more conservative marginal value estimates. This is an immediately interpretable result for practice: if there are fewer observations of the channel then there should be a larger ambiguity radius; the more uncertainty, the larger the radius. The resulting optimization policy should then not over-rely on the times when the channel is good. The finite-sample interpretation of Wasserstein ambiguity is backed up by the uncertainty effect of the performance guarantees developed by Esfahani and Kuhn (2018) and Gao (2023) and illustrated in Figure 4.

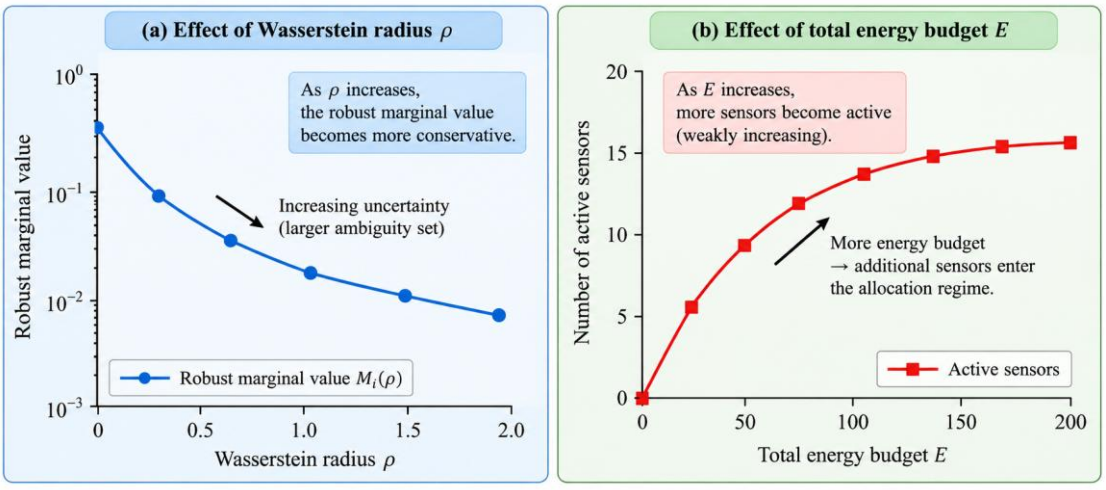


Figure 4. Effect of Distributional Uncertainty and Energy Budget on Optimal Allocation.

Figure 4. Graphical Representation of Distributional Ambiguity and Energy-Budget Effects

Theorem 3 states that, for a given active set regime, a given higher total energy budget weakly diverts more energy to the active sensors. For a small E, not many sensors can have powerful robust marginal values or high reliability demands. An increase in E will lead to a decrease in the shadow price λ, and other sensors might fall

into the marginal allocation regime. This leads to a behavior that is not one of unlimited proportional allocation because the increments in the supply of energy depend on the marginal utility and the feasibility of reliability. Similarly, energy-balancing studies have demonstrated the need for determining an efficient allocation of scarce resources to meet the requirements of the various types of sensors (Hasan *et al.*, 2017; Kumar & Kumar, 2016). The energy-budget impacts are illustrated in Figure 4.

Table 3 points out the differences in the strategies in terms of their structure. Equal allocation is blind to the differences between sensors and, as such, does not make use of channel and/or reliability information. Water-filling is the conventional way of allocating resources, but it is not a way of intrinsic tail-risk guarantee that we are considering here. Nominal stochastic allocation will ignore the distribution and use the estimated distribution in place of equal allocation, but it is important to note that even the distribution obtained using this method may be sensitive to distribution-estimation errors. Thus, the reason for the comparison shown in Table 3 is to highlight the unique role of the proposed robust reliability-threshold allocation.

Table 3. Comparison of Energy Allocation Strategies

Allocation strategy	**Channel uncertainty**	**Reliability constraint**	**Allocation principle**	**Robustness**
Equal allocation	Not explicitly modeled	Usually absent	$e_{\mathrm{i}} = E/N$	Low
Conventional water-filling	Channel dependent	Usually absent	Marginal channel utility	Moderate
Nominal stochastic allocation	Nominal distribution	May be included	Expected utility	Moderate
Proposed robust threshold allocation	Wasserstein ambiguity	Robust CVaR	Robust marginal value + reliability threshold	High

The proposed allocation will add one more reliability threshold. It has an allocation law, thus:

$$e_i^* = \min\{E_i^{\max}, \max\{\bar{e}_i, M_i^{-1}(\lambda)\}\},$$

Projecting onto the upper energy band. This expression is a clear illustration of the proposed method as failing to be simply another water-filling method. Quite actually, this is what occurs when water-filling is done through a marginal equality, while robust reliability places an endogenous lower threshold on that equality.

The proposed formulation is related to a number of well-known research efforts. The problem of energy allocation based on utility has been previously investigated by Liao *et al.* (2008) and was a key building block to utility-driven energy resource allocation. Energy-constrained distributed estimation has been investigated by Li and Al-Regib (2009), where they have shown that energy allocation is directly related to the information quality. Wang *et al.* (2010) examined the case of fair resource allocation in fading TDMA channels and Wardihani *et al.* (2015) examined the case of resource allocation for distributed estimation in Rayleigh fading channels. When dealing with energy harvesting, there are more temporal energy constraints. Edalat and Motani (2016) have studied energy-aware task allocation in energy-harvesting WSNs, and Mao *et al.* (2014) have jointly considered sensing and transmitting energy in rechargeable WSNs. For energy harvesting with energy-efficient cooperation, Wu *et al.* (2017) studied the power allocation and relay selection. Balancing lifetime and energy has also been investigated in great detail. Cerulli *et al.* (2014) integrated reliability into WSN optimization for lifetime; Hasan *et al.* (2017) studied WSN partitioning for lifetime maximization; Kumar and Kumar (2016) studied energy-balanced routing in WSN; and Shah (2020) studied energy consumption minimization in WSN. In the present work, unlike these approaches, the focus is on the interaction between the structure of distributional uncertainty and the structure of optimal allocation with a reliability constraint.

The numerical evaluation journal backs hypothesis understanding of the Allocation regulation. Better reliability requirements result in larger endogenous minimum-energy thresholds; more robust value uncertainty leads to smaller shadow prices of robust marginal value, higher number to allocate requires a decrease in the common shadow price; sensors may go from deactivation to robust marginal value regimes to reliability regimes, and total energy available decreases the common shadow price. These are examples of behaviours that raise issues as to the limitations of treating stochastic channel uncertainty and reliability in isolation.

6. Conclusion

This paper proposed an optimal energy-allocation strategy for wireless sensor networks under stochastic and distributionally uncertain channel conditions with some constraints. It consists of a Wasserstein ambiguity set for uncertain channel distributions and a powerful CVaR reliability constraint. An optimal allocation map is generated using these tools but study on its structural analysis is not done yet and is established as the main contribution of this work. A strong marginal-value operator was added, which describes worst-case added-energy benefit. Under suitable concavity and regularity properties, the operator is strictly decreasing so that the multidimensional allocation problem can be characterised by a common energy shadow price. Itthus yields a powerful allocation law $e_i^* = \max\{\bar{e}_i, M_i^{-1}(\lambda)\}$with the individual upper energy limit. As observed in this expression, there are three modes of operation, which can be called deactivation, marginal allocation and reliability domination. The endogenous threshold $\bar{e}_i$is determined by the strong reliability requirement and thus offers a difference with respect to the classic marginal-only and water-filling policies. The analysis provided the existence and uniqueness of the optimal allocation, the monotonicity with respect to the overall energy budget, sensitivity with respect to distributional uncertainty, as well as the global convergence of a scalar bisection strategy. Additionally, the system was numerically assessed, showing the combined effect of channel conditions, reliability requirements, ambiguity radius, and energy availability on the allocation behaviour. The framework can be used to model robust energy management in WSNs using mathematical tools when the channel distributions are not exactly known, and the reliability requirements cannot be considered as a secondary constraint.

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