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Elliptic Problems with Non Lipschitz Nonlinearities: Some Recent Results and Open Questions

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Let $p \in]1, +\infty[$, let $r, s \in]0, p[$, with $r < s$, and let $\lambda \in]0, +\infty[$. In this paper, we present some recent existence and multiplicity results on the solutions of the Dirichlet problem for the elliptic equation $-\Delta_p u = (\lambda|u|^{s-2}u - |u|^{r-2}u)\chi_{\{u\neq 0\}}$ in a bounded domain $\Omega \subset \mathbb{R}^N$, with 0-boundary data. Some related open questions are also proposed.

Keywords: Elliptic boundary value problem, nonnegative solution, positive solution, least energy solution, least energy nodal solution, variational methods, indefinite nonlinearities.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a nonempty open bounded set with smooth boundary. Let $p \in]1, +\infty[$, $r, s \in]0, p[$, and $\lambda \in]0, +\infty[$. Throughout this paper, we always assume $r < s$.

Let us consider the problem

$$\begin{cases} -\Delta_p u = (\lambda|u|^{s-2}u - |u|^{r-2}u)\chi_{\{u\neq 0\}} & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_\lambda)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $\chi_{\{u\neq 0\}}$ is the characteristic function of the set $\{x \in \Omega : u(x) \neq 0\}$. The convention $\infty \cdot 0 = 0$ is adopted in the expression $(\lambda|u|^{s-2}u - |u|^{r-2}u)\chi_{\{u\neq 0\}}$.

By a weak solution of problem (P_λ) we mean any $u \in W_0^{1,p}(\Omega)$ satisfying

1) $(\lambda|u(\cdot)|^{r-2}u(\cdot) - |u(\cdot)|^{r-2}u(\cdot))\chi_{\{u\neq 0\}}v(\cdot) \in L^1(\Omega)$,

2) $\displaystyle\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx = \int_{\{u\neq 0\}} (\lambda|u(x)|^{s-2} - |u(x)|^{r-2})u(x)v(x)dx$,

for all $v \in W_0^{1,p}(\Omega)$.

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First, we are interested in finding nonzero and nonnegative solutions to problem (P_λ). Then, we want to confront the problem of finding positive solutions to (P_λ). This question is a bit delicate due to the fact that some features of the nonlinearity $f(t) := (\lambda|t|^{s-2}t - |t|^{r-2}t)\chi_{\{t\neq 0\}}$ give rise to some difficulties when one wants to study the positivity of nonnegative and nonzero solutions to problem (P_λ). In particular, we are referring to the following features of the nonlinearity f:

a) f is not Lipschitz continuous in each neighborhood of the origin;
b) f is negative in a right neighborhood of the origin.

Conditions *a)* and *b)* prevent us to use the Strong Maximum Principle in order to obtain positive solutions from nonnegative and nonezero solutions (see [11] for instance). Actually, the existence of positive solutions to problem (P_λ) has been proved by different methods (Implicit Function Theorem ([8]), quadrature method for the 1-dimensional case ([7])). The multiplicity of positive solutions for problem (P_λ) is still an open question, even in the 1-dimensional case when $r \in (0, 1)$.

However, the violation of Strong Maximum Principle suggests to look for solutions with other relevant properties. In this direction, for λ large, the existence of a continuous family of compact support solutions has been proved in [7] for the 1-dimensional case, and in [12] for higher dimension with some additional condition on the exponents r, s. This family is constructed starting from a positive solution u to problem (P_λ) which satisfies the additional Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, where ν is the outer unit normal to $\partial\Omega$.

Finally, we deal with the existence of least energy solutions and least energy nodal solutions to problem (P_λ).

In the next section, we present a list of some known results about problem (P_λ) concerning the above questions, and propose some related open problems.

2. Existence and multiplicity result for problem (P_λ).

2.1. The 1-dimensional case

For the nonsingular case $r > 1$, an exhaustive description of the nonnegative solution set of problem (P_λ) in the 1-dimensional case, with $\Omega = (-1, 1)$, is given by the following result

Theorem 1 (Theorem 1 of [7]) Assume $r > 1$. Then, there exist two positive constants Λ_1, Λ_2, with $\Lambda_1 < \Lambda_2$, such that the problem

$$\begin{cases} -(|u'|^{p-2}u') = \lambda|u|^{s-2}u - |u|^{r-2}u & \text{in } (-1,1), \\ u(-1) = u(1) = 0 \end{cases}$$

admits:

i) no positive solution, for all $\lambda < \Lambda_1$;
ii) a unique positive solution, for $\lambda = \Lambda_1$;

iii) exactly two positive solutions u_λ, v_λ, with $v_\lambda < u_\lambda$ in $]-1,1[$, for all $\lambda \in]\Lambda_1, \Lambda_2]$, and the solution v_{Λ_2} (corresponding to $\lambda = \Lambda_2$) satisfies $v'_{\Lambda_2}(-1) = v'_{\Lambda_2}(1) = 0$.

iv) one positive solution and a continuum of solutions with compact support in $]-1,1[$, for all $\lambda > \Lambda_2$.

The proof of Theorem 1 is based on the fact that the equation $-(|u'|^{p-2}u') = \lambda|u|^{s-2}u - |u|^{r-2}u$ is integrable by quadrature. This allows to obtain an explicit expression for the positive solutions of the equation and then, from this expression, one can find the conditions on the parameter λ in order to one or more solutions satisfy the boundary conditions of the problem. For $\lambda > \Lambda_2$, the solution v_{Λ_2} is used to construct the continuum of solutions with compact support in $]-1,1[$. Indeed, this continuum of solutions is formed by functions u such that:

- on some compact subinterval I of $]-1,1[$, u is a suitable composition of v_{Λ_2} with certain affine functions;
- u is identically zero on $]-1,1[\setminus I$.

In the singular case $r \in]0,1[$, a complete description of the solution set is not known yet. Some partial results can be found in [6].

2.2. The N-dimensional case

In the N-dimensional case, an existence and multiplicity result, comparable with Theorem 1, is the following Theorem:

Theorem 2 (Theorem 1 of [2]) Assume $r > 1$. Then, there exist two positive constants Λ_1, Λ_0, with $\Lambda_1 < \Lambda_0$, such that problem (P_λ) admits:

i) no nonzero solution, for all $\lambda < \Lambda_1$;

ii) at least a nonzero and nonnegative solution, for all $\lambda \in [\Lambda_1, \Lambda_0[$;

iii) at least two nonzero and nonnegative solutions, for all $\lambda \geq \Lambda_0$.

The proof of Theorem 2 is based on variational methods. The energy functional I associated to problem (P_λ) has a local minimum point at 0 and I has, for large λ, the mountain pass geometry as well. This yields the existence of a nonnegative solution of mountain pass type. For larger $\lambda's$, I possesses a global minimum point with negative energy as well. This local minimum point is then a second nonnegative solution. The nonzero nonnegative solution corresponding to $\lambda = \Lambda_1$ is obtained as limit (in $W_0^{1,p}(\Omega)$) of any family of solutions $\{u_\lambda\}_{\lambda > \Lambda_1}$ as $\lambda \to \Lambda_1^+$.

Theorem 2 gives rise to two natural questions:

Question 1

Let $A := \Big\{\lambda \in]0,+\infty[: (P_\lambda)$ *has at least two nonzero and nonnegative solutions*$\Big\}$.

Is $\Lambda_1 = \inf A$?

Question 2

Does problem (P_λ) admit positive solutions?

A positive answer to the above questions has been given in [1] for the semilinear nonsingular case, i.e. when $p = 2$. Indeed, one has the following result:

Theorem 3 (Theorem 1 of [1]) Let $p = 2$ and $r \in]1, 2[$. Then, there exists a positive constant Λ_1 such that problem (P_λ) admits:

i) no nonzero solution, for all $\lambda < \Lambda_1$;

ii) at least a nonzero and nonnegative solution, for $\lambda = \Lambda_1$;

iii) at least a positive solution u_λ satisfying $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, which is a local minimum point of the the energy functional associated to problem (P_λ), and a further nonzero and nonnegative solution, for all $\lambda > \Lambda_1$.

The proof of Theorem 3 exploits Theorem 3.13 [8] which gives the existence of at least a positive solution u_λ satisfying the Hopf's boundary condition $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, for $\lambda > \Lambda_1$. The existence of this solution is proved in [8] using the Implicit Function Theorem. However, this way to find u_λ does not allow to derive any variational property of u_λ. In [1], using a version of the Hopf's Lemma involving unbounded coefficients and a well known result of Brezis-Nirenberg (see [5]) which assures that $C^1_0(\overline{\Omega})$ local minimizers of energy functionals are also a $W^{1,2}_0(\Omega)$ local minimizers, it is proved that the solution u_λ given in [8] is actually a local minimizer of I. Thus, the second solution comes from the Mountain Pass Theorem using the fact that 0 is another (strict) local minimizer of I.

The statement of Theorem 3 leads to the following further open questions, some of them arising from a comparison of Theorem 3 with Theorem 1:

Open questions: Assume $p = 2$ and $r > 1$.

1) Let u be any nonzero and nonnegative local minimum point of the energy functional I associated to problem (P_λ). Is u positive in Ω? Does I admit a unique nonzero and nonnegative local minimum point?
2) Does problem (P_λ) admit multiple positive solutions for each $\lambda > \Lambda_1$ sufficiently near to Λ_1?
3) Is the solution u_λ corresponding to $\lambda = \Lambda_1$ positive in Ω?
4) Does problem (P_λ) admit compact support solutions for λ large enough?

A positive answer to the last question was given in [12] under the additional conditions that Ω is star-shaped and the dimension N is greater than $rs(2 - r)^{-1}(2 - s)^{-1}$.

The above mentioned Theorem 3.13 of [8] gives the existence of at least a positive solution also in the singular case $r \in]0, 1]$. An extension of Theorem 3 to this case, with the assumptions that the power λu^{s-1} is nonsingular, i.e. with $s \in]1, 2[$, was recently given in [4] where the following result was established:

Theorem 4 (Theorem 1 of [1]) Let $p = 2$, $r \in]0,1]$ and $s \in]1,2[$. Then, there exists a positive constant Λ_1 such that problem (P_λ) admits:

i) no positive solution, for all $\lambda < \Lambda_1$;

ii) at least a positive solution u_λ satisfying $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, which is a local minimum point of the energy functional associated to problem (P_λ), and a further nonzero and nonnegative solution, for all $\lambda > \Lambda_1$;

Note that, if compared with Theorem 3, Theorem 4 does not give information about the existence of nonzero and nonnegative solutions in correspondence to $\lambda = \Lambda_1$. So, knowing whether or not problem (P_λ) admits nonzero and nonnegative (or even positive) solutions for $\lambda = \Lambda_1$ remains an open question. The proof of Theorem 4 is again based on variational methods but it is much more delicate than the proof of Theorem 3 due to the presence of the singular term. One of the ingredients of the proof is a gradient estimate of the solutions of a regularized problem given in [10], where the existence of multiple nonzero and nonnegative solutions for problem (P_λ) was established. Theorem 4 improves the multiplicity result of [10] in two aspects: the first one is that the existence of a positive solution (satisfying the Hopf's boundary condition $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\partial\Omega$) entails the existence of a second nonzero and nonnegative solution. The second one concerns the variational property of the positive solution which can be characterized from being a local minimum point of the energy functional.

It is likely that similar techniques used to deal with the case $r \in]0,1]$ and $s \in]1,2[$ work also in the "double singular" case $r, s \in]0,1[$ to prove the existence of multiple nonzero and nonnegative solutions.

2.3. Existence of least energy solutions

An interesting and widely studied topic in the field of the elliptic boundary value problems is to investigate the existence of least energy solutions. By definition, a least energy solution is a non zero solution which minimizes the energy functional over the set of all nonzero solutions of the problem. Thus, with regard to problem (P_λ), a least energy solution is a solution u_λ such that $I(u_\lambda) \leq I(u)$, for all $u \in S_\lambda$, where

$$I(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{s}\int_\Omega |u|^s dx + \frac{1}{r}\int_\Omega |u|^r dx,\ r \in W_0^{1,p}(\Omega),$$

is the energy functional associated to (P_λ), and S_λ is the set of all nonzero solutions of the problem (P_λ). Note that in the nonsingular case $r > 1$, we have that I is of class C^1 and $S_\lambda := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : I'(u) = 0\}$.

In the following, we will consider just the nonsingular case $r > 1$ and, for simplicity, we assume $p = 2$ (however, almost all the results can be proved also for $p > 1$). Moreover, for our convenience, we transform problem (P_λ) into the following equivalent form

$$\begin{cases} -\Delta u = (|u|^{s-2}u - \mu|u|^{r-2}u)\chi_{\{u\neq 0\}} & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_\mu)$$

by means of the substitution $u \to \lambda^{\frac{1}{r-2}}u$ and the change of parameter $\mu = \lambda^{\frac{2-s}{r-2}}$. In [3], it is proved that for $\mu > 0$ the set of all nonzero solutions of the problem (P_μ) is weakly compact. Therefore, taking into account that the energy functional I_μ is coercive and sequentially weakly lower semicontinuous, the existence of a least energy solution u_μ trivially follows by the Weierstrass Theorem. In some case, least energy solutions minimize the energy functional on the associated Nehari manifold (which, by definition, contains the set of all nonzero solutions). For instance, using the Lagrange Multipliers Theorem, one sees that this occurs when $\mu = 0$ and $s \in]2, 2^*[$, where 2^* is the critical exponent for the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$. In our case, the Nehari manifold associated to the energy functional I is

$$\begin{aligned} N_\mu \;&:= \{u \in W_0^{1,2}(\Omega) \setminus \{0\} : I'(u)(u) = 0\} \\ &= \{u \in W_0^{1,2}(\Omega) \setminus \{0\} : \int_\Omega |\nabla u|^2 dx = \int_\Omega |u|^s dx - \mu \int_\Omega |u|^r dx\}. \end{aligned}$$

As well as S_μ, it is proved in [3] that the Nehari manifold N_μ is weakly compact as well. Therefore, there exists a minimizer v_μ of I restricted to N_μ. Nevertheless, differently to the aforementioned case corresponding to $\mu = 0$ and $s \in]2, 2^*[$, in this case it is not clear whether or not v_μ is a solution of (P_μ), i.e. whether or not v_μ is a least energy solution.

Observe that for $\mu = 0$, it is well known the neither S_μ nor N_μ are (strongly) closed. Therefore, the existence of a least energy solution cannot be obtained by applying the Weierstrass Theorem to I restricted to S_μ or N_μ. Nevertheless, it is also well known that, in this case, I admits a positive global minimum point u_μ, with $I(u_\mu) < 0$. Clearly, u_μ is a least energy solution of problem (P_μ).

2.4. Existence of least energy nodal solutions

Besides the existence of least energy solutions, another interesting problem is finding least energy nodal solutions to boundary value elliptic problems. A least energy nodal solution is, by definition, a sign-changing solution which minimizes the energy functional over the set of all sign-changing solutions of the problem. Let us suppose again $p = 2$ and $r > 1$. When $\mu = 0$, it is a well known fact that problem (P_μ) admits infinitely sign-changing solutions. In [3] it is proved that any sequence in the set $S_\mu^\pm$ of all sign-changing solutions of (P_μ) which minimizes the restriction of I to $S_\mu^\pm$ must converge in the $C^1(\overline{\Omega})$-topology to a nonzero solution $u_\mu^\pm$. Then, using the Hopf's Lemma, one sees that u_μ must be sign-changing. Therefore, u_μ is a least energy nodal solution of problem (P_μ).

When $\mu > 0$, the existence of sign-changing solutions for problem (P_μ) is assured for μ small enough by Theorem 1.3 of [9]. As for the case of the least energy solutions, in [3] it is proved that, for $\mu > 0$, the set $S_\mu^\pm$ and the *nodal* Nehari manifold $N_\mu^\pm$ defined by

$$N_\mu^\pm \ := \{u \in W_0^{1,2}(\Omega) : \max\{u,0\}, \max\{-u,0\} \in N_\mu\},$$

are both weakly compact. Therefore, when $S_\mu^\pm \neq \emptyset$, problem (P_μ) admits a least energy nodal solution $u_\mu^\pm$ and a global minimizer $v_\mu^\pm$ of the energy functional I restricted to $N_\mu^\pm$. When $\mu = 0$ and $s \in]2, 2^*[$, using the Miranda Theorem on the existence of zeros for functions of two variables, one can prove that minimizers of I restricted to $N_\mu^\pm$ are least energy nodal solutions. For $\mu > 0$ and $s \in]1, 2[$, knowing whether or not the previous property holds is an open problem. In other words, it is an open question to see whether or not $v_\mu^\pm$ is a solution to (P_μ).

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