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Fermatean Lattice Fuzzy Vector Space of Quantum Subspaces over Hopf Subalgebras

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Abstract
The Main goal of this paper is to introduce the concept of lattice structure of fermatean fuzzy vector space of quantum subspace with respect to Hopf algebras. We have defined fermatean L-fuzzy sub algebras, fermatean L-fuzzy sub co algebras and fermatean L-fuzzy left (right) coideals are defined and the characterizations of fermatean L-fuzzy sub co algebras under homomorphisms of coalgebras are investigated. We have discussed the duality of fermatean L-fuzzy sub co algebras and duality of fermatean L-fuzzy sub coalgebras as well as fermatean L-fuzzy sub algebras are studied. Finally, we studied fermatean L-fuzzy quantum subsets with some suitable examples.

Keywords: Fuzzy set, Intuitionistic fuzzy set, Fermatean L-fuzzy sub algebras, Vector space, Quantum subspace, Fermatean L-fuzzy sub coalgebras

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh [34] and this concept was adapted by Goguen [11] and Sanchez [28] to define and study fuzzy relations. Yuan and Wu [33] introduced the concepts of fuzzy sublattices and fuzzy ideals of a lattice. Ajmal and Thomas [2] defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. As a continuation of these studies, we define a fuzzy lattice as a fuzzy relation and work on fuzzy posets and fuzzy lattices in this note. In 1983, Atanassov [4], introduced the concept of intuitionistic fuzzy sets as a generalization of the concept of fuzzy sets defined by Zadeh [34], to overcome the difficulties in dealing with uncertainties.Fermatean fuzzy environment is the modern tool for handling uncertainty in many decisions making problems. Fermatean fuzzy set gives a modern way to model vagueness and uncertainty with high precision and accuracy compared to intuitionistic fuzzy set. Senapti and Yager [22] coined the Fermatean fuzzy set (FFS) with its comparison measures. FFS can characterize more complex uncertain information by redefining the constraint condition $0 \leq \mu^3+\nu^3 \leq 1$. In other words, IFS and PFS are two special forms of FFS, which means that the FFSs are able to handle higher levels of uncertainties. Senapti and Yager [22] gave an example: For understanding the FFS better, we give an instance to illuminate the understand ability of the FFS: We can definitely get $0.9 + 0.6 > 1$,and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get$(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$,which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get$(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945$,which is good enough to apply the Fermatean fuzzy set to control it. we have developed some new operators for fermatean fuzzy sets [22]. A fuzzy lattice extends conventional mathematical lattice theory by allowing ordered relations between elements to be fuzzy (partial/graded) rather than strictly true or false. A fermatean fuzzy lattice is an algebraic structure that combines fermatean fuzzy sets with lattice theory, designed to manage high degrees of uncertainty, ambiguity and hesitation in decision making. Fermatean fuzzy set are an extension of intuitionistic fuzzy set and Pythagorean fuzzy set, used to model uncertainty by allowing the sum of the cubes of membership and non-membership degrees to be less than or equal to 1. Quantum theory describes the behavior of matter and energy at atomic scales, employing superposition, wave Partech duality, and probabilistic measurement outcomes. These principles have gathered significant attention in recent years, particularly in contexts such as quantum computing.In the works of Faddaeev et.al [6]. Quantum groups were introduced and a new class of Hopf algebras was constructed. Vallin [29] developed Hopf C* algebra theory. Recently, different approaches are contained in the papers Vaksman and Soibelman [30]. Kruszynski and Woronowicz [15], and Brown and

Goodearl [5]. A compact matrix pseudo group is defined in [33] as a non-commutative compact space endowed with a group structure.Manin [16] defined quantum plane as a particular start point to construct quantum groups and introduced multiparametric quantum deformation of the general linear supergroup in [17] and later Wachter [31] discussed analysis on q-deformed quantum spaces. Recently the author [21] defined quantum partial differentiations of Q-analytic functions on quantum super space A_q and investigated relations between them and some automorphisms of O_q. For more details on Hopf algebras, we refer the readers to [1] and [13]. So far algebraic structures to fuzzy quantum spaces and compatibility [21], fuzzy quantum spaces [31], quantum structures and fuzzy set theory [21] and fuzzy sub coalgebras and duality [21] have been established and discussed but no fermatean lattice fuzzy vector space of quantum space has been done. The Main goal of this paper is to introduce the concept of lattice structure of fermatean fuzzy vector space of quantum subspace with respect to Hopf algebras.

We proceed as follows, in section-2, just we recall the definition of fermatean L-fuzzy vector subspaces and dual of them. Fermatean lattice fuzzy structures studied in section-3. In section-4, we discuss fermatean L-fuzzy sub coalgebras as well as fermatean L-fuzzy Hopf algebras. In section-5, fermatean L-fuzzy quantum subsets with some suitable examples are given.

2. Preliminaries

In this section, we study the basic concepts of important Lattice structures and its operations.

Definition 2.1:[2] A lattice is posset $(L, \leq)$ in which every pair of elements $\{a, b\}$ has a unique LUB and a unique GLB.

Example-2.2:[2] The power set of a set under inclusion $(\subseteq)$ is a lattice as are the natural numbers ordered by divisibility where LUB=LCM, GLB=GCD.

Definition 2.3:[34] (Fuzzy set) Let X be a non-empty set. A fuzzy set 'A' drawn from X is defined as $A = \{(x, \mu_A(x))/x \in X\}$, where $\mu_A: X \to [0,1]$ is the member function of the fuzzy set A.

Definition 2.4:[2] Let (X, B) be a posset. (X, B) is a fuzzy lattice if and only if $x \wedge y$ and $x \vee y$ exists for all $x, y \in X$.

Example2.5:[2] Let $X = \{a, b, c\}$ and let $B: X \times X \to [0,1]$ be a fuzzy relation such that $B(a,a) = B(b,b) = B(c,c) = 1$, $B(a,b) = B(a,c) = B(b,c) = 0$, $B(b,a) = 0.5$, $B(c,a) = 0.3$ and $B(c,b) = 0.2$. Then it is easily verified that B is a fuzzy partial order relation. Also $a \vee b = a$, $a \vee c = a$, $b \vee c = b$, $a \wedge b = b$, $a \wedge c = c$ and $b \wedge c = c$. Then (X, B) is a fuzzy lattice.

Definition 2.6:[2] The Lattice is equipped with meet $(\wedge)$ and join $(\vee)$ operators tailored to fermatean numbers which are essential for determining the supremum and infimum of these of these sets, typically defined as

$$B \wedge C = \big(max(J_B, J_C), min(K_B, K_C)\big)$$

$$B \vee C = \big(min(J_B, J_C), max(K_B, K_C)\big)$$

Researchers define subsets of a main lattice that are themselves fermatean fuzzy lattices.

Definition 2.7:[4]Let X be a non-empty set. An intuitionistic fuzzy set A in X is an object having form $A = \{(x, J_A(x), K_A(x))/x \in X\}$, whose the functions $J_A: X \to [0,1]$ and $K_A: X \to [0,1]$define respectively, the degree of membership and non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, $0 \leq J_A(x) + K_A(x) \leq 1$.

Definition 2.8:[32] Let X be a non-empty set. APythagorean fuzzy set A in X is an object having form $A = \{(x, J_A(x), K_A(x))/x \in X\}$, whose the functions $J_A: X \to [0,1]$ and $K_A: X \to [0,1]$define respectively, the degree of membership and non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, $0 \leq {J_A}^2(x) + {K_A}^2(x) \leq 1$.

Definition 2.9:[22]Let X be a non-empty set. An fermatean fuzzy set A in X is an object having form $A = \{(x, J_A(x), K_A(x))/x \in X\}$, whose the functions $J_A: X \to [0,1]$ and $K_A: X \to [0,1]$define respectively, the degree of membership and non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$, $0 \leq {J_A}^3(x) + {K_A}^3(x) \leq 1$.

3. Fermatean Lattice fuzzy structures

Definition 3.1: Let 'P' be a non-empty set. A fermatean L-fuzzy subset 'D' of 'P' is defined as an object having the form

$D = \{(w, J_D(w), K_D(w))/w \in P\}$ or $D = (J_D, K_D)$,

where the functions $J_D: P \to L$ and $K_D: P \to L$ denote the degree of membership and non-membership of each element $w \in P$ to the set 'D', respectively and ${J_D}^3(w) \leq N\left({K_D}^3(w)\right)$ for each $w \in P$.

Definition 3.2: Let 'V' be a d -Vector space where $d = R, C$ or any field with characteristic greater than or equal to 2. A fermatean L-fuzzy subset $D = (J_D, K_D)$ of V is called a fermatean L-fuzzy vector subspace of V if

${J_D}^3(\alpha w_1 + \beta w_2) \geq min\{{J_D}^3(w_1), {J_D}^3(w_2)\}$,

$$ {K_D}^3(\alpha w_1 + \beta w_2) \leq max\{{K_D}^3(w_1), {K_D}^3(w_2)\} $$

for any $w_1, w_2 \in V$ and $\alpha, \beta \in d$.

Definition 3.3: The fermatean L-fuzzy subsets $\tilde{0}$ and $\tilde{1}$ are added by

${J_{\tilde{0}}}^3(w) = 0, {K_{\tilde{0}}}^3(w) = 1$ if $w \neq 0$ and

${J_{\tilde{0}}}^3(0) = 1, {K_{\tilde{0}}}^3(0) = 0$, and

${J_{\tilde{1}}}^3(w) = 0, {K_{\tilde{1}}}^3(w) = 0, \forall\, w \in P$.

Let $A = (J_A, K_A)$ and $B = (J_B, K_B)$ be fermatean L-fuzzy subsets of P. We say that $A \leq B$ if for all $w \in P$, we have ${J_A}^3(w) \leq {J_B}^3(w)$, ${K_A}^3(w) \geq {K_B}^3(w)$. We define fermatean L-fuzzy subsets $A \cap B$, $A \cup B$ by

${J_{A\cap B}}^3(w) = min\{{J_A}^3(w), {J_B}^3(w)\}$, ${K_{A\cap B}}^3(w) = max\{{K_A}^3(w), {K_B}^3(w)\}$

${J_{A\cup B}}^3(w) = max\{{J_A}^3(w), {J_B}^3(w)\}$, ${K_{A\cap B}}^3(w) = min\{{K_A}^3(w), {K_B}^3(w)\}$

Definition 3.4: Let $A = (J_A, K_A)$ and $B = (J_B, K_B)$ be fermatean L-fuzzy vector spaces of a d-vector space V. The fermatean L-fuzzy subsets $A + B$ and αA of V for each $\alpha \in d, w \in X$ are defined by

$$ {J_{A\cap B}}^3(w) = \begin{cases} \sup\limits_{w=a+b} min\{{J_A}^3(a), {J_B}^3(b)\}, & \text{if } w = a + b \\ 0, & \text{otherwise} \end{cases} $$

$$ {K_{A\cap B}}^3(w) = \begin{cases} \inf\limits_{w=a+b} max\{{K_A}^3(a), {K_B}^3(b)\}, & \text{if } w = a + b \\ 0, & \text{otherwise} \end{cases} $$

and

$$ {J_{\alpha A}}^3(w) = \begin{cases} {J_A}^3(\alpha A), & \text{if } \alpha \neq 0 \\ 1, & \text{if } \alpha = 0, w = 0 \\ 0, & \text{if} \alpha = 0, w \neq 0 \end{cases} $$

$$ {K_{\alpha A}}^3(w) = \begin{cases} {K_A}^3(\alpha A), & \text{if } \alpha \neq 0 \\ 1, & \text{if } \alpha = 0, w = 0 \\ 0, & \text{if} \alpha = 0, w \neq 0 \end{cases} $$

Further if $A \cap B = \tilde{0}$, then $A + B$ is said to be the direct sum denote by $A \oplus B$.

Lemma3.5: Let $A = (J_A, K_A)$ and $B = (J_B, K_B)$ be fermatean L-fuzzy vector spaces of a L-fuzzy vector space V. Then

$A + B = (J_{A+B}, K_{A+B})$,

$A \cap B = (J_{A\cap B}, K_{A\cap B})$ and

$$\alpha A = (J_{\alpha A}, K_{\alpha A})$$

for $\alpha \in d$, are also be fermatean L-fuzzy vector spaces of V. $h^{-1}(B)$

Definition 3.6: Let h be a mapping from a d-vector space V to a d-vector space V'. If$A = (J_A, K_A)$ and $B = (J_B, K_B)$ are fermatean L-fuzzy vector spaces of V and V' respectively, then the pre images of $B = (J_B, K_B)$ under 'h' if defined to be an fermatean fuzzy set $h^{-1}[w] = \left(J_{h^{-1}(B)}, K_{h^{-1}(B)}\right)$, where$J_{h^{-1}(B)}(w) = J_B\big(h(w)\big)$, $K_{h^{-1}(B)}(w) = K_B\big(h(w)\big)$for any $w \in V$ and the image of $A = (J_A, K_A)$ under 'h' is defined to be an fermatean L-fuzzy set $h[A] = \left(J_{h[A]}, K_{h[A]}\right)$,where

$$J_{h[A]}(w) = \begin{cases} \sup\limits_{w_1 \in h^{-1}(w_2)} \{{J_A}^3(w_1)\}, & \text{if} w_2 \in h(v) \\ 0, & \text{if} w_2 \notin h(v) \end{cases}$$

$$K_{h[A]}(w) = \begin{cases} \inf\limits_{w_1 \in h^{-1}(w_2)} \{{K_A}^3(w_1)\}, & \text{if} w_2 \in h(v) \\ 0, & \text{if} w_2 \notin h(v) \end{cases}$$

Definition 3.7: Let $A = (J_A, K_A)$ and $B = (J_B, K_B)$ are fermatean L-fuzzy vector spaces of V and V' respectively, and $h: V \to V'$ be a mapping. Then $h^{-1}[B] = \left(J_{h^{-1}[B]}, K_{h^{-1}[B]}\right)$ and$h[A] = \left(J_{h[A]}, K_{h[A]}\right)$ are fermatean L-fuzzy vector subspaces of V and V' respectively.

Remark3.8: From now on, we assume that $L = (L, \leq, \wedge, \vee, ')$ is a complete distributive lattice set with at least two elements and $(L, +)$ is also an additive group.

To obtain the best generalization of notions of fermatean uncertainty sets [22], fuzzy sub coalgebras (Chen. W.2009) and fuzzy quantum spaces. We consider the condition $0 \leq {J_D}^3(w) + {K_D}^3(w) \leq 1$ instead of the condition ${J_D}^3(w) \leq N\left({K_D}^3(w)\right)$ for each $w \in P$ in Definition 3.1.

Throughout the chapter, we denote all fermatean L-fuzzy subsets of P by FFL^P.

4. Standard Results of Fermatean Lattice Fuzzy Vector Space D^*

Preposition4.1: Let V be a d-vector space and $D = (J_D, K_D)$ be fermatean L-fuzzy vector subspaces of V. We define $J_{D^*}, K_{D^*}: V^* \to L$ by

$$J_{D^*}(h) = \begin{cases} 0.5 - 0.5\, sup\{{J_D}^3(w)/w \in V, h(x) \neq 0, \text{if } h \neq 0\} \\ 0.5 - 0.5\, inf\{{J_D}^3(w)/w \in V, \quad \text{if } h = 0\} \end{cases}$$

$$K_{D^*}(h) = \begin{cases} 0.5 - 0.5\, inf\{{K_D}^3(w)/w \in V, h(x) \neq 0, \text{if } h \neq 0\} \\ 0.5 - 0.5\, sup\{{K_D}^3(w)/w \in V, \quad \text{if } h = 0\} \end{cases}.$$

Then $D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy vector subspace of V^*.

Proof: First, we shoe that $D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy subset of V^*.

By remark-3.8, it must be shown that $0 \leq {J_{D^*}}^3(h) + {K_{D^*}}^3(h) \leq 1$ for $h \in V^*$.

Case(i): If $h \neq 0$, then

$$
\begin{aligned}
&{J_{D^*}}^3(h) + {K_{D^*}}^3(h) \\
&\qquad = 1 - 0.5\ sup\left\{\frac{{J_D}^3(w)}{w} \in V, h(x) \neq 0, \text{if } h \neq 0\right\} \\
&\qquad - 0.5\ inf\left\{{K_D}^3(w)/w \in V, h(x) \neq 0, \text{if } h \neq 0\right\} \\
&\qquad \leq 1
\end{aligned}
$$

Case(ii): If $h = 0$, then

$$
\begin{aligned}
{J_{D^*}}^3(h) + {K_{D^*}}^3(h) &= 1 - 0.5\left\{inf\left\{{J_D}^3(w)/w \in V\right\} + sup\left\{{K_D}^3(w)/w \in V\right\}\right\} \\
&\leq 1
\end{aligned}
$$

Hence $D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy subset of V^*.

Now it must be show that

$${K_{D^*}}^3(\alpha h + \beta\varphi) \leq max\left\{{K_{D^*}}^3(h), {K_{D^*}}^3(\varphi)\right\}, \forall h, \varphi \in V^*, 0 \neq \alpha, \beta \in d$$

We consider three parts;

(i) If $h = 0$ and $\varphi \neq 0$, then

$$
\begin{aligned}
{K_{D^*}}^3(\alpha h + \beta\varphi) &= {J_{D^*}}^3(0) \\
&= 0.5 - 0.5\ sup\left\{{K_{D^*}}^3(w)/w \in V\right\} \\
&= {K_{D^*}}^3(h) \\
&= {K_{D^*}}^3(\varphi)
\end{aligned}
$$

(ii) If h $= 0$ and φ $= 0$, then

$$
\begin{aligned}
{K_{D^*}}^3(\alpha h + \beta\varphi) &= 0.5 - 0.5\ inf\left\{{K_{D^*}}^3(w)/w \in V, \beta\varphi(w) \neq 0\right\} \\
&\leq min\left\{\left(0.5 - 0.5sup\left\{{J_{D^*}}^3(w)/w \in V\right\}\right), \left(0.5 - 0.5inf\left\{{J_{D^*}}^3(w)/w \in V, \varphi(w) \neq 0\right\}\right)\right\} \\
&= max\left\{{J_{D^*}}^3(h), {K_{D^*}}^3(\varphi)\right\}
\end{aligned}
$$

(iii) If h $\neq 0$ and φ $\neq 0$, then

$$
\begin{aligned}
{K_{D^*}}^3(\alpha h + \beta\varphi) &= 0.5 - 0.5\ inf\left\{{K_{D^*}}^3(w)/w \in V, (\alpha h + \beta\varphi)(w) \neq 0\right\} \\
&\leq min\left\{\left(0.5 - 0.5inf\left\{{J_{D^*}}^3(w)/w \in V, h(x) \neq 0\right\}\right), \left(0.5 - 0.5inf\left\{{J_{D^*}}^3(w)/w \in V, \varphi(w) \neq 0\right\}\right)\right\} \\
&= max\left\{{J_{D^*}}^3(h), {K_{D^*}}^3(\varphi)\right\}
\end{aligned}
$$

We can prove similar that,

$${J_{D^*}}^3(\alpha h + \beta\varphi) \geq min\left\{{J_{D^*}}^3(h), {J_{D^*}}^3(\varphi)\right\}, \forall h, \varphi \in V^*, 0 \neq \alpha, \beta \in d.$$

Hence $D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy vector subspace of V^*.

Definition 4.2: Let $(A, P, \cup)$ be a d-algebra. A fermatean L-fuzzy subset $D = (J_D, K_D)$ of A is called fermatean L-fuzzy subalgebra of A if it satisfies the following condition

1. $D = (J_D, K_D)$ is fermatean L-fuzzy vector subspace of A.
2. ${J_D}^3(w_1 w_2) \geq min\left\{{J_D}^3(w_1), {J_D}^3(w_2)\right\}$,
${K_D}^3(w_1 w_2) \leq max\left\{{K_D}^3(w_1), {K_D}^3(w_2)\right\}$, for all $w_1, w_2 \in A$.

Definition 4.3: If we set the condition

$J_D{}^3(w_1w_2) \geq J_D{}^3(w_2)$, $K_D{}^3(w_1w_2) \leq K_D{}^3(w_2)$,

$$\left(J_D{}^3(w_1w_2) \geq J_D{}^3(w_1), K_D{}^3(w_1w_2) \leq K_D{}^3(w_1)\right)$$

for $x, y \in A$, instead of condition 2 in Definition 4.2, then $D = (J_D, K_D)$ is called frematean L-fuzzy left(right) ideal of A.

Definition 4.4: Let (T, ∇, ϵ) be a coalgebra and for any $w \in T$,

$$\nabla(w) = \sum_{i=1}^{n} w_{i1} \otimes w_{i2}$$

A frematean L-fuzzy subset $D = (J_D, K_D)$ of T is called a frematean L-fuzzy sub coalgebra of T if it satisfies the following axioms;

(i) $D = (J_D, K_D)$ is frematean L-fuzzy vector subspace of T,
(ii) $J_D{}^3(w) \geq min\{J_D{}^3(w_{i1}), J_D{}^3(w_{i2})\}$,
$K_D{}^3(w) \leq max\{K_D{}^3(w_{i1}), K_D{}^3(w_{i2})\}$, for all $w \in T$, for all i.

Definition 4.5: If we set the condition

$J_D{}^3(w) \leq J_D{}^3(w_{i2})$, $K_D{}^3(w) \geq K_D{}^3(w_{i2})$,

$\left(J_D{}^3(w) \leq J_D{}^3(w_{i1}), K_D{}^3(w) \geq K_D{}^3(w_{i1})\right)$,

for all $w \in T$, for all i. Instead of the condition (ii) in Definition 4.4, then $D = (J_D, K_D)$ is called a frematean L-fuzzy left(right) coideal of T.

Remark4.6:

(i) A frematean L-fuzzy sub coalgebra is a frematean L-fuzzy left(right) coideal.
(ii) If $D = (J_D, K_D)$ is a frematean L-fuzzy left and right coideal, by Definition 4.5, we get

$$J_D{}^3(w) \leq J_D{}^3(w_{i1}), J_D{}^3(w) \leq J_D{}^3(w_{i2})$$

$\Rightarrow J_D{}^3(w) \leq min\{J_D{}^3(w_{i1}), J_D{}^3(w_{i2})\}$ and
$K_D{}^3(w) \geq K_D{}^3(w_{i1}), K_D{}^3(w) \geq K_D{}^3(w_{i2})$
$\Rightarrow K_D{}^3(w) \geq max\{K_D{}^3(w_{i1}), K_D{}^3(w_{i2})\}$
for all $w \in T$, for all i. Hence D is a frematean L-fuzzy sub coalgebra.

Theorem4.7:A frematean L-fuzzy subset $D = (J_D, K_D)$ of the coalgebra T is a frematean L-fuzzy left(right) coideal if and only if the level sets

$$D_{u,v} = \{w \in T / J_D{}^3(w) \geq u, K_D{}^3(w) \leq v\}$$

are left(right) coideals of T where $u + v \in L$, $0 \leq u \leq J_D{}^3(0)$, $K_D{}^3(0) \leq v \leq 1$.

Proof:

Sufficient Part:

Let $D = (J_D, K_D)$ be a frematean L-fuzzy left coideal of T and $u, v \in L$, $0 \leq u \leq J_D(0)$, $K_D(0) \leq v \leq 1$.

Since $0 \in D_{u,v}$, then $D_{u,v} \neq \emptyset$.

Let $w_1, w_2 \in D_{u,v}$ and $\alpha, \beta \in A$.
Using (i) of Definition 4.2, we have

$J_D{}^3(\alpha w_1 + \beta w_2) \geq min\{J_D{}^3(w_1), J_D{}^3(w_2)\} \geq u$,

$K_D{}^3(\alpha w_1 + \beta w_2) \leq max\{K_D{}^3(w_1), K_D{}^3(w_2)\} \leq v.$

So, $\alpha w_1 + \beta w_2 \in D_{u,v}$.

Let $w \in D_{u,v} \subset T$.

By Definition 4.4, we get

$u \leq J_D{}^3(w) \leq J_D{}^3(w_{i2})$ and $v \geq K_D{}^3(w) \geq K_D{}^3(w_{i2})$, then $w_{i2} \in D_{u,v}$.

Hence, we will have $\nabla(D_{u,v}) \subset T \otimes D_{u,v}$ is a left coideal of T.

Necessary Part:

Let $w_1, w_2 \in T$. We have that $J_D{}^3(w_1) = u$, $K_D{}^3(w_1) = v$ and $J_D{}^3(w_2) = u'$, $K_D{}^3(w_2) = v'$.

Hence $u, v, u', v' \in L$ such that $u + v \in L$, $u' + v' \in L$, $0 \leq u, u' \leq J_D{}^3(0)$,

$J_D{}^3(0) \leq v, v' \leq 1$. Then $w_1 \in D_{u,v}$ and $w_2 \in D_{u',v'}$.

Without loss of generality, we assume one two cases;

(i) $u \leq u'$ and $v \leq v'$. Hence $J_D{}^3(w_1), J_D{}^3(w_2) \geq v$ and $K_D{}^3(w_1), K_D{}^3(w_2) \leq v'$.
Therefore $w_1, w_2 \in D_{u,v'}$.
Since $D_{u,v'}$ is a subspace of T, hence $\alpha w_1 + \beta w_2 \in D_{u,v'}$, for each $\alpha, \beta \in A$.
Thus, $J_D{}^3(\alpha w_1 + \beta w_2) \geq u \Rightarrow min\{u, u'\} = min\{J_D{}^3(w_1), J_D{}^3(w_2)\}$,
$K_D{}^3(\alpha w_1 + \beta w_2) \leq v' \Rightarrow max\{v, v'\} = max\{K_D{}^3(w_1), K_D{}^3(w_2)\}$.

(ii) $u \leq u'$ and $v \leq v'$. Hence $J_D{}^3(w_1), J_D{}^3(w_2) \geq u$ and $K_D{}^3(w_1), K_D{}^3(w_2) \leq v$.
Therefore $w_1, w_2 \in D_{u,v}$.
Since $D_{u,v}$ is a subspace of T, hence $\alpha w_1 + \beta w_2 \in D_{u,v}$, for each $\alpha, \beta \in A$.
Thus, $J_D{}^3(\alpha w_1 + \beta w_2) \geq u \Rightarrow min\{u, u'\} = min\{J_D{}^3(w_1), J_D{}^3(w_2)\}$,
$K_D{}^3(\alpha w_1 + \beta w_2) \leq v \Rightarrow max\{v, v'\} = max\{K_D{}^3(w_1), K_D{}^3(w_2)\}$.

Therefore, D satisfies the condition(1) of Definition 4.2.

To prove condition (2), we apply the assumption that $D_{u,v}$ is a left coideal of T.

We have $\nabla(D_{u,v}) \subset T \otimes D_{u,v}$. Hence $w_{i2} \in D_{u,v}$.

Thus, $J_D{}^3(w_{i2}) \geq u = J_D{}^3(w_1)$, $K_D{}^3(w_{i2}) \leq v = K_D{}^3(w_1)$.

Therefore, D is a fermatean L-fuzzy left coideal of T.

The proof of fermatean L-fuzzy right coideal is similar.

Definition 4.8: Let X and Y be two coalgebras. The linear map $h: X \to Y$ is a morphism of coalgebras, if $\nabla_Y \circ h = (h \otimes h) \circ \nabla_Y$, $\varepsilon_Y \circ h = \varepsilon_X$

$\sum_{i=1}^{n} h(w)_{i1} \otimes h(w)_{i2} = \sum_{j=1}^{m} h(w_{j1}) \otimes h(w_{j2})$.

Preposition4.9: Let $D = (J_D, K_D)$ be fermatean L-fuzzy sub coalgebras(respectively, left/right coideal) of Y and $h: X \to Y$ be a morphism of coalgebras. Then$h^{-1}[D] = (J_{h^{-1}[D]}, K_{h^{-1}[D]})$is fermatean L-fuzzy sub coalgebras(respectively, left/right coideal) of X.

Proof: Let $w_1, w_2 \in X$ and $\alpha, \beta \in A$. Then

$$J_{h^{-1}[D]}{}^3(\alpha w_1 + \beta w_2) = J_D{}^3(h(\alpha w_1 + \beta w_2))$$

$$= J_D{}^3(\alpha h(w_1) + \beta h(w_2))$$

$$\geq min\{{J_D}^3(h(w_1)), {J_D}^3(h(w_2))\}$$

$$= min\{{J_{h^{-1}[D]}}^3(w_1), {J_{h^{-1}[D]}}^3(w_2)\}$$

$${K_{h^{-1}[D]}}^3(\alpha w_1 + \beta w_2) = {K_D}^3(h(\alpha w_1 + \beta w_2))$$

$$= {K_D}^3(\alpha h(w_1) + \beta h(w_2))$$

$$\leq max\{{K_D}^3(h(w_1)), {K_D}^3(h(w_2))\}$$

$$= max\{{K_{h^{-1}[D]}}^3(w_1), {K_{h^{-1}[D]}}^3(w_2)\}$$

Let $\nabla_X(w) = \sum_{j=1}^{m} w_{j1} \otimes w_{j2}$.

We have $\nabla_Y(h(w)) = \sum_{i=1}^{n} h(w)_{i1} \otimes h(w)_{i2} = \sum_{j=1}^{m} w_{j1} \otimes w_{j2}$.

$${J_{h^{-1}[D]}}^3(w) = {J_D}^3(h(w))$$

$$\leq min\{{J_D}^3(h(w)_{i1}), {J_D}^3(h(w)_{i2})\}$$

$$= min\left\{{J_D}^3\left(h(w_{j1})\right), {J_D}^3\left(h(w_{j2})\right)\right\}$$

$= min\{{J_{h^{-1}[D]}}^3(w_{j1}), {J_{h^{-1}[D]}}^3(w_{j2})\}$.

Similarly, we will prove ${K_{h^{-1}[D]}}^3(w) \geq max\{{K_{h^{-1}[D]}}^3(w_{j1}), {K_{h^{-1}[D]}}^3(w_{j2})\}$

Preposition4.10: Let $A = (J_A, K_A)$ be fermatean L-fuzzy sub coalgebras (respectively, left/right coideal) of X and $h: X \to Y$ be a morphism of coalgebras. Then $h[A] = (J_{h[A]}, K_{h[A]})$ is fermatean L-fuzzy sub coalgebras(respectively, left/right coideal) of Y.

Proof: Let $w_1, w_2 \in Y$. We can see that

$$\left.\begin{array}{l} {J_{h[A]}}^3(w_1 + w_2) \geq min\{{J_{h[A]}}^3(w_1), {J_{h[A]}}^3(w_2)\} \\ {K_{h[A]}}^3(w_1 + w_2) \leq max\{{K_{h[A]}}^3(w_1), {K_{h[A]}}^3(w_2)\} \end{array}\right\} \quad \rightarrow(1)$$

If $h^{-1}(w_1) = \emptyset$ and $h^{-1}(w_2) = \emptyset$, then

$0 = min\{{J_{h[A]}}^3(w_1), {J_{h[A]}}^3(w_2)\}$ and

$1 = max\{{K_{h[A]}}^3(w_1), {K_{h[A]}}^3(w_2)\}$.

So, equation (1) holds.

Now we assume that there exists $u, v \in X$ such that $h(u) = w_1$, $h(v) = w_2$. So, by linearity of h, we have $h(u + v) = w_1 + w_2$. Hence $u + v \in h^{-1}(w_1 + w_2)$ and

we can have,

$${J_{h[A]}}^3(w_1 + w_2) = Sup\{{J_A}^3(w_3)/w_3 \in h^{-1}(w_1 + w_2)\}$$

$$\geq Sup\{{J_A}^3(u + v)/u + v \in h^{-1}(w_1 + w_2)\}$$

$$\geq Sup\{(min\{{J_A}^3(u), {J_A}^3(v)\})/u + v \in h^{-1}(w_1 + w_2)\}$$

$$\geq min\left\{Sup\{{J_A}^3(u)/u \in h^{-1}(w_1)\}, Sup\{{J_A}^3(v)/u \in h^{-1}(w_2)\}\right\}$$

$\geq min\{{J_{h[A]}}^3(w_1), {J_{h[A]}}^3(w_2)\}$.

$${K_{h[A]}}^3(w_1 + w_2) = inf\{{K_A}^3(w_3)/w_3 \in h^{-1}(w_1 + w_2)\}$$

$$\leq \inf\{K_A{}^3(u+v)/u+v \in h^{-1}(w_1+w_2)\}$$
$$\leq Sup\{(max\{K_A{}^3(u), K_A{}^3(v)\})/u+v \in h^{-1}(w_1+w_2)\}$$
$$\leq max\{Sup\{K_A{}^3(u)/u \in h^{-1}(w_1)\}, Sup\{K_A{}^3(v)/u \in h^{-1}(w_2)\}\}$$
$$\leq max\{K_{h[A]}{}^3(w_1), K_{h[A]}{}^3(w_2)\}.$$

Let $w_1 \in Y$ and $\alpha \in A$.

For each $w_3 \in h^{-1}(w_1)$, we get $h(\alpha w_3) = \alpha h(w_3) = \alpha w_1$.

So, $J_{h[A]}{}^3(\alpha w_1) = Sup\{J_A{}^3(z)/z \in h^{-1}(\alpha w_1)\}$

$$\geq Sup\{J_A{}^3(\alpha w_3)/\alpha w_3 \in h^{-1}(\alpha w_1)\}$$
$$\geq Sup\{J_A{}^3(w_3)/w_3 \in h^{-1}(w_1)\}$$
$$= J_{h[A]}{}^3(w_1)$$

$$K_{h[A]}{}^3(\alpha w_1) = \inf\{K_A{}^3(z)/z \in h^{-1}(\alpha w_1)\}$$

$$\leq \inf\{K_A{}^3(\alpha w_3)/\alpha w_3 \in h^{-1}(\alpha w_1)\}$$
$$\leq \inf\{K_A{}^3(w_3)/w_3 \in h^{-1}(w_1)\}$$

$= K_{h[A]}{}^3(w_1)$.

Let $w_1 \in Y$ and $\nabla_Y(w) = \sum_{j=1}^m w_{j1} \otimes w_{j2}$.

We claim that

$$\left.\begin{array}{l} J_{h[A]}{}^3(w_1) \leq min\{J_{h[A]}{}^3(w_{j1}), J_{h[A]}{}^3(w_{j2})\} \\ K_{h[A]}{}^3(w_1) \geq max\{K_{h[A]}{}^3(w_{j1}), K_{h[A]}{}^3(w_{j2})\}\end{array}\right\} \quad \rightarrow(2)$$

If $h^{-1}(w_1) = \emptyset$, then $0 = J_{h[A]}{}^3(w_1)$ and $1 = K_{h[A]}{}^3(w_1)$.

So, equation (2) holds. Then there exists $w_3 \in Y$ such that $h(w_3) = w_1$.

Since, $\sum_{j=1}^m w_{j1} \otimes w_{j2} = \nabla_Y(w_1) = \nabla_Y(h(w_3))$

$$= (h \otimes h)\nabla_X(w_3)$$
$$= \sum_{i=1}^n h(w_{i1}) \otimes h(w_{i2}).$$

So $w_{i1} \in h^{-1}(w_{j1})$ and $w_{i2} \in h^{-1}(w_{j2})$. Hence

$$J_{h[A]}{}^3(w_1) = Sup\{J_A{}^3(w_3)/w_3 \in h^{-1}(w_1)\}$$
$$\geq Sup\{min\{J_A{}^3(w_{i1}), J_A{}^3(w_{i2})\}/w_{i1} \in h^{-1}(w_{j1}), w_{i2} \in h^{-1}(w_{j2})\}$$
$$= Sup\{min\{J_A{}^3(w_{i1})/w_{i1} \in h^{-1}(w_{j1})\}, min\{J_A{}^3(w_{i2})/w_{i2} \in h^{-1}(w_{j2})\}\}$$
$$= min\{J_{h[A]}{}^3(w_{j1}), J_{h[A]}{}^3(w_{j2})\}.$$

Other result can be obtained similar.

Therefore, $h[A] = (J_{h[A]}, K_{h[A]})$ is fermatean L-fuzzy sub coalgebra of Y.

The case of fermatean L-fuzzy left(right) coideal can be proved in similar manner.

Let (T, ∇, ε) be a d −algebra. We define the map $B: T^* \rightarrow T^* \otimes T^*$, $B = \nabla \circ \rho$, where ρ is expected as $\rho: T^* \otimes T \rightarrow (T \otimes T^*)$ by

$$\rho(h_1 \otimes h_2)(w_1 \otimes w_2) = h_1(w_1)h_2(w_2)$$

and $U: d \to T^*$ by $U = \varepsilon^* \circ \varphi$ where $\varphi: d \to d^*$ is the canonical isomorphism. Therefore, we have (T^*, B, U) is an algebra

The multiplication is denoted by $B(h_1 \otimes h_2) = h_1 * h_2$ and we have defined by

$$(h_1 * h_2)(w) = \sum_{i=1}^{n} h_1(w_{i1}) \otimes h_2(w_{i2})$$

Preposition4.11: Let (T, ∇, ε) be a coalgebra.

(i) Let $D = (J_D, K_D)$ be fermatean L-fuzzy sub coalgebra of X. Then $D^* = (J_{D^*}, K_{D^*})$is fermatean L-fuzzy ideal of T^*.

(i) Let $D = (J_D, K_D)$ be fermatean L-fuzzy left(right) coideal of X. Then$D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy left(right) coideal of T^*.

Proof: (i) Using proposition-4.1, $D^* = (J_{D^*}, K_{D^*})$ is a fermatean L-fuzzy vector space of T^*.

We will show that ${K_{D^*}}^3(h_1 * h_2) \leq min\{{K_{D^*}}^3(h_1), {K_{D^*}}^3(h_2)\}$ for any h_1,$h_2 \in T^*$.

Let $h_1 \neq 0$ and $h_2 \neq 0$. We have

$${K_{D^*}}^3(h_1 * h_2) = 0.5 - 0.5\ inf\{{K_D}^3(w)/w \in V, (h_1 * h_2)(w) \neq 0\}$$

$$= 0.5 - 0.5\ inf\left\{min\{{K_D}^3(w_{i1}), {K_D}^3(w_{i2})\}/w_{i1}, w_{i2} \in V, \sum_{i=1}^{n} h_1(w_{i1})h_2(w_{i2}) \neq 0\right\}$$

$$= 0.5 - 0.5\ inf\{min\{{K_D}^3(w_{i1}), {K_D}^3(w_{i2})\}/w_{i1}, w_{i2} \in V, h_1(w_{i1})h_2(w_{i2}) \neq 0\}$$

$$= 0.5 - 0.5\ inf\{min\{{K_D}^3(w_{i1}), {K_D}^3(w_{i2})\}/w_{i1}, w_{i2} \in V, h_1(w_{i1}) \neq 0, h_2(w_{i2}) \neq 0\}$$

$$\leq 0.5 - 0.5\ min\begin{Bmatrix} inf\{{K_D}^3(w_{i1})/w_{i1} \in V, h_1(w_{i1}) \neq 0\}, \\ inf\{{K_D}^3(w_{i2})/w_{i2} \in V, h_2(w_{i2}) \neq 0\} \end{Bmatrix}$$

$$\leq 0.5 - 0.5\ min\begin{Bmatrix} inf\{{K_D}^3(l)/l \in V, h_1(l) \neq 0\}, \\ inf\{{K_D}^3(m)/m \in V, h_2(m) \neq 0\} \end{Bmatrix}$$

$\leq min\{{K_{D^*}}^3(h_1), {K_{D^*}}^3(h_2)\}$.

Let $h_1 = 0$ and $h_2 \neq 0$. Then we have,

$$(h_1 * h_2)(w) = \sum_{i=1}^{n} h_1(w_{i1})h_2(w_{i2}) = 0$$

Thus,${K_{D^*}}^3(h_1 * h_2) = {K_{D^*}}^3(0) = {J_{D^*}}^3(h_1)$

$$= 0.5 - 0.5\ sup\{{K_{D^*}}^3(w)/w \in V\}$$

$$\leq 0.5 - 0.5\ sup\{{K_{D^*}}^3(w)/w \in V, h_2(w) \neq 0\}$$
$$\leq 0.5 - 0.5\ inf\{{K_{D^*}}^3(w)/w \in V, h_2(w) \neq 0\}$$
$$= {K_{D^*}}^3(h_2)$$

Hence, we have, ${K_{D^*}}^3(h_1 * h_2) \leq min\{{K_{D^*}}^3(h_1), {K_{D^*}}^3(h_2)\}$.

Let $h_1 = 0$ and $h_2 = 0$.

Then we have, ${K_{D^*}}^3(h_1 * h_2) = {K_{D^*}}^3(0) = {K_{D^*}}^3(h_1) = {J_{D^*}}^3(h_2)$ and the result is proved.

We can show similarly, ${J_{D^*}}^3(h_1 * h_2) \geq max\{{J_{D^*}}^3(h_1), {J_{D^*}}^3(h_2)\}$ for any $h_1, h_2 \in T^*$.

Hence, $D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy ideal of T^*.

(ii) In a similar way, we can prove it.

Let the d-algebra (A, P, U) be finite dimensional algebra. We define the mappings $\nabla: A^* \to A^* \otimes A^*$ and $\epsilon: A^* \to d$ by $\nabla = \rho^{-1} \circ P^*$ and $\epsilon = \varphi_2 \circ U^*$ where $\rho: A^* \otimes A^* \to (A \otimes A)^*$ by $\rho(h_1 \otimes h_2)(w_1 \otimes w_2) = h_1(w_1)h_2(w_2)$ and $\varphi_2: d^* \to d$ by $\varphi_2(h_1) = 1$. Then (A^*, ∇, ϵ).

Preposition4.12: Let (A, P, U) be finite dimensional algebra.

(i) Let $D = (J_D, K_D)$ be fermatean L-fuzzy ideal of A. Then $D^* = (J_{D^*}, K_{D^*})$is fermatean L-fuzzy sub coalgebra of A^*.

(ii) Let $D = (J_D, K_D)$ be fermatean L-fuzzy left(right) ideal of A. Then$D^* = (J_{D^*}, K_{D^*})$ is fermatean L-fuzzy left(right) coideal of A^*.

Proof: (i) Using proposition-4.1, $D^* = (J_{D^*}, K_{D^*})$ is a fermatean L-fuzzy vector subspace of T^*. Let $\nabla(h) = \sum_{j=1}^{n} h_{j1} \otimes h_{j2}$,We will show that

${K_{D^*}}^3(h) \geq max\{{K_{D^*}}^3(h_{j1}), {K_{D^*}}^3(h_{j2})\}$.

Let $h \neq 0$. Then we get

$$
\begin{aligned}
{K_{D^*}}^3(h) &= 0.5 - 0.5\, inf\{{K_D}^3(w)/w \in A, h(w) \neq 0\} \\
&\geq 0.5 - 0.5\, inf\{{K_D}^3(lm)/lm \in A, h(lm) \neq 0\} \\
&\geq 0.5 - 0.5\, inf\left\{max\{{K_D}^3(l), {K_D}^3(m)\}/lm \in V, \sum_{j=1}^{n} h_{j1}(l) \otimes h_{j2}(m) \neq 0\right\} \\
&\geq 0.5 - 0.5\, inf\{max\{{K_D}^3(l), {K_D}^3(m)\}/lm \in V, h_{j1}(l)h_{j2}(m) \neq 0\} \\
&\geq 0.5 - 0.5\, min\begin{Bmatrix} inf\{{K_D}^3(l)/l \in V, h_{j1}(l) \neq 0\}, \\ inf\{{K_D}^3(m)/m \in V, h_{j2}(m) \neq 0\} \end{Bmatrix}
\end{aligned}
$$

If $inf\{{K_D}^3(l)/l \in V, h_{j1}(l) \neq 0\} \geq inf\{{K_D}^3(m)/m \in V, h_{j2}(m) \neq 0\}$, then

${K_{D^*}}^3(h_{j1}) \leq {K_{D^*}}^3(h_{j2})$.

Hence ${K_{D^*}}^3(h) \geq {K_{D^*}}^3(h_{j2})$.

If $inf\{{K_D}^3(l)/l \in V, h_{j1}(l) \neq 0\} \leq inf\{{K_D}^3(m)/m \in V, h_{j2}(m) \neq 0\}$, then

${K_{D^*}}^3(h_{j1}) \geq {K_{D^*}}^3(h_{j2})$. Hence ${K_{D^*}}^3(h) \geq {K_{D^*}}^3(h_{j1})$.

Therefore, ${K_{D^*}}^3(h) \geq max\{{K_{D^*}}^3(h_{j1}), {K_{D^*}}^3(h_{j2})\}$.

Let $h = 0$. Then we get, $h_{j1} = h_{j2} = 0$. So,${K_{D^*}}^3(h) \geq max\{{K_{D^*}}^3(h_{j1}), {K_{D^*}}^3(h_{j2})\}$.

We can also show that, ${J_{D^*}}^3(h) \leq min\{{J_{D^*}}^3(h_{j1}), {J_{D^*}}^3(h_{j2})\}$.

The proof is completed.

(ii) This will be proved similarly to (i).

Definition 4.13: Let $(H, M, \cup, \nabla, \epsilon)$ be a d-algebra. If $D = (J_D, K_D)$ is either a fermatean L-fuzzy subalgebra or fermatean L-fuzzy sub coalgebra of H, then it is called a fermatean L-fuzzy sub algebra of H. Let $Q: H \to H$ be the antipode of H. Then for each, $w \in H$, we get

$$(Q \otimes F)\nabla(w) = (F \otimes Q)\nabla(w) = \cup\left(\epsilon(w)\right).$$

If,

$$K_D{}^3\left(\sum_{(w)} Q(w_{i1})w_{i2}\right) = K_D{}^3\left(\sum_{(w)} w_{i1}Q(w_{i2})\right) = K_D{}^3\left(\cup\left(\epsilon(w)\right)\right)$$

and

$$J_D{}^3\left(\sum_{(w)} Q(w_{i1})w_{i2}\right) = J_D{}^3\left(\sum_{(w)} w_{i1}Q(w_{i2})\right) = J_D{}^3\left(\cup\left(\epsilon(w)\right)\right)$$

then, $D = (J_D, K_D)$ is called a fermatean L-fuzzy Hopf subalgebra of H.

Example4.14: Let A be the algebra generated by an inverse element x and an element y such that $y^n = 0$ and $xy = \lambda yx$ where λ is a primitive 2^n-th root of unity. It is explained that A is a Hoph algebra with the comultiplication, counit and antipode explained as

$\nabla(x) = x \otimes x, \nabla(y) = x \otimes y + y \otimes x^{-1}, \epsilon(x) = 1, \epsilon(y) = 0.$

$Q(x) = x^{-1}, Q(y) = -x^{-1}yx.$

"Let $C = \langle x \rangle$ be an infinite cyclic group and $x^L(x) = \sqrt{\lambda}$. Clearly $A \approx H(T, \cap, x^L, x^R)$.

So A has the properties of one extension Hopf algebras of the form $H(T, \cap, Z, Z^*, x, y)$".

Now we define a fermatean L-fuzzy subset $D = (J_D, K_D)$ of A given by

$J_D{}^3(x) = J_D{}^3(x^{-1}) = 0.4.$

$K_D{}^3(x) = K_D{}^3(x^{-1}) = 0.5$

$J_D{}^3(y) = 0.3, K_D{}^3(y) = 0.7,$

$J_D{}^3(1) = 1, K_D{}^3(1) = 0, J_D{}^3(0) = 0.6, K_D{}^3(0.1) = 0.4.$

It is easy to prove that $D = (J_D, K_D)$ is fermatean L-fuzzy sub bi algebra of A.

Also, $J_D{}^3\left((Q \otimes F)\nabla(x)\right) = J_D{}^3\left((Q \otimes F)(x \otimes x)\right)$

$$= J_D{}^3(Q(x)x)$$

$$= J_D{}^3(x^{-1}x)$$

$$= J_D{}^3(1) = 1$$

$= J_D{}^3\left(\cup\left(\epsilon(x)\right)\right),$

$$J_D{}^3\left((Q \otimes F)\nabla(y)\right) = J_D{}^3\left((Q \otimes F)(x \otimes y + y \otimes x^{-1})\right)$$

$$= J_D{}^3(Q(x)y + Q(y)x^{-1})$$

$$= J_D{}^3(x^{-1}y + (-x^{-1}yx)x^{-1})$$

$$= J_D{}^3(0) = 0.6$$

$= J_D{}^3\left(\cup\left(\epsilon(y)\right)\right)$.

Similarly, we will show that $J_D{}^3\big((F \otimes Q)\nabla(w)\big) = J_D{}^3\left(\cup\left(\epsilon(x)\right)\right)$, $\forall\, w \in A$.

Hence, $D = (J_D, K_D)$ satisfies the condition (4.7) and similarly satisfies (3.6).

Therefore $D = (J_D, K_D)$ is frematean L-fuzzy Hopf subalgebra of A.

Example4.15: Let H be a d-vector space with basis$\{a_i\}_{i=0}^{\infty}$. Define two d-linear mappings $M: H \otimes H \to H$ and $U: d \to H$ by $M(a_i \otimes a_j) = a_i a_j \forall i, j \geq 0$ and $U(1) = a_0$.Also define two d-linear mappings $\nabla: H \to H \otimes H$ and $\epsilon: H \to d$ by

$$\nabla(a_n) = \sum_{i=0}^{n} a_i \otimes a_{n-i}\,, \epsilon(a_n) = \delta_{n,0}\ \forall n \geq 0.$$

And define the antipode $Q: H \to H^*$, $Q(a_n) = \delta_{n,0} a_n$. It is showed in $(H, M, U, \nabla, \epsilon, T)$ is a Hopf algebra.

Let $F = [0,1]$. We define frematean L-fuzzy subset $D = (J_D, K_D)$ of H by

$$J_D{}^3(a_n) = \frac{1}{n+1}\,, K_D{}^3(a_n) = \frac{n}{n+1}, \forall n \geq 0.$$

$J^3(\alpha w_1 + \beta w_2) = max\{J^3(w_1), J^3(w_2)\}$,

$$K^3(\alpha w_1 + \beta w_2) = min\{K^3(w_1), K^3(w_2)\},$$

$$J^3(w_1 w_2) = min\{J^3(w_1), J^3(w_2)\}$$

$K^3(w_1 w_2) = min\{K^3(w_1), K^3(w_2)\}$, for all $w_1, w_2 \in H$

and for all $\alpha, \beta \in d$.

Therefore D is frematean L-fuzzy subalgebra of H.

Since, we have $i = 0,1,2, \ldots, \infty$;

$$\frac{1}{n+1} \leq min\left\{\frac{1}{i+1}\,, \frac{1}{n-i+1}\right\}$$

$$\Rightarrow J_D{}^3(a_n) \leq min\{J_D{}^3(a_i), J_D{}^3(a_{n-i})\}$$

$$\Rightarrow J_D{}^3(a_n) \leq min\{J_D{}^3(w_{i1}), J_D{}^3(w_{i2})\}.$$

$$\frac{n}{n+1} \geq max\left\{\frac{i}{i+1}\,, \frac{n}{n-i+1}\right\}$$

$$\Rightarrow K_D{}^3(a_n) \geq max\{K_D{}^3(a_i), K_D{}^3(a_{n-i})\}$$

$$\Rightarrow K_D{}^3(a_n) \geq min\{K_D{}^3(w_{i1}), K_D{}^3(w_{i2})\}.$$

Hence D is frematean L-fuzzy sub coalgebra of H.

Now,

$$J_D{}^3\left(\sum_{i=0}^{n} Q(a_i)a_{n-i}\right) = \bigvee_{i=0}^{n} min\{J_D{}^3(\delta_{i,0,c_i}), J_D{}^3(\delta_{n-i})\}$$

$$= \bigvee_{i=0}^{n} min\left\{1, \frac{1}{n-i+1}\right\} = 1$$

$$= \bigvee_{i=0}^{n} min\left\{\frac{1}{i+1}, 1\right\}$$

$$= \bigvee_{i=0}^{n} min\{J_D{}^3(a_i), J_D{}^3(\delta_{n-i,0,c_{n-i}})\}$$

$$= J_D{}^3\left(\sum_{i=0}^{n} a_i Q(a_{n-i})\right)$$

$$J_D{}^3((U \circ \epsilon)(a_0)) = J_D{}^3\left(U(\delta_{0,0})\right) = J_D{}^3(1) = 1,$$

$$J_D{}^3((U \circ \epsilon)(a_0)) = J_D{}^3\left(U(\delta_{n,0})\right) = J_D{}^3(0) = 1.$$

Hence D satisfies (4.7) and we can similarly show that (4.6) holds.

Hence D is fermatean L-fuzzy Hopf subalgebra of H.

5. Fermatean Lattice-fuzzy Quantum Subset

Let 'q' be an inverse real(complex) number less than 1, and F_q be the two sided ideal of the free algebra $\lambda[w_1, w_2]$ generated by the element $w_2w_1 - qw_1w_2$.

The quantum plane is defined as quotient-algebra $\lambda_q[w_1, w_2] = \frac{\lambda[w_1,w_2]}{F_q}$.

Definition5.1: Let $D = (J_D, K_D)$ be fermatean L-fuzzy Hopf subalgebra of $\lambda[w_1, w_2]$. If is satisfies the condition

$$J_D{}^3(w_2w_1) = J_D{}^3(qw_1w_2)$$

$$K_D{}^3(w_2w_1) = K_D{}^3(qw_1w_2),$$

then D is called fermatean L-fuzzy quantum subspace of $\lambda_q[w_1, w_2]$.

Example5.2: We will consider $H = \lambda[w_1, w_2]$ as a d-vector space with the basis $\{w_1{}^m w_2{}^n\}_{m,n=0}^{\infty}$ and define $M(w_1{}^m w_2{}^n \otimes w_1{}^k w_2{}^l) = w_1{}^{m+k} w_2{}^{n+l}$, $U(1) = x^0y^0 = 1$.

$$\nabla(w_1{}^m w_2{}^n) = \sum_{i=0}^{m}\sum_{j=0}^{n} w_1{}^i w_2{}^j \otimes w_1{}^{m-i} w_2{}^{n-j}, \epsilon(w_1{}^m w_2{}^n) = \delta_{m+n,0}.$$

$$\mathrm{Q}(w_1{}^m w_2{}^n) = \delta_{m+n,0} w_1{}^m w_2{}^n, \forall m, n, k, l \geq 0.$$

Then, similarly to example-4.15, we will prove that $\lambda[w_1, w_2]$ is a Hopf algebra.

Define a fermatean L-fuzzy subset D of H by

$$J_D{}^3(w_1{}^m w_2{}^n) = \frac{1}{m+n+1}, K_D{}^3(w_1{}^m w_2{}^n) = \frac{m+n}{m+n+1}, \forall\, m, n \geq 0.$$

Then,$D = (J_D, K_D)$ is fermatean L-fuzzy Hopf subalgebra of H. Also, we have

$$J_D{}^3(w_1w_2) = 0.33 = J_D{}^3(qw_2w_1)$$

$$K_D{}^3(w_1w_2) = 0.66 = K_D{}^3(qw_2w_1)$$

Hence,D is fermatean L-fuzzy quantum subset of $\lambda_q[w_1, w_2]$.

6. Conclusion

We have discussed a lattice structure of fermatean fuzzy vector space of quantum subspace with respect to Hopf algebras, fermatean L-fuzzy sub algebras, fermatean L-fuzzy sub coalgebras and fermatean L-fuzzy left (right) coideals and the characterizations of fermatean L-fuzzy sub coalgebras under homomorphisms of coalgebras. The duality of fermatean L-fuzzy sub coalgebras and duality of fermatean L-fuzzy sub coalgebras as well as fermatean L-fuzzy sub algebras are studied.

7. Future work

One can obtain the similar concept of quantum theory in various other algebraic structures like Neutrosophic group, complex fuzzy subgroups.

Author Contributions

Conceptualization, Methodology, formal analysis, investigation, writing – original draft (Ramar Bhuvaneshwari). Conceptualization, Methodology, formal analysis, writing, supervision, project administration – original draft (Samidurai Vaiyapuri Manemaran), Validation, Investigation, writing, review and editing-(Samidurai Vaiyapuri Manemaran). Resources, Writing, review and Editing-(Rathinam Nagarajan).

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The authors declare no conflicts of interest.

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