Minimax Theory and its Applications
Volume 4 (2019), No. 1, 1–19

On a Fractional $p\&q$ Laplacian Problem with Critical Growth[*]

Vincenzo Ambrosio
Department of Mathematics, EPFL SB CAMA, Station 8, 1015 Lausanne, Switzerland
vincenzo.ambrosio2@unina.it

Teresa Isernia
Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche 12, 60131 Ancona, Italy
teresa.isernia@unina.it

Gaetano Siciliano
Departamento de Matemática, Universidade de Sao Paulo, Rua do Matao 1010, 05508-090 São Paulo SP, Brazil
sicilian@ime.usp.br

Received: July 8, 2018
Accepted: November 12, 2018

We deal with a class of nonlocal problems of the type

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u = \lambda |u|^{r-2}u + |u|^{q^*_s-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $1 < p < q < N/s$, $(-\Delta)^s_\alpha$, with $\alpha \in \{p,q\}$, is the fractional α-Laplacian, Ω is a bounded domain of $\mathbb{R}^N$ and $\lambda > 0$ is a parameter. Roughly speaking, when r is "large" we prove the existence of a solution for large values of λ and when r is "small" we prove the existence of infinitely many solutions for small values of λ.

Keywords: Fractional Laplacians, variational methods, critical exponent.

1991 Mathematics Subject Classification: 47G20, 35R11, 35A15.

1. Introduction

In [14] the authors considered the following problem involving the critical Sobolev exponent:

$$\begin{cases} -\Delta_p u = \lambda |u|^{r-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $r < p^*$ (the Sobolev critical exponent) and $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. In particular they showed the existence of solutions depending on the values of r and the parameter λ.

---

[*] V. Ambrosio and T. Isernia are partially supported by INdAM-GNAMPA Project 2017 entitled: *Teoria e modelli per problemi non locali.* G. Siciliano is partially supported by Capes, CNPq and Fapesp, Brazil.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50 © Heldermann Verlag

Later, in [18], the authors studied a problem involving the $p\&q$-Laplacian, proving the existence of infinitely many solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{r-2}u + |u|^{q^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

whenever λ is small enough.

In [13] the above results have been extended to a more general class of quasilinear operators of the type $-\nabla\cdot(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)$ under suitable assumptions on the C^1-function a. Indeed the author obtained two results: existence of a nontrivial solution for large $\lambda > 0$ whenever $r \in (q, q^*)$, and infinitely many solutions for $\lambda > 0$ small provided that $r \in (1, q)$.

On the other hand, in recent years great attention have received differential problems involving fractional and nonlocal operators. Indeed it is commonly established that fractional operators give a better description of several models and appears in many physical sciences. For example the fractional Laplacian appears in some problems in Physics and Chemistry, see e.g. [19, 20], in phase transition [1, 26], in material science [4], in the Fractional Quantum Dynamics developed by Laskin in [15, 16, 17], optimization and finance [10, 12], in obstacle problems [21, 25], in conformal geometry and minimal surfaces [6, 7, 8]. The list may continue with crystal dislocation, soft thin films, multiple scattering, anomalous diffusion, quasi-geostrophic flows, water waves, and so on.

Thus, motivated by the previous comments, we asked if the results proved in [14, 13, 18] have a fractional counterpart, and indeed we are able to give a positive answer.

More precisely this paper is concerned with existence and multiplicity results for a differential operator involving the sum of two fractional Laplacians:

$$\begin{cases} (-\Delta)^s_p u + (-\Delta)^s_q u = \lambda |u|^{r-2}u + |u|^{q^*_s-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain in $\mathbb{R}^N$ and

- $s \in (0, 1)$,
- $1 < p < q < q^*_s$, $sq < N$ and $q^*_s = Nq/(N - sq)$ is the critical Sobolev exponent,
- $\lambda > 0$ is a parameter,
- the exponent r varies in the ranges $(1, q)$ and (q, q^*_s).

We recall that for $\alpha \in \{p, q\}$, $(-\Delta)^s_\alpha$ is the α-fractional Laplacian which can be defined, up to a multiplicative constant, by

$$(-\Delta)^s_\alpha u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{\alpha-2}(u(x) - u(y))}{|x - y|^{N+s\alpha}} dy, \quad x \in \mathbb{R}^N,$$

where P.V. stands for *principle value.*

To state our result we introduce few basic objects. $W^{s,\alpha}(\mathcal{D})$ denotes the fractional Sobolev space with norm

$$\|u\|_{W^{s,\alpha}(\mathcal{D})} = |u|_{L^\alpha(\mathcal{D})} + [u]_{W^{s,\alpha}(\mathcal{D})},$$

where $$[u]_{W^{s,\alpha}(\mathcal{D})} = \left(\int_{\mathcal{D}}\int_{\mathcal{D}} \frac{|u(x)-u(y)|^\alpha}{|x-y|^{N+s\alpha}}dxdy\right)^{1/\alpha}$$

is the *Gagliardo semi-norm.* When $\mathcal{D} = \mathbb{R}^N$, we set $[u]_{s,\alpha} = [u]_{W^{s,\alpha}(\mathbb{R}^N)}$ and

$$W_0^{s,\alpha}(\Omega) = \left\{u \in L^\alpha(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \text{ and } [u]_{s,\alpha} < +\infty\right\}$$

with dual space $W^{-s,\alpha}(\Omega)$. Note that it is

$$W_0^{s,\alpha}(\Omega) = \left\{u \in L^{\alpha^*}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega,\ [u]_{s,\alpha} < \infty\right\}$$

and that $[\cdot]_{s,\alpha}$ is actually a norm on $W_0^{s,\alpha}(\Omega)$ for which it is a Banach space.

The nonlinear operator $(-\Delta)^s_\alpha$ can then be seen as $(-\Delta)^s_\alpha \colon W_0^{s,\alpha}(\Omega) \to W^{-s,\alpha'}(\Omega)$, where, as usual, α' is the conjugate exponent of α, acting by duality

$$\langle(-\Delta)^s_\alpha u, v\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\alpha-2}(u(x)-u(y))}{|x-y|^{N+s\alpha}}(v(x)-v(y))\,dx\,dy,$$

where $u, v \in W_0^{s,\alpha}(\Omega)$. This is a variational characterization of $(-\Delta)^s_\alpha$.

The "boundary" condition in (1) is given in the whole complementary of Ω, reflecting the fact that we are dealing with nonlocal operators. For more details on fractional Laplacians the interested reader may consult [11, 22] and the references therein.

For a *weak solution* of problem (1), we mean a function $u \in W_0^{s,q}(\Omega)$ such that

$$\sum_{\alpha\in\{p,q\}} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\alpha-2}(u(x)-u(y))}{|x-y|^{N+s\alpha}}(v(x)-v(y))dxdy$$
$$= \lambda \int_\Omega |u|^{r-2}uvdx + \int_\Omega |u|^{q_s^*-2}uvdx$$

for all $v \in W_0^{s,q}(\Omega)$. Our results can be stated as follows.

Theorem 1.1. *Assume that* $r \in (q, q_s^*)$. *Then there exists* $\lambda^* > 0$ *such that problem* (1) *admits a nontrivial solution for all* $\lambda \in [\lambda^*, +\infty)$.

Theorem 1.2. *Assume that* $r \in (1, q)$. *Then there exists* $\lambda^{**} > 0$ *such that problem* (1) *has infinitely many solutions for all* $\lambda \in (0, \lambda^{**}]$.

To prove our theorems we use Variational Methods and Critical Point Theory. Indeed the solutions of (1) can be seen as critical points of a suitable C^1 functional, said *energy functional*, defined on $W_0^{s,q}(\Omega)$. *En passant,* we anticipate here that the solution found in Theorem 1.1 is at positive level of the energy functional in contrast to the solutions given in Theorem 1.2 which are at negative levels.

The main difficulty of our paper is exactly due to the critical growth of the nonlinearity which implies that the functional does not satisfy the Palais-Smale condition at every level. To this respect we recall once for all the basic definition. We say that a C^1 functional $\mathcal{I}$ defined on the Banach space $\mathcal{X}$ satisfies the Palais-Smale condition at level c (we also write $(PS)_c$ for short) if every sequence $(u_n) \subset \mathcal{X}$ such that

$$\mathcal{I}(u_n) \to c \in \mathbb{R} \quad \text{and} \quad \mathcal{I}'(u_n) \to 0$$

admits a convergent subsequence. If this condition holds for c in a certain interval $I \subsetneq \mathbb{R}$ we say also that $\mathcal{I}$ satisfies a *local* (PS) *condition.* It is known that this compactness condition is a fundamental tool in order to apply Variational Methods and the Ljusternick-Schnirelmann Theory and obtain existence and multiplicity of critical points of functionals. Then our main work consists in proving that, once we have candidates to critical levels, they are actually in a range where the Palais-Smale condition holds.

We remark that, to our knowledge, in the literature there are only two papers [3, 9] dealing with nonlocal problems in $\mathbb{R}^N$ involving the sum of fractional p-Laplacians. For this reason, we would like to go further in this direction, but considering now fractional $p\&q$ problems in bounded domains.

The paper is organised as follows.

In Section 2 we give few preliminaries introducing the variational framework for the problem. In Section 3 , after ensuring a local (PS) condition, we give the proof of Theorem 1.1 by using the Mountain Pass Theorem. In Section 4, by using the Ljusternick-Schnirelmann theory we show the existence of infinitely many solutions proving Theorem 1.2.

Notation. We use the following notations:

- We denote by $C_1, C_2, \dots$ or $C, C', C'', \dots$ suitable but irrelevant positive constants which may change also from line to line;
- We use the standard notation $|u|_r^r = \displaystyle\int_\Omega |u|^r dx$ for the L^r-norm of u.
- $a_n \rightharpoonup a$ and $a_n \to a$ mean the weak and strong convergence as $n \to \infty$.
- $S_* = \inf\{[u]_{s,q}^q : u \in W_0^{s,q}(\Omega), |u|_{q_s^*} = 1\}$.

2. Preliminaries

Before introducing the variational setting in which we study our problem, we can observe that:

Lemma 2.1. *If $p \le q$ then $W_0^{s,q}(\Omega) \subseteq W_0^{s,p}(\Omega)$ with continuous embedding.*

Proof. Take $u \in W_0^{s,q}(\Omega)$. Let us note that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy = \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + 2\int_{\Omega}\int_{\Omega^c} \frac{|u(x)|^p}{|x-y|^{N+sp}} dxdy.$$

Since $W^{s,q}(\Omega) \subseteq W^{s,p}(\Omega)$, we can see that

$$\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \leq C[u]^p_{W^{s,q}(\Omega)}. \tag{2}$$

Now, $\Omega^c = \Omega_1 \cup \Omega_1^c$, where $\Omega_1 = \{x \in \Omega^c : dist(x, \partial\Omega) \leq 1\}$. Then we have

$$\begin{aligned}\int_{\Omega}\int_{\Omega^c} \frac{|u(x)|^p}{|x-y|^{N+sp}} dxdy &= \int_{\Omega}\int_{\Omega_1} \frac{|u(x)|^p}{|x-y|^{N+sp}} dxdy + \int_{\Omega}\int_{\Omega_1^c} \frac{|u(x)|^p}{|x-y|^{N+sp}} dxdy \\ &=: I_1 + I_2. \end{aligned}\tag{3}$$

Regarding I_1 we can see that

$$\begin{aligned} I_1 &= \int_{\Omega}\int_{\Omega_1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \leq \int_{\Omega_1}\int_{\Omega_1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \\ &\leq C[u]^p_{W^{s,q}(\Omega_1)} \leq C[u]^p_{s,q}. \end{aligned}\tag{4}$$

Concerning I_2, we can use a change of variable to infer that

$$I_2 \leq \int_{\Omega}\int_{B_1^c} \frac{|u(x)|^p}{|z|^{N+sp}} dzdx \leq C|u|_p^p \leq C|u|_q^p. \tag{5}$$

Putting together (2) – (5) we get the desired result. □

It is easily seen that solutions of (1) can be found as critical critical points of the following C^1 functional $\mathcal{J}_\lambda : W_0^{s,q}(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{p}[u]^p_{s,p} + \frac{1}{q}[u]^q_{s,q} - \frac{\lambda}{r}|u|_r^r - \frac{1}{q_s^*}|u|_{q_s^*}^{q_s^*}.$$

The space $W_0^{s,q}(\Omega)$ is endowed with the norm

$$\|u\| = [u]_{s,p} + [u]_{s,q}$$

which, by Lemma 2.1, is equivalent to $[\cdot]_{s,q}$. We observe that $\mathcal{J}_\lambda$ is unbounded below on $W_0^{s,q}(\Omega)$, since fixed $v \in W_0^{s,q}(\Omega) \setminus \{0\}$, we have

$$\mathcal{J}_\lambda(tv) = \frac{t^p}{p}[v]^p_{s,p} + \frac{t^q}{q}[v]^q_{s,q} - \lambda\frac{t^r}{r}|v|_r^r - \frac{t^{q_s^*}}{q_s^*}|v|_{q_s^*}^{q_s^*} \longrightarrow -\infty \quad \text{as } t \to \infty. \tag{6}$$

Throughout the paper we will frequently make use of the following results.

Theorem 2.2. (see [11]) *$W_0^{s,p}(\Omega)$ is continuously embedded into $L^t(\mathbb{R}^N)$ for any $t \in [p, p_s^*]$ and compactly into $L^t(\Omega)$ for any $t \in [1, p_s^*)$.*

Lemma 2.3. (see [3]) *Let* $(u_n) \subset D^{s,p}(\mathbb{R}^N)$ *be a bounded sequence and assume* $\psi \in C^\infty_c(\mathbb{R}^N)$ *such that* $0 \leq \psi \leq 1$, $\psi = 1$ *in* B_1, $\psi = 0$ *in* B_2^c *with* $|\nabla \psi|_\infty \leq 2$. *Set* $\psi_\rho(x) = \psi(\frac{x-\overline{x}}{\rho})$ *where* $\overline{x} \in \mathbb{R}^N$ *is a fixed point. Then we have*

$$\lim_{\rho\to 0}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} |u_n(x)|^p \frac{|\psi_\rho(x)-\psi_\rho(y)|^p}{|x-y|^{N+sp}}\, dxdy = 0.$$

3. Proof of Theorem 1.1

In this section we provide the proof of Theorem 1.1 by using the Mountain Pass Theorem [2].

Lemma 3.1. *For each* $\lambda > 0$ *the functional* $\mathcal{J}_\lambda$ *satisfies the following conditions:*

(i) *there exist* $\alpha, \beta > 0$ *such that* $\mathcal{J}_\lambda(u) \geq \beta$ *if* $\|u\| = \alpha$,

(ii) *there exists* $e \in W^{s,q}_0(\Omega)$ *such that* $\|e\| > \alpha$ *and* $\mathcal{J}_\lambda(e) < 0$.

Proof. Let us note that making use of Theorem 2.2 it follows that

$$\begin{aligned}\mathcal{J}_\lambda(u) &= \frac{1}{p}[u]^p_{s,p} + \frac{1}{q}[u]^q_{s,q} - \frac{\lambda}{r}|u|^r_r - \frac{1}{q^*_s}|u|^{q^*_s}_{q^*_s} \\ &\geq C_1\|u\|^q - \frac{\lambda}{r}|u|^r_r - \frac{1}{q^*_s}|u|^{q^*_s}_{q^*_s} \geq C_1\|u\|^q - \lambda C_2\|u\|^r - C_3\|u\|^{q^*_s}.\end{aligned}$$

Since $r \in (q, q^*_s)$ there exist $\alpha, \beta > 0$ such that $\mathcal{J}_\lambda(u) \geq \beta$ for all $u \in W^{s,q}_0(\Omega)$ such that $\|u\| = \alpha$ and (i) is proved. Item (ii) follows by (6). □

In view of Lemma 3.1 we can define the Mountain Pass value

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], W^{s,q}_0(\Omega)) : \gamma(0) = 0 \text{ and } \mathcal{J}_\lambda(\gamma(1)) < 0\}$.

Since we are dealing with the critical exponent, the Palais-Smale condition is not satisfied at every level; indeed we have the following local Palais-Smale condition.

Lemma 3.2. *For every* $c < \left(\frac{1}{r} - \frac{1}{q^*_s}\right) S_*^{N/sq}$ *the functional* $\mathcal{J}_\lambda$ *satisfies the* $(PS)_c$ *condition.*

Proof. Let $(u_n) \subset W^{s,q}_0(\Omega)$ be a sequence such that

$$\mathcal{J}_\lambda(u_n) \to c \text{ and } \mathcal{J}'_\lambda(u_n) \to 0. \tag{7}$$

We divide the proof in several steps.

Step 1: There exists $u \in W^{s,q}_0(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{s,q}_0(\Omega)$.

By using (7) and the fact that $p < q < r < q_s^*$ we have

$$\begin{aligned}C(1+\|u_n\|) &\geq \mathcal{J}_\lambda(u_n) - \frac{1}{r}\langle \mathcal{J}_\lambda'(u_n), u_n\rangle \\ &geq \left(\frac{1}{p}-\frac{1}{r}\right)[u_n]_{s,p}^p + \left(\frac{1}{q}-\frac{1}{r}\right)[u_n]_{s,q}^q + \left(\frac{1}{r}-\frac{1}{q_s^*}\right)|u_n|_{q_s^*}^{q_s^*} \\ &\geq \left(\frac{1}{q}-\frac{1}{r}\right)([u_n]_{s,p}^p + [u_n]_{s,q}^q) \geq C'\|u_n\|^p\end{aligned}$$

which gives the boundedness of (u_n). Eventually passing to a subsequence Step 1 is proved.

Step 2: $u_n \to u$ in $L^{q^*}(\Omega)$.

We may assume that $\quad u_n \to u \quad$ in $L^t(\Omega)$ for all $t \in [1, q_s^*)$.

By applying [23, Theorem 2.5] we can see that there exist two Borel regular measures μ and ν, Λ denumerable, $(x_i) \subset \overline{\Omega}$, $\nu_i \geq 0$, $\mu_i \geq 0$ with $\mu_i + \nu_i > 0$ for all $i \in \Lambda$ such that

$$|D^s u_n|^q \rightharpoonup \mu \text{ and } |u_n|^{q_s^*} \rightharpoonup \nu, \quad \text{and} \tag{8}$$

$$d\nu = |u|^{q_s^*} + \sum_{i\in\Lambda}\nu_i\delta_{x_i}, \quad d\mu \geq |D^s u|^q + \sum_{i\in\Lambda}\mu_i\delta_{x_i}, \quad S_*\nu_i^{q/q_s^*} \leq \mu_i \quad \forall i \in \Lambda. \tag{9}$$

We aim to show that $\nu_i = 0$ for all $i \in \Lambda$.

Assume by contradiction that x_i is a singular point of measures μ and ν. For any $\rho > 0$, we set $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in B_1 and $\psi = 0$ in B_2^c and $|\nabla\psi|_\infty \leq 2$. Since $(u_n\psi_\rho)$ is bounded in $W_0^{s,q}(\Omega)$, we get $\langle \mathcal{J}_\lambda'(u_n), u_n\psi_\rho\rangle = o_n(1)$, or equivalently

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}}(u_n(x)\psi_\rho(x) - u_n(y)\psi_\rho(y))dxdy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+sq}}(u_n(x)\psi_\rho(x) - u_n(y)\psi_\rho(y))dxdy \\ &= \lambda\int_{\mathbb{R}^N}|u_n|^r\psi_\rho dx + \int_{\mathbb{R}^N}|u_n|^{q_s^*}\psi_\rho dx + o_n(1).\end{aligned} \tag{10}$$

Now, we note that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}}(u_n(x)\psi_\rho(x) - u_n(y)\psi_\rho(y))dxdy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\psi_\rho(x)dxdy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}}u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy,\end{aligned}$$

so (10) becomes

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+sq}} u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy \\
&= -\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p}}{|x-y|^{N+sp}} \psi_\rho(x)dxdy - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q}}{|x-y|^{N+sq}} \psi_\rho(x)dxdy \\
&\quad + \lambda \int_{\mathbb{R}^N} |u_n|^r \psi_\rho dx + \int_{\mathbb{R}^N} |u_n|^{q_s^*} \psi_\rho dx + o_n(1). \qquad (11)
\end{aligned}$$

At this point, we can observe that the Hölder inequality and Step 1 give

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy \right| \\
&\leq [u_n]_{s,p}^{p-1} \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\rho(x)-\psi_\rho(y)|^p}{|x-y|^{N+sp}} |u_n(y)|^p dxdy \right)^{1/p} \\
&\leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\psi_\rho(x)-\psi_\rho(y)|^p}{|x-y|^{N+sp}} |u_n(y)|^p dxdy \right)^{1/p},
\end{aligned}$$

so by using Lemma 2.3 we get

$$\begin{aligned}
\lim_{\rho\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} & \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} \\
& \cdot u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy = 0 \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\rho\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} & \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+sq}} \\
& \cdot u_n(y)(\psi_\rho(x)-\psi_\rho(y))dxdy = 0. \qquad (13)
\end{aligned}$$

Clearly $u_n \to u$ in $L^t(\Omega)$ for all $t \in [1, q_s^*)$ and ψ_ρ has compact support, hence

$$\lim_{\rho\to 0} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^r \psi_\rho dx = 0. \qquad (14)$$

Taking into account (11)-(14) and using (8) we have $\nu_i \geq \mu_i$. This and (9) imply $\nu_i \geq S_*^{N/sq}$. Then

$$\begin{aligned}
c &= \mathcal{J}_\lambda(u_n) - \frac{1}{r}\langle \mathcal{J}_\lambda'(u_n), u_n\rangle + o_n(1) \\
&\geq \left(\frac{1}{r} - \frac{1}{q_s^*}\right) \int_{\mathbb{R}^N} |u_n|^{q_s^*} dx + o_n(1) \geq \left(\frac{1}{r} - \frac{1}{q_s^*}\right) \int_{B_\rho(x_i)} \psi_\rho |u_n|^{q_s^*} dx + o_n(1),
\end{aligned}$$

and by passing to the limit as $n \to \infty$ we arrive at

$$c \geq \left(\frac{1}{r} - \frac{1}{q_s^*}\right) \sum_{i \in \Lambda} \psi_\rho(x_i)\nu_i = \left(\frac{1}{r} - \frac{1}{q_s^*}\right) \sum_{i \in \Lambda} \nu_i \geq \left(\frac{1}{r} - \frac{1}{q_s^*}\right) S_*^{N/sq}$$

which contradicts the assumption of Lemma 3.2. As a consequence

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q_s^*} dx = \int_\Omega |u|^{q_s^*} dx.$$

From the Brezis-Lieb Lemma [5] we obtain $u_n \to u$ in $L^{q_s^*}(\Omega)$ and consequently

$$u_n \to u \text{ in } L^t(\Omega) \quad \forall t \in [1, q_s^*],$$

proving Step 2.

Step 3: $u_n \to u$ in $W_0^{s,q}(\Omega)$.

By the Dominated Convergence Theorem it follows that

$$\int_\Omega |u_n|^{t-2} u_n u \, dx \to \int_\Omega |u|^t dx \quad \forall t \in [q, q_s^*]. \tag{15}$$

Let us define

$$A_n := \mathcal{J}'_\lambda(u_n)[u_n] + |u_n|_r^r + |u_n|_{q_s^*}^{q_s^*} - \langle \mathcal{J}'_\lambda(u_n), u\rangle - \int_\Omega |u_n|^{r-2} u_n u \, dx - \int_\Omega |u_n|^{q_s^*-2} u_n u \, dx$$

and

$$\begin{aligned} B_n := [u]_{s,p}^p + [u]_{s,q}^q &- \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(u_n(x)-u_n(y)) \, dxdy \\ &- \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{N+sq}}(u_n(x)-u_n(y)) \, dxdy. \end{aligned}$$

Since $\mathcal{J}'_\lambda(u_n) \to 0$, (u_n) is bounded in $W_0^{s,q}(\Omega)$ and (15) we infer that

$$A_n = o_n(1), \tag{16}$$

whereas the fact that $u_n \rightharpoonup u$ in $W_0^{s,q}(\Omega)$ implies that

$$B_n = o_n(1). \tag{17}$$

Let us note that

$$\begin{aligned} A_n + B_n = \Bigg[& [u_n]_{s,p}^p + [u_n]_{s,q}^q \\ & - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+sq}}(u(x)-u(y))dxdy \end{aligned}$$

$$- \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}}(u(x)-u(y))dxdy\Bigg]$$

$$+ \Bigg[[u]^p_{s,p} + [u]^q_{s,q}$$

$$- \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(u_n(x)-u_n(y))dxdy$$

$$- \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{N+sq}}(u_n(x)-u_n(y))dxdy\Bigg]$$

or equivalently

$$A_n + B_n = \tag{18}$$

$$= \iint_{\mathbb{R}^{2N}} \left[\frac{|v_n(x,y)|^{p-2}v_n(x,y)}{|x-y|^{N+sp}} - \frac{|v(x,y)|^{p-2}v(x,y)}{|x-y|^{N+sp}} \right] [v_n(x,y) - v(x,y))]\, dxdy$$
$$+ \iint_{\mathbb{R}^{2N}} \left[\frac{|v_n(x,y)|^{q-2}v_n(x,y)}{|x-y|^{N+sq}} - \frac{|v(x,y)|^{q-2}v(x,y)}{|x-y|^{N+sq}} \right] [v_n(x,y) - v(x,y))]\, dxdy,$$

where $v_n(x,y) := u_n(x) - u_n(y)$ and $v(x,y) := u(x) - u(y)$.

Now, we recall the following useful inequalities: for all $x, y \in \mathbb{R}^N$

$$C'_p|x-y|^p \le (|x|^{p-2}x - |y|^{p-2}y, x-y), \quad p \ge 2, \quad \text{and} \tag{19}$$

$$C''_p \frac{|x-y|^p}{(|x|^p + |y|^p)^{(2-p)/2}} \le [(|x|^{p-2}x - |y|^{p-2}y, x-y)]^{p/2}, \quad 1 < p < 2, \tag{20}$$

and distinguish two cases depending on the values of p.

Case 1: $2 \le p < q$. Then (18) and (19) give

$$A_n + B_n \ge C'_p \iint_{\mathbb{R}^{2N}} \frac{|(u_n-u)(x) - (u_n-u)(y)|^p}{|x-y|^{N+sp}} dxdy$$
$$+ C'_q \iint_{\mathbb{R}^{2N}} \frac{|(u_n-u)(x) - (u_n-u)(y)|^q}{|x-y|^{N+sq}} dxdy. \tag{21}$$

In view of (16), (17) and (21) we can see that $[u_n - u]_{s,p} \to 0$ and $[u_n - u]_{s,q} \to 0$, showing that $u_n \to u$ in $W^{s,q}_0(\Omega)$.

Case 2: $1 < p < 2$.

Taking into account (20) and applying the Hölder inequality with exponents $2/(2-p)$ and $2/p$ we get

$$C''_p \iint_{\mathbb{R}^{2N}} \frac{|v_n(x,y) - v(x,y)|^p}{|x-y|^{N+sp}} dxdy \le \left[\iint_{\mathbb{R}^{2N}} \frac{|v_n(x,y)|^p}{|x-y|^{N+sp}} + \frac{|v(x,y)|^p}{|x-y|^{N+sp}} dxdy \right]^{(2-p)/2}$$
$$\times \left[\iint_{\mathbb{R}^{2N}} \left[\frac{|v_n(x,y)|^{p-2}v_n(x,y)}{|x-y|^{N+sp}} - \frac{|v(x,y)|^{p-2}v(x,y)}{|x-y|^{N+sp}} \right] [v_n(x,y) - v(x,y)]\, dxdy \right]^{p/2}$$

which gives
$$C''_p[u_n-u]^2_{s,p} \le ([u_n]^p_{s,p}+[u]^p_{s,p})^{(2-p)/p} \tag{22}$$
$$\times\left[\iint_{\mathbb{R}^{2N}}\left[\frac{|v_n(x,y)|^{q-2}v_n(x,y)}{|x-y|^{N+sq}}-\frac{|v(x,y)|^{q-2}v(x,y)}{|x-y|^{N+sq}}\right][v_n(x,y)-v(x,y)]\,dxdy\right]$$
$$\le C\left[\iint_{\mathbb{R}^{2N}}\left[\frac{|v_n(x,y)|^{q-2}v_n(x,y)}{|x-y|^{N+sq}}-\frac{|v(x,y)|^{q-2}v(x,y)}{|x-y|^{N+sq}}\right][v_n(x,y)-v(x,y)]\,dxdy\right],$$

where we have used the fact that (u_n) is bounded in $W^{s,q}_0(\Omega)$.

If $1<p<2\le q$ we can argue as in (21) to deduce that
$$C_q[u_n-u]^q_{s,q} \le \tag{23}$$
$$\le \iint_{\mathbb{R}^{2N}}\left[\frac{|v_n(x,y)|^{q-2}v_n(x,y)}{|x-y|^{N+sq}}-\frac{|v(x,y)|^{q-2}v(x,y)}{|x-y|^{N+sq}}\right][v_n(x,y)-v(x,y)]\,dxdy.$$

Taking into account (16), (17), (18), (22) and (23) we get
$$o_n(1)=A_n+B_n\ge C_p[u_n-u]^2_{s,p}+C_q[u_n-u]^q_{s,q}$$
that is $u_n\to u$ in $W^{s,q}_0(\Omega)$.

If $1<p<q<2$, from the above arguments we can infer that
$$o_n(1)=A_n+B_n\ge C_p[u_n-u]^2_{s,p}+C_q[u_n-u]^2_{s,q},$$
and again the convergence $u_n\to u$ in $W^{s,q}_0(\Omega)$ holds, completing the proof. □

Lemma 3.3. *There exists $\lambda_*>0$ with $c_\lambda\in\left(0,\left(\frac{1}{r}-\frac{1}{q^*_s}\right)S_*^{N/sq}\right)$ for all $\lambda\ge\lambda_*$.*

Proof. Take $v\in C^\infty_c(\mathbb{R}^N)$ such that $v>0$ in $\mathbb{R}^N$. Then there exists $t_\lambda>0$ such that $\mathcal{J}_\lambda(t_\lambda v)=\max_{t\ge 0}\mathcal{J}_\lambda(tv)$. As a consequence $\langle\mathcal{J}'_\lambda(t_\lambda v),t_\lambda v\rangle=0$ that is
$$t^p_\lambda[v]^p_{s,p}+t^q_\lambda[v]^q_{s,q}=\lambda t^r_\lambda|v|^r_r+t^{q^*_s}_\lambda|v|^{q^*_s}_{q^*_s}, \tag{24}$$
which implies that
$$t^p_\lambda[v]^p_{s,p}+t^q_\lambda[v]^q_{s,q}>t^{q^*_s}_\lambda|v|^{q^*_s}_{q^*_s}.$$

Since $p<q<q^*_s$ we can infer that (t_λ) is bounded and that there exists a sequence $\lambda_n\to\infty$ such that $t_{\lambda_n}\to\bar t\ge 0$.

Now we aim to prove that $\bar t=0$. Assume by contradiction that $\bar t>0$. Taking into account the fact that (t_{λ_n}) is bounded and (24), we can see that there exists a positive constant K such that
$$K\ge t^p_{\lambda_n}[v]^p_{s,p}+t^q_{\lambda_n}[v]^q_{s,q}\ge\lambda_n t^r_{\lambda_n}|v|^r_r\to\infty,$$
which is impossible. Then $\bar t=0$.

Now, set $\gamma(t) = te$ with $t \in [0,1]$, where e is given in Lemma 3.1. Since $\gamma \in \Gamma$ we can see that

$$0 < c_\lambda \leq \max_{t\in[0,1]} \mathcal{J}_\lambda(tv) = \mathcal{J}_\lambda(t_\lambda e) \leq \frac{t_\lambda^p}{p}[e]_{s,p}^p + \frac{t_\lambda^q}{q}[e]_{s,q}^q. \tag{25}$$

Taking λ sufficiently large, let us say $\lambda \geq \lambda_*$, and recalling that $t_\lambda \to 0$ when $\lambda \to \infty$, we can infer that

$$t_\lambda^p[v]_{s,p}^p + t_\lambda^q[v]_{s,q}^q < \left(\frac{1}{r} - \frac{1}{q_s^*}\right) S_*^{N/sq},$$

which together with (25) yields $0 < c_\lambda < \left(\frac{1}{r} - \frac{1}{q_s^*}\right) S_*^{N/sq}$. □

We can now complete the proof of Theorem 1.1.

In virtue of Lemma 3.3, there is a $\lambda_* > 0$ such that $c_\lambda \in \left(0, \left(\frac{1}{r} - \frac{1}{q_s^*}\right) S_*^{N/sq}\right)$ for any $\lambda \geq \lambda_*$. Then, by Lemma 3.2 the (PS) condition holds at level c_λ and the conclusion is achieved by the classical Mountain Pass Theorem [2].

4. Proof of Theorem 1.2

In this section we study the existence of infinitely many solutions to (1) when $r \in (1, q)$. We begin by recalling some useful notions regarding the genus. For more details we refer the reader to [24].

Let E be a Banach space and let

$$\Gamma = \{A \subset E : A \text{ is closed in } E \text{ symmetric with respect to the origin }\}.$$

For $A \in \Gamma$, we define the *genus*

$$\gamma(A) = \inf\{k \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^k \setminus \{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

If there is no mapping φ as above for any $k \in \mathbb{N}$, then $\gamma(A) = \infty$. Moreover $\gamma(\emptyset) = 0$. Then we recall the following result.

Proposition 4.1. *Let A and B be closed symmetric subset of E which do not contain the origin. Then we have*

(i) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.*

(ii) *If there is an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$.*

(iii) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*

(iv) *The n-dimensional sphere $\mathbb{S}^n$ has genus $n+1$ by the Borsuk-Ulam Theorem.*

(v) *If $\gamma(A) \geq 2$, then A has infinitely many points.*

(vi) *If A is compact, then $\gamma(A) < \infty$ and there exist $\delta > 0$ and a closed and symmetric neighborhood $N_\delta(A) = \{x \in E : \|x - A\| \leq \delta\}$ of A such that $\gamma(N_\delta(A)) = \gamma(A)$.*

In order to implement the Ljusternick-Schnirelmann Theory, some preliminaries are in order. Indeed a first difficulty concerning the functional $\mathcal{J}_\lambda$ is that it is unbounded below on $W_0^{s,q}(\Omega)$, see (6).

By using the embeddings $W_0^{s,q}(\Omega) \subset L^r(\Omega)$ and $W_0^{s,p}(\Omega) \subset L^{q_s^*}(\mathbb{R}^N)$ we can see that, for $\lambda > 0$,

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q - \frac{\lambda}{rC^{r/2}}[u]_{s,q}^r - \frac{1}{q_s^* S_*^{q_s^*/2}}[u]_{s,q}^{q_s^*} \\ &\geq \frac{1}{q}[u]_{s,q}^q - \frac{\lambda}{r}C(s,q,r)[u]_{s,q}^r - \frac{1}{q_s^* S_*^{q_s^*/q}}[u]_{s,q}^{q_s^*},\end{aligned}$$

so that, setting $g_\lambda(t) := \frac{1}{q}t^q - \frac{\lambda}{r}C(s,q,r)t^r - \frac{1}{q_s^* S_*^{q_s^*/q}}t^{q_s^*}$, $t \geq 0$, we have

$$\mathcal{J}_\lambda(u) \geq g_\lambda([u]_{s,q}). \tag{26}$$

Let us define the number

$$\begin{aligned}\kappa_\lambda = \Big(\frac{1}{q} - \frac{1}{q_s^*}\Big)\Big\{S_*^{N/sq} - \lambda^{q_s^*/(q_s^*-r)}\Big[\frac{\left(\frac{1}{r}-\frac{1}{q}\right)|\Omega|^{(q_s^*-r)/q_s^*}}{\left(\frac{1}{q}-\frac{1}{q_s^*}\right)}\Big]^{q_s^*/(q_s^*-r)} \times \\ \times \Big[\Big(\frac{r}{q_s^*}\Big)^{r/(q_s^*-r)} - \Big(\frac{r}{q_s^*}\Big)^{q_s^*/(q_s^*-r)}\Big]\Big\}.\end{aligned}$$

Then there exists $\lambda^{**} > 0$ such that

(a) $g_{\lambda^{**}}$ achieves its positive maximum and has just two positive zeroes satisfying $0 < R_0 < R_1$. Note that actually they depends on λ^{**}, but we will omit this dependence.

(b) for every $\lambda \in (0, \lambda^{**}]$ it holds: $\kappa_\lambda \geq 0$.

Let us define $\phi \in C_c^\infty([0,\infty))$ such that it is non increasing and $0 \leq \phi \leq 1$, $\phi = 1$ in $[0, R_0]$, $\phi = 0$ in $[R_1, \infty)$. Then we introduce the following truncated functional

$$\widetilde{J}_\lambda(u) = \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q - \frac{\lambda}{r}|u|_r^r - \phi([u]_{s,q})\frac{1}{q_s^*}|u|_{q_s^*}^{q_s^*}.$$

We observe that $\widetilde{J}_\lambda \in C^1(W_0^{s,q}(\Omega), \mathbb{R})$ and as before we obtain, for every $\lambda > 0$,

$$\widetilde{J}_\lambda(u) \geq \frac{1}{q}[u]_{s,q}^q - \frac{\lambda}{r}C(s,q,r)[u]_{s,q}^r - \phi([u]_{s,q})\frac{1}{q_s^* S_*^{q_s^*/q}}[u]_{s,q}^{q_s^*} \tag{27}$$

so that, setting $\widetilde{g}_\lambda(t) = \frac{1}{q}t^q - \frac{\lambda}{r}C(s,q,r)t^r - \phi(t)\frac{1}{q_s^* S_*^{q_s^*/q}}t^{q_s^*}$, $t \geq 0$, we deduce

$$\widetilde{J}_\lambda(u) \geq \widetilde{g}_\lambda([u]_{s,q}) \tag{28}$$

and consequently

$$\forall\lambda \in (0,\lambda^{**}):\ \widetilde{J}_\lambda(u) \geq \widetilde{J}_{\lambda^{**}}(u) \geq \widetilde{g}_{\lambda^{**}}([u]_{s,q}). \tag{29}$$

Observe now that the function $\widetilde{g}_{\lambda^{**}}$ is bounded below on $[0,\infty)$ and is coercive, since for $t \geq R_1$ it is

$$\widetilde{g}_{\lambda^{**}}(t) = \frac{1}{q}t^q - \frac{\lambda^{**}}{r}C(s,q,r)t^r \tag{30}$$

hence $\lim_{t\to+\infty}\widetilde{g}_{\lambda_{**}}(t) = +\infty$ being $r < q$. From (29) and (30) it follows that $\widetilde{J}_\lambda$ is bounded below on $W_0^{s,q}(\Omega)$.

Furthermore $\widetilde{J}_\lambda$ is also coercive. Indeed if $\|u\| \to +\infty$ then it has to be necessarily $[u]_{s,q} \to +\infty$ and the coercivity follows from (29) and (30).

Summing up we have proved the following

Lemma 4.2. *For all $\lambda \in (0,\lambda^{**}]$ the functional $\widetilde{J}_\lambda$ is bounded below and coercive on $W_0^{s,q}(\Omega)$.*

Now we prove that $\widetilde{J}_\lambda$ verifies the following local (PS).

Lemma 4.3. *For every $\lambda \in (0,\lambda^{**}]$ the functional $\widetilde{J}_\lambda$ satisfies the (PS) condition in the range $(-\infty,\kappa_\lambda)$.*

Proof. Let $(u_n) \subset W_0^{s,q}(\Omega)$ be such that

$$\widetilde{J}_\lambda(u_n) \to c \in (-\infty,\kappa_\lambda) \quad \text{and} \quad \widetilde{J}'_\lambda(u_n) \to 0. \tag{31}$$

Since, for what we have seen before, when $\lambda \in (0,\lambda^{**})$ the functional is coercive, the sequence (u_n) is bounded and by [23, Theorem 2.5] we may assume that there exist two Borel regular measures μ and ν, Λ denumerable, $(x_i) \subset \overline{\Omega}$, $\nu_i \geq 0$, $\mu_i \geq 0$ with $\mu_i + \nu_i > 0$ for all $i \in \Lambda$ such that

$$|D^s u_n|^q \rightharpoonup \mu, \quad |u_n|^{q_s^*} \rightharpoonup \nu, \quad \text{and, for all } i \in \Lambda,$$
$$d\nu = |u|^{q_s^*} + \sum_{i\in\Lambda}\nu_i\delta_{x_i}, \quad d\mu \geq |D^s u|^q + \sum_{i\in\Lambda}\mu_i\delta_{x_i}, \quad S_*\nu_i^{q/q_s^*} \leq \mu_i.$$

We aim to show that $\nu_i = 0$ for all $i \in \Lambda$. Assume by contradiction that x_i is a singular point of measures μ and ν. For any $\rho > 0$, we set $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in B_1 and $\psi = 0$ in B_2^c and $|\nabla\psi|_\infty \leq 2$. Then,

$$\begin{aligned} c &= \mathcal{J}_\lambda(u_n) - \frac{1}{q}\langle \mathcal{J}'_\lambda(u_n), u_n\rangle + o_n(1) \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)[u_n]_{s,p}^p - \lambda\left(\frac{1}{r} - \frac{1}{q}\right)|u_n|_r^r + \left(\frac{1}{q} - \frac{1}{q_s^*}\right)|u_n|_{q_s^*}^{q_s^*} + o_n(1) \end{aligned}$$

$$\geq -\lambda\left(\frac{1}{r}-\frac{1}{q}\right)|u_n|_r^r+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)\int_\Omega \psi_\rho u_n^{q_s^*}dx+o_n(1).$$

Taking the limit as $n\to\infty$, recalling that $\nu_i\geq S_*^{N/sq}$ and by using the Hölder inequality, we get

$$\begin{aligned} c &\geq -\lambda\left(\frac{1}{r}-\frac{1}{q}\right)|u|_r^r+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)\int_\Omega u^{q_s^*}dx+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)\sum_{i\in\Lambda}\psi_\rho(x_i)\nu_i \\ &\geq \left(\frac{1}{q}-\frac{1}{q_s^*}\right)S_*^{N/sq}-\lambda\left(\frac{1}{r}-\frac{1}{q}\right)|u|_r^r+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)|u|_{q_s^*}^{q_s^*} \\ &\geq \left(\frac{1}{q}-\frac{1}{q_s^*}\right)S_*^{N/sq}-\lambda\left(\frac{1}{r}-\frac{1}{q}\right)|\Omega|^{(q_s^*-r)/q_s^*}|u|_{q_s^*}^r+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)|u|_{q_s^*}^{q_s^*}. \end{aligned} \tag{32}$$

Now, observing that

$$\begin{aligned} \min_{t\geq 0}\left\{-\lambda\left(\frac{1}{r}-\frac{1}{q}\right)|\Omega|^{(q_s^*-r)/q_s^*}t^r+\left(\frac{1}{q}-\frac{1}{q_s^*}\right)t^{q_s^*}\right\} &= -\left(\frac{1}{q}-\frac{1}{q_s^*}\right)\lambda^{q_s^*/(q_s^*-r)}\times \\ \times\left[\frac{\left(\frac{1}{r}-\frac{1}{q}\right)|\Omega|^{(q_s^*-r)/q_s^*}}{\left(\frac{1}{q}-\frac{1}{q_s^*}\right)}\right]^{q_s^*/(q_s^*-r)}&\left[\left(\frac{r}{q_s^*}\right)^{r/(q_s^*-r)}-\left(\frac{r}{q_s^*}\right)^{q_s^*/(q_s^*-r)}\right] \end{aligned} \tag{33}$$

we can see that (32) and (4) give a contradiction in view of (31).

As a consequence $\Lambda=\emptyset$ and $u_n\to u$ in $L^{q_s^*}(\Omega)$. Then, we can argue as in the proof of Lemma 3.2, Step 3, to deduce that $u_n\to u$ in $W_0^{s,q}(\Omega)$. □

Lemma 4.4. *Let* $\lambda\in(0,\lambda^{**}]$ *and* $u\in W_0^{s,q}(\Omega)$ *such that* $\widetilde{J}_\lambda(u)<0$. *Then for any* v *in a neighbourhood of* u *we have* $\widetilde{J}_\lambda(v)=\mathcal{J}_\lambda(v)$.

Proof. By (28) and taking into account that

$$g_{\lambda^{**}}([u]_{s,q})\leq\widetilde{g}_{\lambda^{**}}([u]_{s,q})<\widetilde{g}_\lambda([u]_{s,q})\leq\widetilde{J}_\lambda(u)<0,$$

we get $[u]_{s,q}<R_0$ hence $\phi([u]_{s,q})=1$ and $\widetilde{J}_\lambda(u)=\mathcal{J}_\lambda(u)$. From the continuity of $\widetilde{J}_\lambda$ we also have $\widetilde{J}_\lambda(v)=\mathcal{J}_\lambda(v)$ for any $v\in B_{R_0/2}(u)$. □

We will apply now the Ljusternick-Schnirelmann Theory to the functional $\widetilde{J}_\lambda$ taking into account that, by Lemma 4.3 and Lemma 4.4, it satisfies the (PS) condition at negative levels and that at such levels it coincide with the originary functional $\mathcal{J}_\lambda$.

Lemma 4.5. *Let* $\lambda>0$. *Given* $k\in\mathbb{N}$, *there exists* $\varepsilon_k>0$ *such that* $\gamma(\widetilde{J}_\lambda^{-\varepsilon_k})\geq k$, *where* $\widetilde{J}_\lambda^a=\left\{u\in W_0^{s,q}(\Omega):\widetilde{J}_\lambda(u)\leq a\right\}$ *are the sublevel sets of* $\widetilde{J}_\lambda$.

Proof. Take $k \in \mathbb{N}$ and we consider a k-dimensional subspace X_k of $W_0^{s,q}(\Omega)$. Therefore we can find a constant $C_k > 0$ such that

$$C_k \|u\|^r \leq |u|_r^r \quad \forall u \in X_k. \tag{34}$$

Choose $\rho \in (0,1)$ such that $\|u\| = \rho$. Then, by using (34), we get

$$\widetilde{J}_\lambda(u) \leq \left(\frac{1}{p} + \frac{1}{q}\right) \rho^p - \frac{\lambda}{r} C_k \rho^r,$$

and taking $$\rho \in \left(0, \min\left\{1, \left(\left(\frac{1}{p} + \frac{1}{q}\right)^{-1} \frac{\lambda}{r} C_k\right)^{1/(p-r)}\right\}\right)$$

we can find $\varepsilon_k > 0$ such that $\widetilde{J}_\lambda(u) < -\varepsilon_k$ for any $u \in \mathbb{S}_\rho = \{v \in X_k : \|v\| = \rho\}$. Therefore, $\mathbb{S}_\rho \subset \widetilde{J}_\lambda^{-\varepsilon_k}$ and, being $\widetilde{J}_\lambda^{-\varepsilon_k}$ symmetric and closed, we can apply (vi) and (iv) of Proposition 4.1 to deduce that $\gamma(\widetilde{J}_\lambda^{-\varepsilon_k}) \geq \gamma(\mathbb{S}_\rho) = k$. □

For any $\lambda > 0$ and $k \in \mathbb{N}$ we define the minimax levels of $\widetilde{J}_\lambda$ as

$$c_k(\lambda) = \inf_{C \in \Gamma_k} \sup_{u \in C} \widetilde{J}_\lambda(u)$$

where Γ_k denotes the family of closed symmetric subsets A of $W_0^{s,q}(\Omega)$ such that $0 \notin A$ and $\gamma(A) \geq k$. Of course they satisfy $c_k(\lambda) \leq c_{k+1}(\lambda)$ and for $\lambda \in (0, \lambda^{**})$:

$$-\infty < c_1(\lambda) \leq c_2(\lambda) \leq \dots.$$

Define also the set of critical points at level c of the functional $\widetilde{J}_\lambda$ as

$$K_c(\lambda) = \left\{u \in W_0^{s,q}(\Omega) : \widetilde{J}_\lambda'(u) = 0 \text{ and } \widetilde{J}_\lambda(u) = c\right\}.$$

Lemma 4.6. *For every $\lambda > 0$ and $k \in \mathbb{N}$ we have $c_k(\lambda) < 0$.*

Proof. Fix $k \in \mathbb{N}$. In view of Lemma 4.5 we can find $\varepsilon_k > 0$ with $\gamma(\widetilde{J}_\lambda^{-\varepsilon_k}) \geq k$. Taking into account that $0 \notin \widetilde{J}_\lambda^{-\varepsilon_k}$ and $\widetilde{J}_\lambda^{-\varepsilon_k} \in \Gamma_k$ we infer that

$$-\infty < c_k(\lambda) = \inf_{C \in \Gamma_k} \sup_{u \in C} \widetilde{J}_\lambda(u) \leq \sup_{u \in \widetilde{J}_\lambda^{-\varepsilon_k}} \widetilde{J}_\lambda(u) \leq -\varepsilon_k < 0,$$

concluding the proof. □

The next result is standard in the Ljusternick-Schnirelmann Theory, nevertheless we give the details for the reader's convenience.

Lemma 4.7. *Let $\lambda > 0$. If, for some $k_0, m \in \mathbb{N}$, we have*

$$c(\lambda) := c_{k_0}(\lambda) = c_{k_0+1}(\lambda) = \cdots = c_{k_0+m}(\lambda) > -\infty,$$

then $\gamma(K_{c,\lambda}) \geq m+1$, where for brevity $K_{c,\lambda} := K_{c(\lambda)}(\lambda)$.

Proof. Taking into account Lemma 4.3 and Lemma 4.6, we can see that

$$c(\lambda) = c_{k_0}(\lambda) = c_{k_0+1}(\lambda) = \cdots = c_{k_0+m}(\lambda) < 0 \quad \text{and} \quad K_{c,\lambda} \text{ is compact.}$$

Clearly, $K_{c,\lambda} = -K_{c,\lambda}$. Now, assume by contradiction that $\gamma(K_{c,\lambda}) \leq m$. Then, by using (vi) of Proposition 4.1 we can find a closed and symmetric set U such that $K_{c,\lambda} \subset U$ and $\gamma(U) = \gamma(K_{c,\lambda}) \leq m$. Since by Lemma 4.6 $c(\lambda) < 0$, we can suppose that $U \subset \widetilde{J}_\lambda^0$. By using the Deformation Lemma (see e.g. [24]) there exists an odd homeomorphism $\eta : W_0^{s,q}(\Omega) \to W_0^{s,q}(\Omega)$ such that $\eta\left(\widetilde{J}_\lambda^{c(\lambda)+\delta} - U\right) \subset \widetilde{J}_\lambda^{c(\lambda)-\delta}$ for some $\delta > 0$ with $0 < \delta < -c(\lambda)$. Hence, $\widetilde{J}_\lambda^{c(\lambda)+\delta} \subset \widetilde{J}_\lambda^0$ and being $c(\lambda) = c_{k_0+m}(\lambda)$, there exists $A \in \Gamma_{k_0+m}$ such that $\sup_{u\in A} \widetilde{J}_\lambda(u) < c(\lambda) + \delta$. So,

$$\gamma(A) \geq k_0 + m, \quad A \subset \widetilde{J}_\lambda^{c(\lambda)+\delta} \quad \text{and} \quad \eta\,(A - U) \subset \eta\left(\widetilde{J}_\lambda^{c(\lambda)+\delta} - U\right) \subset \widetilde{J}_\lambda^{c(\lambda)-\delta}.$$

Consequently, by using (vi) and (iii) of Proposition 4.1, we get

$$k_0 \leq \gamma(A) - m \leq \gamma(A) - \gamma(U) \leq \gamma\left(\overline{A-U}\right) \leq \gamma\left(\eta(\overline{A-U})\right) \leq \gamma(\widetilde{J}_\lambda^{c(\lambda)-\delta})$$

meaning that $\widetilde{J}_\lambda^{c(\lambda)-\delta} \subset \Gamma_{k_0}$. But then

$$c(\lambda) = c_{k_0}(\lambda) = \inf_{C\in\Gamma_k} \sup_{u\in C} \widetilde{J}_\lambda(u) \leq \sup_{u\in\widetilde{J}_\lambda^{c-\delta}} \widetilde{J}_\lambda^{c(\lambda)-\delta}(u) \leq c(\lambda) - \delta$$

giving a contradiction which proves the lemma. □

We are noe able to conclude the proof of Theorem 1.2:

Let $\lambda \in (0, \lambda^{**}]$, then $c_k(\lambda) \in (-\infty, 0)$.

If $c_j(\lambda) \neq c_i(\lambda)$ for all $i \neq j$, then we obtain infinitely many critical points of $\widetilde{J}_\lambda$ and then of $\mathcal{J}_\lambda$ by Lemma 4.4. Therefore (1) has infinitely many solutions.

If there exist $k_0, m \in \mathbb{N}$ such that $c_{k_0}(\lambda) = c_{k_0+1}(\lambda) = \cdots = c_{k_0+m}(\lambda)$, by applying Lemma 4.7, $\gamma(K_{c,\lambda}) \geq m + 1 \geq 2$ and then, by (v) of Proposition 4.1, $\widetilde{J}_\lambda$ has infinitely many critical points at negative level. Then again by Lemma 4.4 they are critical points of $\mathcal{J}_\lambda$ and also in this case (1) has infinitely many solutions.

References

[1] G. Alberti, G. Bouchitté, P. Seppecher: *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. 144 (1998) 1–46.

[2] A. Ambrosetti, P. H. Rabinowitz: *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973) 349–381.

[3] V. Ambrosio: *Fractional p&q Laplacian problems in $\mathbb{R}^N$ with critical growth*, arXiv:1801.10449.

[4] P. W. Bates: *On some nonlocal evolution equations arising in materials science*, Nonlinear Dynamics and Evolution Equations 48 (2006) 13–52.

[5] H. Brézis, E. H. Lieb: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88(3) (1983) 486–490.

[6] L. Caffarelli, J. M. Roquejoffre, O. Savin: *Nonlocal minimal surfaces*, Comm. Pure Apple. Math. 63 (2012) 1111–1144.

[7] L. Caffarelli, E. Valdinoci: *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations 32 (2007) 1245–1260.

[8] S.-Y. A. Chang, M. del Mar González: *Fractional Laplacian in conformal geometry*, Adv. Math. 226 (2011) 1410–1432.

[9] C. Chen, J. Bao: *Existence, nonexistence, and multiplicity of solutions for the fractional p&q-Laplacian equation in* $\mathbb{R}^N$, Bound. Value Probl., paper no. 153 (2016) 16 pp.

[10] R. Cont, P. Tankov: *Financial Modeling with Jump Processes*, Chapman & Hall, CRC Financial Mathematics Series, Boca Raton (2004).

[11] E. Di Nezza, G. Palatucci, E. Valdinoci: *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012) 521–573.

[12] G. Duvaut, J. L. Lions: *Inequalities in Mechanics and Physics*, Springer, Berlin et al. (1976).

[13] G. M. Figueiredo: *Existence and multiplicity of solutions for a class of p&q elliptic problems with critical exponent*, Math. Nachr. 11-12 (2013) 1129–1141.

[14] G. Garcia Azorero, I. Peral Alonso: *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. 323 (1991) 877–895.

[15] N. Laskin: *Fractals and quantum mechanics*, Chaos 10 (2000) 780–790.

[16] N. Laskin: *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A 268 (2000) 298–305.

[17] N. Laskin: *Fractional Schrödinger equation*, Phys. Rev. E 66 (2002) 056108.

[18] G. Li, G. Zhan: *Multiple solutions for the p&q-Laplacian problem with critical exponent*, Acta Math. Sci. 29B (2009) 903–918.

[19] R. Metzler, J. Klafter: *The random walk's guide to anomalous diffusion: a fractional dynamic approach,* Phys. Rep. 339 (2000) 77.

[20] R. Metzler, J. Klafter: *The restaurant at the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A 37 (2004) 161–208.

[21] E. Milakis, L. Silvestre: *Regularity for the nonlinear Signorini problem*, Adv. Math. 217 (2008) 1301–1312.

[22] G. Molica Bisci, V. Rădulescu, R. Servadei: *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications 162, Cambridge University Press, Cambridge (2016).

[23] S. Mosconi, M. Squassina: *Nonlocal problems at nearly critical growth*, Nonlinear Anal. 136 (2016) 84–101.

[24] P. H. Rabinowitz: *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBME Regional Conf. Ser. in Math. 65, American Mathematical Society, Providence (1986).

[25] L. Silvestre: *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 13 (1960) 457–468.

[26] Y. Sire, E. Valdinoci: *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetric result*, J. Funct. Anal. 60 (2007) 67–112.