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A Multiplicity Result for a Non-Autonomous Sublinear Elliptic Problem Involving Nonlinearities Indefinite in Sign

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Let Ω be a bounded smooth domain in $\mathbb{R}^N$, let $\alpha, \beta \colon \Omega \to \mathbb{R}$ be two measurable functions, and let $s \in]1,2[$ and $r \in]1,s[$. We deal with the following non autonomous elliptic problem

$$\begin{cases} -\Delta u = \alpha(x)u^{s-1} - \mu\beta(x)u^{r-1}, & \text{in } \Omega \\ u \geq 0, & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

where $\mu \in \mathbb{R}$ is a parameter. We establish, via minimax methods, a multiplicity result under suitable summability conditions on the weight functions α, β.

Keywords: Sublinear elliptic problem, weight function, nonnegative solution, positive solution, minimax method, mountain pass, multiplicity.

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1. Introduction

Throughout this paper, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a nonempty open bounded set with regular boundary $\partial\Omega$, and $\Delta := \operatorname{div}(\nabla(\cdot))$ is the Laplacian operator.

Let $r, s \in]1,2[$, with $r < s$, and let $\lambda > 0$. Consider the following boundary value problem

$$\begin{cases} -\Delta u = \lambda u^{s-1} - u^{r-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem is a model for physical diffusion phenomena with a stronger absorption, represented by the reaction term "$\lambda u^{s-1} - u^{r-1}$", than diffusion, represented

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by the differential term " $-\Delta u$" (see [3]). From a mathematical point of view, we are in presence of a nonlinear reaction term which is negative and not Lipschitz continuous near 0. In this situation, some difficulties arise when one looks for positive solutions. Indeed, the particular type of nonlinearity prevents us to apply the classical Strong Maximum Principle in order to get positive solutions from the nonzero and nonnegative ones. However, using a different approach based on the implicit function theorem and comparison results, the existence of positive solutions was proved in [4]. More precisely, Theorem 3.13 of [4] states the existence of $\lambda^* > 0$ such that problem (1) admits a positive solution satisfying the Hopf's boundary condition for each $\lambda \in]\lambda^*, +\infty[$ and no solution for $\lambda \in]0, \lambda^*[$. The question of the existence of multiple solutions was addressed in [1] where, for $\lambda \in]\lambda^*, +\infty[$, the existence of a second nonzero solution was proved via minimax methods, by using classical Mountain Pass Theorems. To realize that the associated energy functional I has the mountain pass geometry, in [1] it is proved that I has the following properties:

(1) the positive solution found in [1] is a local minimum point of I;

(2) the functional I has a (strict) local minimum point at $u = 0$.

The proofs of both properties 1) and 2) rely on a classical result by Brezis-Nirenberg which ensures that $C^1_0(\overline{\Omega})$ local minimum points of I are also $W^{1,2}_0(\Omega)$ local minimum points of the same functional (see [2]).

In the present paper, we will study the following non-autonomous version of the above problem

$$\begin{cases} -\Delta v = \lambda\alpha(x)v^{s-1} - \beta(x)v^{r-1} & \text{in } \Omega \\ v \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\alpha, \beta \colon \Omega \to \mathbb{R}$ are two measurable functions and $\lambda \in \mathbb{R}$ is a parameter. To avoid some technical complications, we will consider the following equivalent statement of the previous problem

$$(P_\mu) \qquad \begin{cases} -\Delta u = \alpha(x)u^{s-1} - \mu\beta(x)u^{r-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

which we can easily obtain from (1) by putting $u = \lambda^{\frac{1}{2-s}}v$ and $\mu = \lambda^{\frac{r-2}{2-s}}$. The main question in dealing with problem (P_μ) is that, even when the summability of the weight functions α, β allows to define the energy functional in the whole space $W^{1,2}_0(\Omega)$, the Brezis-Nirenberg result quoted above could be no longer applicable. Therefore, the argument used in [1] could not work in this case.

As we will see, to get a mountain pass geometry for the energy functional, we need to impose suitable summability conditions on the positive part of the weight α jointly to a comparison condition involving the weight functions α, β. Differently to [1], we will avoid to use the Brezis-Nirenberg result. Instead, we will make use of appropriate Hölder estimates.

Solutions to problem (P_μ) will be understood in the weak sense. By a *weak solution* of problem (P_μ) we mean any $u \in W_0^{1,2}(\Omega)$, with $u \geq 0$ a.e. in Ω, satisfying the equation

$$\int_\Omega \big(\nabla u(x)\nabla\varphi(x) - \alpha(x)u(x)^{s-1}\varphi(x) + \mu\beta(x)u(x)^{r-1}\varphi(x)\big)\,dx = 0,$$

for all $\varphi \in W_0^{1,2}(\Omega)$. A weak solution u to problem (P_μ) is said positive if $u > 0$ a.e. in Ω.

2. Notations and preliminary results

Let $p \geq 1$. In what follows, we will use the symbol

$$\|u\|_p \stackrel{\text{def}}{=} \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}, \quad u \in L^p(\Omega),$$

to denote the standard norm in the space $L^p(\Omega)$. Moreover, we denote by

$$\|u\|_\infty \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|, \quad u \in L^\infty(\Omega),$$

the standard norm in the space $L^\infty(\Omega)$. We equip the space $W_0^{1,2}(\Omega)$ with the usual Poincarè norm

$$\|u\| \stackrel{\text{def}}{=} \|\nabla u\|_2.$$

Let $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ and finally, for each $p \in [1, 2^*]$, we denote by

$$c_p \stackrel{\text{def}}{=} \sup_{u\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|_p}{\|u\|}$$

the best embedding constant.

Assume $\alpha \in L^{\frac{2^*}{2^*-s}}(\Omega)$ and $\beta \in L^{\frac{2^*}{2^*-r}}(\Omega)$. Then, by the Sobolev embedding theorems, it is easy to see that the functional

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{s}\int_\Omega \alpha(x)u_+^s dx + \frac{\mu}{r}\int_\Omega \beta(x)u_+^r dx, \quad u \in W_0^{1,2}(\Omega), \tag{2}$$

is well defined and Gâteaux differentiable in $W_0^{1,2}(\Omega)$, for each $\mu \in \mathbb{R}$. Here, given a function $h\colon \Omega \to \mathbb{R}$, the symbol h_+ denotes the function defined by

$$h_+(x) \stackrel{\text{def}}{=} \max\{h(x), 0\}, \quad x \in \Omega. \tag{3}$$

If $\alpha \in L^q(\Omega)$ and $\beta \in L^m(\Omega)$, for some $q > \frac{2^*}{2^*-s}$ and $m > \frac{2^*}{2^*-r}$, by the compact embedding theorems, the functional I_μ turns out to be sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$. Of course, I_μ is strongly continuous in $W_0^{1,2}(\Omega)$ as well. Moreover one has

$$I'_\mu(u)(v) = \int_\Omega \big(\nabla u(x)\nabla v(x) - \alpha(x)u_+(x)^{s-1}v(x) + \mu\beta(x)u_+(x)^{r-1}v(x)\big)\,dx$$

for all $v \in W_0^{1,2}(\Omega)$. It is an easy matter to see that the critical points of I_μ are exactly the weak solutions of (P_μ). For $\mu = 0$, I_μ takes the form

$$I_\mu(u) = I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u_+(x)^s\,dx, \quad u \in W_0^{1,2}(\Omega).$$

Our first result gives a sufficient condition (which is also necessary, as can easily be checked) on the weigh function α in order to get a global minimum point u_0 of I_0 with negative energy.

Lemma 2.1. *Let* $q > \frac{2^*}{2^*-s}$ *and* $\alpha \in L^q(\Omega)$, *with* $\operatorname{ess\,sup}_\Omega \alpha > 0$. *Then, there exists* $u_0 \in W_0^{1,2}(\Omega)$ *such that*

$$I_0(u_0) = \inf_{u \in W_0^{1,2}(\Omega)} I_0(u) < 0.$$

Moreover, one has $\|u_0\| = \left(\sup_{\|u\|=1}\int_\Omega \alpha(x)u(x)^s dx\right)^{\frac{1}{2-s}}$.

Proof. At first, observe that

$$\begin{aligned}
\inf_{u \in W_0^{1,2}(\Omega)} I_0(u) &= \inf_{\sigma>0}\inf_{\|u\|=\sigma}\left(\frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u_+^s\,dx\right) \\
&= \inf_{\sigma>0}\left(\frac{1}{2}\sigma^2 + \inf_{\|u\|=\sigma}\left(-\frac{1}{s}\int_\Omega \alpha(x)u_+^s\,dx\right)\right) \\
&= \inf_{\sigma>0}\left(\frac{1}{2}\sigma^2 - \frac{1}{s}\sup_{\|u\|=\sigma}\int_\Omega \alpha(x)u_+^s\,dx\right) \\
&= \inf_{\sigma>0}\left(\frac{1}{2}\sigma^2 - \frac{\sigma^s}{s}\sup_{\|u\|=1}\int_\Omega \alpha(x)u_+^s\,dx\right)
\end{aligned}$$

Then, since $s \in]1,2[$, one has $\inf_{u \in W_0^{1,2}(\Omega)} I_0(u) < 0$, if and only if

$$\sup_{\|u\|=1}\int_\Omega \alpha(x)u_+^s\,dx > 0. \tag{4}$$

Let us show that the condition $\operatorname{ess\,sup}_\Omega \alpha > 0$, which is equivalent to $\alpha_+ \neq 0$, implies (4). To this end, note that, if $\alpha_+ \neq 0$, one has

$$\int_\Omega \alpha_+(x)\alpha_+(x)^{\frac{s}{2^*-s}}\, dx = \int_\Omega \alpha_+(x)^{\frac{2^*}{2^*-s}}\, dx > 0. \tag{5}$$

Moreover, the functional

$$J(u) \stackrel{\text{def}}{=} \int_\Omega \alpha(x)u_+(x)^s dx, \quad u \in L^{2^*}(\Omega)$$

is (well defined and) continuous in $L^{2^*}(\Omega)$. Since $\alpha_+(x)^{\frac{1}{2^*-s}} \in L^{2^*}(\Omega)$, we can evaluate J at $u = \alpha_+(x)^{\frac{1}{2^*-s}}$. Thanks to (5), one has

$$J(\alpha_+^{\frac{1}{2^*-s}}) = \int_\Omega \alpha_+(x)^{\frac{2^*}{2^*-s}}\, dx > 0.$$

Then, using the density of $W_0^{1,2}(\Omega)$ in $L^{2^*}(\Omega)$ and (5), we can find a function $v \in W_0^{1,2}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \alpha_+(x)v_+(x)dx > 0.$$

Therefore, proving (4),

$$\sup_{\|u\|=1} \int_\Omega \alpha_+(x)u_+(x)dx \geq \int_\Omega \alpha_+(x)\frac{v_+(x)}{\|v\|}dx > 0\,.$$

Finally, note that being I_0 coercive and sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$, there exists a global minimum point $u_0 \in W_0^{1,2}(\Omega)$ of I_0 in $W_0^{1,2}(\Omega)$. Since u_0 is a critical point of I_0, one has

$$I_0'(u_0)(u_0) = \|u_0\|^2 - \int_\Omega \alpha(x)u_0(x)^s dx = 0.$$

Thus, if we consider the Nehari manifold $\mathcal{N}_0$ of I_0, defined by

$$\begin{aligned}\mathcal{N}_0 :&= \left\{u \in W_0^{1,2}(\Omega) \setminus \{0\} : \|u\|^2 = \int_\Omega \alpha(x)u(x)^s dx\right\} \\ &= \left\{\left(\frac{\int_\Omega \alpha(x)u(x)^s dx}{\|u\|^2}\right)^{\frac{1}{2-s}} u : u \in W_0^{1,2}(\Omega) \setminus \{0\}, \int_\Omega \alpha(x)u(x)^s dx \geq 0\right\},\end{aligned}$$

we get $$\left(\frac{1}{2} - \frac{1}{s}\right)\|u_0\|^2 = I_0(u_0) = \inf_{u \in W_0^{1,2}(\Omega)} I_0(u) = \left(\frac{1}{2} - \frac{1}{s}\right)\sup_{u \in \mathcal{N}_0} \|u\|^2$$

$$= \left(\frac{1}{2} - \frac{1}{s}\right)\left[\sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \|u\|^{-1}\left(\int_\Omega \alpha(x)u(x)^s dx\right)^{\frac{1}{s}}\right]^{\frac{2s}{2-s}}$$

$$= \left(\frac{1}{2} - \frac{1}{s}\right)\left(\sup_{\|u\|=1}\int_\Omega \alpha(x)u(x)^s dx\right)^{\frac{2}{2-s}},$$

that is $\|u_0\| = \left(\sup_{\|u\|=1} \int_\Omega \alpha(x)u(x)^s dx\right)^{\frac{1}{2-s}}$. This concludes the proof. □

Now, we will apply Lemma 2.1 to obtain, for μ sufficiently small, a global minimum point of I_μ with negative energy. More precisely, we will prove the following lemma

Lemma 2.2. *Let* $q > \frac{2^*}{2^*-s}$, $m > \frac{2^*}{2^*-r}$, $\alpha \in L^q(\Omega)$, *with* $\operatorname{ess\,sup}_\Omega \alpha > 0$, *and* $\beta \in L^m(\Omega)$. *Then, there exist* $\mu_0, \rho, c_0 \in]0, +\infty[$ *such that, for each* $\mu \in [-\mu_0, \mu_0]$, *there exists a (nonnegative) function* $u_\mu \in W_0^{1,2}(\Omega)$ *satisfying*

$$I_\mu(u_\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u) \le -\rho < 0, \quad \textit{and} \tag{6}$$

$$\|u_\mu\| \ge c_0. \tag{7}$$

Proof. For each $\mu > 0$, the functional I_μ is coercive e sequentially lower semicontinuous. Hence, there exists $u_\mu \in W_0^{1,2}(\Omega)$ such that

$$I_\mu(u_\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u).$$

Now, consider the function $g \colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u), \quad \text{for each } \mu \in \mathbb{R}.$$

Since g is an inferior envelope of affine functions, then it is concave in $\mathbb{R}$, and thus continuous there as well. Moreover, by Lemma 2.1, one has $g(0) < 0$. Therefore, if we fix $\rho \in]g(0), 0[$, there exists $\mu_0 > 0$ such that $g(\mu) < \rho$, for each $\mu \in [-\mu_0, \mu_0]$. Therefore, (6) holds.

It remains to prove (7). Let $\mu \in [-\mu_0, \mu_0]$ and let $u_\mu \in W_0^{1,2}(\Omega)$ be satisfying (6). Using the Sobolev embedding theorems, we can find two constants $c_1, c_2 > 0$ (depending only on $N, s, r, \Omega, \mu_0, \alpha, \beta,$) such that

$$\begin{aligned}
-\rho \ge I_\mu(u_\mu) &= \frac{1}{2}\|u_\mu\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u_\mu^s dx + \frac{\mu}{r}\int_\Omega \beta(x)u_\mu^r dx \\
&\ge \frac{1}{2}\|u_\mu\|^2 - \frac{1}{s}\int_\Omega |\alpha(x)|u_\mu^s dx - \frac{\mu_0}{r}\int_\Omega |\beta(x)|u_\mu^r dx \\
&\ge \frac{1}{2}\|u_\mu\|^2 - c_1\|u_\mu\|^s - c_2\|u_\mu\|^r \ge \frac{1}{4}\|u_\mu\|^2 - \varepsilon_0\|u_\mu\|,
\end{aligned}$$

where $\varepsilon_0 = \max_{t>0}(c_1 t^{s-1} + c_2 t^{r-1} - \frac{1}{4}t) > 0$. To conclude, it is sufficient to observe that the inequality $\frac{1}{4}\|u_\mu\|^2 - \varepsilon_0\|u_\mu\| + \rho < 0$ entails $\varepsilon_0^2 > \rho$ and

$$\|u_\mu\| \ge 2(\varepsilon_0 - \sqrt{\varepsilon_0^2 - \rho}) := c_0 > 0.$$

□

At this point, our goal is to find sufficient conditions on the weight functions α, β to get a mountain pass geometry for the functional I_μ, at least for μ small enough. The next two lemmas give such conditions.

Let us consider two measurable sets $\Omega_{\alpha_+}, \Omega_{\alpha_-} \subset \Omega$ such that $\Omega_{\alpha_+} \cup \Omega_{\alpha_-} = \Omega$, $\alpha(x) > 0$ for almost all $x \in \Omega_{\alpha_+}$, and $\alpha(x) \le 0$ for almost all $x \in \Omega_{\alpha_-}$;

Lemma 2.3. *Let α, β be as in Lemma 2.2. Assume that*

(i) $\quad \lambda_{\alpha\beta} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega_{\alpha_+}} \dfrac{\beta(x)}{\alpha(x)} > 0;$

(ii) $\quad \alpha_+ \in L^q(\Omega)$, *for some* $q > \frac{N}{2}$.

Then, for each $\mu > 0$, there exists $\sigma_0 > 0$ such that

$$\inf_{\|u\|=\sigma} I_\mu(u) \ge \frac{1}{4}\sigma^2 > 0, \quad \text{for all } \ \sigma \in]0, \sigma_0[\tag{8}$$

In particular, $u = 0$ is a (strict) local minimum point for the functional I_μ.

Proof. Let $q > \frac{N}{2}$ be such that $\alpha_+ \in L^q(\Omega)$ and let $\mu > 0$. We may assume $q < \infty$. Let us denote $q' = \frac{q}{q-1}$. Then, $q' < \frac{N}{N-2}$ and

$$p_0 \stackrel{\text{def}}{=} \frac{2^*}{q'} \in]2, 2^*[.$$

By an easy calculation we see that for almost all $x \in \Omega_{\alpha_+}$, one has $\alpha(x) > 0$ and

$$\begin{aligned}
&\sup_{t>0} \left\{ \frac{\frac{1}{s}\alpha(x)t^s - \frac{\mu}{r}\beta(x)t^r}{t^{p_0}} \right\} \\
&= \left(\frac{\mu}{r}\frac{\beta(x)}{\alpha(x)}\right)^{-\frac{p_0-s}{s-r}} \left(\frac{s(p_0-r)}{p_0-s}\right)^{-\frac{p_0-r}{s-r}} \left(\frac{s-r}{p_0-s}\right) \alpha(x) \le M\,\alpha(x)
\end{aligned} \tag{9}$$

where $\quad M = \lambda_{\alpha\beta}^{-\frac{p_0-s}{s-r}} \left(\dfrac{r}{\mu}\right)^{\frac{p_0-s}{s-r}} \left(\dfrac{p_0-s}{s(p_0-r)}\right)^{\frac{p_0-r}{s-r}} \left(\dfrac{s-r}{p_0-s}\right) > 0$

Moreover, by the Hölder inequality, we see that $\alpha \cdot u^{p_0} \in L^1(\Omega)$ for each $u \in W_0^{1,2}(\Omega)$. Thus, keeping in mind (9), for $\sigma > 0$ one has:

$$\begin{aligned}
\inf_{\|u\|=\sigma} I_\mu(u) &= \inf_{\|u\|=\sigma} \left(\frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u^s\,dx + \frac{\mu}{r}\int_\Omega \beta(x)u^r\,dx\right) \\
&= \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \left(\int_\Omega \frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r\,dx\right) \\
&\ge \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \int_{\Omega_{\alpha_+}} \left(\frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r\right) dx \\
&\ge \frac{1}{2}\sigma^2 - M \sup_{\|u\|=\sigma} \int_{\Omega_{\alpha_+}} \alpha(x)u^{p_0}\,dx
\end{aligned}$$

$$\geq \frac{1}{2}\sigma^2 - M \sup_{\|u\|=\sigma} \left(\int_{\Omega_{\alpha_+}} \alpha(x)^q dx\right)^{\frac{1}{q}} \left(\int_\Omega u^{2^*} dx\right)^{\frac{1}{q'}}$$

$$\geq \frac{1}{2}\sigma^2 - M c_{2^*}^{\frac{2^*}{q'}} \sup_{\|u\|=\sigma} \left(\int_{\Omega_{\alpha_+}} \alpha(x)^q dx\right)^{\frac{1}{q}} \|u\|^{p_0}$$

$$= \frac{1}{2}\sigma^2 - M c_{2^*}^{\frac{2^*}{q'}} \left(\int_{\Omega_{\alpha_+}} \alpha(x)^q dx\right)^{\frac{1}{q}} \sigma^{p_0}.$$

Since $p_0 > 2$, one has $\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{4}\sigma^2 > 0 = I_\mu(0)$, provided that σ is small enough. This concludes the proof. □

In view of Lemma 2.2 and Lemma 2.3, we realize that, under the assumptions of Lemma 2.3, I_μ has the mountain pass geometry for all $\mu \in]0, \mu_0]$. Note that, the key ingredient to guarantee the mountain pass geometry for I_μ is that, for each $\mu \in]0, \mu_0]$, an inequality like (8) holds at least for some $\sigma \in]0, c_0[$, where c_0 is as in Lemma 2.2. This inequality can be also obtained under different assumptions, as stated by the next lemma

Lemma 2.4. *Let $\alpha, \beta, c_0, \mu_0$ be as in Lemma 2.2. Assume that*

(i) $\lambda_{\alpha\beta} \stackrel{\text{def}}{=} \operatorname{essinf}_{x\in\Omega_{\alpha_+}} \frac{\beta(x)}{\alpha(x)} > 0$;

(ii) $\liminf_{k\to+\infty} \left(k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha(x)\geq k} \alpha(x)^{\frac{2^*}{2^*-s}} dx\right) = 0$

Then, for each $\mu \in]0, \mu_0]$, there exists $\sigma \in]0, c_0[$ such that

$$\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{6}\sigma^2 > 0. \tag{10}$$

Proof. Similar as in the proof of Lemma 2.3, we see that for almost all $x \in \Omega_{\alpha_+}$, one has

$$\sup_{t>0} \left\{ \frac{\frac{1}{s}\alpha(x)t^s - \frac{\mu}{r}\beta(x)t^r}{t^{2^*}} \right\} =$$

$$= \left(\frac{\mu}{r}\frac{\beta(x)}{\alpha(x)}\right)^{-\frac{2^*-s}{s-r}} \left(\frac{s(2^*-r)}{2^*-s}\right)^{-\frac{2^*-r}{s-r}} \left(\frac{s-r}{2^*-s}\right) \alpha(x) \leq M\,\alpha(x) \tag{11}$$

where $M = \lambda_{\alpha\beta}^{-\frac{2^*-s}{s-r}} \left(\frac{r}{\mu}\right)^{\frac{2^*-s}{s-r}} \left(\frac{2^*-s}{s(2^*-r)}\right)^{\frac{2^*-r}{s-r}} \left(\frac{s-r}{2^*-s}\right) > 0.$

Keeping in mind (11) and using the Hölder's inequality, for $K, \sigma > 0$ one has:

$$\inf_{\|u\|=\sigma} I_\mu(u) = \inf_{\|u\|=\sigma} \left(\frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u^s\,dx + \frac{\mu}{r}\int_\Omega \beta(x)u^r\,dx\right)$$

$$= \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \left(\int_\Omega \frac{1}{s}\,\alpha(x)u^s - \frac{\mu}{r}\,\beta(x)u^r\,dx \right)$$

$$\geq \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \left[\int\limits_{0<\alpha(x)\leq k} \left(\frac{1}{s}\,\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r\right)dx + \frac{1}{s} \int\limits_{\alpha(x)\geq k} \alpha(x)u^s\,dx \right]$$

$$\geq \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \int_{0<\alpha(x)\leq k} M\alpha(x)u^{2^*}dx - \frac{1}{s}\sup_{\|u\|=\sigma} \int_{\alpha(x)\geq k} \alpha(x)u^s\,dx$$

$$\geq \frac{1}{2}\sigma^2 - Mkc_{2^*}^{2^*}\sigma^{2^*} - \frac{c_{2^*}^s}{s}\left(\int_{\alpha(x)\geq k} \alpha(x)^{\frac{2^*}{2^*-s}}\,dx\right)^{\frac{2^*-s}{2^*}} \sigma^s \tag{12}$$

Now, thanks to assumption (ii), we can find $k_0 > 0$ such that

$$(6Mc_{2^*}^{2^*})^{-\frac{1}{2^*-2}} k_0^{-\frac{1}{2^*-2}} < c_0, \quad \text{and}$$

$$k_0^{\frac{1}{2^*-2}} \left(\int_{\alpha(x)\geq k_0} \alpha^{\frac{2^*}{2^*-s}} \right)^{\frac{2^*-s}{2^*(2-s)}} \leq (6Mc_{2^*}^{2^*})^{\frac{1}{2-2^*}} \left(\frac{6}{s}c_{2^*}^s\right)^{\frac{1}{s-2}}.$$

Hence, we can fix $\sigma > 0$ such that

$$\left(\frac{6}{s}c_{2^*}^s\right)^{\frac{1}{2-s}} \left(\int_{\alpha(x)\geq k_0} \alpha^{\frac{2^*}{2^*-s}} \right)^{\frac{(2^*-s)}{2^*(2-s)}} \leq \sigma \leq (6Mc_{2^*}^{2^*})^{\frac{1}{2-2^*}} k_0^{\frac{1}{2-2^*}} < c_0$$

As a consequence, we obtain

$$\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{6}\sigma^2 + \left(\frac{1}{6}\sigma^2 - Mk_0c_{2^*}^{2^*}\sigma^{2^*}\right) + \left[\frac{1}{6}\sigma^2 - \frac{c_{2^*}^s}{s}\Big(\int\limits_{\alpha\geq k_0} \alpha(x)^{\frac{2^*}{2^*-s}}\,dx\Big)^{\frac{2^*-s}{2^*}}\sigma^s\right]$$

$$\geq \frac{1}{6}\sigma^2 > 0 \qquad \square$$

Remark 2.5. A sufficient condition for the validity of assumption (ii) can be given in terms of the symmetric-decreasing rearrangement α_+^* of the function α_+, defined by

$$\alpha_+^*(x) = \int_0^\infty \chi_{\{\alpha_+>t\}^*}(x)dt, \quad \text{for all} \quad x \in \Omega^*,$$

where Ω^* is the open ball centered at 0 with the same measure as that of Ω, $\{\alpha_+ > t\}^*$ is the open ball centered at 0 with the same measure as that of $\{y \in \Omega : \alpha_+(y) > t\}$, and $\chi_{\{\alpha_+>t\}^*}$ is the characteristic function of $\{\alpha_+ > t\}^*$ (see [5] page 80). More precisely, let us to show that if

$$\lim_{|x|\to 0} |x|^2\alpha_+^*(x) = 0$$

then, condition (ii) of Lemma 2.4 holds. Let $\varepsilon > 0$ and let $\delta_\varepsilon > 0$ be such that

$$|x|^2\alpha_+^*(x) < \varepsilon, \quad \text{for each } x \in \mathbb{R}^N \text{ such that } |x| \le \delta_\varepsilon.$$

Put $k_\varepsilon = \delta_\varepsilon^{-2}\varepsilon$ and let $k \in \mathbb{R}$ be such that $k \ge k_\varepsilon$. Then, for each $x \in \mathbb{R}^N$ such that $\varepsilon|x|^{-2} \ge k$ one has $|x| \le \delta_\varepsilon$, and so $|x|^2\alpha_+^*(x) < \varepsilon$. Consequently, after noticing that $\frac{N}{2} = \frac{2^*}{2^*-2}$, one has

$$\int_{\alpha_+^* \ge k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}}dx \le \varepsilon \int_{\varepsilon|x|^{-2}\ge k} (\varepsilon|x|^{-2})^{\frac{2^*}{2^*-s}}dx \le \varepsilon^{\frac{2^*}{2^*-s}} \int_0^{\sqrt{\frac{\varepsilon}{k}}} t^{N-1-\frac{22^*}{2^*-s}}dt$$
$$= \left(N - \frac{22^*}{2^*-s}\right)^{-1} \varepsilon^{\frac{2^*}{2^*-s}}\varepsilon^{N-\frac{2^*}{2^*-s}}k^{\frac{N}{2}-\frac{2^*}{2^*-s}} = 2\frac{2^*-2}{2^*}\frac{2^*-s}{2-s}\varepsilon^N k^{\frac{2^*}{2^*-2}\frac{s-2}{2^*-s}}.$$

Therefore, for $k \ge k_\varepsilon$, one has

$$k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha_+^*\ge k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}}dx < 2\frac{2^*-2}{2^*}\frac{2^*-s}{2-s}\varepsilon^N.$$

This means that

$$\lim_{k\to+\infty} k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha_+^*\ge k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}}dx = 0. \tag{13}$$

Then, the validity of assumption (ii) of Lemma 2.4 follows by the identity (see [5], pag 81)

$$\int_{\alpha_+^*\ge k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}}dx = \int_{\alpha_+\ge k} \alpha_+(x)^{\frac{2^*}{2^*-s}}dx = \int_{\alpha\ge k} \alpha(x)^{\frac{2^*}{2^*-s}}dx.$$

It is worth noticing that, by similar arguments, one can show that the limit (13) implies $\lim_{|x|\to 0} |x|^2\alpha_+^*(x) := 0$, but this latter might not hold if in (13) "lim" is replaced by "liminf", as in assumption (ii) of Lemma 2.4. □

An easy consequence of the above lemmas is the main result of this paper:

Theorem 2.6. *Let* $r, s \in]1,2[$, *with* $r < s$, $q > \frac{2^*}{2^*-s}$, $m > \frac{2^*}{2^*-r}$, $\alpha \in L^q(\Omega)$, *and* $\beta \in L^m(\Omega)$. *Assume that*

(a) $\operatorname{ess\,sup}_\Omega \alpha > 0$.

Then, there exists $\mu_0 > 0$ *such that problem* (P_μ) *admits at least a nonzero solution for each* $\mu \in [-\mu_0, \mu_0]$. *Suppose, in addition, that* α, β *satisfy*

(b) $\beta(x) \ge 0$ *for a.e.* $x \in \Omega$; (c) $\lambda_{\alpha\beta} \overset{\text{def}}{=} \operatorname*{essinf}_{x\in\Omega_{\alpha_+}} \frac{\beta(x)}{\alpha(x)} > 0$;

(d) *either* $\alpha_+ \in L^p(\Omega)$, *for some* $p > \frac{N}{2}$,

or $$\liminf_{k\to+\infty}\left(k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha(x)\ge k} \alpha(x)^{\frac{2^*}{2^*-s}}dx\right) = 0.$$

Then, for each $\mu \in]0, \mu_0]$, *problem* (P_μ) *admits at least two nonzero solutions.*

Proof. Under assumption (a), the first part of the thesis follows directly from Lemma 2.1. Under the additional assumptions (b), (c) and (d), Lemmas 2.2–2.4 say that I_μ satisfies the mountain pass geometry for all $\mu \in]0, \mu_0]$. Since I_μ satisfies the Palais-Smale condition (see Example 38.25 of [6]) as well, then there exists a second solution of mountain pass type. This second solution is nonzero in view of inequalities (8), (10). □

Remark 2.7. Assume $0 \in \Omega$ and suppose that $\alpha(x) = \beta(x) = |x|^{-\eta}$, for all $x \in \Omega$, where $\eta > 0$. In this case, problem (P_μ) takes the form

$$(P_{\mu,\eta}) \qquad \begin{cases} -\Delta u = \dfrac{1}{|x|^\eta}(u^{s-1} - \mu u^{r-1}) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where the equation

$$-\Delta u = \frac{1}{|x|^\eta}(u^{s-1} - \mu u^{r-1})$$

can be seen as a perturbation of the Henon equation $-\Delta u = \frac{1}{|x|^\eta} u^{s-1}$. It is easy to see that Theorem 2.5 applies to this case if and only if $\eta \in]0, 2[$. Indeed, if $\eta \in]0, 2]$, the function $x \in \Omega \setminus \{0\} \to |x|^{-\eta}$ is p-summable in Ω for each $p \in [1, N/2[$, and

$$N/2 = \frac{2^*}{2^* - 2} > \frac{2^*}{2^* - s} > \frac{2^*}{2^* - r}.$$

Moreover, assumptions $a), b), c)$ are clearly satisfied for each $\eta > 0$ and assumption (d) is just satisfied if and only if $\eta < 2$.

In view of the above considerations, it would be interesting to examine the case $\eta = 2$. In this case, we can apply Lemma 2.2 to get, for each $\mu \in \mathbb{R}$ sufficiently small, a solution which is a global minimum of the energy functional, but we cannot apply Theorem 2.5 to get a second solution. However, we conjecture that a second solution exists if $\mu \in I \subset]0, +\infty[$, where I is a suitable interval. It would be also interesting to investigate the boundedness/unboundedness of solutions in this case. Indeed, when $\eta = 2$, the right hand side of the equation is not q-summable for $q > \frac{N}{2}$, so the boundedness of the solutions cannot be obtained by using a standard Moser iteration scheme. □

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