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Perturbed Problems Involving the Square Root of the Laplacian

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We prove multiplicity of solutions for perturbed problems involving the square root of the Laplacian $\mathcal{A} = (-\Delta)^{1/2}$. More precisely, we consider the problem

$$\begin{cases} \mathcal{A}u = \lambda u + f(x,u) + \varepsilon g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\varepsilon \in \mathbb{R}$, $N > 1$, f is a subcritical function with asymptotic linear behavior at infinity, and g is a continuous function. We also show the invariance under small perturbations of the number of distinct critical levels of the associated energy functional to the unperturbed problem, in both resonant and non-resonant case.

Keywords: Fractional Laplacian, variational methods, multiplicity of solutions.

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1. Introduction

In this paper we deal with a perturbed asymptotically (at infinity) linear problem for the square root of the Laplacian operator, which can be defined as a Dirichlet to Neumann map via an extension problem to the upper half space. In the last years many mathematicians have been interested in the study of nonlinear problems involving the fractional powers $(-\Delta)^s, 0 < s < 1$, of the Laplace operator. Mainly, motivated by the applications in several physical phenomena like flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics and also in probability, American option in finance and in α-stable Lévy processes (cf., e.g., [3, 7, 13, 30]).

Due to the fact that the fractional Laplacian is a nonlocal operator, local PDE techniques do not apply, however these difficulties can be overcome by means of operators that map a Dirichlet boundary condition to a Neumann-type one via an extension problem, see for instance [11] (Section 2 for details).

Motivated by the discussion above and paper [6] by Bartolo-Candela-Salvatore where were studied problems such as

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u = p(x,u) + \varepsilon g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\Omega \subset \mathbb{R}^N$, $N > 2$, $\varepsilon \in \mathbb{R}$, p subcritical and asymptotically linear at infinity, and g is a continuous function. More precisely, in [6] it was proved that the number of distinct critical levels of the functional associated to the unperturbed problem (i.e., when $\varepsilon = 0$) is "stable" under small perturbations and it is also given a multiplicity result when p is odd, both in the resonant and in the non-resonant case.

In this manuscript we consider the following problem

$$(P_{\lambda,\varepsilon}) \qquad \begin{cases} \mathcal{A}u = \lambda u + f(x,u) + \varepsilon g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$, $\lambda \in \mathbb{R}$, f, g are given functions on $\Omega \times \mathbb{R}$ and $\mathcal{A}$ is the square root of the Laplacian (we refer to Section 2 for its definition).

Hereafter we assume that

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;

(f_2) the following limits hold uniformly with respect to $x \in \Omega$:

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t} = 0 \tag{1}$$

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = \alpha \in \mathbb{R}. \tag{2}$$

Remark 1.1. We could replace (cf., e.g., [6]) assumption (f_1) with the assumption that f is a Carathéodory function and

$$\sup_{|t|\leq\delta} |f(\cdot,t)| \in L^\infty(\Omega) \quad \text{for all } \delta > 0.$$

Remarks 1.2. (i) We denote by $\sigma(\mathcal{A})$ the spectrum of $\mathcal{A}$ with zero Dirichlet boundary conditions and $(\lambda_n)_n$ the non-decreasing sequence of the distinct eigenvalues of $\mathcal{A}$ with zero Dirichlet boundary conditions.

(ii) The natural energy space is denoted by $H_0^{1/2}(\Omega)$, the fractional Sobolev space. See Section 2 for more details.

First we deal with the non-resonant case and state a result dealing with the existence of solutions for (1); moreover, if we allow both f and g to be odd, we obtain multiple solutions.

Theorem 1.3. *Assume that $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, (f_1), $(f_2)(1)$ hold and $\lambda \notin \sigma(\mathcal{A})$. Then there exists $\bar{\varepsilon} > 0$ such that problem (1) has at least one weak solution, provided $|\varepsilon| \leq \bar{\varepsilon}$.*

Theorem 1.4. *Assume that (f_1) and (f_2) hold, $\lambda \notin \sigma(\mathcal{A})$, $f(x,\cdot)$, $g(x,\cdot)$ are odd for all $x \in \Omega$ and*

(Λ) *there exist $h, k \in \mathbb{N}$ such that $\alpha + \lambda < \lambda_h < \lambda_k < \lambda$.*

If we denote by $\bar{m}$, $1 \leq \bar{m} \leq \dim(M_h \oplus \ldots \oplus M_k)$, the number of distinct mini-max critical levels of the unperturbed functional $\mathcal{J}_\lambda$ in (11), then there exists $\bar{\varepsilon} > 0$ such that problem (1) has at least $\bar{m}$ distinct pairs of solutions for any $|\varepsilon| \leq \bar{\varepsilon}$.

In order to deal with the so-called resonant case, we establish the following additional hypotheses:

(f_3) $\exists\, M > 0$ such that $|f(x,t)| \leq M$ for all $(x,t) \in \Omega \times \mathbb{R}$;

(f_4) $\lim_{|t|\to\infty} F(x,t) = l \in \{\pm\infty\}$ uniformly with respect to $x \in \Omega$,
where $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$.

Theorem 1.5. *Assume that $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $(f_1), (f_3), (f_4)$ hold and there exists an integer $k \geq 1$ such that $\lambda = \lambda_k \in \sigma(\mathcal{A})$. Then there exists $\bar{\varepsilon} > 0$ such that problem (1) has at least one solution for all $|\varepsilon| \leq \bar{\varepsilon}$.*

Moreover, if $f(x,\cdot)$, $g(x,\cdot)$ are odd for all $x \in \Omega$ and $(f_2)(2)$ holds, under the further assumption

(Λ') *there exists $h \geq 1$ such that $\alpha + \lambda_k < \lambda_h < \lambda_k$,*

then the same multiplicity result as in Theorem 1.4 holds.

Our last results dealing with non-odd perturbations are weaker than our previous results, in the sense that we have to work with nonlinearities such that the associated functional has only critical levels arising for topological reasons, named *topologically relevant* (cf. Definition 2.11).

Theorem 1.6. *Assume that* $f, g \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, (f_1), (f_2) *and* (Λ) *hold,* $\lambda \notin \sigma(\mathcal{A})$ *and* $f(x,\cdot)$ *is odd for all* $x \in \Omega$.

If all the critical levels of the functional $\mathcal{J}_\lambda$ *in* (11) *are topologically relevant, then the same multiplicity result as in Theorem* 1.4 *holds.*

We can extend the previous theorem to the resonant case as follows.

Theorem 1.7. *Under the assumptions of Theorem* 1.6*, if there exists an integer* $k \geq 1$ *such that* $\lambda = \lambda_k \in \sigma(\mathcal{A})$*, just replacing hypotheses* $(f_2)(1)$, (Λ) *respectively by* (f_3), (Λ') *and adding condition* (f_4)*, the same multiplicity result as in Theorem* 1.4 *holds.*

Remark 1.8. We point out that assumption (Λ) and (Λ') can be replaced by

$$\lambda < \lambda_h < \lambda_k < \alpha + \lambda \quad \text{and} \quad \lambda_k < \lambda_h < \alpha + \lambda_k,$$

cf. [6, Remark 4.2]

The paper is structured as follows. In Section 2 we introduce some notations, preliminary definitions and the variational setting. Section 3 deals with the symmetric case in both the non-resonant case (Subsection 3.1) and the resonant one (Subsection 3.2). Finally, in Section 4, we prove our main results concerning with the perturbed problems.

Notations. Throughout this paper we denote by $(X, \|\cdot\|_X)$ a Hilbert space, by $(X', \|\cdot\|_{X'})$ its dual space, by I a C^1 functional on X and by

- $I^b = \{u \in X : I(u) \leq b\}$ the sublevel of I corresponding to the point $b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$;
- $K_c = \{u \in X : I(u) = c, \ \mathrm{d}I(u) = 0\}$ the set of the critical points of the functional I in X at the critical level $c \in \mathbb{R}$.
- $|\cdot|_s$ the classical norm in the Lebesgue space $L^s(\Omega)$, $1 \leq s \leq +\infty$;
- M_j the eigenspace corresponding to the eigenvalue λ_j of $\mathcal{A}$ in $H_0^{1/2}(\Omega)$ and u_j the projection of u in M_j, for any $j \in \mathbb{N}$ and for each $u \in H_0^{1/2}(\Omega)$;
- $H^-(j) = \bigoplus_{i\leq j} M_i$ and $H^+(j) = \overline{\bigoplus_{i\geq j} M_i}$, for any $i, j \in \mathbb{N}$.

2. Preliminaries

We start this section by recalling the spectral definition of the square root of the Laplacian operator $\mathcal{A} = (-\Delta)^{1/2}$. We point out that, in order to define $\mathcal{A}$, one way to define the fractional powers of a positive operator is through its spectral decomposition, taking the fractional powers of the corresponding eigenvalues, i.e., let $\{\mu_n\}_{n\in\mathbb{N}}$ be the non-decreasing sequence of (positive) eigenvalues with associated eigenfunctions ϕ_n of the Laplace operator in a bounded domain Ω with zero Dirichlet boundary condition,

$$\begin{cases} -\Delta\phi_k = \mu_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We assume that the eigenfunctions ϕ_k are normalized by $\|\phi_k\|_{L^2(\Omega)} = 1$. Let us consider the space of functions

$$H_0^{1/2}(\Omega) = \left\{ u = \sum_{k=1}^{\infty} a_k\phi_k \ : \ \left(\sum_{k=1}^{\infty} a_k^2\mu_k^{1/2}\right)^{1/2} < \infty \right\},$$

with the norm
$$\|u\|_{H_0^{1/2}(\Omega)} = \left(\sum_{k=1}^{\infty} a_k^2\mu_k^{1/2}\right)^{1/2}.$$

The square root of the Laplacian $\mathcal{A}$ is defined by

$$\mathcal{A}u = \sum_{k=1}^{\infty} a_k\mu_k^{1/2}\phi_k\,,$$

for every $u = \sum_{k=1}^{\infty} a_k\phi_k \in H_0^{1/2}(\Omega)$. It is clear that $\mathcal{A}$ maps $H_0^{1/2}(\Omega)$ into its dual space $H^{-1/2}(\Omega)$, being $\{\mu_k^{1/2}, \phi_k\}_{k\in\mathbb{N}}$ the eigenvalues and eigenfunctions of $\mathcal{A}$, with zero Dirichlet boundary conditions. From now on we will denote $\lambda_k = \mu_k^{1/2}$, $k \in \mathbb{N}$. We notice that $\|u\|_{H_0^{1/2}(\Omega)} = \|\mathcal{A}^{1/2}u\|_{L^2(\Omega)}$.

There exists another way of computing the square root of the Laplacian in $\mathbb{R}^N$. It proceeds through the so-called Dirichlet to Neumann operator by Stein [29], which is based on the harmonic extension in one more variable in $\mathbb{R}^{N+1}_+$. We also refer to the work by Caffarelli and Silvestre [11] for a generalized procedure when one consider other fractional powers of the Laplacian. For bounded domains, as in this work, cf., e.g., [8, 9, 10, 12, 22] for the corresponding harmonic extension.

Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain and let us define the associated cylinder $\mathcal{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$. The points in $\mathcal{C}_\Omega$ are denoted by $(x,y) \in \Omega \times (0,\infty)$. The lateral boundary of the cylinder will be denoted by $\partial_L\mathcal{C}_\Omega = \partial\Omega \times (0,\infty)$.

Now, for a function $u \in H_0^{1/2}(\Omega)$, we define its *harmonic extension* $w = \mathrm{E}(u)$ to the cylinder $\mathcal{C}_\Omega$ as the unique solution to the problem

$$\begin{cases} -\Delta w = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L\mathcal{C}_\Omega, \\ w = u & \text{on } \Omega \times \{y = 0\}. \end{cases} \tag{4}$$

The extension function w belongs to the space $X_0(\mathcal{C}_\Omega)$ defined as the completion of $\mathcal{C}_0^\infty(\Omega \times [0,\infty))$ endowed with the norm

$$\|w\|_{X_0(\mathcal{C}_\Omega)} = \left(\int_{\mathcal{C}_\Omega} |\nabla w|^2 \mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

The trace of $w = \mathrm{E}(u)$ is nothing but u, i.e., for $x \in \Omega$,

$$\mathrm{Tr}(\mathrm{E}(u))(x) = \mathrm{E}(u)(x,0) = u(x),$$

which is a continuous operator from $X_0(\mathcal{C}_\Omega)$ into $H_0^{1/2}(\Omega)$ (see [10, Lemma 2.6]). The extension operator E is an isometry between $H_0^{1/2}(\Omega)$ and $X_0(\mathcal{C}_\Omega)$, that is,

$$\|\mathrm{E}(u)\|_{X_0(\mathcal{C}_\Omega)} = \|u\|_{H_0^{1/2}(\Omega)}, \quad \forall\, u \in H_0^{1/2}(\Omega).$$

Even more, for any function $w \in X_0(\mathcal{C}_\Omega)$, we have the following inequality for the trace $w(\cdot, 0) = \mathrm{Tr}(w)(\cdot)$

$$\|\mathrm{Tr}(w)\|_{H_0^{1/2}(\Omega)} \le \|w\|_{X_0(\mathcal{C}_\Omega)}.$$

The relevance of the extension function w is that it is related to the fractional Laplacian of the original function u through the formula

$$\frac{\partial w}{\partial \nu}(x,0) := -\lim_{y\to 0^+} \frac{\partial w}{\partial y}(x,y) = \mathcal{A}u(x), \tag{5}$$

see for instance [8, 10, 11, 12, 29] for more details.

Another tool which is very useful in what follows is the fractional Sobolev inequality: namely, for every function $v \in H_0^{1/2}(\Omega)$ we have

$$\left(\int_\Omega |v|^r \,\mathrm{d}x\right)^{2/r} \le C \int_\Omega |\mathcal{A}^{1/2} v|^2 \,\mathrm{d}x, \tag{6}$$

for any $1 \le r \le 2^\sharp$, where $2^\sharp = \frac{2N}{N-1}$ denotes the critical Sobolev exponent. Furthermore we have the following trace inequality

$$\left(\int_\Omega |\mathrm{Tr}(z)(x)|^r \,\mathrm{d}x\right)^{2/r} \le C \int_{\mathcal{C}_\Omega} |\nabla z(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{7}$$

for any $1 \le r \le 2^\sharp$, and any $z \in X_0(\mathcal{C}_\Omega)$, for a universal constant $C = C(r, N, \Omega) > 0$. We recall that when $r = 2^\sharp$, the best constant in (6)-(7) is not achieved in any bounded domain; see for instance [21].

Note that by the Sobolev inequality (6) we have the continuous embedding $H_0^{1/2}(\Omega) \hookrightarrow L^r(\Omega)$ for any $1 \le r \le 2^\sharp$, which clearly is compact for $r < 2^\sharp$.

Remark 2.1. Let us point out that the operator $\mathcal{A}$ should not be confused with the integro-differential operator defined, up to a constant, by

$$(-\Delta)^{1/2} u(x) := -\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+1}} \mathrm{d}y, \quad x \in \mathbb{R}^N.$$

A detailed analysis of the differences between these operators can be found in [23, 24, 28]. See also the papers [17, 25, 26] for some recent interesting results related to this subject.

Moreover, we would like to notice that techniques analogous to those used in this paper can be applied to problem (1) led by the operator $\mathcal{A}_{a/2}$ for some $0 < a < 2$ (cf., e.g., [4, 8]). □

If f belongs to the dual space $H^{-1/2}(\Omega)$ of $H_0^{1/2}(\Omega)$, we consider the Dirichlet problem

$$\begin{cases} \mathcal{A}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Since the definition of the fractional Laplacian allows one to integrate by parts in the proper spaces, a natural definition of energy solution to problem (8) is the following.

Definition 2.2. We say that $u \in H_0^{1/2}(\Omega)$ is an *energy solution* of (8) if the identity

$$\int_\Omega \mathcal{A}^{1/2}u\, \mathcal{A}^{1/2}\varphi = \int_\Omega f\varphi$$

holds for every test function $\varphi \in H_0^{1/2}(\Omega)$. □

In [10] it is proved the existence and uniqueness of energy solution to problem (8) provided $f \in H^{-1/2}(\Omega)$. This means that the inverse operator $K = (\mathcal{A})^{-1}$ is well defined from $H^{-1/2}(\Omega)$ into $H_0^{1/2}(\Omega)$.

On the other hand, by using the Dirichlet to Neumann operator, we can also characterize the energy solutions of (8) through the problem

$$\begin{cases} -\Delta w = 0 & \text{in } \mathcal{C}_\Omega, \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial w}{\partial \nu} = f & \text{on } \Omega \times \{y = 0\}. \end{cases}$$

Indeed, if $w \in X_0(\mathcal{C}_\Omega)$ is an energy solution of this problem, i.e., if

$$\int_{\mathcal{C}_\Omega} \langle \nabla w, \nabla \varphi \rangle \, \mathrm{d}x\, \mathrm{d}y = \int_\Omega f\, \mathrm{Tr}(\varphi)\, \mathrm{d}x, \qquad \forall\, \varphi \in X_0(\mathcal{C}_\Omega),$$

then the trace function $u(\cdot) = w(\cdot, 0) = \mathrm{Tr}(w)(\cdot)$ belongs to the space $H_0^{1/2}(\Omega)$ and it is an energy solution of problem (8). The converse is also true.

As a consequence of the arguments above, given f satisfying (f_1) and (f_2), we say that $u = \mathrm{Tr}(w) \in H_0^{1/2}(\Omega)$ is an *energy solution* of $(P_{\lambda,0})$ if $w \in X_0(\mathcal{C}_\Omega)$ is an energy solution of

$$\begin{cases} -\div(\nabla w) = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu}(x,0) = \lambda \mathrm{Tr}(w)(x) + f(x, \mathrm{Tr}(w)(x)); & x \in \Omega, \end{cases} \tag{9}$$

that is, for all $\varphi \in X_0(\mathcal{C}_\Omega)$,

$$\int_{\mathcal{C}_\Omega} \langle \nabla w, \nabla \varphi \rangle \,\mathrm{d}x\,\mathrm{d}y = \lambda \int_\Omega \mathrm{Tr}(w)(x)\mathrm{Tr}(\varphi)(x)\,\mathrm{d}x + \int_\Omega f(x, \mathrm{Tr}(w)(x))\mathrm{Tr}(\varphi)(x)\,\mathrm{d}x,$$

By (f_1) and $(f_2)(1)$, for all $\sigma > 0$ there exists $a_\sigma > 0$ such that

$$|f(x,t)| \le \sigma|t| + a_\sigma \quad \text{for all } t \in \mathbb{R} \quad \text{and } x \in \Omega. \tag{10}$$

Therefore, the energy functional $\mathcal{J}_\lambda : X_0(\mathcal{C}_\Omega) \to \mathbb{R}$ associated to (9) is defined by

$$\mathcal{J}_\lambda(w) := \frac{1}{2}\int_{\mathcal{C}_\Omega} |\nabla w(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{2}\int_\Omega |\mathrm{Tr}(w)(x)|^2\,\mathrm{d}x - \int_\Omega F(x, \mathrm{Tr}(w)(x))\,\mathrm{d}x, \tag{11}$$

for every $w \in X_0(\mathcal{C}_\Omega)$, where $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$ and it is of class C^1 on $X_0(\mathcal{C}_\Omega)$ (cf., e.g., [2, Theorem 4]). Moreover, the traces of its critical points are energy solutions of $(P_{\lambda,0})$, therefore we look for critical points of functional $\mathcal{J}_\lambda$.

Following [6, Section 2], we recall some standard notions and known results that we will use below.

Besides some existence critical point theorems, sharper multiplicity results can be stated when one deals with symmetric functionals on Hilbert spaces (cf., e.g., [1, 5, 27]). In this section we recall [5, Theorem 2.9], therefore we need to introduce a pseudo-index theory and the notion of the index theory for an even functional with symmetry group $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ (cf., e.g., [27]).

Let us set

$$\Sigma = \Sigma(X) = \left\{ A \subseteq X : \begin{array}{l} A \text{ closed and symmetric with respect} \\ \text{to the origin, i.e., } -u \in A \text{ if } u \in A \end{array} \right\}$$

and

$$\mathcal{H} = \{h \in C(X,X) : h \text{ odd}\}.$$

For $A \in \Sigma$, $A \neq \emptyset$, the *genus* of A is defined by

$$\gamma(A) = \inf\{k \in \mathbb{N} : \exists\, \psi \in C(A, \mathbb{R}^k \setminus \{0\}) \text{ such that } \psi(-u) = -\psi(u) \text{ for all } u \in A\},$$

if the infimum above exists; on the contrary $\gamma(A) = +\infty$. We assume that $\gamma(\emptyset) = 0$. The index theory $(\Sigma, \mathcal{H}, \gamma)$ related to $\mathbb{Z}_2$ is also called *genus*.

The pseudo-index related to the genus, for an even functional $I : X \to \mathbb{R}$ and $S \in \Sigma$ is the triplet $(S, \mathcal{H}^*, \gamma^*)$ such that $\mathcal{H}^*$ is a group of bounded homeomorphisms h with $h(u) = u$ if $u \notin I^{-1}(]0,\infty[)$, and $\gamma^* : \Sigma \longrightarrow \mathbb{N} \cup \{+\infty\}$ is the map defined by

$$\gamma^*(A) = \min_{h \in \mathcal{H}^*} \gamma(h(A) \cap S) \quad \text{for all } A \in \Sigma.$$

It results that $\qquad \gamma(h(A) \cap S) = \gamma(A \cap h^{-1}(S)) \quad$ for all $h \in \mathcal{H}^*$,

then $\qquad \gamma^*(A) = \min_{h\in\mathcal{H}^*} \gamma(A \cap h(S)) \quad$ for all $A \in \Sigma$.

We enunciate the following *mini-max* theorem, that is essentially stated by [5, Theorem 2.9] in a more general setting.

Theorem 2.3. [5, Theorem 2.9] *Consider $a, b, c_0, c_\infty \in \bar{\mathbb{R}}$ such that we have $-\infty \le a < c_0 < c_\infty < b \le +\infty$. Let I be an even functional, $(\Sigma, \mathcal{H}, \gamma)$ the genus theory on X, $S \in \Sigma$, $(S, \mathcal{H}^*, \gamma^*)$ the pseudo-index theory related to the genus, I and S, with*

$$\mathcal{H}^* = \{h \in \mathcal{H} : h \text{ bounded homeomorphism s.t. } h(u) = u \text{ if } u \notin I^{-1}(]a, b[)\}.$$

Assume that:

(i) *the functional I satisfies the Palais-Smale condition in $]a, b[$;*

(ii) *$S \subseteq I^{-1}([c_0, +\infty[)$;*

(iii) *there exist $\tilde{k} \in \mathbb{N}$ and $\tilde{A} \in \Sigma$ such that $\tilde{A} \subseteq I^{c_\infty}$ and $\gamma^*(\tilde{A}) \ge \tilde{k}$.*

Then the numbers $\qquad c_i = \inf_{A\in\Sigma_i^*} \sup_{u\in A} I(u), \qquad i \in \{1, \dots, \tilde{k}\}, \qquad (12)$

with $\Sigma_i^* = \{A \in \Sigma : \gamma^*(A) \ge i\}$, *are critical values for I and*

$$c_0 \le c_1 \le \ldots \le c_{\tilde{k}} \le c_\infty.$$

Furthermore, if $c = c_i = \ldots = c_{i+r}$, with $i \ge 1$ and $i + r \le \tilde{k}$, then $\gamma(K_c) \ge r + 1$.

In order to apply the theorem above, we need the following result, which allows us to obtain a lower bound for the pseudo-index of a suitable $\tilde{A}$ as in (iii); for the proof we refer to [5, Theorem A.2].

Theorem 2.4. [5, Theorem A.2] *Let $(\Sigma, \mathcal{H}, \gamma)$ be the genus theory on X and V, W two closed subspaces of X. Assume that*

$$\dim V < +\infty \qquad \textit{and} \qquad \operatorname{codim} W < +\infty.$$

Then, for every odd bounded homeomorphism h on X and every open bounded symmetric neighbourhood B of 0 in X, it results

$$\gamma(V \cap h(\partial B \cap W)) \ge \dim V - \operatorname{codim} W.$$

Since in our main theorems we may deal with problems (1), possibly without a variational structure on $H_0^{1/2}(\Omega)$, following [20], we use the auxiliary notion of *essential value*, as it is introduced in [15] (cf. also [14]) in the study of perturbations of non-smooth functionals. Even more, for even functionals we introduce the definition of *odd-essential value*, which is useful in order to prove multiplicity results for odd perturbations.

Definition 2.5. Assume that $I\colon X \to \mathbb{R}$ is a continuous functional (resp. I even continuous) and $a, b \in \bar{\mathbb{R}}$, with $a \le b$. We say that the pair (I^b, I^a) is *trivial* (resp. *odd-trivial*) if, for each neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b in $\overline{\mathbb{R}}$, there exists a continuous (resp. an odd continuous) map $\varphi\colon I^{\beta'} \times [0,1] \to I^{\beta''}$ satisfying

(i) $\varphi(x, 0) = x$ for each $x \in I^{\beta'}$;

(ii) $\varphi(I^{\beta'} \times \{1\}) \subseteq I^{\alpha''}$;

(iii) $\varphi(I^{\alpha'} \times [0,1]) \subseteq I^{\alpha''}$.

Definition 2.6. Suppose that $I\colon X \to \mathbb{R}$ is a continuous functional (resp. I even continuous). We say that $c \in \mathbb{R}$ is an *essential value* (resp. an *odd-essential value*) of I if for each $\varepsilon > 0$ there exist $a, b \in]c - \varepsilon, c + \varepsilon[$, $a < b$, such that the pair (I^b, I^a) is not trivial (resp. not odd-trivial).

The next result establishes that small perturbations of a continuous functional preserve the essential values (cf. [15, Theorem 3.1] and [14, Theorem 2.6]); in particular this holds for the odd ones, just up to some small variations in the proof of [15, Theorem 3.1].

Theorem 2.7. *Let $c \in \mathbb{R}$ be an essential value (resp. odd-essential value) of $I\colon X \to \mathbb{R}$ continuous (resp. I even continuous). Then, for every $\eta > 0$ there exists $\delta > 0$ such that every functional (resp. even functional) $G \in C(X, \mathbb{R})$ with*

$$\sup\{|I(u) - G(u)| : u \in X\} < \delta$$

admits an essential value (resp. odd-essential value) in $]c - \eta, c + \eta[$.

On the setting of smooth functionals we recall some results which link critical and essential values. In particular, the critical values arising from mini-max procedures are essential, if all the involved deformations are of the "same kind" (cf. [15, Theorems 3.7 and 3.9]).

Theorem 2.8. *Let $c \in \mathbb{R}$ be an essential value of $I \in C^1(X, \mathbb{R})$. If the Palais-Smale condition at level c holds, then c is a critical value of I.*

Note that in general the reverse implication does not hold: even if the Palais-Smale condition at level c holds, a critical value is not necessarily an essential one (see e.g. [15, Example 3.12]).

Theorem 2.9. *Suppose that Γ is a non empty family of non empty subsets of X, $I \in C^1(X, \mathbb{R})$ and $d \in \mathbb{R} \cup \{-\infty\}$. Suppose also that for every $C \in \Gamma$ and for every deformation $\varphi\colon X \times [0,1] \longrightarrow X$ with $\varphi(u, t) = u$ on $I^d \times [0,1]$, there holds $\overline{\varphi(C \times \{1\})} \in \Gamma$. Setting*

$$c = \inf_{C \in \Gamma} \sup_{u \in C} I(u), \tag{13}$$

if $d < c < +\infty$, then c is an essential value of I.

It is relevant to point out that, if I is even, the previous theorem does not apply to the critical values c_i given by (12); indeed, if φ is a deformation as in Theorem

2.9, the set $\overline{\varphi(C \times \{1\})}$, $C \in \Gamma$, does not necessarily belong to Γ because φ could be not odd. As a consequence, we cannot assert that the c_i values are essential values of I. However, slight modifications in the proof of [15, Theorem 3.9] allow us to state the following result concerning odd-essential values.

Corollary 2.10. *Assume that Γ is a non empty family of non empty symmetric subsets of X, $I \in C^1(X, \mathbb{R})$ even and $d \in \mathbb{R} \cup \{-\infty\}$. Let us suppose that, for every $C \in \Gamma$ and for every odd deformation $\varphi\colon X \times [0,1] \longrightarrow X$ with $\varphi(u,t) = u$ on $I^d \times [0,1]$, there holds $\overline{\varphi(C \times \{1\})} \in \Gamma$. Then, choosing c as in* (13)*, if $d < c < +\infty$, we have that c is an odd-essential value of I.*

Since we also deal with perturbation from symmetric problems, we restrict ourselves to consider a subset of the critical values of the functional defined by (11), hence a stability result does still hold (we also refer to [19] for a related result concerning sign-changing solutions of some elliptic problems).

Actually, in the following Corollary 2.12 we consider just the preservation under small perturbations of some critical levels of a smooth functional satisfying the Palais-Smale condition. From the Deformation Lemma, if $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level c and c is not a critical value, then for any $\bar{\eta} > 0$ there exists $\eta \in]0, \bar{\eta}[$ and $\varphi \in C(X \times [0,1], X)$, which is odd if I is even, satisfying $\varphi(u,1) = u$ if $I(u) \notin [c - \bar{\eta}, c + \bar{\eta}]$ and $\varphi(I^{c+\eta}, 1) \subset I^{c-\eta}$. In some sense now we require that also the other implication is true, i.e., if $I^{c-\eta}$ is a strong deformation retract of $I^{c+\eta}$, then c is not critical.

Definition 2.11. Let $I \in C^1(X, \mathbb{R})$ be a functional. We say that a critical level c of I is *topologically relevant* if it is an essential value of I.

As we work with the special class of critical levels which are essential too, according to Definition 2.11, we point out some consequences of Theorems 2.7 and 2.8 in the following corollary.

Corollary 2.12. *Let $c \in \mathbb{R}$ be a topologically relevant critical value of a functional $I \in C^1(X, \mathbb{R})$. Then, for every $\eta > 0$ there exists $\delta > 0$ such that every functional $G \in C^1(X, \mathbb{R})$ satisfying (PS) in $]c - \eta, c + \eta[$ with*

$$\sup\{|J(u) - G(u)| : u \in X\} < \delta$$

admits a critical value in $]c - \eta, c + \eta[$.

3. The symmetric case

This section is devoted to study the symmetric case. It is divided in two subsections, the first one deals with the non-resonant problem, while in the second one is studied the resonant case.

3.1. The non-resonant case

In order to simplify the notations, we set from now on $X := X_0(\mathcal{C}_\Omega)$.

As recalled in Section 2 the operator $\mathcal{A}$ has a countable, increasing, diverging sequence of distinct eigenvalues $(\lambda_j)_j$ and, if we denote by $(\mu_j)_j$ the sequence of eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, we have $\lambda_j = \mu_j^{1/2}$. The sequence $(\varphi_j)_j$ of eigenfunctions corresponding to λ_j is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^{1/2}(\Omega)$.

Next we recall some relations between the eigenvalues $(\lambda_j)_j$ of the square root of the Laplacian and of the corresponding extended eigenvalue problem in the half cylinder $\mathcal{C}_\Omega$:

$$\begin{cases} -\Delta w = 0 & \text{in } \mathcal{C}_\Omega \\ w = 0 & \text{on } \partial_L \mathcal{C}_\Omega \\ \dfrac{\partial w}{\partial \nu} = \lambda_j w & \text{on } \Omega \times \{y = 0\}, \end{cases} \tag{14}$$

where we remind that $\frac{\partial w}{\partial \nu}$ is defined by (5).

Setting with an abuse of notation $H^-(k) := \text{span}\{w_1, \dots, w_k\}$, with w_j solution of (14) for all $j \in \{1, \dots, k\}$, it follows that $\text{Tr}(w_j) = \varphi_j$, being $(\varphi_j)_j$ the basis of eigenfunctions of $\mathcal{A}$ in $H_0^{1/2}(\Omega)$. Then, we denote for every $k \in \mathbb{N}$ by $H^+(k+1)$ its orthogonal complement in X. Therefore $X = H^-(k) \oplus H^+(k+1)$. Let us observe that $(w_j)_j$ is an orthogonal system in X since $(\text{Tr}(w_j))_j$ is an orthonormal system in $L^2(\Omega)$. The following inequalities hold (we refer to [2, Section 2] for more details):

$$\|w\|_X^2 \le \lambda_k |\text{Tr}(w)|_2^2 \quad \text{on } H^-(k) \tag{15}$$

and

$$\lambda_{k+1} |\text{Tr}(w)|_2^2 \le \|w\|_X^2 \text{ on } H^+(k+1). \tag{16}$$

At first, we deal with problem (1) for $\varepsilon = 0$, i.e., in the symmetric case.

Theorem 3.1. *Assume that* (f_1) *and* $(f_2)(1)$ *hold. Then, if* $\lambda \notin \sigma(\mathcal{A})$*, problem* $(P_{\lambda,0})$ *has at least a solution. Moreover, if* $f(x,\cdot)$ *is odd for all* $x \in \Omega$ *and assumptions* $(f_2)(2)$*,* (Λ) *hold, then problem* $(P_{\lambda,0})$ *has at least* $\dim(M_h \oplus \ldots \oplus M_k)$ *distinct pairs of non-trivial solutions.*

In order to apply variational methods, we need the following lemma.

Lemma 3.2. *Assume that* (f_1) *and* $(f_2)(1)$ *hold. Then, if* $\lambda \notin \sigma(\mathcal{A})$*, the functional* $\mathcal{J}_\lambda$ *satisfies the Palais-Smale condition in* $\mathbb{R}$*, i.e., for any* $c \in \mathbb{R}$ *and any sequence* $(w_n)_n$ *in* X *satifying*

$$\mathcal{J}_\lambda(w_n) \longrightarrow c \quad \textit{and} \quad \sup\{|\langle \mathcal{J}'_\lambda(w_n), \varphi\rangle| : \varphi \in X,\ \|\varphi\|_X = 1\} \to 0 \textit{ as } n \to \infty,$$

there exists a strongly convergent subsequence in X*.*

Proof. Let $c \in \mathbb{R}$ and $(w_n)_n$ be a sequence in X such that

$$\mathcal{J}_\lambda(w_n) \to c \quad \text{and} \quad \|\mathcal{J}'_\lambda(w_n)\|_{X'} \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

Then, we have, for all $\varphi \in X$,

$$\int_{\mathcal{C}_\Omega} \langle \nabla w_n, \nabla \varphi \rangle \, \mathrm{d}x \, \mathrm{d}y - \lambda \int_\Omega \mathrm{Tr}(w_n)\mathrm{Tr}(\varphi) \, \mathrm{d}x - \int_\Omega f(x, \mathrm{Tr}(w_n))\mathrm{Tr}(\varphi) \, \mathrm{d}x = o(1), \tag{18}$$

where $o(1) \to 0$ as $n \to \infty$.

As a first step we prove that the Palais-Smale sequence $(w_n)_n$ is bounded in X. To do so, we argue by contradiction, i.e., we assume that

$$\|w_n\|_X \to \infty \qquad \text{as } n \to \infty. \tag{19}$$

Without loss of generality we can assume that $\|w_n\|_X > 0$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. We set $v_n = w_n/\|w_j\|_X$; plainly $(v_n)_n$ is bounded in X and there exists $v \in X$ such that, up to subsequences, for $n \to \infty$ we have

$$v_n \rightharpoonup v \ \text{ weakly in } X \ \text{ and } \ \mathrm{Tr}(v_n) \to \mathrm{Tr}(v) \ \text{ strongly in } L^2(\Omega). \tag{20}$$

By (18) taking $\varphi := v_n - v$, and dividing by $\|w_n\|_X$ we infer that

$$\int_{\mathcal{C}_\Omega} \langle \nabla v_n, \nabla(v_n - v) \rangle \, \mathrm{d}x \, \mathrm{d}y = \tag{21}$$

$$= \lambda \int_\Omega \mathrm{Tr}(v_n)(\mathrm{Tr}(v_n) - \mathrm{Tr}(v)) \, \mathrm{d}x + \int_\Omega \frac{f(x, \mathrm{Tr}(w_n))}{\|w_n\|_X}(\mathrm{Tr}(v_n) - \mathrm{Tr}(v)) \, \mathrm{d}x + o(1).$$

On the other hand, by (20) it follows that

$$\left| \int_\Omega \mathrm{Tr}(v_n)(\mathrm{Tr}(v_n) - \mathrm{Tr}(v)) \, \mathrm{d}x \right| \leq |\mathrm{Tr}(v_n)|_2 |\mathrm{Tr}(v_n) - \mathrm{Tr}(v)|_2 = o(1), \tag{22}$$

so that (10), (19) and (20) imply that

$$\left| \int_\Omega \frac{f(x, \mathrm{Tr}(w_n))}{\|w_n\|_X}(\mathrm{Tr}(v_n) - \mathrm{Tr}(v)) \, \mathrm{d}x \right| \leq$$

$$\leq \sigma \frac{|\mathrm{Tr}(w_n)|_2 |\mathrm{Tr}(v_n) - \mathrm{Tr}(v)|_2}{\|w_n\|_X} + a_\sigma \frac{|\mathrm{Tr}(v_n) - \mathrm{Tr}(v)|_1}{\|w_n\|_X} = o(1). \tag{23}$$

From (21)–(23) it results that

$$\int_{\mathcal{C}_\Omega} \langle \nabla v_n, \nabla(v_n - v) \rangle \, \mathrm{d}x\mathrm{d}y = o(1)$$

which implies $\qquad v_n \to v \quad$ strongly in $X \quad$ as $n \to \infty$, (24)

and even more using that $\|v_n\|_X = 1,$ then $v \neq 0$.

Now, dividing (18) by $\|w_n\|_X$, it follows that

$$\int_{\mathcal{C}_\Omega} \langle \nabla v_n, \nabla \varphi \rangle \, \mathrm{d}x\mathrm{d}y =$$

$$= \lambda \int_\Omega \mathrm{Tr}(v_n)\mathrm{Tr}(\varphi) \, \mathrm{d}x + \int_\Omega \frac{f(x, \mathrm{Tr}(w_n))}{\|w_n\|_X} \mathrm{Tr}(\varphi) \, \mathrm{d}x + o(1). \tag{25}$$

From (19) and $(f_2)(1)$ we infer that

$$\lim_{n\to\infty} \int_\Omega \frac{f(x, \mathrm{Tr}(w_n))}{\|w_n\|_X} \mathrm{Tr}(\varphi) \, \mathrm{d}x = 0 \quad \text{for all } \varphi \in X,$$

therefore by (24) passing to the limit in (25) as $n \to \infty$ we have that

$$\int_{\mathcal{C}_\Omega} \nabla v \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}y = \lambda \int_\Omega \mathrm{Tr}(v)\mathrm{Tr}(\varphi) \, \mathrm{d}x \quad \text{for all } \varphi \in X,$$

i.e., $z := \mathrm{Tr}(v)$ is an energy solution of the problem

$$\begin{cases} \mathcal{A}z = \lambda z & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

This is a contradiction because $\lambda \notin \sigma(\mathcal{A})$. Therefore the sequence $(\|w_n\|_X)_n$ is bounded. In order to conclude the proof, we demonstrate that every Palais-Smale sequence strongly converges, up to subsequences. Let us point out that, up to subsequences, there exists $w \in X$ such that, as $n \to \infty$

$$w_n \rightharpoonup w \text{ weakly in } X \quad \text{and} \quad \mathrm{Tr}(w_n) \to \mathrm{Tr}(w) \text{ strongly in } L^2(\Omega). \tag{26}$$

By (17) and (26)

$$J'_\lambda(w_n)[w_n - w] \to 0 \quad \text{as } n \to \infty; \tag{27}$$

moreover, (10) implies

$$\int_\Omega |f(x, \mathrm{Tr}(w_n))| \mathrm{Tr}(w_n) - \mathrm{Tr}(w)| \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{28}$$

Then, by (18), (26), (27) and (28) we obtain

$$\|w_n\|_X^2 - \int_{\mathcal{C}_\Omega} \langle \nabla w_n, \nabla w \rangle \, \mathrm{d}x \, \mathrm{d}y \to 0 \quad \text{as } n \to \infty. \tag{29}$$

Finally, (26) jointly with (29) imply that

$$w_n \to w \quad \text{strongly in } X \quad \text{as } n \to \infty$$

which completes the proof. □

Proof of Theorem 3.1. Here we focus on the multiplicity result. By Lemma 3.2 the functional $\mathcal{J}_\lambda$ satisfies that Palais-Smale condition at any level $c \in \mathbb{R}$. Moreover, (Λ) implies $\alpha < 0$. Notice that by (f_2) it follows that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = 0 \in \mathbb{R} \quad \text{uniformly with respect to } x \in \Omega, \text{ and}$$

$$\lim_{t\to 0} \frac{F(x,t)}{|t|^2} = \frac{\alpha}{2} \in \mathbb{R} \quad \text{uniformly with respect to } x \in \Omega;$$

hence, for any $\sigma > 0$, there exist $R_\sigma, \delta_\sigma > 0$ (without loss of generality $R_\sigma \geq 1$) such that

$$|F(x,t)| \leq \frac{\sigma}{2}|t|^2 \quad \text{if } |t| > R_\sigma, \text{ for all } x \in \Omega \tag{30}$$

and
$$\left|F(x,t) - \frac{\alpha}{2}|t|^2\right| \leq \frac{\sigma}{2}|t|^2 \quad \text{if } |t| < \delta_\sigma, \text{ for all } x \in \Omega.$$

Moreover, by (f_1), taking any $q \in [0, 2^\sharp - 2[$ there exists $a_{R_\sigma} > 0$ such that, if $\delta_\sigma \leq |t| \leq R_\sigma$, for all $x \in \Omega$, we have

$$|F(x,t)| \leq a_{R_\sigma}|t|^{q+2}. \tag{31}$$

Now, (30)–(31) imply that for any $\sigma > 0$ there exists $a_\sigma > 0$ large enough such that for all $x \in \Omega$ and for all $t \in \mathbb{R}$

$$F(x,t) \leq \frac{\sigma + \alpha}{2}|t|^2 + a_\sigma |t|^{q+2},$$

which implies

$$\int_\Omega F(x, \mathrm{Tr}(w))\, \mathrm{d}x \leq \frac{(\sigma+\alpha)}{2}|\mathrm{Tr}(w)|_2^2 + a_\sigma |\mathrm{Tr}(w)|_{q+2}^{q+2} \quad \text{for all } w \in X.$$

By the Sobolev embeddings, for a suitable $a'_\sigma > 0$ it results that

$$\mathcal{J}_\lambda(w) \geq \frac{1}{2}\|w\|_X^2 - \frac{\lambda + \alpha + \sigma}{2}|\mathrm{Tr}(w)|_2^2 - a'_\sigma \|w\|_X^{q+2} \quad \text{for all } w \in X.$$

Hence (16) implies

$$\mathcal{J}_\lambda(w) \geq \frac{1}{2}\left(1 - \frac{\lambda+\alpha+\sigma}{\lambda_h}\right)\|w\|_X^2 - a'_\sigma \|w\|_X^{q+2} \quad \text{for all } w \in H^+(h).$$

Therefore, by (Λ) for σ small enough, there exists $a''_\sigma > 0$ such that

$$\mathcal{J}_\lambda(w) \geq a''_\sigma \|w\|_X^2 - a'_\sigma \|w\|^{q+2} \quad \text{for all } w \in H^+(h).$$

So, taking $\rho > 0$ sufficiently small, there exists $c_0 > 0$ such that

$$\mathcal{J}_\lambda(w) \geq c_0 \quad \text{for all } w \in S_\rho \cap H^+(h),$$

where $S_\rho = \{w \in X : \|w\|_X = \rho\}$.

On the other hand we notice that by (10), fixed $\sigma > 0$ there exists $a'_\sigma > 0$ such that

$$\mathcal{J}_\lambda(w) \leq \frac{1}{2}\|w\|_X^2 - \frac{\lambda}{2}|\mathrm{Tr}(w)|_2^2 + \frac{\sigma}{2}|\mathrm{Tr}(w)|_2^2 + a'_\sigma|\mathrm{Tr}(w)|_2. \tag{32}$$

Taking λ_k as in (Λ) and $\sigma > 0$ such that $\lambda_k + \sigma < \lambda$, it results that

$$\mathcal{J}_\lambda(w) \leq \frac{1}{2}\left(\lambda_k + \sigma - \lambda\right)|\mathrm{Tr}(w)|_2^2 + a'_\sigma|\mathrm{Tr}(w)|_2 \quad \text{for all } w \in H^-(k).$$

Then functional $\mathcal{J}_\lambda$ tends to $-\infty$ as $\|w\|_X$ diverges in $H^-(k)$, so there exists $c_\infty > c_0$ such that

$$\mathcal{J}_\lambda(w) \leq c_\infty \quad \text{for all } u \in H^-(k).$$

Finally, considering the pseudo-index theory $(S_\rho \cap H^+(h), \mathcal{H}^*, \gamma^*)$ related to the genus, $S_\rho \cap H^+(h)$ and $\mathcal{J}_\lambda$, by Theorem 2.4 (applied to $V = H^-(k), \partial B = S_\rho$ and $W = H^+(h)$) we get

$$\gamma\left(H^-(k) \cap h\left(S_\rho \cap H^+(h)\right)\right) \geq \dim H^-(k) - \text{ codim } H^+(h) \quad \text{for all } h \in \mathcal{H}^*.$$

As a consequence, Theorem 2.3 applies with $\tilde{A} := H^-(k)$ and $S := S_\rho \cap H^+(h)$, hence $\mathcal{J}_\lambda$ has at least $\dim(M_h \oplus \ldots \oplus M_k)$ distinct pairs of critical points corresponding to $\dim(M_h \oplus \ldots \oplus M_k)$ distinct critical values c_i, where c_i are as in (12).

Now, let us observe that, if Z denotes the set of $\dim(M_h \oplus \ldots \oplus M_k)$ distinct pairs of critical points of $\mathcal{J}_\lambda$, the set $\mathrm{Tr}(Z)$ contains $\dim(M_h \oplus \ldots \oplus M_k)$ distinct pairs of energy solutions of problem $(P_{\lambda,0})$ and the proof is complete. □

3.2. The resonant case

In this subsection we deal again with problem (1) for $\varepsilon = 0$, but we also admit the resonant case. More precisely, we have the following.

Theorem 3.3. *Assume that there exists an integer $k \geq 1$ such that $\lambda = \lambda_k \in \sigma(\mathcal{A})$. If moreover (f_1), (f_3) and (f_4) hold, then problem $(P_{\lambda,0})$ has at least a solution. Moreover, if $f(x,\cdot)$ is odd for all $x \in \Omega$ and assumptions $(f_2)(2)$, (Λ') hold, then problem $(P_{\lambda,0})$ has at least* $\dim(M_h \oplus \ldots \oplus M_k)$ *distinct pairs of non-trivial solutions.*

Proof. The existence result follows by [27, Theorem 4.12] up to slight modifications. Indeed, by standard computations (cf., e.g., [16, Lemma 3.1, Lemma 3.2] it follows that

$$\frac{1}{2}\|w\|_X^2 - \frac{\lambda_k}{2}|\mathrm{Tr}(w)|_2^2 \geq \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|w\|_X^2 \quad \text{for all } w \in H^+(k+1)$$

and that there exists $\bar{M} > 0$ such that

$$\|w\|_X^2 - \lambda_k|\mathrm{Tr}(w)|_2^2 \leq -\bar{M}\|w^-\|_X^2 \quad \text{for all } w \in H^-(k)$$

where $w = w^0 + w^-$, with $w^0 \in M_k$ and $w^- \in H^-(k-1)$. Moreover by (f_3) and (f_4) it follows that for a suitable positive constant M^*

$$\left|\int_\Omega F(x, \mathrm{Tr}(w))\, \mathrm{d}x\right| \le M^* \|w\|_X \qquad \text{for all } w \in X,$$

and

$$\int_\Omega F(x, \mathrm{Tr}(w))\, \mathrm{dx} \to \infty$$

as $\|w\|_X \to \infty$, uniformly for $w \in H^-(k-1)$ (cf., e.g, [16, Lemma 3.3 and Lemma 3.4]. Therefore we easily get that

$$\liminf_{w \in H^+(k+1), \|w\|_X \to \infty} \frac{\mathcal{J}_{\lambda_k}(w)}{\|w\|_X} > 0 \qquad \text{and} \qquad \lim_{w \in H^-(k), \|w\|_X \to \infty} \mathcal{J}_{\lambda_k}(w) = -\infty$$

(cf., e.g, [16, Proposition 4.1 and Proposition 4.2]). Finally, in order to prove the boundedness of every Palais-Smale sequence, we can follow [16, Proposition 4.3], then we can conclude as in [16, Proposition 4.4] (cf. also here the proof of Lemma 3.2). Passing to the trace of the found solution, we conclude the proof. Up to some small modifications in the proof of Theorem 3.1, we obtain also the multiplicity result. □

4. Perturbed problems

Proof of Theorem 1.3. For $\varepsilon = 0$, the existence of at least one solution is a consequence of the Saddle Point Theorem (cf. [27, Theorem 4.6]). Indeed, by Lemma 3.2 we have just to study the geometric structure of the problem in order to find a critical point of the unperturbed functional $\mathcal{J}_\lambda$ in (11). By (10), with fixed $\sigma > 0$ there exists $a'_\sigma > 0$, we set $\lambda_h < \lambda$ and take $\sigma > 0$ such that $\lambda_h + \sigma < \lambda$; then by (15) and (32), we get

$$\mathcal{J}_\lambda(w) \le \frac{1}{2}\,(\lambda_h + \sigma - \lambda)\, |\mathrm{Tr}(w)|_2^2 + a'_\sigma |\mathrm{Tr}(w)|_2 \qquad \text{on } H^-(h).$$

By (15) it follows also that $|\mathrm{Tr}(w)|_2 \to \infty$ when $w \in H^-(h)$ and $\|w\|_X \to \infty$, which implies that

$$\mathcal{J}_\lambda(w) \to -\infty \qquad \text{as } \|w\|_X \to \infty \qquad \text{and } w \in H^-(h),$$

then there exists $\delta > 0$ such that

$$\mathcal{J}_\lambda(w) \le -\delta \qquad \text{on } w \in H^-(h).$$

Again by (10) for all $w \in X$ it follows that

$$\frac{\lambda}{2} \int_\Omega |\mathrm{Tr}(w)|^2\, \mathrm{d}x + \int_\Omega F(x, \mathrm{Tr}(w))\, \mathrm{d}x \le \frac{1}{2}(\lambda + \sigma)|\mathrm{Tr}(w)|_2^2 + a_\sigma |\mathrm{Tr}(w)|_1,$$

which, jointly with (16), shows that, on $H^+(h+1)$,

$$\frac{\lambda}{2}\int_\Omega |\mathrm{Tr}(w)|^2\,\mathrm{d}x + \int_\Omega F(x,\mathrm{Tr}(w))\,\mathrm{d}x \le \frac{1}{2\lambda_{h+1}}(\lambda+\sigma)\|w\|_X^2 + a_\sigma|\mathrm{Tr}(w)|_1.$$

On the other hand by (7) it follows that $|\mathrm{Tr}(w)|_1 \le c_1\|w\|_X$, hence it results that

$$\mathcal{J}_\lambda(w) \ge \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{h+1}}-\frac{\sigma}{\lambda_{h+1}}\right)\|w\|_X^2 - c_1a_\sigma\|w\|_X \quad \text{on } H^+(h+1). \tag{33}$$

Assumption (Λ) and (33) imply the existence of $\gamma > 0$ such that, for h large enough,

$$\mathcal{J}_\lambda(w) \ge \gamma \quad \text{on } H^+(h+1).$$

Therefore the Saddle Point Theorem implies the existence of a critical point of $\mathcal{J}_\lambda$. By Theorem 2.9 its corresponding critical value is an essential critical value of $\mathcal{J}_\lambda$, therefore by Theorems 2.7, 2.8, taking the trace, we conclude the proof. □

Proof of Theorem 1.4. Fixing any $j\in\mathbb{N}$, following [20] we consider a continuous cut function

$$\beta_j(t) = \begin{cases} 0 & \text{if } |t| \ge j+1 \\ 1 & \text{if } |t| \le j \end{cases}$$

such that $0 < \beta_j(t) < 1$ if $j < |t| < j+1$. Then, let us set

$$g_j(x,t) = \beta_j(t)g(x,t) \qquad \text{and} \qquad G_j(x,t) = \int_0^t g_j(x,s)\,\mathrm{d}s.$$

Let us remark that, if $g(x,\cdot)$ is odd, then choosing β_j even, it results that $g_j(x,\cdot)$ and $G_j(x,\cdot)$ are odd and even respectively, for $x \in \Omega$. Furthermore, there exists $\varepsilon_1(j) > 0$ such that

$$\varepsilon_1(j)|g_j(x,t)| < 1, \qquad \varepsilon_1(j)|G_j(x,t)| < 1 \quad \text{for all } x \in \Omega, t \in \mathbb{R}, \tag{34}$$

thus for any $|\varepsilon| \le \varepsilon_1(j)$ we consider the functionals

$$\mathcal{J}_{\lambda,j,\varepsilon}(w) = \mathcal{J}_\lambda(w) - \varepsilon\int_\Omega G_j(x,w)\,\mathrm{d}x \quad \text{on } X.$$

Let $\bar{m}$ be the number of the distinct critical levels c_i of $\mathcal{J}_\lambda$ found in Theorem 3.1. Clearly $1 \le \bar{m} \le \dim(M_h \oplus \ldots \oplus M_k)$ and $0 < c_0 < c_{i_1} < \ldots < c_{i_{\bar{m}}} \le c_\infty$, where c_0 and c_∞ are as in the proof of Theorem 3.1. These critical levels are also odd-essential levels for $\mathcal{J}_\lambda$. Indeed, in order to prove this it suffices to apply Corollary 2.10. Namely, we take $X = X_0(\mathcal{C}_\Omega)$, $\Gamma = \Sigma_i$ defined in Theorem 2.3, $d = 0$. Then, for any odd homeomorphism $\varphi : X \times [0,1] \longrightarrow X$ such that $\varphi(w,t) = w$ if $\mathcal{J}_\lambda(w) \le 0$, we have that the set $\overline{\varphi(C \times \{1\})}$ is closed and symmetric, for each $C \in \Sigma_i$. Moreover, from the supervariancy property of γ^*, we have

$$\gamma^*\left(\overline{\varphi(C\times\{1\})}\right) = \gamma^*\left(\varphi\left(\overline{C\times\{1\}}\right)\right) = \gamma^*\left(\overline{C\times\{1\}}\right) = \gamma^*(C) \ge i,$$

hence $\overline{\varphi(C \times \{1\})}$ belongs to Σ_i and the conclusion follows.

So, by Theorem 2.7 there exists $\varepsilon_2(j) \in]0, \varepsilon_1(j)[$ such that, if $|\varepsilon| \leq \varepsilon_2(j)$, then $\mathcal{J}_{\lambda,j,\varepsilon}$ has at least $\bar{m}$ odd-essential values $d_i^{j,\varepsilon}$, with $i \in \{1, \dots, \bar{m}\}$, such that

$$\frac{c_0}{2} < d_1^{j,\varepsilon} < \dots < d_{\bar{m}}^{j,\varepsilon} < c_\infty + 1. \tag{35}$$

Since $\mathcal{J}_{\lambda,j,\varepsilon}$ satisfies the Palais-Smale condition for any level $c \in \mathbb{R}$, by Theorem 2.8, for each $i \in \{1, \dots, \bar{m}\}$ $\mathcal{J}_{\lambda,j,\varepsilon}$ has a critical point $w_i^{j,\varepsilon}$ such that

$$(P_{\lambda,j,\varepsilon}) \qquad \begin{cases} -\div(\nabla w_i^{j,\varepsilon}) = 0 & \text{in } \mathcal{C}_\Omega, \\ w_i^{j,\varepsilon} = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ \dfrac{\partial w_i^{j,\varepsilon}}{\partial \nu} = \lambda \mathrm{Tr}(w_i^{j,\varepsilon}) + f(x, \mathrm{Tr}(w_i^{j,\varepsilon})) + \varepsilon g_j(x, \mathrm{Tr}(w_i^{j,\varepsilon})) & \text{in } \Omega \end{cases}$$

and

$$d_i^{j,\varepsilon} = \frac{1}{2}\int_{\mathcal{C}_\Omega} |\nabla w_i^{j,\varepsilon}(x,y)|^2 \,\mathrm{d}x, \mathrm{d}y - \frac{\lambda}{2}\int_\Omega |\mathrm{Tr}(w_i^{j,\varepsilon})(x)|^2 \,\mathrm{d}x - \int_\Omega \left(F(x, \mathrm{Tr}(w_i^{j,\varepsilon})) + \varepsilon G_j(x, \mathrm{Tr}(w_i^{j,\varepsilon}))\right) \,\mathrm{d}x.$$

We claim that

$$\|w_i^{j,\varepsilon}\|_X \leq C_1 \quad \text{for all } j \in \mathbb{N}, |\varepsilon| \leq \varepsilon_2(j), i \in \{1, \dots, \bar{m}\}. \tag{36}$$

Firstly, for all $w \in X$, as $w = w^+ + w^-$ with $w^+ \in H^+(k+1)$ and $w^- \in H^-(k)$, standard computations show that there exists $\delta > 0$ such that

$$\|w^+\|_X^2 - \lambda |\mathrm{Tr}(w^+)|_2^2 \geq \delta \|w^+\|_X^2, \tag{37}$$

$$\lambda |\mathrm{Tr}(w^-)|_2^2 - \|w^-\|_X^2 \geq \delta \|w^-\|_X^2. \tag{38}$$

Clearly, from $(P_{\lambda,j,\varepsilon})$ and (34) we have

$$\|(w_i^{j,\varepsilon})^+\|_X^2 - \lambda |\mathrm{Tr}((w_i^{j,\varepsilon})^+)|_2^2 \leq \int_\Omega |f(x, \mathrm{Tr}(w_i^{j,\varepsilon}))||(\mathrm{Tr}(w_i^{j,\varepsilon}))^+| \,\mathrm{d}x + |(\mathrm{Tr}(w_i^{j,\varepsilon}))^+|_1,$$

$$\lambda |(\mathrm{Tr}(w_i^{j,\varepsilon}))^-|_2^2 - \|(w_i^{j,\varepsilon})^-\|_X^2 \leq \int_\Omega |f(x, \mathrm{Tr}(w_i^{j,\varepsilon}))||(\mathrm{Tr}(w_i^{j,\varepsilon}))^-| \,\mathrm{d}x + |(\mathrm{Tr}(w_i^{j,\varepsilon}))^-|_1.$$

Hence, by (10) and (37), respectively (38), for suitable $\tilde{\varepsilon}, C_2 > 0$, easily we obtain

$$(\delta - \tilde{\varepsilon})\|(w_i^{j,\varepsilon})^+\|_X^2 \leq \tilde{\varepsilon}\|w_i^{j,\varepsilon}\|_X^2 + C_2\|w_i^{j,\varepsilon}\|_X,$$

respectively

$$(\delta - \tilde{\varepsilon})\|(w_i^{j,\varepsilon})^-\|_X^2 \leq \tilde{\varepsilon}\|w_i^{j,\varepsilon}\|_X^2 + C_2\|w_i^{j,\varepsilon}\|_X.$$

Putting together the two previous inequalities and choosing $\tilde{\varepsilon}$ sufficiently small, we get that (36) holds. From (36) and [18, Theorem 8.15] it follows that

$$|w_i^{j,\varepsilon}|_\infty \leq C_3 \quad \text{for all } j \in \mathbb{N}, |\varepsilon| \leq \varepsilon_2(j), i \in \{1, \dots, \bar{m}\}.$$

Therefore, for $j > C_3$, passing to the traces, problem $(P_{\lambda,\varepsilon})$ has at least $\bar{m}$ pairs of solutions. □

Proof of Theorem 1.5. Let us consider the cut functions β_j and the notations as in the proof of Theorem 1.3. By assumption the functionals $\mathcal{J}_\lambda$ and $\mathcal{J}_{\lambda,j,\varepsilon}$ satisfy the Palais-Smale condition. Then $\mathcal{J}_{\lambda,j,\varepsilon}$ has at least $\bar{m}$ odd-essential values $d_i^{j,\varepsilon}$ verifying (35). Now each $w \in X$ can be written as $w = w^+ + w^- + w^0$, with $w^+ \in H^+(k+1), w^- \in H^-(k-1)$ and $w_0 \in M_k$. Again standard computations show that there exists $\delta > 0$ such that

$$\|w^+\|_X^2 - \lambda_k |(\mathrm{Tr}(w))^+|_2^2 \geq \delta \|w^+\|_X^2, \tag{39}$$

$$\lambda_k |(\mathrm{Tr}(w))^-|^2 - \|w^-\|_X^2 \geq \delta \|w^-\|_X^2. \tag{40}$$

From (f_3), (34), $(P_{\lambda,j,\varepsilon})$ and (39)–(40), it follows that

$$\|(w_i^{j,\varepsilon})^\pm\|_X \leq C_1 \quad \text{for all } j \in \mathbb{N}, |\varepsilon| \leq \varepsilon_2(j), i \in \{1, \dots, \bar{m}\}.$$

We claim that also $\|(w_i^{j,\varepsilon})^0\|$ is bounded. Indeed, by (f_3) it follows that

$$\left| \int_\Omega F(x, \mathrm{Tr}(w))\, \mathrm{d}x \right| \leq C_2 \|w\|_X \quad \text{for all} \quad w \in X;$$

thus, as

$$\mathcal{J}_{\lambda,j,\varepsilon}(w_i^{j,\varepsilon}) \leq c_\infty + 1 \quad \text{for all } j \in \mathbb{N}, |\varepsilon| \leq \varepsilon_2(j), i \in \{1, \dots, \bar{m}\},$$

the thesis follows by (34) proceeding as in the proof of Theorem 3.3. □

Proof of Theorem 1.6. In this case, once found the critical values of $\mathcal{J}_\lambda$ by Theorem 3.1, as they are assumed to be topologically relevant, we can apply Corollary 2.12, so there exists $\varepsilon_2(j) \in]0, \varepsilon_1(j)]$ such that, if $|\varepsilon| \leq \varepsilon_2(j)$, then $\mathcal{J}_{\lambda,j,\varepsilon}$ has at least $\bar{m}$ critical values $d_k^{j,\varepsilon}$, with $k \in \{1, \dots, \bar{m}\}$, such that (35) holds. Then we proceed as in the proof of Theorem 1.3. □

Proof of Theorem 1.7. It is enough to combine the arguments in the proofs of Theorems 1.5 and 1.6. □

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