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On the Existence and Uniqueness of Dirichlet Problems on a Positive Half-Line

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We consider Dirichlet problems on a positive half-line. The methods which we apply and compare are the usage of a global invertibility theorem and of a direct variational approach. For both approaches the solution is the limit of a strongly convergent minimizing sequence to a suitably chosen action functional. Moreover, we show that the Euler action functional satisfies the PS condition at the infimal level.

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1. Introduction

In this paper we are concerned with the existence and uniqueness result for a Dirichlet problem on a positive half-line which reads as follows. Let $f\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$, $f'_u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ be Carathéodory functions, where f_x denoted the derivative of f with respect to the second variable. We consider the following Dirichlet problem

$$\begin{cases} -\ddot{u}+u=f(x,u), \\ u(0)=u(\infty)=0. \end{cases} \tag{1}$$

In considering the above problem, we will compare the global invertibility method described for example in [6] with a direct variational approach (see [10] for a background) showing that both lead to existence and uniqueness result however the assumptions differ a bit. Moreover, it appears that any minimizing sequence, obtained via a direct variational approach, is not only weakly convergent in $H_0^1(0,\infty)$ but norm convergent which means that also the following theorem can be applied for the existence:

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Theorem 1.1. [10] *Let $J\colon X \to \mathbb{R}$ be a C^1 functional which satisfies the $(PS)_{\inf}$ condition. Suppose in addition that J is bounded from below. Then the infimum of J is achieved at some $u_0 \in X$ and u_0 is a critical point of J.*

Our results show, by the example of a considered problem, that without assuming the Ambrosetti-Rabinowitz condition, the classical Euler action functional $J\colon H_0^1(0,\infty) \to \mathbb{R}$ given by

$$J(u) = \tfrac{1}{2}\int_0^\infty |\dot{u}(t)|^2\, dt + \tfrac{1}{2}\int_0^\infty |u(t)|^2 dt - \int_0^\infty F(t,u(t))dt \tag{2}$$

corresponding to (1) satisfies the Palais-Smale condition at the infimal level. Here $F\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is defined by a formula

$$F(x,u) = \int_0^u f(x,s)ds.$$

This seems to hold for some other problems whose existence is investigated by the direct method. By the global invertibility method, we obtain also that the minimizing sequence is norm convergent. Moreover, both methods lead to obtaining classical solutions and not only weak ones.

Problems on a half line received lately some attention but the main approach concerning the existence issue was by fixed point theorems and the method of lower and upper solution. We mention the following works which pertain to the existence results for problems on the half line: [1], [3], [5] and references therein. We would like to mention that neither of these results has been reached by the method which we suggest.

2. A new proof of the global inversion theorem

The idea of using global invertibility to solving a nonlinear boundary value problem is closely related to transferring the given problem to some abstract nonlinear equation which we next solve obtaining not only the unique solvability of the problem under consideration but also the fact that the solution operator is a diffeomorphism. Hence the given problems fulfills the conditions of Hadamard's programme.

To be more precise let E, F be Banach spaces and let $f\colon E\to F$ be a C^1-mapping which is locally invertible, i.e. $f'(x) \in \operatorname{Isom}(E,F)$ for any $x\in X$. Namely, we consider mappings with conditions which guarantee that these are local diffeomorphisms. Therefore, a natural question is to ask under which conditions we can assure that f is a global diffeomorphism which was first considered by Hadamard [7], who obtained a sufficient condition in terms of the growth of $(f'(x))^{-1}$ via some integral condition. The theorem, which we provide is given in the setting of Banach spaces to which it was extended by Lévy (see [9], [13], [15]).

Theorem 2.1. (Hadamard-Lévy theorem) *Let $f\colon E \to F$ be a local diffeomorphism of class C^1 which satisfies the following integral condition*

$$\int_0^\infty \min_{\|x\|=r} \|f\,'(x)^{-1}\|^{-1} dr = \infty.$$

Then f is a global diffeomorphism.

Since then many other tools have been introduced and applied to various problems mainly for integral equation. The problem in question also allows a variational approach for its proof with suitably modified assumptions. In a finite dimensional setting the result reads:

Theorem 2.2. *Let X, Y be finite dimensional Euclidean spaces. Assume that $f\colon X \to Y$ is a C^1-mapping such that*

(a) *$f'(x)$ is invertible for any $x \in X$,*

(b) *$x \to \|f(x)\|$ is coercive, i.e. $\|f(x)\| \to \infty$ as $\|x\| \to \infty$.*

Then f is a C^1-diffeomorphism.

The application of variational methods to global invertibility in a Hilbert space that has been inspired by Theorem 2.2 and stated in infinite dimensional spaces on [8] is as follows:

Theorem 2.3. *Let X be a real Banach space and H a real Hilbert space. If $f\colon X \to H$ is a C^1-mapping such that:*

(A1) *for any $y \in H$ the functional $\varphi\colon X \to \mathbb{R}$ given by the formula*

$$\varphi(x) = \frac{1}{2}\|f(x) - y\|^2$$

satisfies the Palais-Smale condition,

(A2) *for any $x \in X$, $f'(x) \in \operatorname{Isom}(X, H)$,*

then f is a diffeomorphism.

There have been some attempts to simplify the original proof from [8], see [6]. Here we provide another proof which in our opinion best reflects the structure of the critical set to functional φ. We note that the functional φ defined as above can serve as a sort of action functional related to problem $f(x) = y$ with fixed y. We will need:

Theorem 2.4. ([14, Theorem 2]) *Let $J\colon X \to \mathbb{R}$ be a C^1 functional satisfying the Palais-Smale condition. If every critical point of J is a local minimum, then every one is an absolute minimum, and the critical set is connected.*

Proof of Theorem 2.3. By Theorem 1.1 for a fixed y functional φ has an argument of a minimum x_0 which is a critical point, i.e. it satisfies

$$\varphi'(x_0) = (f(x_0) - y) \circ f'(x_0) = 0.$$

Since $f'(x_0) \in \mathrm{Isom}(X, H)$ we see that $f(x_0) = y$ and due to that fact that $f'(x_0)$ is invertible this solution is locally unique. Note that any other critical point x also satisfies $f(x) - y = 0$. This means that, by definition of φ, that any critical point is a strict local minimizer. Hence, it is a global minimizer by Theorem 2.4 and it is unique due to remarks we have already put. □

Hence the problem of solvability of $f(x) = y$ with fixed y is reduced to finding an argument of a minimum of φ hence it is in some sense comparable with the direct variational approach but differs in that we do not need a variational formulation of the problem under consideration for the global invertibility to work.

3. Remarks on the space setting

We say that $u \in H^1(0,\infty)$ if $u \in L^2(0,\infty)$ and if there exists a function $g \in L^2(0,\infty)$, called a *weak derivative*, and such that

$$\int_0^\infty u(x)\dot{\varphi}(x)dx = -\int_0^\infty g(x)\varphi(x)dx$$

for all $\varphi \in C_0^\infty[0,\infty)$, where $C_0^\infty[0,\infty)$ is the space of those functions $u(x)$ from $C^\infty[0,\infty)$ which satisfy $u(0) = u(\infty) = 0$. We denote $g := \dot{u}$ and endow the space $H^1(0,\infty)$ with a usual norm

$$\|u\|_{H^1} = \left(\int_0^\infty |u(x)|^2 dx + \int_0^\infty |\dot{u}(x)|^2 dx \right)^{\frac{1}{2}}$$

associated with the scalar product

$$\langle u, v \rangle_{H^1} = \int_0^\infty u(x)v(x)dx + \int_0^\infty \dot{u}(x)\dot{v}(x)dx,$$

which makes it a Hilbert space. Moreover we denote $H_0^1(0,\infty) := \overline{C_c^\infty[0,\infty)}_{H^1}$, where closure is taken in $H^1(0,\infty)$.

If $u \in H^1(0,\infty)$ and $\dot{u} \in H^1(0,\infty)$ then we say that $u \in H^2(0,\infty)$. The weak derivative of $\dot{u}$ is denoted by $\ddot{u}$. The space $H^2(0,\infty)$ endowed with the inner product

$$\langle u, v \rangle_{H^2} = \int_0^\infty u(x)v(x)dx + 2\int_0^\infty \dot{u}(x)\dot{v}(x)dx + \int_0^\infty \ddot{u}(x)\ddot{v}(x)dx.$$

becomes a Hilbert space. We denote $\widetilde{H}_0^2(0,\infty) = H_0^1(0,\infty) \cap H^2(0,\infty)$.

Proposition 3.1. *Take* $u \in \widetilde{H}_0^2(0,\infty)$. *Then*

$$\|u\|_{H^2}^2 = \int_0^\infty |\ddot{u}(x)|^2 + 2|\dot{u}(x)|^2 + |u(x)|^2 dx = \int_0^\infty |\ddot{u}(x) - u(x)|^2 dx.$$

Proof. Let $u \in \widetilde{H}_0^2(0,\infty)$. Bearing in mind that $u(0) = u(\infty) = 0$, and using the integration by parts we obtain

$$\begin{aligned}\int_0^\infty |\ddot{u}(x) - u(x)|^2 dx &= \int_0^\infty |\ddot{u}(x)|^2 dx - 2\int_0^\infty \ddot{u}(x)u(x)dx + \int_0^\infty |u(x)|^2 dx \\ &= \int_0^\infty |\ddot{u}(x)|^2 dx + 2\int_0^\infty \dot{u}(x)\dot{u}(x)dx + \int_0^\infty |u(x)|^2 dx \\ &= \int_0^\infty |\ddot{u}(x)|^2 dx + 2\int_0^\infty |\dot{u}(x)|^2 dx + \int_0^\infty |u(x)|^2 dx. \qquad \square\end{aligned}$$

Let us recall that embedding $H_0^1(0,\infty) \hookrightarrow C_0[0,\infty)$ is continuous, where $C_0[0,\infty)$ is a space of continuous functions satisfying $u(0) = u(\infty) = 0$ endowed with a standard supremum norm $\|\cdot\|_\infty$.

Proposition 3.2. *For every* $u \in H_0^1(0,\infty)$ *we have* $\sqrt{2}\|u\|_\infty \leq \|u\|_{H^1}$.

Proof. Take any $u \in C_0^1[0,\infty)$. Denote $g(u) = |u|u$. Then we have $g'(u) = 2|u|$ and hence for every $\xi \in [0,\infty)$ it holds

$$\begin{aligned}|u(\xi)|^2 = |g(u(\xi))| &= \left|\int_0^\xi (g(u(x)))'_x \, dx\right| = \left|\int_0^\xi 2|u(x)|\dot{u}(x)dx\right| \\ &\leq \int_0^\xi 2|u(x)||\dot{u}(x)|dx \leq \int_0^\xi |u(x)|^2 dx + \int_0^\xi |\dot{u}(x)|^2 dx. \qquad (3)\end{aligned}$$

Analogously for every $\xi \in [0,\infty)$ we have

$$|u(\xi)|^2 \leq \int_\xi^\infty |u(x)|^2 dx + \int_\xi^\infty |\dot{u}(x)|^2 dx.$$

Hence for every $\xi \in [0,\infty)$ we have

$$\begin{aligned}2|u(\xi)|^2 &= |u(\xi)|^2 + |u(\xi)|^2 \\ &\leq \int_0^\xi |u(x)|^2 dx + \int_0^\xi |\dot{u}(x)|^2 dx + \int_\xi^\infty |u(x)|^2 dx + \int_\xi^\infty |\dot{u}(x)|^2 dx\end{aligned}$$

$$= \int_0^\infty |u(x)|^2 dx + \int_0^\infty |\dot{u}(x)|^2 dx = \|u\|_{H^1}.$$

By the density of $C_0^1[0,\infty)$ in $H_0^1(0,\infty)$ the assertion holds. □

Remark 3.3. One may take a sequence of functions

$$\rho_n(x) := \begin{cases} x^n & \text{for } x \in [0,n], \\ (2n-x)^n & \text{for } x \in (n,2n] \end{cases}$$

to check that the constant in Proposition 3.2 can not be improved. □

However, for the space $\widetilde{H}_0^2(0,\infty)$ we have

Proposition 3.4. *For every $u \in \widetilde{H}_0^2(0,\infty)$ we have $K\|u\|_\infty \leq \|u\|_{\widetilde{H}_0^2}$, where $K > \sqrt{2}$.*

Proof. We will use Proposition 3.2. Fix $u \in \widetilde{H}_0^2(0,\infty)$ and $\xi \in [0,\infty)$. Using (3) we have

$$|u(\xi)|^2 \leq \int_0^\xi 2|u(x)||\dot{u}(x)|dx \leq \tfrac{1}{\sqrt{2}} \left(\int_0^\xi |u(x)|^2 dx + 2\int_0^\xi |\dot{u}(x)|^2 dx \right)$$
$$\leq \tfrac{1}{\sqrt{2}} \left(\int_0^\xi |u(x)|^2 dx + 2\int_0^\xi |\dot{u}(x)|^2 dx + \int_0^\xi |\ddot{u}(x)|^2 dx \right).$$

Using same argumentation as in the proof of Proposition 3.2 we finally obtain

$$2|u(\xi)| \leq \tfrac{1}{\sqrt{2}} \left(\int_0^\xi |u(x)|^2 dx + 2\int_0^\xi |\dot{u}(x)|^2 dx + \int_0^\xi |\ddot{u}(x)|^2 dx \right) = \tfrac{1}{\sqrt{2}}\|u\|_{\widetilde{H}_0^2}$$

for every $\xi \in [0,\infty)$. Hence the assertion follows since $K \geq \sqrt[4]{8} > \sqrt{2}$. □

Remark 3.5. As a consequence of $(C_0[0,\infty))^* \hookrightarrow (H_0^1(0,\infty))^*$, we see that if $u_n \rightharpoonup u_0$ in $H_0^1(0,\infty)$ then $u_n(x) \to u_0(x)$ for all $x \in [0,\infty)$.

4. Existence results by a direct method

We begin with providing the assumptions which we employ for the direct method.

(V1) *$f\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function differentiable with respect to the second coordinate;*

(V2) *there exist nonnegative functions $a, b \in L^2(0,\infty) \cap L^1(0,\infty)$ such that $\|a\|_{L^2} < \sqrt{2}$ and $|f(x,u)| \leq a(x)|u| + b(x)$ for all $u \in \mathbb{R}$ and a.e. $x \in [0,\infty)$;*

(V3) *there exists a positive function* $\alpha \in L^\infty(0,\infty)$ *such that* $\|\alpha\|_{L^\infty} \le 1$ *and* $f'_u(x,u) \le \alpha(x)$ *for all* $u \in \mathbb{R}$ *and a.e.* $x \in [0,\infty)$ *;*

For every $u \in \mathbb{R}$ and a.e. $x \in [0,\infty)$ we obtain

$$\begin{aligned} |F(x,u)| &= \left| \int_0^u f(x,s)ds \right| \le \left| \int_0^u |f(x,s)|ds \right| \\ &\le \left| \int_0^u a(x)|s|ds \right| + \left| \int_0^u b(x)ds \right| = \tfrac{1}{2}a(x)|u|^2 + b(x)|u|. \end{aligned} \tag{4}$$

As is common with variational problems for ordinary differential equation (1) admits two types of solutions, namely a weak and a classical one. A function $u \in H_0^1(0,\infty)$ is a *weak solution* to (1) if

$$\int_0^\infty \dot{u}(x)\dot{v}(x)dx + \int_0^\infty u(x)v(x)dx - \int_0^\infty f(x,u(x))v(x)dx = 0 \tag{5}$$

for all $v \in H_0^1(0,\infty)$. A function $u \in H_0^1(0,\infty)$ is a *classical solution* to (1) if both u and $\dot{u}$ are locally absolutely continuous functions on $[0,\infty)$,

$$-\ddot{u}(x) + u(x) = f(t,u(x))$$

for a.e. $x \in [0,\infty)$ and the boundary conditions $u(0) = u(\infty) = 0$ are satisfied. We would like to recall, following [2], that any function $u \in H_0^1(0,\infty)$ is locally absolutely continuous, i.e. absolutely continuous on any closed bounded interval contained in $[0,\infty)$. However it is not in general absolutely continuous on the whole half line which makes the problem different from the classical bounded one.

We have a counterpart to a du Bois Reymond Lemma which we present with a proof, although it mimics the one given in [5] and which we cite for the reader's convenience since now it is performed without assuming a special type of a non-linear function f and with a different growth assumption.

Proposition 4.1. *Assume that* (V1) *and* (V2) *hold. If* $u \in H_0^1(0,\infty)$ *is a weak solution of* (1)*, then* u *is a classical solution of problem* (1).

Proof. Let u be a weak solution of problem (1). Since $C_0^\infty[0,\infty) \subset H_0^1(0,\infty)$ we can use space $C_0^\infty[0,\infty)$ in (5).

Let the us define $Y\colon [0,\infty) \to \mathbb{R}$ by $Y(x) = u(x) - f(x,u(x))$, and $Z\colon [0,\infty) \to \mathbb{R}$ by $Z(x) = \int_0^x Y(s)ds$. Since $Y \in L^1_{loc}(0,\infty)$, it follows that Z is locally absolutely continuous function on $[0,\infty)$. By using Dirichlet Formula from [11], we obtain by definition of Z

$$\int_0^\infty Z(x)\dot{v}(x)dx = \int_0^\infty Y(s)\left(\int_s^\infty \dot{v}(x)dx\right)ds = -\int_0^\infty Y(s)v(s)ds.$$

Thus using (5), we get

$$\int_0^\infty (\dot{u}(x) - Z(x))\, \dot{v}(x)dx = 0$$

for all $v \in C_0^\infty[0,\infty)$. Since $u \in H_0^1(0,\infty)$, we see that $\dot{u} \in L^1_{loc}(0,\infty)$. Thus by the Fundamental Theorem of the Calculus of Variations, we see that there exists $c \in \mathbb{R}$ such that

$$\dot{u}(x) = Z(x) + c = \int_0^x (u(s) - f(s,u(s)))ds + c,$$

for a.e. $x \in [0,\infty)$. This means that $\dot{u}$ is locally absolutely continuous function on $[0,\infty)$ which implies that for a.e. $x \in [0,\infty)$

$$(\dot{u}(x))'_x = Y(x) = u(x) - f(x,u(x))$$

for a.e. $x \in [0,\infty)$. On the other hand, as $u \in H_0^1(0,\infty)$, then we obtain $u(0) = u(\infty)$, which means that u is a classical solution of Problem (1) . □

It is standard to prove

Proposition 4.2. *Assume conditions* (V1) *and* (V3)*. Then the functional J is well-defined and continuously differentiable on $H_0^1(0,\infty)$. The derivative of J at any $u \in H_0^1(0,\infty)$ has the following form*

$$J'(u)v = \int_0^\infty \dot{u}(x)\dot{v}(x)dx + \int_0^\infty u(x)v(x)dx - \int_0^\infty f(x,u(x))v(x)dx$$

for all $v \in H_0^1(0,\infty)$. Moreover, the functional J is weakly l.s.c. on $H_0^1(0,\infty)$.

Lemma 4.3. *Assume conditions* (V1)−(V3)*. Then functional J is coercive and strictly convex on $H_0^1(0,\infty)$.*

Proof. Note that $\frac{1}{2}\int_0^\infty (\dot{u}(x))^2\, dx + \frac{1}{2}\int_0^\infty u^2(x)dx = \frac{1}{2}\|u\|^2_{H^1}$ and that, using (4) and Proposition 3.2, we obtain that for every $u \in H_0^1(0,\infty)$ we have

$$\int_0^\infty |F(x,u(x))|\, dx \le \int_0^\infty \tfrac{1}{2}a(x)|u(x)|^2 dx + \int_0^\infty b(x)|u(x)|dx$$

$$\le \tfrac{1}{2}\int_0^\infty |a(x)||u(x)|^2 dx + \int_0^\infty |b(x)||u(x)|dx$$

$$\le \tfrac{1}{2}\|u\|_\infty \|a\|_{L^2}\|u\|_{L^2} + \|b\|_{L^2}\|u\|_{L^2} \le \tfrac{1}{2\sqrt{2}}\|a\|_{L^2}\|u\|^2_{H^1} + \|b\|_{L^2}\|u\|_{H^1}.$$

This proves the first assertion.

Observe that under (V3) the function $g(u) = \frac{1}{2}|u|^2 - F(x,u)$ is convex for a.e. $x \in [0,\infty)$. Hence by definition for all $\lambda \in [0,1]$ and every $u, v \in H_0^1(0,\infty)$ we have

$$\lambda \int_0^\infty g(u(x))dx + (1-\lambda)\int_0^\infty g(v(x))dx \geq \int_0^\infty g\left(\lambda u(x) + (1-\lambda)v(x)\right)dx.$$

Therefore J is strictly convex as a sum of convex and strictly convex functions. □

From the above results it is easy to see that

Theorem 4.4. *Assume conditions* (V1)−(V3). *Then problem* (1) *has exactly one solution* u_0 *such that*

$$\inf_{u \in H_0^1(0,\infty)} J(u) = J(u_0) = \inf_n J(u_n),$$

where $u_n \rightharpoonup u_0$ *in* $H_0^1(0,\infty)$.

We shall prove that in fact $u_n \to u_0$ and $J'(u_n) \to 0$, i.e. every minimizing sequence is a Palais-Smale sequence for J. Since J has exactly one critical point, we see that J satisfies the Palais-Smale condition at level $\inf_{u \in H_0^1(0,\infty)} J(u)$.

Proposition 4.5. *Under assumptions of the above theorem, the minimizing sequence* (u_n) *is strongly convergent (possibly up to a subsequence) and* $J'(u_n) \to 0$.

Proof. Indeed, from the Ekeland variational principle we see that for a minimizing sequence it follows that $J'(u_n) \to 0$. Since $J'(u_k) \longrightarrow 0$, we see that for some $\epsilon > 0$ there exists k_0 with $\|J'(u_k)\| \leq \epsilon$ for $k \geq k_0$. Note that for $k \geq k_0$

$$|J'(u_k)u_k| \leq \epsilon \|u_k\|.$$

Observe further that by direct calculation

$$J'(u_k)u_k = \int_0^\infty |\dot{u}_k(x)|^2 dx + \int_0^\infty |u_k(x)|^2 dx - \int_0^\infty f(x, u_k(x))u_k(x)dx.$$

Next, we prove that (u_k) converges strongly to some $\overline{u}$ in $H_0^1(0,\infty)$. Since (u_k) is bounded in $H_0^1(0,\infty)$, there exists a subsequence of (u_k), still denoted (u_k), such that (u_k) converges weakly to some $\overline{u}$ in $H_0^1(0,\infty)$ with $\|\overline{u}\| \leq M$. Note that

$$\int_0^\infty |f(x,u_k(x))|\,dx \leq \left(\int_0^\infty |a(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^\infty |u_k(x)|^2 dx\right)^{\frac{1}{2}} + \int_0^\infty b(x)dx$$

$$\leq M \left(\int_0^\infty |a(x)|^2 dx\right)^{\frac{1}{2}} + \int_0^\infty b(x)dx$$

This means that by the Lebesgue Dominated Convergence Theorem we now have

$$\lim_{k\to\infty}\int_0^\infty f(x,u_k(x))dx = f(x,\overline{u}(x))dx. \tag{6}$$

Since $\lim\limits_{k\to\infty} J'(u_k) = 0$ and since (u_k) converges weakly to some $\overline{u}$, we see that

$$\lim_{k\to\infty}\left(J'(u_k) - J'(\overline{u})\right)(u_k - \overline{u}) = 0. \tag{7}$$

Calculating in (7) directly we see that

$$\begin{aligned}&\left(J'(u_k) - J'(\overline{u})\right)(u_k - \overline{u}) = \\ &\quad = \|u_k - \overline{u}\|^2 - \int_0^\infty \left(f(x,u_k(x)) - f(x,\overline{u}(x))\right)(u_k(x) - \overline{u}(x))\, dx.\end{aligned}$$

Then (7) and (6) imply that (u_k) is strongly convergent. □

It is seen from these results that both methods result in the existence and uniqueness of a solutions with similar assumptions (up to some mild additional condition in the direct method). The advantage of the method based on the diffeomorphism relies on the fact the we know the exact lower bound for the functional.

5. Existence of solution by the global diffeomorphism theorem

We start with the assumptions which we employ in our considerations:

(D1) $f\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ *is a* C^1*-Caratheodory function;*

(D2) *there exist nonnegative functions* $a, b \in L^2(0,\infty)$ *such that* $\|a\|_{L^2} < \frac{1}{K}$*, where* K *is taken from Proposition* 3.4*, and* $|f(x,u)| \le a(x)|u| + b(x)$ *for all* $u\in\mathbb{R}$ *and a.e.* $x\in[0,\infty)$*;*

(D3) *there exist nonnegative functions* $\alpha\in L^\infty(0,\infty)$*,* $\beta\in L^2(0,\infty)$ *and* $\gamma\in C(\mathbb{R},(0,\infty))$ *such that* $\|\alpha\|_{L^\infty} < 1$ *and*

$$-\beta(x)\gamma(u) \le f'_u(x,u) \le \min\{\alpha(x), \beta(x)\gamma(u)\}$$

for all $u\in\mathbb{R}$ *and a.e.* $x\in[0,\infty)$.

Let us equip the space $\widetilde{H}_0^2(0,\infty)$ with the norm

$$\|u\|_{\widetilde{H}_0^2} = \left(\int_0^\infty |\ddot{u}(x) - u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Remember that by Proposition 3.1 this norm makes $\widetilde{H}_0^2(0,\infty)$ into a Hilbert space. Recall also that $\widetilde{H}_0^2(0,\infty)\hookrightarrow C_0[0,\infty)$ continuously and, for all $u\in\widetilde{H}_0^2$,

$$\|u\|_\infty \le \|u\|_{H^1} \le \|u\|_{\widetilde{H}_0^2}.$$

We define $T\colon \widetilde{H}_0^2(0,\infty) \to L^2(0,\infty)$ by the equation

$$T(u)(x) = \ddot{u}(x) - u(x) - f(x, u(x)) \quad \text{for a.e. } x \in [0,\infty). \tag{8}$$

Proposition 5.1. *Assume* (D1) – (D3) *hold. Then the operator T given by* (8) *is of class C^1 with a derivative given by the formula*

$$T'(u)v(x) = \ddot{v}(x) - v(x) - f'_u(x, u(x))v(x)$$

for all $v \in \widetilde{H}_0^2(0,\infty)$ and a.e. $x \in [0,\infty)$.

Proof. We see that it is enough to show that $N\colon \widetilde{H}_0^2(0,\infty) \to L^2(0,\infty)$ given by the formula

$$N(u)(x) = f(x, u(x))$$

is C^1. Since $\widetilde{H}_0^2(0,\infty) \subset C_0[0,\infty) \cap L^2(0,\infty)$ then for every $u, v \in \widetilde{H}_0^2(0,\infty)$ the function $x \mapsto f'_u(x, u(x))v(x)$ belongs to $L^2(0,\infty)$. Moreover $v \mapsto f'_u(\cdot, u(\cdot))v(\cdot)$ is a linear and bounded operator and hence it is the Gâteaux derivative of N at the point u.

Assume $u_n \to u_0$ in $\widetilde{H}_0^2(0,\infty)$. Then $u_n \to u_0$ in $C_0[0,\infty)$. Hence

$$\|N'(u_n) - N'(u_0)\|_{\mathcal{L}(\widetilde{H}_0^2, L^2)} = \sup_{\|v\|_{\widetilde{H}_0^2}=1} \left(\int_0^\infty |f'_u(x, u_n(x))v(x) - f'_u(x, u_0(x))v(x)|^2 dx \right)^{\frac{1}{2}}$$

Using the Lebesgue Dominated Convergence Theorem we obtain that for every fixed $v \in \widetilde{H}_0^2$

$$\left(\int_0^\infty |f'_u(x, u_n(x))v(x) - f'_u(x, u_0(x))v(x)|^2 dx \right)^{\frac{1}{2}} \to 0\,.$$

Therefore N is of class C^1 and hence T is also C^1. □

We will show that the assumptions of Theorem 2.3 are satisfied.

To verify assumption (A1) we define a functional $\varphi\colon \widetilde{H}_0^2(0,\infty) \to \mathbb{R}$ by the formula

$$\varphi(u) = \int_0^\infty |\ddot{u}(x) - u(x) + f(x, u(x))|^2 dx.$$

We prove that under (D1) – (D3) the functional φ satisfies the Palais-Smale condition.

Proposition 5.2. *Assume that* (D1) *and* (D2) *hold. Then the functional φ is C^1 and coercive on $\widetilde{H}_0^2$.*

Proof. Obviously φ is C^1 as a composition of C^1 mappings. To show coercivity observe that

$$\begin{aligned}
(\varphi(u))^{\frac{1}{2}} &= \left(\int_0^\infty |\ddot{u}(x) - u(x) + f(x,u(x))|^2 dx\right)^{\frac{1}{2}} \\
&\geqslant \left(\int_0^\infty |\ddot{u}(x) - u(x)|^2 dx\right)^{\frac{1}{2}} - \left(\int_0^\infty |f(x,u(x))|^2 dx\right)^{\frac{1}{2}} \\
&\geqslant \|u\|_{\widetilde{H}_0^2} - \left(\int_0^\infty |a(x)u(x)|^2 dx\right)^{\frac{2}{2}} - \left(\int_0^\infty |b(x)|^2 dx\right)^{\frac{1}{2}} \\
&\geqslant \|u\|_{\widetilde{H}_0^2} - \|u\|_\infty \left(\int_0^\infty |a(x)|^2 dx\right)^{\frac{2}{2}} - \left(\int_0^\infty |b(x)|^2 dx\right)^{\frac{1}{2}} \\
&\geqslant (1 - K\|a\|_{L^2}) \|u\|_{\widetilde{H}_0^2} - \|b\|_{L^2}.
\end{aligned}$$

Hence φ is coercive. □

Lemma 5.3. *Assume that (D1)-(D3) hold. Then functional φ satisfies Palais-Smale condition.*

Proof. Let (u_n) be the Palais-Smale sequence for φ. Then it is bounded by Proposition 5.2 and therefore we can choose a subsequence which converge weakly to some $u_0 \in \widetilde{H}_0^2(0,\infty)$. Denote this subsequence by (u_n) and observe that $u_n(x) \to u_0(x)$ for a.e. $x \in [0,\infty)$ and also $|u_n(x)| \le M$ for a.e. $x \in [0,\infty)$ for some $M > 0$. Using (D2) and (D3) we obtain

$$|f(x,u_0(x)) - f(x,u_n(x))|^2 \le 4M^2|a(x)|^2 + 8M|a(x)b(x)| + 4|b(x)|^2$$

and $$|f'_u(x,u_n(x))\,(u_0(x) - u_n(x))\,|^2 \le 4M^2 \sup_{-M \le v \le M} |\gamma(v)|^2|\beta(x)|^2$$

for every $n \in \mathbb{N}$ and a.e. $x \in [0,\infty)$. Since $u_n(x) \to u_0(x)$ a.e. on $[0,\infty)$ by the Lebesgue Dominated Convergence Theorem we see

$$\lim_{n\to\infty} \int_0^\infty |f(x,u_0(x)) - f(x,u_n(x))|^2 dx = 0 \tag{9}$$

and $$\lim_{n\to\infty} \int_0^\infty |f'_u(x,u_n(x))\,(u_n(x) - u_0(x))\,|^2 dx = 0. \tag{10}$$

A direct calculation yields

$$\varphi'(u_n)(u_n - u_0) - \varphi'(u_0)(u_n - u_0) = \|u_n - u_0\|^2_{\widetilde{H}_0^2} + \sum_{k=1}^{5} \int_0^\infty \psi_k(x)dx, \tag{11}$$

where $\psi_1(x) = (\ddot{u}_0(x) - u_0(x))f'_u(x, u_0(x))(u_n(x) - u_0(x)),$
$\psi_2(x) = (\ddot{u}_n(x) - u_n(x))f'_u(x, u_n(x))(u_0(x) - u_n(x)),$
$\psi_3(x) = f(x, u_0(x))f'_u(x, u_0(x))(u_0(x) - u_n(x)),$
$\psi_4(x) = f(x, u_n(x))f'_u(x, u_n(x))(u_n(x) - u_0(x)),$
$\psi_5(x) = (f(x, u_0(x)) - f(x, u_n(x)))(\ddot{u}_n(x) - u_n(x) - \ddot{u}_0(x) + u_0(x)).$

Since $u_n \rightharpoonup u_0$ in $\widetilde{H}_0^2(0,\infty)$, we have $u_n \rightharpoonup u_0$ in $L^2(0,\infty)$ and hence for $k = 1, 3$ $\int_0^\infty \psi_k(x)dx \to 0$. Moreover, using (9), (10) and assumption (D2) we obtain

$$\int_0^\infty |\psi_2(x)|dx \le \left(\int_0^\infty |\ddot{u}(x) - u(x)|^2\right)^{\frac{1}{2}} \left(\int_0^\infty |f'_u(x, u_n(x))(u_0(x) - u_n(x))|^2\right)^{\frac{1}{2}},$$

$$\int_0^\infty |\psi_4(x)|dx \le \left(\int_0^\infty |f(x, u_n(x))|^2\right)^{\frac{1}{2}} \left(\int_0^\infty |f'_u(x, u_n(x))(u_0(x) - u_n(x))|^2\right)^{\frac{1}{2}},$$

$$\int_0^\infty |\psi_5(x)|dx \le \left(\int_0^\infty |f(x, u_0(x)) - f(x, u_n(x))|^2\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty |\ddot{u}_n(x) - u_n(x) - \ddot{u}_0(x) + u_0(x)|^2\right)^{\frac{1}{2}}.$$

Moreover since (u_n) is a Palais-Smale sequence then

$$|\varphi'(u_n)(u_n - u_0)| \le \|\varphi'(u_n)\|_{\left(\widetilde{H}_0^2\right)^*} \|u_n - u_0\|_{L^2} \to 0\,.$$

Finally $\varphi'(u_0)(u_n - u_0) \to 0$ since $u_n \rightharpoonup u_0$ and by (11) we obtain that $u_n \to u_0$ strongly in $\widetilde{H}_0^2(0,\infty)$. □

Observe that for every $y \in L^2(0,\infty)$ functional

$$\varphi_y(x) = \int_0^\infty |\ddot{u}(x) - u(x) + f(x, u(x)) - y(x)|^2 dx$$

satisfies the Palais-Smale condition. Hence assumption (A1) of Theorem 2.3 is satisfied.

In order to prove invertibility of T' at every point we fix $u, y \in \widetilde{H}_0^2(0,\infty)$ and consider an auxiliary problem

$$\begin{cases} \ddot{v} - v - f'_u(x, u(x))v = y, \\ v(0) = v(\infty) = 0. \end{cases} \tag{12}$$

In view of Proposition 5.1 problem (12) is the explicit form of $T'(u)v = y$. Having u and y fixed arbitrary, we see that the unique solvability of (12) provides that the assumption (A2) of Theorem 2.3 is satisfied. Problem (12) is solved by a direct method applied to the functional $J\colon H_0^1(0,\infty) \to \mathbb{R}$ defined by the formula

$$J(v) = \int_0^\infty |\dot{v}(x)|^2 dx + \int_0^\infty |v(x)|^2 dx - \int_0^\infty f'_u(x,u(x))|v(x)|^2 dx - \int_0^\infty y(x)v(x)dx. \quad (13)$$

Then we easily obtain

Proposition 5.4. *Under the assumptions* (D1) – (D3) *the functional J given by* (13) *is coercive and strictly convex.*

Proof. See that under (D3) we have

$$\begin{aligned} J(v) &= \int_0^\infty |\dot{v}(x)|^2 dx + \int_0^\infty \left(1 - f'_u(x,u(x))\right)|v(x)|^2 dx - \int_0^\infty y(x)v(x)dx \\ &\geq \|\dot{v}\|_{L^2}^2 + (1 - \|\alpha\|_{L^\infty})\|v\|_{L^2}^2 - \|y\|_{L^2}\|v\|_{L^2}. \end{aligned}$$

Hence J is coercive. It is also strictly convex since, for every $u \in \mathbb{R}$ and a.e. $x \in [0,\infty)$, the function $w \mapsto w^2 - f'_u(x,u)w^2$ is strictly convex. □

Using the previous proposition we obtain that J possesses a unique critical point which is a unique solution of (12). As a conclusion we have

Theorem 5.5. *Assume* (D1) – (D3) *hold. Then problem* (1) *has a unique solution. Moreover the solution operator T given by* (8) *is a diffeomorphism.*

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