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Odd- Even Congruence Labeling of Certain Classes of Graphs

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Abstract
In graph theory, graph labeling involves the systematic assignment of positive integers to the structural components of a graph, specifically its vertices and edges, under predefined constraints. When modular arithmetic is utilized to assign odd values to vertices and even values to edges, the scheme is identified as **odd-**even congruence labeling. Formally, a graph G=(V,E)admits an odd-even congruence labeling if there exist injective mappings h: V(G) → $\mathbb{Z}^+$and k: E(G) → $\mathbb{Z}^+$such that for every edge e=wz ∈ E(G), the relation h(w) ≡h(z) (mod k(e))holds. This study investigates the implementation of this technique across several unique architectures. Specifically, we formally prove that the ice cream graph, umbrella graph, double fan graph, rectangular book graph, pentagonal book graph, path graph, triangular ladder graph, quadrilateral snack graph, jewel graph, Vanessa graph, helm graph, closed helm graph, jellyfish graph, andsunlet graphall admits an odd-even congruence labeling.

Keywords: odd-even congruence labeling, ice cream graph, umbrella graph,jewel graph, jellyfish graph

1.Introduction

Leonhard Euler, a Swiss mathematician, developed graph theory in 1736[3] to depict the Koenigsberg bridge problem as a graph and concluded that there is no solution. A. Rosa first proposed the concept of graph labeling in 1966 [8]. Every mathematical truth can be labeled differently and displayed as a graph. Considered the most important area of graph theory in recent years, labeling techniques have gained several labeling notions and impacted a number of mathematicians. There are many different domains in which graph labeling is used, including coding theory, physical cosmology, database administration, and secret sharing schemes.

K.Kanakambika and G. Thamizhendhi [4] established the formal definitions for congruence and prime congruence labelling. Their work involves assigning labels to vertices and edges based on the congruence modulo of integers. They successfully demonstrated that several graph families qualify as congruence graphs, including path graphs, cycle graphs, friendship graphs, cycle graphs with zigzag chords. Furthermore, they verified that fan, star, and gear graphs satisfy the conditions for prime congruence graphs.

G.Thamizhendhi and K.Kanakambika [5] pioneered the study of odd-even congruence labeling. Their research established this labeling property for various graph classes, including star graphs, comb graphs, complete bipartite graphs, and shadow graphs of paths. Furthermore, they demonstrated that

odd–even congruence graphs can be formed by joining two even cycles C_r via a path $P_t$as well as though the tensor product of $K_{1,}$and P_2.

The concept of odd-even congruence labeling was expanded by Rituraj and Shambhu Kumar Mishra [6],applied it to several structures including ladder graphs,triangular snakes, coconut trees,and triangular books, as well as other tree- based graphs. Their subsequent research [7] explored this labeling in friendship, shell, fan, and generalized butterfly graphs, alongside vertex switchingP_2+mK_1configurations.

In this paper, we investigate odd-even congruence labelingon the ice cream graph, umbrella graph, double fan graph, rectangular book graph, pentagonal book graph, path graph,triangular ladder graph,quadrilateral snack graph, jewel graph, helm graph, closed helm graph, sunlet graph, jellyfish graph, and sunflower graphs.

2. Preliminaries

Definition 2.1[14]An **icecream graph**$I_n(n\geq 3)$is obtained by joining a shell graph and a path graph $P_n$such that they share a common endpoint called the apex vertex w, with w_1 and w_{n-1} identified accordingly.

Definition 2.2[3]For integers $m \geq 2$ *and* $n \geq 1$,an**umbrella graph**$U_{(m,n)}$ is asimple, connected graphobtained by attachinga path graph P_n to the centralapex vertex of a fan graph F_m.An umbrella graph $U_{(m,n)}$ has m + n+1vertices and 2m +n-1 edges.

Definition 2.3[8]Foran integer $n\geq 2$, a**fan graph**F_n is obtained by joining all vertices of a path graph P_n to an additional vertex called the center vertex. Formally $F_n = K_1+P_n$, with vertex count $|V(F_n)| = n+1$ and edge count $|E(F_n)| = 2n-1$

Definition 2.4 For an integer $n\geq 2$, a **double fan graph** $F_{2,n}$ is a graph obtained by joining two distinct apex vertices to all vertices of a path graph P_n. Formally,$F_{2,n}= \overline{K_2} + P_n$.

Definition 2.5 For an integer $n \geq 1$, the **rectangular book graph** $B_{4,n}$of order n is a graph obtained by taking n copies of the 4-cycle C_4 and identifying them along a single common edge. This shared edge

e=wz is designated as the spine, and each cycle $C_4^{(k)}$=w- w_k- z_k-z forms a page of the book.

Definition 2.6 For an integer $n \geq 1$, a**pentagonal book graph** $B_{5,n}$ of order n is a graph obtained by taking n copies of the 5-cycle C_5 and identifying them along a single common edge. This shared edge

e=wz is designated as the spine, and each cycle $C_5^{(k)}$=w- w_k- x_k- z_k-z forms a page of the book.

Definition 2.7A **Path graph**P_n(also known as a linear graph) is a tree graph whose vertices can be listed in a distinct sequential order $w_1, w_2, w_3,\ldots..,w_n$ such that the edge set is given by

$E(P_n)= \{w_k, w_{k+1}|1 \leq k \leq n-1\}$.

Definition 2.8 [10]Foran integer $n\geq 2$, a**triangular ladder graph**$TL_n$is a graph obtained by adding diagonal edges to each of the n-1 rectangular faces of a ladder graph L_n.

Definition 2.9 [15]For an integer $n\geq 2$, **aquadrilateral Snake graph**QS_n is a graph obtained from a path graph $P_n$with vertices $w_1,w_2,w_3,\ldots\ldots,w_n$by replacing each edge w_kw_{k+1} (for $1 \leq k \leq n-1$)with a 4-cycle C_4.

Definition 2.10[13] The **jewel graph**, J_n is obtained from a 4-cycle with vertices graph x, y, w, z by joining x and y with a prime edge and also by appending an edge from w and z which meets at common vertices $z_k$for $1 \leq k \leq n$.

Definition 2.11 [11] A Vanessa graph V_n,(n≥ 3) is constructed from two fan graphs F_n, of the same order sharing a common vertex, along with a pendant vertex attached to the shared center. Mathematically, it can be represented as $V_n = 2F_n + K_{1,n}$

Definition 2.12 [12]The **Helm graph**$H_n$is obtained by attaching a single pendant edge to each rim vertex of the wheel graph W_n.

Definition 2.13 [12] The **Closed helm graph**$CH_n$is obtained by connecting all the pendant vertices of the helm graph H_n to form an outer cycle.

Definition 2.14[13] The **jellyfish graph**$J_{m,n}$ is obtained from a 4-cycle with vertices x, w, y, z (in cyclic order) by joining x and y with a chord edge, and appending m pendant edges to w, and n pendant edges to z.

Definition 2.15 The **sunlet graph** $S_n$is obtained by attaching a single pendant vertex to each vertex of the cycle C_n. Formally, its vertex set is $V(S_n) = \{w_1, w_2, w_3, \ldots., w_n, z_1, z_2, z_3, \ldots, z_n\}$ and its edge set is $E(S_n) = \{w_1w_{i+1}, w_nw_1, w_iz_i|1 \leq k \leq n\}$.

3. Main Results

Definition 3.2[17]Let *G=(V,E)* be a graph and d = min {2|V|, 2|E|}. A graph *G* admits a **congruence labeling** if there exist bijections h:V(*G*) → {1,2,..., d} and k: E(*G*) → {1,2,…, d}such that for every edge e=wz∈ *E*, the incident endpoint labels satisfy the divisibility condition h (w) ≡ h (z)(mod k(e)).

Definition 3.3[18]Let *G =(V, E)* with |V(*G*)|=m and |E(*G*)|=n. The graph*G* is called an **odd-even congruence graph** if there existinjective mappingsh:V(*G*) → {1, 3, 5, ..., 2n-1} for the vertices and

k: E(*G*) → {2, 4, 6, …, 2n} such that for every edge e=wz∈ *E*, the relation h (w) ≡ h (z)(mod k(e)) holds.

Theorem 3.1:

Theicecream graph $I_n$admits an odd-even congruence labeling for all integers n≥3.

Proof:

Let $I_n$representan icecream graph.

We define the vertex set V(I_n)and the edge set E(I_n)are as follows:

$V(I_n) = \{w\} \cup \{w_k | 1 \leq k \leq n\}$

$E(I_n) = E_a \cup E_b \cup E_c$,where the respective edge subsets are specified as follows:

$E_a = \{ww_k | 1 \leq k \leq n\}$, $E_b = \{w_k w_{k+1} | 1 \leq k \leq n-1\}$, $E_c = \{w_1 w_n\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as:$|V(I_n)| = n+1$ and $|E(I_n)|=2n$

We introduce the vertex labeling function $h:V(I_n) \to \mathbb{Z}^+$ such that $h(V(I_n))$consists exclusively of positive odd integers:

$h(w) =1$

$h(w_k) =2^{k+1}-1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K:E(I_n) \to \mathbb{Z}^+$such that K(E($I_n$)) consists exclusively of positive even integers:

$K(ww_k)= 2^{k+1}- 2$ for $1 \leq k \leq n$

$K(w_k w_{k+1})=2^{k+1}$for $1 \leq k \leq \leq n-1$

$K(w_1 w_n)=2^{n+1}- 4$

Verification of Congruence Conditions:

1. For edgesbelonging toE_a($e=ww_k$):

$K(ww_k) =2^{k+1} -2$ for $1 \leq k \leq n$

Verify:

$|h(w_k) - h(w)| = |(2^{k+1} -1) -1|= 2^{k+1} -2$

Since$K(ww_k) =2^{k+1} -2$, it divides $|h(w_k) - h(w)|$

2. For edges in E_b($e=w_k w_{k+1}$):

$K(w_k w_{k+1}) =2^{k+1}$for $1 \leq k \leq n-1$

Verify:

$|h(w_{k+1}) - h(w_k)| = |(2^{k+1+1} -1) - (2^{k+1} -1)| = |2^{k+2}-2^{k+1}|=|2^{k+1}(2-1)| =2^{k+1}$

Since$K(w_k w_{k+1}) =2^{k+1}$, it divides $|h(w_{k+1}) - h(w_k)|$

3. For edges in$E_c(e=w_1w_n)$:

$K(w_1w_n) =2^{n+1}– 4$

Verify:

$| h(w_n) – h(w_1) | = | (2^{n+1} -1) – (2^2-1) |= 2^{n+1}- 4$

Since$K (w_1w_n) =2^{n+1}- 4$, it divides $| h(w_n) – h(w_1) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that

anicecream graph $I_n$admitsan odd-even congruence labeling.

Example 3.2

Consider theicecream graph $I_n$with n=5

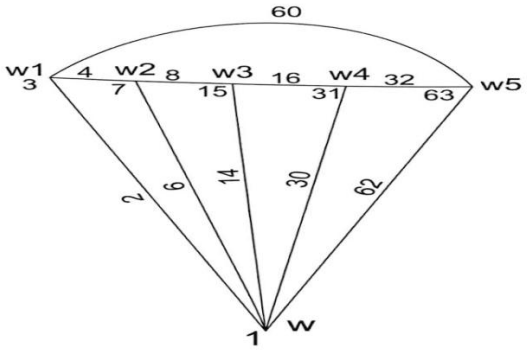


Fig.1

Fig.1demonstrates that anicecream graph I_5 admits an odd-even congruence labeling.

Theorem 3.3:

An umbrella graph $U_{(m,n)}$ admits an odd-even congruence labeling for all $m\geq 2$, $n \geq 1$.

Proof:

Let $U_{(m,n)}$represent theumbrella graph.

We define the vertex set $V(U_{(m,n)})$ and the edge set$E(U_{(m,n)})$ areas follows:

$$V(U_{(m,n)}) = \{w\}\cup\{w_k \mid 1\leq k \leq m\} \cup \{z_r \mid 1\leq r \leq n\}$$

$E(U_{(m,n)}) =E_a\cup E_b\cup E_c$, where the respective edge subsets are specified as follows:

$E_a= \{ww_k \mid 1\leq k \leq m\}$, $E_b= \{w_kw_{k+1} \mid 1\leq k \leq m-1\}$, $E_c= \{wz_1\} \cup\{z_rz_{r+1} \mid 1\leq r \leq n-1\}$

Based on this framework, the total number of vertices and the total number of edges are calculatedas:$|V(U_{(m,n)})| = m + n +1$ and $|E(U_{(m,n)})|=2m+n-1$

We introduce the vertex labeling function $h:V(U_{(m,n)}) \rightarrow \mathbb{Z}^+$ such that $h(V(U_{(m,n)}))$

consistsexclusively of positive odd integers:

$h(w) = 1$

$h(w_k) =2^{k+1}-1$ for $1 \leq k \leq m$

$h(z_r) = 2^{m+r+1} - 1 \quad \text{for } 1 \leq r \leq n$

Next, we define the edge labeling function $K:E(U_{(m,n)}) \rightarrow \mathbb{Z}^+$ such that $K(E(U_{(m,n)}))$ consists exclusively of positive even integers:

$K(ww_k) = 2^{k+1} - 2$ for $1 \leq k \leq m$

$K(w_k w_{k+1}) = 2^{k+1}$ for $1 \leq k \leq m-1$

$K(wz_1) = 2^{m+2} - 2$

$K(z_r z_{r+1}) = 2^{m+r+1}$ for $1 \leq r \leq n-1$

Verification of Congruence Conditions:

1. For edges belonging to $E_a(e=ww_k)$:

$K(ww_k) = 2^{k+1} - 2$ for $1 \leq k \leq m$

Verify:

$| h(w_k) - h(w) | = | (2^{k+1} - 1) - 1| = 2^{k+1} - 2$

Since $K(ww_k) = 2^{k+1} - 2$, it divides $| h(w_k) - h(w) |$

2. For edges belonging to $E_b(e=w_k w_{k+1})$:

$K(w_k w_{k+1}) = 2^{k+1}$ for $1 \leq k \leq m-1$

Verify:

$| h(w_{k+1}) - h(w_k) | = | (2^{k+2} - 1) - (2^{k+1} - 1) | = |2^{k+2} - 2^{k+1}| = |2^{k+1}(2-1)| = 2^{k+1}$

Since $K(w_k w_{k+1}) = 2^{k+1}$, it divides $| h(w_{k+1}) - h(w_k) |$

3. For edges belonging to E_c:

*For the initial handle edge wz_1:

$K(wz_1) = 2^{m+2} - 2$

Verify:

$| h(z_1) - h(w) | = | (2^{m+2} - 1) - 1) | = 2^{m+2} - 2$

Since $K(wz_1) = 2^{m+2} - 2$, it divides $| h(z_1) - h(w) |$

*For the subsequent handle edges $z_r z_{r+1}$

$K(z_r z_{r+1}) = 2^{m+r+1}$ for $1 \leq r \leq n-1$

Verify:

$| h(z_{r+1}) - h(z_r) | = | (2^{m+r+2} - 1) - (2^{m+r+1} - 1) | = 2^{m+r+1}$

Since $K(z_r z_{r+1}) = 2^{m+r+1}$, it divides $| h(z_{r+1}) - h(z_r) |$

As a result, the divisibility criteria are satisfiedacross all edge categories, confirming that anumbrella graph $U_{(m,n)}$ admits an odd-even congruence labeling.

Example 3.4

Consider an umbrella graph $U_{(m,n)}$ with m=6 and n=4.

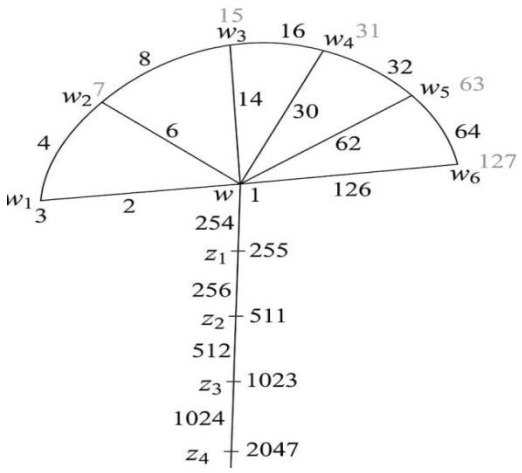


Fig.2

Fig.2 demonstrates that an umbrella graph $U_{(6,4)}$ admits an odd-even congruence labeling.

Theorem 3.5:

A double fan graph $F_{2,n}$admits an odd-even congruence labelingfor all n≥ 2.

Proof:

Let $F_{2,n}$represent thedouble fan graph.

We define the two apex vertices as w_1 and w_2, and the path vertices as z_1, z_2, z_3,…, z_n.

We define thevertex set $V(F_{2,n})$ and the edge set $E(F_{2,n})$ are as follows:

$V(F_{2,n}) = \{w_1, w_2\} \cup \{z_k | 1 \leq k \leq n\}$

$E(F_{2,n}) = E_a \cup E_b \cup E_c$, where the respective edge subsets are specified as follows:

$E_a = \{w_1 z_k | 1 \leq k \leq n\}$

$E_b = \{w_2 z_k | 1 \leq k \leq n\}$

$E_c = \{z_k z_{k+1} | 1 \leq k \leq n-1\}$

Based on this framework, the total number of vertices and thetotal number of edges are calculated as :$|V(F_{2,n})| = n+2$ and $|E(F_{2,n})| = 3n-1$.

We introduce the vertex labeling function $h: V(F_{2,n}) \rightarrow \mathbb{Z}^+$ such that $h(V(F_{2,n}))$ consists exclusively of positive odd integers:

$h(w_1)=1$, $h(w_2)=3$

$h(z_k)=2^{k+2}-1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K: E(F_{2,n}) \rightarrow \mathbb{Z}^+$such that $K(E(F_{2,n}))$ consists exclusively of positive even integers:

$K(w_1 z_k) = 2^{k+2}-2$ for $1 \leq k \leq n$

$K(w_2 z_k) = 2^{k+2}-4$ for $1 \leq k \leq n$

$K(z_k z_{k+1}) = 2^{k+2}$for $1 \leq k \leq n-1$

Verification of Congruence Conditions:

1. For edges belonging to $E_a(e=w_1z_k)$:

$K(w_1z_k) = 2^{k+2}-2$ for $1 \leq k \leq n$

Verify:

$| h(z_k) - h(w_1) | = | (2^{k+2} -1) -1| = 2^{k+2}-2$

Since$K(w_1z_k) = 2^{k+2}-2$,it divides $| h(z_k) - h(w_1) |$

2. For edges belonging $E_b(e=w_2z_k)$:

$K(w_2z_k) = 2^{k+2}-4$ for $1 \leq k \leq n$

Verify:

$|h(z_k) - h(w_2) | = | (2^{k+2} -1) -3) | = 2^{k+2}- 4$

Since$K(w_2z_k) = 2^{k+2}- 4$, it divides $| h(z_k) - h(w_2) |$

3. For path edges belonging to$E_c(e=z_kz_{k+1})$:

$K(z_kz_{k+1}) = 2^{k+2}$for $1 \leq k \leq n-1$

Verify:

$|h(z_{k+1}) - h(z_k) | = | (2^{k+3} -1) - (2^{k+2} -1) | = 2^{k+2}$

Since$K(z_kz_{k+1}) = 2^{k+2}$, it divides $| h(z_{k+1}) - h(z_k) |$

As a result, the divisibility requirements are satisfied across all edge categories,confirming that thedouble fan graph $F_{2,n}$admits an odd-even congruence labeling.

Example 3.6:

Consider a double fan graph $F_{2,n}$ with n=5

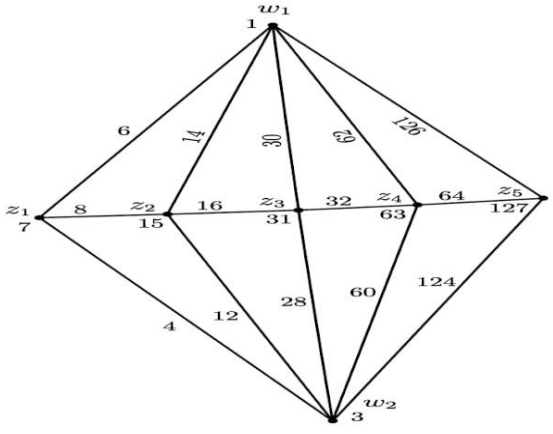


Fig. 3

Fig. 3 demonstrates that the double fan graph $F_{2,n}$ admits an odd-even congruence labeling.

Theorem 3.7:

The rectangular book graph $B_{4,n}$admits an odd-even congruencelabelingfor all $n \geq 1$.

Proof:

Let $B_{4,n}$represent the rectangular book graph consisting of n pages,where each page is formed by a 4-cycle C_4 sharing a single common spine edge. We denote the common spine vertices as w andz,and the remaining outer path vertices for each page as w_k and z_k.

We define the vertex set V ($B_{4,n}$) and the edge set E ($B_{4,n}$) as follows:

$$V(B_{4,n}) = \{w, z\} \cup (w_k | 1 \leq k \leq n\} \cup \{z_k \mid 1 \leq k \leq n\}$$

$E(B_{4,n}) = E_a \cup E_b \cup E_c \cup E_d$, where the respective edge subsets are specified as follows:

$E_a = \{wz\}$

$E_b = \{ww_k \mid 1 \leq k \leq n\}$

$E_c = \{w_k z_k \mid 1 \leq k \leq n\}$

$E_d = \{z_k z \mid 1 \leq k \leq n\}$

Based on this framework, the total number of verticesand the total number of edges are calculated as: $|V(B_{4,n})| = 2n+2$ and $|E(B_{4,n})| = 3n+1$.

We now introduce the vertex labeling function $h: V(B_{4,n}) \to \mathbb{Z}^+$ such that $h(V(B_{4,n}))$ consists exclusively of positive odd integers:

$h(w) = 1$

$h(z) = 3$

$h(w_k) = 10(2^k)+1$ for $1 \leq k \leq n$

$h(z_k) = 4(2^k)+1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K: E(B_{4,n}) \to \mathbb{Z}^+$such that $K(E(B_{4,n}))$ consists exclusively of positive even integers:

$K(wz) = 2$

$K(ww_k) = 10(2^k)$ for $1 \leq k \leq n$

$K(w_k z_k) = 6(2^k)$for $1 \leq k \leq n$

$K(z_k z) = 4(2^k) - 2$for $1 \leq k \leq n$

Verification of congruence conditions:

1. For the spine edge belonging toE_a($e=wz$):

$K(wz) = 2$

Verify:

$|h(z) - h(w)| = |3 - 1| = 2$

SinceK(wz) =2, it divides $|h(z) - h(w)|$

2. For edges connecting to vertex x belonging to E_b($e=ww_k$):

$K(ww_k) = 10(2^k)$ for $1 \leq k \leq n$

Verify:

$$| h(w_k) - h(w) | = | (10(2^k)+1) - 1) | = 10(2^k)$$

Since $K(ww_k) = 10(2^k)$, it divides $| h(w_k) - h(w) |$

3. For outer path edges belonging to $E_c(e=w_kz_k)$:

$K(w_kz_k) = 6(2^k)$ for $1 \leq k \leq n$

Verify:

$|h(z_k) - h(w_k) | = | (4(2^k) +1) - (10(2^k) +1) | = 6(2^k)$

Since $K(w_kz_k) = 6(2^k)$, it divides $| h(z_k) - h(w_k) |$

4. For edges connecting to vertex y belonging to E_d $(e=z_kz)$:

$K(z_kz) = 4(2^k)-2$ for $1 \leq k \leq n$

Verify:

$| h(z_k) - h(z) | = | (4(2^k) +1) - 3 | = 4(2^k)-2$

Since $K(z_kz) = 4(2^k)-2$, it divides $| h(z_k) - h(z) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the rectangular book graph $B_{4,n}$ admits an odd-even congruence labeling.

Example 3.8:

Consider the rectangular book graph $B_{4,n}$ with n=4

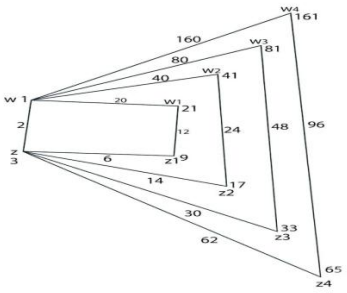


Fig. 4

Fig. 4 demonstrates that the rectangular book graph $B_{4,n}$ admits an odd-even congruence labeling.

Theorem 3.9:

A pentagonal book graph $B_{5,n}$ admits an odd-even congruence labeling for all $n \geq 1$.

Proof:

Let $B_{5,n}$ represent the pentagonal book graph consisting of n pages, where each page is a 5-cycle sharing common spine edge. We denote the common spine vertices as w and z, and the remaining three outer path vertices completing the 5-cycle pages as w_k, x_k and z_k.

We define the vertex set $V(B_{5,n})$ and the edge set $E(B_{5,n})$ are as follows:

$V(B_{5,n}) = \{w, z\} \cup (w_k | 1 \leq k \leq n\} \cup (x_k | 1 \leq k \leq n\} \cup (z_k | 1 \leq k \leq n\}$

$E(B_{5,n}) = E_a \cup E_b \cup E_c \cup E_d \cup E_f$,where the respective edgesubsets are specified as follows:

$E_a = \{wz\}$

$E_b = \{ww_k \mid 1 \leq k \leq n\}$

$E_c = \{w_k x_k | 1 \leq k \leq n\}$

$E_d = \{x_k z_k | 1 \leq k \leq n\}$

$E_f = \{z_k z \mid 1 \leq k \leq n\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as: $|V(B_{5,n})| = 3n+2$ and $|E(B_{5,n})| = 4n+1$

We now introduce the vertex labeling function $h: V(B_{5,n}) \rightarrow \mathbb{Z}^+$ such that $h(V(B_{5,n}))$ consistsexclusively of positive odd integers:

$h(w) = 1$

$h(z) = 3$

$h(w_k) = 28(2^k) + 1$ for $1 \leq k \leq n$

$h(x_k) = 12(2^k) + 1$ for $1 \leq k \leq n$

$h(z_k) = 4(2^k) + 1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K: E(B_{5,n}) \rightarrow \mathbb{Z}^+$such that $K(E(B_{5,n}))$ consists exclusively of positive even integers:

$K(wz) = 2$

$K(ww_k) = 28(2^k)$ for $1 \leq k \leq n$

$K(w_k x_k) = 16(2^k)$for $1 \leq k \leq n$

$K(x_k z_k) = 8(2^k)$for $1 \leq k \leq n$

$K(z_k z) = 4(2^k)$-2for $1 \leq k \leq n$

Verification of Congruence conditions:

1. For the spine edge belonging to E_a(e=wz):

$K(wz) = 2$

Verify:

$$| h(z) - h(w) | = | 3 - 1 | = 2$$

Since$K(wz) = 2$, it divides $| h(z) - h(w) |$

2. For edges belonging to E_b (e= ww_k):

$K(ww_k) = 28(2^k)$for $1 \leq k \leq n$

Verify:

$| h(w_k) - h(w) | = | (28(2^k) +1) -1) |= 28(2^k)$

Since $K(ww_k) =28(2^k)$, it divides $| h(w_k) - h(w) |$

3. For outer path edges belonging to $E_c(e=w_kx_k)$:

$K(w_kx_k) = 16(2^k)$for $1 \leq k \leq n$

Verify:

$$| h(x_k) - h(w_k) | = | (28(2^k) +1) - (12(2^k) +1) |= 16(2^k)$$

Since$K(w_kx_k) =16(2^k)$, it divides $| h(x_k) - h(w_k) |$

4. For outer page edges belonging to E_d $(e=x_kz_k)$:

$K(x_kz_k) = 8(2^k)$for $1 \leq k \leq n$

Verify:

$$| h(z_k) - h(x_k) | = | (12(2^k) +1) - (4(2^k) +1) |= 8(2^k)$$

Since$K(z_kx_k) =8(2^k)$,it divides $| h(z_k) - h(x_k) |$

5. For edges connecting back to the spine belonging to E_f $(e=z_kz)$:

$K(z_kz) = 4(2^k) -2$ for $1 \leq k \leq n$

Verify:

$$| h(z_k) - h(z) | = | (4(2^k) +1)-3 |= 4(2^k) -2$$

Since$K(z_kz) =4(2^k) -2$, it divides $| h(z_k) - h(z) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the pentagonal book graph $B_{5,n}$admits an odd-even congruence labeling.

Example 3.10:

Consider the pentagonal book graph $B_{5,n}$ with n=4

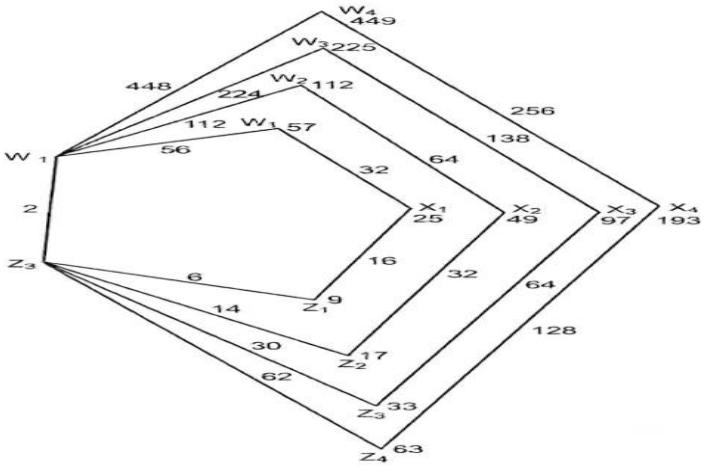


Fig. 5

Fig. 5 demonstrates that the pentagonal book graph $B_{5,n}$ admits an odd-even congruence labeling.

Theorem 3.11:

Apath graph $P_n$admits an odd-even congruence labeling for any integer $n \geq 2$.

Proof:

Let $P_n$represent a path graph consisting of n vertices and n-1 edges.

We define the sequential vertices along the path as $w_1, w_2, w_3, \ldots, w_n$.

The vertex set $V(P_n)$ and edge set $E(P_n)$ are formally constructedin the following manner:

$V(P_n) = \{w_k | 1 \leq k \leq n\}$

$E(P_n) = \{w_k w_{k+1} | 1 \leq k \leq n-1\}$

Based on this framework, the total number of verticesand the total number of edges are

calculatedas: $|V(P_n)| = n$ and $|E(P_n)| = n-1$

We now introduce the vertex labeling function$h$:$V(P_n)$: $\rightarrow \mathbb{Z}^+$ such that $h(V(P_n))$ consists

exclusively of positive odd integers:

$h(w_k) = 2^k-1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K:E(P_n) \rightarrow \mathbb{Z}^+$such that $K(E(P_n))$ consists

exclusively of positive even integers:

$K(w_k w_{k+1}) = 2^k$for $1 \leq k \leq n-1$

To verify that the congruence mapping condition holds, we compute the absolute difference of

the labels for any pair of adjacent vertices incident to an arbitrary edge $w_k w_{k+1}$:

$|h(w_{k+1}) - h(w_k)| = |(2^{k+1} -1) - (2^k - 1)| = |2^{k+1} - 2^k| = 2^k$

Since the assigned edge label is exactly $K(w_k w_{k+1}) = 2^k$, it divides $|h(w_{k+1}) - h(w_k)|$.

This mathematical divisibility ensures that the equivalence condition $h(w_k) \equiv h(w_{k+1}) \pmod{K(w_k w_{k+1})}$ is satisfied uniformly across all edges in the graph. Hence, the path graph $P_n$admits an odd-even congruencelabeling graph.

Example 3.12:

Consider the path graph $P_n$with n=5.



Fig. 6

Fig. 6 demonstrates that the path graph P_5 admits an odd-even congruence labeling.

Theorem 3.13:

The triangular ladder graph TL_nadmits an odd- even congruence labeling for all $n \geq 2$.

Proof:

Let TL_nrepresent the triangular ladder graph. We define the vertices of the two parallel paths as$w_1, w_2, w_3, \ldots, w_nandz_1, z_2, z_3, \ldots, z_n$. The vertex set $V(TL_n)$ and the edge set $E(TL_n)$are as follows:

$V(TL_n) = \{w_k | 1 \leq k \leq n\} \cup \{z_k | 1 \leq k \leq n\}$

$E(TL_n) = E_a \cup E_b \cup E_c \cup E_d$,where the respective edge subsetsarespecified as follows:

$E_a = \{w_k w_{k+1} \mid 1 \leq k \leq n-1\}$

$E_b = \{z_k z_{k+1} \mid 1 \leq k \leq n-1\}$

$E_c = \{w_k z_k | 1 \leq k \leq n\}$

$E_d = \{w_k z_{k+1} \mid 1 \leq k \leq n-1\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as:$|V(TL_n)| = 2n$ and $|E(TL_n)| = 4n-3$.

We now introduce the vertex labeling function$h : V(TL_n) \rightarrow \mathbb{Z}^+$such that $h(V(TL_n))$consists exclusively of positive odd integers:

$h(w_k) = 7(2^k) - 9$for $1 \leq k \leq n$

$h(z_k) = 5(2^k) - 9$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K : E(TL_n) \rightarrow \mathbb{Z}^+$such that K (E(TLn))consists exclusively of positive even integers:

$K(w_k w_{k+1}) = 7(2^k)$for $1 \leq k \leq n-1$

$K(z_k z_{k+1}) = 5(2^k)$for $1 \leq k \leq n-1$

$K(w_k z_k) = 2^{k+1}$for$1 \leq k \leq n$

$K(w_k z_{k+1}) = 3(2^k)$for $1 \leq k \leq n-1$

Verification of Congruence conditions:

1. Upper path edges belonging to$E_a(e=w_k w_{k+1})$:

$K(w_k w_{k+1}) = 7(2^k)$for $1 \leq k \leq n-1$

Verify:

$|h(w_{k+1}) - h(w_k)| = |(7(2^{k+1})-9) - (7(2^k)-9)| = 7(2^k)$

Since$K(w_k w_{k+1}) = 7(2^k)$, it divides $|h(w_{k+1}) - h(w_k)|$

2. Lower path edges belonging to $E_b(e= z_k z_{k+1})$:

$K(z_k z_{k+1}) = 5(2^k)$for $1 \leq k \leq n-1$

Verify:

$|h(z_{k+1}) - h(z_k)| = |(5(2^{k+1}) - 9) - (5(2^k) - 9)| = 5(2^k)$

SinceK(z_kz_{k+1}) =5(2^k), it divides | $h(z_{k+1}) - h(z_k)$ |

3. Vertical Rung edges belonging toE_c($e=w_kz_k$):

K(w_kz_k) =$2^{k+1}$for $1 \leq k \leq n$

Verify:

| $h(z_k) - h(w_k)$ | = | (5(2^k)-9) – (7(2^k)-9) |= 2(2^k) =2^{k+1}

SinceK(w_kz_k) =2^{k+1}, it divides | $h(z_k) - h(w_k)$ |

4. diagonal cross-Rung edges belonging to E_d ($e=w_kz_{k+1}$):

K(w_kz_{k+1}) =3(2^k) for $1 \leq k \leq n-1$

Verify:

| $h(z_{k+1}) - h(w_k)$ | = |(5(2^{k+1})-9)- (7(2^k)-9) |=3(2^k)

SinceK(w_kz_{k+1}) =3(2^k), it divides | $h(z_{k+1}) - h(w_k)$ |

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the triangular ladder graph TL_nadmits an odd-even congruence labeling.

Example 3.14:

Consider the triangular ladder graph TL_n with n=6.

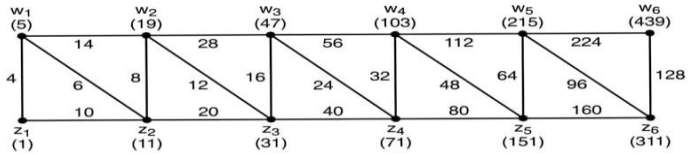


Fig. 7

Fig. 7 demonstrates that the triangular ladder graph TL_6 admits an odd-even congruence labeling.

Theorem 3.15:

Aquadrilateral snack graph QS_nadmits an odd-even congruence labeling for any integer n≥ 2.

Proof:

Let QS_nrepresent aquadrilateral snack graphobtained by replacing every edge of a primary spine pathw$_1$, w$_2$, w$_3$, …., w$_n$with a cycle of length 4 (C_4), using newly added upper and lower parallel vertex paths denoted by x_k and z_k.

We define the vertex set V(QS_n) and the edge set E(QS_n) as follows:

V(QS_n) = $\{w_k | 1 \leq k \leq n\} \cup \{x_k | 1 \leq k \leq n\} \cup \{z_k | 1 \leq k \leq n-1\}$

E(QS_n) = $E_a \cup E_b \cup E_c \cup E_d$, Where the respective edge partitions forming the snake-like cycle components are defined as follows:

E_a= $\{w_kx_k | 1 \leq k \leq n-1\}$

E_b= $\{x_kz_k | 1 \leq k \leq n-1\}$

$E_c = \{z_k w_{k+1} \mid 1 \le k \le n-1\}$

$E_d = \{w_k w_{k+1} \mid 1 \le k \le n-1\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as: $|V(QS_n)| = 3n-2$ and $|E(QS_n)| = 4n-4$.

We now introduce the vertex labeling function $h: V(QS_n) \to \mathbb{Z}^+$ such that $h(V(QS_n))$ consists exclusively of positive odd integers:

$h(w_k) = 2^{3k-2} - 1$ for $1 \le k \le n$

$h(x_k) = 2^{3k-1} - 1$ for $1 \le k \le n-1$

$h(z_k) = 2^{3k} - 1$ for $1 \le k \le n-1$

Next, we define the edge labeling function $K: E(QS_n) \to \mathbb{Z}^+$ such that $K(E(QS_n))$ consists exclusivelyof positive even integers:

$K(w_k x_k) = 2^{3k-2}$ for $1 \le k \le n-1$

$K(x_k z_k) = 2^{3k-1}$ for $1 \le k \le n-1$

$K(z_k w_{k+1}) = 2^{3k}$ for $1 \le k \le n-1$

$K(w_k w_{k+1}) = 14(8^{k-1})$ for $1 \le k \le n-1$

1. For edges belonging to $E_a (e = w_k x_k)$:

$K(w_k x_k) = 2^{3k-2}$ for $1 \le k \le n-1$

Verify:

$$| h(w_k) - h(x_k) | = | (2^{3k-2} -1) - (2^{3k-1} -1) | = |2^{3k-2} - 2^{3k-1}| = |2^{3k-2}(2-1)| = 2^{3k-2}$$

Since $K(w_k x_k) = 2^{3k-2}$, it divides $| h(w_k) - h(x_k) |$

2. For edges belonging to $E_b (e = x_k z_k)$:

$K(x_k z_k) = 2^{3k-1}$ for $1 \le k \le n-1$

Verify:

$$| h(z_k) - h(x_k) | = | (2^{3k} -1) - (2^{3k-1} - 1) | = |2^{3k} - 2^{3k-1}| = |2^{3k-1}(2-1)| = 2^{3k-1}$$

Since $K(z_k x_k) = 2^{3k-1}$, it divides $| h(z_k) - h(x_k) |$

3. For edges belonging to $E_c (e = z_k w_{k+1})$:

$K(z_k w_{k+1}) = 2^{3k}$ for $1 \le k \le n-1$

Verify:

$$| h(w_{k+1}) - h(z_k) | = | (2^{3(k+1)-2} -1) - (2^{3k-1} -1) | = |(2^{3k+1} -1) - (2^{3k} -1)| = |2^{3k}(2-1)| = 2^{3k}$$

Since $K(z_k w_{k+1}) = 2^{3k}$, it divides $| h(w_{k+1}) - h(z_k) |$

4. For edges belonging to $E_d (e = w_k w_{k+1})$:

$K(w_k w_{k+1}) = 14(8^{k-1})$ for $1 \le k \le n-1$

Verify:

$| h(w_{k+1}) - h(w_k) | = | (2^{3(k+1)-2} -1) - (2^{3k-2} -1) |= |(2^{3k+1} -1)-(2^{3k-2} -1)|=14(8^{k-1})$

Since$K(w_kw_{k+1}) =14(8^{k-1})$, it divides $| h(w_{k+1}) - h(w_k) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the quadrilateral snake graph QS_nadmits an odd-even congruence labeling.

Example 3.16:

Consider the Quadrilateral snake graph QS_n with n=3.

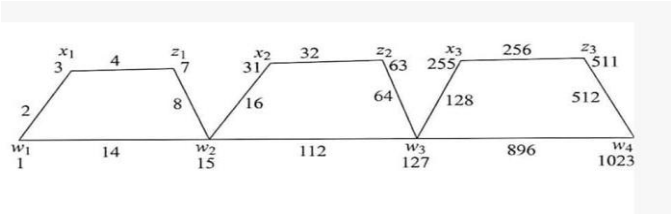


Fig. 8

Fig. 8 demonstrates that the Quadrilateral snack graph QS_3admits an odd-even congruence labeling.

Theorem 3.17:

Ajewel graph $J_n$admits an odd-even congruence labeling for any integer $n \geq 1$.

Proof:

Let $J_n$representa jewel graph.

We definethe vertex set $V(J_n)$ and the edge set $E(J_n)$ are as follows:

$V(J_n) = \{w,z,x,y\} \cup \{z_k \mid 1 \leq k \leq n\}$

$E(J_n) = E_a \cup E_b \cup E_c$,Where the respective edge subsets are specified as follows:

$E_a = \{wx, xy, yz, wy, xz\}$

$E_b = \{wz_k \mid 1 \leq k \leq n\}$

$E_c = \{zz_k \mid 1 \leq k \leq n\}$

Based on this framework, the total number of vertices and the total number of edges are calculated

as: $| V(J_n)| = n+4$ and $| E(J_n)| = 2n+5$.

We now introduce the Vertex labeling function $h: V(J_n) \rightarrow \mathbb{Z}^+$ such that $h(V(J_n))$ consists

exclusively of positive odd integers:

$h(w) = 1$

$h(z) = 3$

$h(x) = 7$

$h(y) = 15$

$h(z_k) = 2^{k+4} - 1$ for $1 \leq k \leq n$

Next, we define the edge labeling function $K: E(J_n) \rightarrow \mathbb{Z}^+$such that $K(E(J_n))$ consists exclusively

of positive even integers:

1. Verification of E_a:

* For edge wx:

K(wx) =6

Verify:

$| h(x) - h(w) | = | 7-1 | = 6$

Since K(wx) =6, itdivides $| h(x) - h(w) |$

* For edge yz:

K(yz) =12

Verify:

$| h(z) - h(y) | = | 3-15 | = 12$

Since K(yz) =12, it divides $|h(z) - h(y) |$

*For edge wy:

K(wy) =14

Verify:

$| h(y) - h(w) | = | 15-1 | = 14$

Since K(wy) =14, it divides $| h(y) - h(w) |$

* For edge xy:

K(xy) =8

Verify:

$| h(y) - h(x) | = | 15-7 | = 8$

Since K(xy) =8, it divides $| h(y) - h(x) |$

* For edge xz:

K(xz) =4

Verify:

$| h(z) - h(x) | = | 3-7 | = 4$

Since K(xv) =4, it divides $| h(z) - h(x) |$

2. Verification of E_b:

* For edges in E_b(e= wz_k):

$K(wz_k) = 2^{k+4} - 2$ for $1 \leq k \leq n$

Verify:

$|h(z_k) - h(w) | = | (2^{k+4}-1)-1| = |2^{k+4} -2| = 2^{k+4} -2$

Since $K(wz_k) = 2^{k+4} - 2$, it divides $|h(z_k) - h(w)|$

3. Verification of E_c:

* For edges in E_c($e = zz_k$):

$K(zz_k) = 2^{k+4} - 4$ for $1 \leq k \leq n$

Verify:

$|h(z_k) - h(z)| = |(2^{k+4} - 1) - 3| = |2^{k+4} - 4| = 2^{k+4} - 4$

Since $K(zz_k) = 2^{k+4} - 4$, it divides $|h(z_k) - h(z)|$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the jewel graph $J_n$admits an odd-even congruence labeling.

Example 3.18:

Consider the jewel graph J_n with n=3.

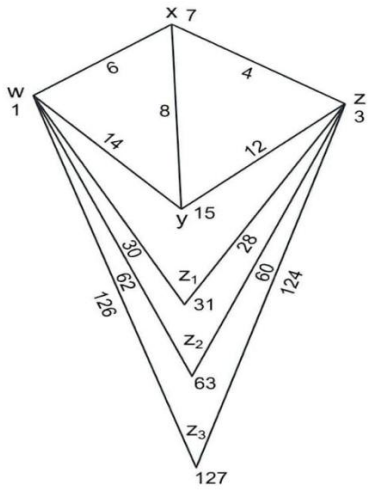


Fig. 9

Fig. 9 demonstrates that the jewel graph J_3admits an odd-even congruence labeling.

Theorem 3.19:

The vanessa graph $V_n$admits an odd-even congruence labeling for any integer $n \geq 2$.

Proof:

Let V_n representa vanessagraph with apex vertex w.

We define the vertex set and edge set as follows:

$V(V_n) = \{w\} \cup \{w_k \mid 1 \leq k \leq n\} \cup \{z_k \mid 1 \leq k \leq 2n\}$

$E(V_n) = E_a \cup E_b \cup E_c \cup E_d \cup E_f$, where the respective edge subsets are specified as follows:

$E_a = \{ww_k \mid 1 \leq k \leq n\}$

$E_b = \{w_k w_{k+1} \mid 1 \leq k \leq n-1\}$

$E_c = \{wz_k \mid 1 \leq k \leq 2n\}$

$E_d = \{z_k z_{k+1} \mid 1 \leq k \leq n-1\}$

$E_f = \{z_k z_{k+1} | n+1 \leq k \leq 2n\}$

Based on this frame work the total number of vertices and the total number of edges are calculated as:$|V(V_n)| = 3n+1$ and $|E(V_n)| = 5n - 2$.

Define the vertex labeling function $h: V(V_n) \rightarrow \mathbb{Z}^+$ such that $h(V(V_n))$ consists exclusively of positive odd integers:

$h(w) = 1$

$h(w_k) = 2^{k+1} - 1$ for $1 \leq k \leq n$

$h(z_k) = 2^{k+n+1} - 1$ for $1 \leq k \leq 2n$

Define the edge labeling function $K: E(V_n) \rightarrow \mathbb{Z}^+$ such that $K(E(V_n))$ consists exclusively of positive even integers:

$K(ww_k) = 2^{k+1} - 2$ for $1 \leq k \leq n$

$K(w_k w_{k+1}) = 2^{k+1}$ for $1 \leq k \leq n-1$

$K(wz_k) = 2^{n+k+1} - 2$ for $1 \leq k \leq 2n$

$K(z_k z_{k+1}) = 2^{k+n+1}$ for $1 \leq k \leq n-1$

$K(z_k z_{k+1}) = 2^{k+n+1}$ for $n+1 \leq k \leq 2n-1$

Verification of Congruence Conditions:

We must verify that for every edge $e=wz \in E(V_n)$, $K(e)$ divides $|h(z)-h(w)|$

1. For edges belonging to $E_a(e=ww_k)$:

$K(ww_k) = 2^{k+1} - 2$ for $1 \leq k \leq n$

Verify:

$|h(w_k) - h(w)| = |(2^{k+1}-1) - 1| = 2^{k+1} - 2$

Since $K(ww_k) = 2^{k+1} - 2$, it divides $|h(w_k) - h(w)|$

2. For edges in $E_b(e=wz_k)$:

$K(w_k w_{k+1}) = 2^{k+1}$ for $1 \leq k \leq n-1$

Verify:

$|h(w_{k+1}) - h(w_k)| = |(2^{k+2}-1) - (2^{k+1}-1)| = 2^{k+1}$

Since $K(w_k w_{k+1}) = 2^{k+1}$, it divides $|h(w_{k+1}) - h(w_k)|$

3. For edges in $E_c(e=wz_k)$:

$K(wz_k) = 2^{k+n+1} - 2$ for $n+1 \leq k \leq 2n$

Verify:

$|h(z_k) - h(w)| = |(2^{k+n+1}-1) - 1| = 2^{k+n+1} - 2$

Since $K(wz_k) = 2^{k+n+1} - 2$, it divides $|h(z_k) - h(w)|$

4. For edges in $E_d(e=z_kz_{k+1})$:

$K(z_kz_{k+1}) = 2^{k+n+1}$ for $1 \leq k \leq n-1$

Verify:

$$| h(z_{k+1}) - h(z_k) | = | (2^{k+n+1+1}-1) - (2^{k+n+1}-1) | = 2^{k+n+1}$$

Since $K(z_kz_{k+1}) = 2(3^{k+n})$, it divides $| h(z_{k+1}) - h(z_k)|$

5. For edges in $E_f(e=z_kz_{k+1})$:

$K(z_kz_{k+1}) = 2^{k+n+1}$ for $n+1 \leq k \leq 2n-1$

Verify:

$$| h(z_{k+1}) - h(z_k) | = | (2^{k+n+1+1}-1) - (2^{k+n+1}-1) | = 2^{k+n+1}$$

Since $K(z_kz_{k+1}) = 2^{k+n+1}$,it divides $| h(z_{k+1}) - h(z_k)|$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the Vanessa graph V_n admits an odd-even congruence labeling.

Example 3.20:

Consider the Vanessa graph $V_n$with n=5.

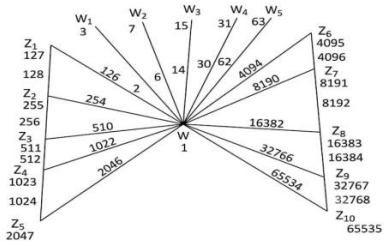


Fig. 10

Fig. 10 demonstrates that the Vanessa graph V_5 admits an odd-even congruence labeling.

Theorem 3.21:

Ahelm graph $H_n$admits an odd-even congruence labeling for all $n \geq 3$.

Proof:

Let $H_n$represent a helm graph obtained from a wheel graph $W_n$formed by attaching a pendant

edge at each vertex of its cycle boundary.

We define the vertex set $V(H_n)$ and the edge set $E(H_n)$ as follows:

$$V(H_n) = \{w\} \cup \{w_k | 1 \leq k \leq n\} \cup \{z_k | 1 \leq k \leq n\}$$

$E(H_n) = E_a \cup E_b \cup E_c \cup E_d$,where the respective edge subsets are specified as follows:

$E_a = \{ww_k | 1 \leq k \leq n\}$

$E_b = \{w_kw_{k+1} | 1 \leq k \leq n-1\}$

$E_c = \{w_nw_1\}$

$E_d = \{w_kz_k | 1 \leq k \leq n\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as: $| V(H_n)| = 2n+1$ and $| E(H_n)| = 3n$.

We now introduce the vertex labeling function h:$V(H_n) \rightarrow \mathbb{Z}^+$such that $h(V(H_n))$consists exclusivelyof positive odd integers:

$h(w) = 1$

$h(w_k) = 7(2^k)-9$ for $1 \leq k \leq n$

$h(z_k) = 10(2^k)-9$ for $1 \leq k \leq n$

Next, we establish the edge labeling function K: $E(H_n) \rightarrow \mathbb{Z}^+$such that $K(E(H_n))$consists exclusively of positive evenintegers:

$K(ww_k) = 7(2^k)-10$for $1 \leq k \leq n$

$K(w_kw_{k+1}) = 7(2^k)$for $1 \leq k \leq n-1$

$K(w_nw_1) = 7(2^n-2)$

$K(w_kz_k) = 3(2^k)$for $1 \leq k \leq n$

Verification of Congruence Conditions:

1. For edges belonging to $E_a(e=ww_k)$:

$K(ww_k) = 7(2^k)-10$for $1 \leq k \leq n$

Verify:

$| h(w_k) - h(w) | = | (7(2^k)-9) -1| = 7(2^k)-10$

Since $K(ww_k) = 7(2^k)-10$, it divides $| h(w_k) - h(w) |$

2. For edges in $E_b(e=w_kw_{k+1})$:

$K(w_kw_{k+1}) = 7(2^k)$for $1 \leq k \leq n-1$

Verify:

$| h(w_{k+1}) - h(w_k) | = | (7(2^{k+1})-9) - (7(2^k)-9) | = 7(2^k)$

Since $K(w_kw_{k+1}) = 7(2^k)$, it divides $| h(w_{k+1}) - h(w_k) |$

3. For edges in $E_c(e=w_nw_1)$:

$K(w_nw_1) = 7(2^n-2)$

Verify:

$| h(w_n) - h(w_1) | = |(7(2^n)-9) - (7(2^1)-9) | = 7(2^n-14) = 7(2^n-2)$

Since $K(w_nw_1) = 7(2^n-2)$, it divides $| h(w_n) - h(w_1) |$

4. For edges in $E_d(e=w_kz_k)$:

$K(w_kz_k) = 3(2^k)$for $1 \leq k \leq n$

Verify:

$| h(z_k) - h(w_k) | = |10\ (2^k)-9) - (7(2^k)-9) | = 3(2^k)$

Since $K(z_k w_k) = 3(2^k)$, it divides $| h(z_k) - h(w_k) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the helm graph H_n admits an odd-even congruence labeling.

Example 3.22:

Consider the helm graph $H_n$with n=8.

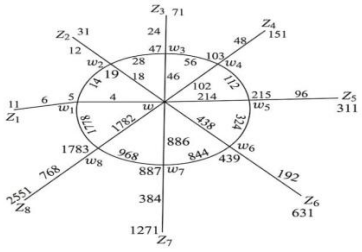


Fig. 11

Fig. 11 demonstrates that the helm graph H_8 admits an odd-even congruence labeling.

Theorem 3.23:

A closed helm graph CH_nadmits an odd -even congruence labeling for all n≥3.

Proof:

Let CH_nrepresent aclosed helm graph consisting of a central apex vertex w, an inner cycle boundary with vertices $w_1, w_2, w_3, \ldots, w_n$and an outer cycle boundary with vertices $z_1, z_2, z_3, \ldots, z_n$.

We define the vertex set $V(CH_n)$ and edge set $E(CH_n)$ are as follows:

$V(CH_n) = \{w\} \cup \{w_k | \ 1 \leq k \leq n\} \cup \{z_k | \ 1 \leq k \leq n\}$

$E(CH_n) = E_a \cup E_b \cup E_c \cup E_d \cup E_f \cup E_g$, where the respective edge subsets represent as:

$E_a = \{ww_k | \ 1 \leq k \leq n\}$

$E_b = \{w_k w_{k+1} | 1 \leq k \leq n-1\}$

$E_c = \{w_n w_1\}$

$E_d = \{w_k z_k | \ 1 \leq k \leq n\}$

$E_f = \{z_k z_{k+1} | 1 \leq k \leq n-1\}$

$E_g = \{z_n z_1\}$

Based on this framework, the total number of vertices and the total number of edges are calculated as$| V(CH_n)| = 2n+1$ and $| E(CH_n)| = 4n$.

We now introduce the vertex labeling function h:$V(CH_n) \rightarrow \mathbb{Z}^+$such that h $(V(CH_n))$consists exclusively of positive odd integers:

$h(w) = 1$

$h(w_k) = 7(2^k)-9$ for $1 \leq k \leq n$

$h(z_k) = 10(2^k)-9$ for $1 \leq k \leq n$

Next, we establish the edge labeling function K: $E(CH_n) \rightarrow \mathbb{Z}^+$ such that $K(E(CH_n))$ consists exclusively of positive even integers:

$K(ww_k) = 7(2^k)-10$ for $1 \leq k \leq n$

$K(w_kw_{k+1}) = 7(2^k)$ for $1 \leq k \leq n-1$

$K(w_nw_1) = 7(2^n-2)$

$K(z_kw_k) = 3(2^k)$ for $1 \leq k \leq n$

$K(z_kz_{k+1}) = 10(2^k)$ for $1 \leq k \leq n-1$

$K(z_nz_1) = 10(2^n-2)$

Verification of Congruence Conditions:

1. For edges belonging to $E_a(e=ww_k)$:

   $K(ww_k) = 7(2^k)-10$ for $1 \leq k \leq n$

Verify:

$|h(w_k) - h(w)| = |(7(2^k)-9) - 1| = 7(2^k)-10$

Since $K(ww_k) = 7(2^k)-10$, it divides $|h(w_k) - h(w)|$

2. For edges in $E_b(e=w_kw_{k+1})$:

$K(w_kw_{k+1}) = 7(2^k)$ for $1 \leq k \leq n$

Verify:

$|h(w_{k+1}) - h(w_k)| = |(7(2^{k+1})-9) - (7(2^k)-9)| = 7(2^k)$

Since $K(w_kw_{k+1}) = 7(2^k)$, it divides $|h(w_{k+1}) - h(w_k)|$

3. For edges in $E_c(e=w_nw_1)$:

$K(w_nw_1) = 7(2^n-2)$

Verify:

$|h(w_n) - h(w_1)| = |(7(2^n)-9) - (7(2^1)-9)| = 7(2^n-2)$

Since $K(w_nw_1) = 7(2^n-2)$, it divides $|h(w_n) - h(w_1)|$

4. For edges in $E_d(e=w_kz_k)$:

$K(w_kz_k) = 3(2^k)$ for $1 \leq k \leq n$

Verify:

$|h(z_k) - h(w_k)| = |(10(2^k)-9) - (7(2^k)-9)| = 3(2^k)$

Since $K(z_kw_k) = 3(2^k)$, it divides $|h(z_k) - h(w_k)|$

5. For edges in $E_f(e=z_kz_{k+1})$:

$K(z_kz_{k+1}) = 10(2^k)$ for $1 \leq k \leq n-1$

Verify:

$|h(z_{k+1}) - h(z_k)| = |(10(2^{k+1})-9) - (10(2^k)-9)| = 10(2^k)$

Since $K(z_kz_{k+1}) = 10(2^k)$, it divides $|h(z_{k+1}) - h(z_k)|$

6. For edges in $E_g(e=z_nz_1)$:

$K(z_nz_1) = 10(2^n - 2)$

Verify:

$|h(z_n) - h(z_1)| = |(10(2^n)-9) - (10(2^1)-9)| = 10(2^n-2)$

Since $K(z_nz_1) = 10(2^n-2)$, it divides $|h(z_n) - h(z_1)|$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the closed helm graph CH_n admits an odd-even congruence labeling.

Example 3.24:

Consider the Closed Helm graph CH_n with n=8.

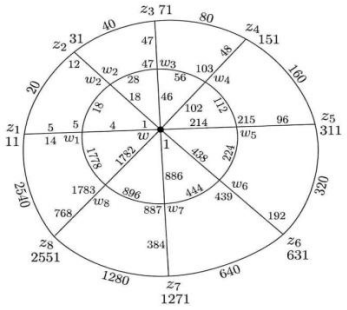


Fig. 12

Fig. 12 demonstrates that the closed helm graph CH_8 admits an odd-even congruence labeling.

Theorem 3.25:

The jellyfish graph $J_{m,n}$admitsan odd-even congruence labeling for all $n \geq 2$.

Proof:

Let $G= J_{m,n}$be ajellyfish graph consisting of a central 4-cycle structure with a spanning cross-chord, where m pendant vertices w_k are attached to vertex w, and n pendant vertices $z_k$are attached to vertex z.

We define the vertex set V(G) and the edge set E(G) are as follows:

$V(G) = \{x, y, w, z\} \cup \{w_k \mid 1 \leq k \leq m\} \cup \{z_k \mid 1 \leq k \leq n\}$

$E(G) = E_a \cup E_b \cup E_c \cup E_d \cup E_f \cup E_g \cup E_h$,where the respective edge subsets are defined as:

$E_a = \{wx\}$, $E_b = \{xz\}$,$E_c = \{wy\}$,$E_d = \{zy\}$, $E_f = \{xy\}$,

E_g= {ww_k|$1 \leq k \leq m$}, E_h= {zz_r | $1 \leq r \leq n$ }

Based on this framework, the total number of vertices and the total number of edges are calculated as:| V(G)| = m+n+4 and | E(G)| = m+n+5.

We introduce the vertex labeling function h: V(G) →$\mathbb{Z}^+$such that h(V(G)) consists exclusively of positiveodd integers:

h(w) = 1, h(z) = 3, h(x)=7, h(y) = 15

$h(w_k) = 10(2^k)- 9$ for $1 \leq k \leq m$

$h(z_r) = 14(2^r)– 9$ for $1 \leq r \leq n$

Next, we establish the edge labeling function K: E(G) →$\mathbb{Z}^+$such that K(E(G)) consists exclusively of positive even integers:

K(wx) = 6

K(xz) =4

K(wy) = 14

K(zy) = 12

K(xy) = 8

$K(ww_k) =10(2^k)- 10$ for $1 \leq k \leq m$

$K(zz_r) = 28(2^{r-1}) -12$ for $1 \leq r \leq n$

Verification of Congruence Conditions:

K(wx) = | h(w)-h(x)| =|1-7|=6

K(xz) = |h(x)-h(z)| =|7-3|=4

K(wy) = | h(w)-h(y)| =|1-15|=14

K(zy) = | h(z)-h(y)| =|3-15|=12

K(xy) = | h(x)-h(y)| =|7-15|=8

1. For edges belonging to E_g($e=ww_k$):

$K(ww_k) =10(2^k) -10$ for $1 \leq k \leq m$

Verify:

$1|\ h(w_k) – h(w)\ | = |\ (10(2^k)-9) – 1\ |=10(2^k) -10$

Since $K(ww_k) =10(2^k) -10$, it divides | $h(w_k) – h$(w) |

2. For pendant edges belonging to E_h($e=zz_r$):

$K(zz_r) =28(2^{r-1})-12$ for $1 \leq r \leq n$

Verify:

$|\ h(z_r) – h(z)\ | = |\ (14(2^r) -9) – 3\ |=14(2^r)-12=28(2^{r-1})-12$

Since $K(zz_r) = 28(2^{r-1})-12$, it divides $| h(z_r) - h(z) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the jellyfish graph $J_{m,n}$admits an odd-even congruence labeling.

Example 3.26:

Consider the jellyfish graph $J_{m,n}$ with m=4 and n=4

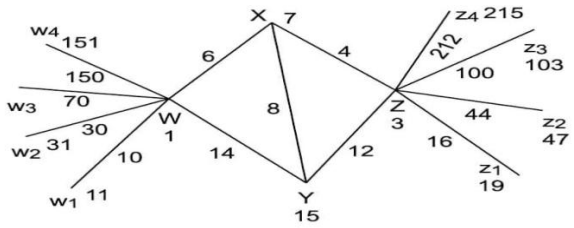


Fig. 13

Fig. 13 demonstrates that the jellyfish graph $J_{4,4}$ admits an odd-even congruence labeling.

Theorem 3.27:

The sunlet graph S_n admits an odd-even congruence labeling for all $n \geq 3$.

Proof:

Let $S_n$representasunlet graph consisting of an inner cycle C_n, and n pendant vertices.

We denote the inner cycle vertices as w_k and the outer pendant vertices as z_k for $1 \leq k \leq n$

We define the vertex set V(G) and the edge set E(G) as follows:

$V(S_n) = \{w_k \mid 1 \leq k \leq n\} \cup \{z_k \mid 1 \leq k \leq n\}$

$E(S_n) = E_a \cup E_b$, where the respective edge subsets are specified as follows:

$E_a = \{w_k z_k \mid 1 \leq k \leq n\}$

$E_b = \{w_k w_{k+1} \mid 1 \leq k \leq n-1\} \cup \{w_n w_1\}$

Based on this frame work, the total number of vertices and the total number of edges are calculated as:$|V(S_n)| = 2n$ and $| E(S_n)| = 2n$.

We introduce the vertex labeling function h: $V(S_n) \rightarrow \mathbb{Z}^+$such that $h(V(S_n))$consists exclusively of positive odd integers:

$h(w_k) = 5(2^k)-9 \; for \; 1 \leq k \leq n$

$h(z_k) = 7(2^k)-9 \; for \; 1 \leq k \leq n$

Next, we establish the edge labeling function K: $E(S_n) \rightarrow \mathbb{Z}^+$such that $K(E(S_n))$ consists exclusively of positive even integers:

$K(w_k z_k) = 2^{k+1} \; for \; 1 \leq k \leq n$

$K(w_k w_{k+1}) = 5(2^k) \; for \; 1 \leq k \leq n-1$

$K(w_n w_1) = 5(2^n)-10$

Verification of Congruence Conditions:

1. For edges belonging to $E_a(e=w_kz_k)$:

$$K(w_kz_k) = 2^{k+1} \; for \; 1 \leq k \leq n$$

Verify:

$$| h(z_k) - h(w_k) | = | (7(2^k)-9) - (5(2^k)-9) | = |2(2^k)| = 2^{k+1}$$

Since $K(w_kz_k) = 2^{k+1}$, it divides $| h(z_k) - h(w_k) |$

2. For edges belonging to $E_b(e=w_kw_{k+1})$:

$K(w_kw_{k+1}) = 5(2^k) \; for \; 1 \leq k \leq n-1$

Verify:

$$| h(w_{k+1}) - h(w_k) | = | (5(2^{k-+1})-9) - (5(2^k)-9) | = 5(2^k)$$

Since $K(w_kw_{k+1}) = 5(2^k)$, it divides $| h(w_{k+1}) - h(w_k) |$

3. For edges belonging to $E_c(e=w_nw_1)$:

$K(w_nw_1) = 5(2^n) - 10$

Verify:

$$| h(w_n) - h(w_1) | = | (5(2^n)-9) - (5(2^1)-9) | = 5(2^n)-10$$

Since $K(z_nz_1) = 5(2^n) - 10$, it divides $| h(z_n) - h(z_1) |$

As a result, the divisibility criteria are satisfied across all edge categories, confirming that the sunlet graph $S_n$admits an odd-even congruence labeling.

Example 3.28:

Consider the sunlet graph S_n with n=7

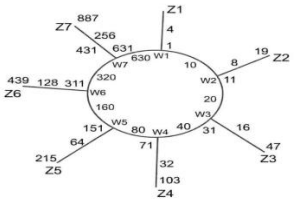


Fig. 14

Fig. 14 demonstrates that the sunlet graph S_7 admits an odd-even congruence labeling.

Conclusion

The study of graph concepts and the introduction of new classes of secure collaboration for significant edge structures are crucial for real-world applications in secure cryptographic frameworks. The findings demonstrate that odd-even congruence labelingis advantageous for various structures includingicecream, umbrella, double fan, triangular ladder, rectangular book, pentagonal book, jewel, Vanessa, helm, closed helm, jellyfish, and sunletgraphs; all are shown to accommodate the proposed labeling scheme.

In the future, this work can be extended to accommodate proposed labeling classes, providing a fruitful ground for research across other families of graphs.

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