Minimax Theory and its Applications
Volume 4 (2019), No. 1, 71–85

Sequences of Weak Solutions for a Navier Problem Driven by the $p(x)$-Biharmonic Operator

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Received: October 2, 2108
Accepted: November 22, 2018

We derive the existence of infinitely many solutions for an elliptic problem involving both the $p(x)$-biharmonic and the $p(x)$-Laplacian operators under Navier boundary conditions. Our approach is of variational nature and does not require any symmetry of the nonlinearities. Instead, a crucial role is played by suitable test functions in some variable exponent Sobolev space, of which we provide the abstract structure better suited to the framework.

Keywords: p(x)-biharmonic operator, p(x)-Laplacian operator, Navier problem, multiplicity.

2010 Mathematics Subject Classification: 35J35, 35J60.

1. Introduction

The aim of this paper is to explore the multiplicity of solutions to the following fourth-order elliptic problem with Navier boundary conditions,

$$(P_{\lambda,\mu}) \qquad \begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, $p \in C^0(\overline{\Omega})$ satisfies $\max\{1, n/2\} < \inf_{\overline{\Omega}} p \leq \sup_{\overline{\Omega}} p$, $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $V \in C^0(\overline{\Omega})$ with $\inf_{\overline{\Omega}} V > 0$, $f, g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions

*Because of a surprising coincidence of names within the same Department, we have to point out that the author was born on August 4, 1968.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50 © Heldermann Verlag

with suitable integrability conditions, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}^+$. The operators $\Delta^2_{p(x)}$ and $\Delta_{p(x)}$ are the natural generalization to the variable exponent framework of the standard p-biharmonic and p-Laplacian ($p > 1$ constant).

Nonlinear problems governed by such a type of operators represent a captivating subject which has attracted great attention during the latest years, also for the very interesting applications (see for instance [9, 18] and the survey paper [16]). Some recent existence results for Navier problems driven by the $p(x)$-biharmonic operator can be found in [1, 2, 13, 14, 20]. In [1] the authors derive the existence of infinitely many eigenpair sequences for an eigenvalue problem via Ljusternik-Schnirelmann's principle on C^1-manifolds, pointing also out the meaningful differences with respect to the constant case. Addressing the $p(x)$-sublinear version of the previous problem, the same authors [2] establish the existence of a sequence of weak solutions by using a version of the symmetric mountain pass theorem, proving the existence of continuous families of eigenvalues. Well-established variational tools are used to deduce the existence of one non-trivial solution: in [3] for two classes of problems with power-type nonlinearities and in [13] for a problem with singular weights.

For the sake of completeness, when p is constant it is worth mentioning [4, 8] where approaches similar to the one of this paper lead to the existence of infinitely many solutions for the problems under examination (in [8] it is also provided an explicit estimate of the parameters for which one has the existence). Finally, the reader can consult the further sources [5, 6, 7, 19] where several classes of variable exponent problems (and related multiplicity results) are dealt with, still from a variational perspective.

In this paper we address the problem of finding conditions on the data sufficient to ensure the existence of infinitely many solutions to $(P_{\lambda,\mu})$. Unlike much of the existing literature on the subject, our approach, variational, does not require any symmetry on the nonlinearities, but rather a judicious behaviour of the term μg at infinity, expressed in terms of its primitive. More specifically, the nonlinearity f having a global $p(x)$-sublinear growth, we distinguish several situations according to the behaviour at $+\infty$ of the functions

$$t \mapsto \int_\Omega \max_{|\xi|\le t} \left(\int_0^\xi g(x,s)ds \right) dx, \quad t \mapsto \int_{B(x_0,\varrho)} \left(\int_0^t g(x,s)ds \right) dx, \quad t \ge 0,$$

for some $x_0 \in \Omega$, $\varrho > 0$, proving the existence of a sequence of non-trivial weak solutions to $(P_{\lambda,\mu})$, with increasing norm in some generalized Lebesgue-Sobolev space. This result is obtained for any λ in $\mathbb{R}$ and for a range of μ prescribed by the aforementioned asymptotic behaviour (Theorem 3.1). The abstract tool behind our result is an alternative theorem for perturbed abstract functionals of [17].

As better clarified in Section 3, in this approach the choice of a suitable test function at which we reasonably evaluate the functionals involved is strategical. This function intervenes in the formulation itself of the multiplicity result, in the sense that different choices of it lead to different hypotheses on the data of the problem. In this regard, we tried to get the broadest generality possible by constructing an abstract class of test functions serving our purpose (see the definition of the space S in Section 2). A direct consequence of Theorem 3.1 is provided in the subsequent Theorem 3.2, upon which all the examples the conclude the paper are modelled.

The paper is then structured as follows. Section 2 includes all the basic results about Lebesgue and Sobolev variable exponent spaces which will be needed, besides the variational set-up of Problem $(P_{\lambda,\mu})$. In Section 3 we exhibit our multiplicity results and concrete examples of nonlinearities which fulfill them.

2. Variational framework

We collect here some preliminary results about Lebesgue and Sobolev variable exponent spaces, which are used in our investigations. The reader is invited to consult [10, 11, 12, 15] and the references therein for a more detailed account on this topic.

To begin with, we fix some notation for the sequel. Given a measurable function $h\colon \Omega \to \mathbb{R}$, we set

$$h^- := \operatorname*{essinf}_{\Omega} h, \quad h^+ := \operatorname*{esssup}_{\Omega} h.$$

We denote by $\omega := \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$ the measure of the unit ball in $\mathbb{R}^n$. If X is a Banach space, the symbol $B(x,r)$ stands for the open ball centered at $x \in X$ and of radius $r > 0$.

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 1$, and let $p \in C^0(\overline{\Omega})$ satisfy $1 < p^- \leq p^+$.

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega |u|^{p(x)} dx < +\infty \right\}.$$

It is a reflexive Banach space when endowed with the Luxemburg norm

$$|u|_{p(x)} = \inf \left\{ \sigma > 0 : \int_\Omega \left| \frac{u}{\sigma} \right|^{p(x)} dx \leq 1 \right\}.$$

If we denote by $\rho_{p(x)}\colon L^{p(x)}(\Omega) \to \mathbb{R}$ the functional defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$$

for all $u \in L^{p(x)}(\Omega)$ (the so-called *modular* of the space $L^{p(x)}(\Omega)$), we can compare it to the Luxemburg norm by means of the following proposition.

Proposition 2.1. *Let $u \in L^{p(x)}(\Omega)$ and let $\{u_k\}$ be a sequence in $L^{p(x)}(\Omega)$; then*

(1) $|u|_{p(x)} < 1 \ (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1 \ (= 1; > 1)$;

(2) $|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;

(3) $|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;

(4) $|u_k - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_k - u) \to 0$.

For any $k \in \mathbb{N}$, define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega) \text{ for any } |\alpha| \leq k \right\},$$

where $D^\alpha u$ is the partial derivative of u with respect to the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. The space $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space under the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}.$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ and then by $X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ the space naturally associated with $(P_{\lambda,\mu})$. On X the functional

$$\|u\|_X := \inf \left\{ \sigma > 0 : \int_\Omega \left| \frac{\Delta u}{\sigma} \right|^{p(x)} dx \leq 1 \right\}$$

for any $u \in X$, defines a norm equivalent to $\|\cdot\|_{2,p(x)}$ (see for instance [21]).

Clearly the embedding $X \hookrightarrow W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega)$ is continuous and, by Rellich-Kondrachov's theorem, $W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ compactly when Ω is bounded and $p^- > \frac{n}{2}$. This leads immediately to the following important result.

Proposition 2.2. *The embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact provided that $p^- > \frac{n}{2}$.*

From now on we consider the case $k = 2$, strictly related to $(P_{\lambda,\mu})$, and consider the following functional

$$\|u\|_V = \inf \left\{ \sigma > 0 : \int_\Omega \left(\left| \frac{\Delta u}{\sigma} \right|^{p(x)} + \left| \frac{\nabla u}{\sigma} \right|^{p(x)} + V(x) \left| \frac{u}{\sigma} \right|^{p(x)} \right) dx \leq 1 \right\}$$

for any $u \in X$. Since $V^- > 0$, $\|\cdot\|_V$ represents a norm on X, equivalent to the previous ones. Obviously, one has $\|u\|_X \leq \|u\|_V$ for every $u \in X$ and therefore there exists a constant $c_\infty > 0$ such that

$$\|u\|_\infty \leq c_\infty \|u\|_V \tag{1}$$

for every $u \in X$. In addition, defining the modular $\rho_{p(x),V} \colon X \to \mathbb{R}$ associated with V by

$$\rho_{p(x),V}(u) = \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx$$

for all $u \in X$, in the same way as Proposition 2.1, it holds:

Proposition 2.3. *Let $u \in X$ and let $\{u_k\}$ be a sequence in X. Then*

(1) $\|u\|_V < 1 \ (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x),V}(u) < 1 \ (= 1; > 1)$;

(2) $\|u\|_V \geq 1 \quad \Rightarrow \quad \|u\|_V^{p^-} \leq \rho_{p(x),V}(u) \leq \|u\|_V^{p^+}$;

(3) $\|u\|_V \leq 1 \quad \Rightarrow \quad \|u\|_V^{p^+} \leq \rho_{p(x),V}(u) \leq \|u\|_V^{p^-}$;

(4) $\|u_k - u\|_V \to 0 \quad \Leftrightarrow \quad \rho_{p(x),V}(u_k - u) = 0$.

Now, for the motivations illustrated in the Introduction, let us introduce the following class of functions.

If $\{a_k\}, \{b_k\}, \{d_k\}$ are three real sequences with $0 < a_k < b_k$ and $d_k > 0$ for all $k \in \mathbb{N}$, let us denote by $S(\{a_k\}, \{b_k\}, \{d_k\})$ the space of all sequences $\{\eta_k\} \subset W^{2,p^+}(a_k, b_k)$ satisfying

(i) $0 \leq \eta_k(x) \leq d_k$ for a.e. $x \in (a_k, b_k)$;

(ii) $\lim_{x \to a_k^+} \eta_k(x) = d_k, \quad \lim_{x \to b_k^-} \eta_k(x) = 0$;

(iii) $\lim_{x \to a_k^+} \eta_k'(x) = \lim_{x \to b_k^-} \eta_k'(x) = 0$;

(iv) for all $i \in \{1, 2\}$ there exists $\kappa_i > 0$, independent of k, such that

$$|\eta_k^{(i)}(x)| \leq \kappa_i \frac{d_k}{(b_k - a_k)^i} \tag{2}$$

for a.e. $x \in (a_k, b_k)$ and for all $k \in \mathbb{N}$.

If $\bar{x} \in \Omega$ and $\{\eta_k\} \in S(\{a_k\}, \{b_k\}, \{d_k\})$, consider the function $u_k \colon \Omega \to \mathbb{R}$ defined as follows:

$$u_k(x) := \begin{cases} 0 & \text{in } \Omega \setminus B(\bar{x}, b_k) \\ d_k & \text{in } B(\bar{x}, a_k) \\ \eta_k(|x - \bar{x}|) & \text{in } B(\bar{x}, b_k) \setminus B(\bar{x}, a_k). \end{cases} \tag{3}$$

Owing to the embedding $W^{2,p^+}(\Omega) \hookrightarrow W^{2,p(x)}(\Omega)$, it is clear that $\{u_k\} \subset X$. Moreover, a simple computation shows that, fixed $k \in \mathbb{N}$, for any $i = 1, 2, \ldots, n$ one has

$$\frac{\partial u_k}{\partial x_i}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(\bar{x}, b_k) \\ 0 & \text{in } B(\bar{x}, a_k) \\ \eta_k'(|x - \bar{x}|) \dfrac{x_i - \bar{x}_i}{|x - \bar{x}|} & \text{in } B(\bar{x}, b_k) \setminus B(\bar{x}, a_k) \end{cases} \tag{4}$$

and

$$\frac{\partial^2 u_k}{\partial x_i^2}(x) = \begin{cases} 0 & \text{in } \Omega \setminus B(\bar{x}, b_k) \\ 0 & \text{in } B(\bar{x}, a_k) \\ \eta_k''(|x - \bar{x}|) \dfrac{(x_i - \bar{x}_i)^2}{|x - \bar{x}|^2} + & \\ \quad + \eta_k'(|x - \bar{x}|) \dfrac{|x - \bar{x}|^2 - (x_i - \bar{x}_i)^2}{|x - \bar{x}|^3} & \text{in } B(\bar{x}, b_k) \setminus B(\bar{x}, a_k) \end{cases} \tag{5}$$

From (2), (4) and (5) we obtain the following inequalities:

$$
\begin{aligned}
|\nabla u_k(x)| &\le |\eta_k'(|x-\bar{x}|)| \le \kappa_1 \frac{d_k}{b_k-a_k}, && (6)\\
|\Delta u_k(x)| &= \eta_k''(|x-\bar{x}|) + (n-1)\frac{\eta_k'(|x-\bar{x}|)}{|x-\bar{x}|} \\
&\le \kappa_2 \frac{d_k}{(b_k-a_k)^2} + \kappa_1(n-1)\frac{d_k}{a_k(b_k-a_k)}, && (7)
\end{aligned}
$$

thanks to which we are able to estimate the modular $\varrho_{p(x),V}$ at u_k, for k large enough, as follows:

$$
\begin{aligned}
\varrho_{p(x),V}(u_k) &\le \left(\kappa_2 \frac{d_k}{(b_k-a_k)^2} + \kappa_1(n-1)\frac{d_k}{a_k(b_k-a_k)}\right)^{p^+} \omega(b_k^n - a_k^n) \\
&\quad + \left(\kappa_1 \frac{d_k}{b_k-a_k}\right)^{p^+} \omega(b_k^n - a_k^n) + \left(\max_{B(\bar{x},b_k)} V\right) d_k^{p^+} \omega b_k^n \\
&\le 2^{p^+-1}\kappa_2^{p^+} \frac{d_k^{p^+}}{(b_k-a_k)^{2p^+}} \omega(b_k^n-a_k^n) + 2^{p^+-1}\kappa_1^{p^+}(n-1)^{p^+} \frac{d_k^{p^+}}{a_k^{p^+}(b_k-a_k)^{p^+}} \omega(b_k^n-a_k^n) \\
&\quad + \kappa_1^{p^+} \frac{d_k^{p^+}}{(b_k-a_k)^{p^+}} \omega(b_k^n-a_k^n) + \left(\max_{B(\bar{x},b_k)} V\right) d_k^{p^+} \omega b_k^n \\
&\le \omega d_k^{p^+} \left(2^{p^+-1}\kappa_2^{p^+} \frac{b_k^n-a_k^n}{(b_k-a_k)^{2p^+}} + \left(2^{p^+-1}\frac{(n-1)^{p^+}}{a_k^{p^+}} + 1\right)\kappa_1^{p^+} \frac{b_k^n-a_k^n}{(b_k-a_k)^{p^+}} \right. \\
&\quad \left. + \left(\max_{B(\bar{x},b_k)} V\right) b_k^n \right). && (8)
\end{aligned}
$$

From now on, without further mentioning, we always assume that $p \in C^0(\overline{\Omega})$ satisfies

$$
\max\left\{1, \frac{n}{2}\right\} < p^- \le p^+.
$$

Let us denote by $\mathcal{C}$ the class of all Carathéodory functions $\zeta\colon \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying $\sup_{|t|\le\xi} |\zeta(\cdot,t)| \in L^1(\Omega)$ for all $\xi > 0$ and let $f, g \in \mathcal{C}$.

By a *weak solution* to $(P_{\lambda,\mu})$ we mean any function $u \in X$ such that

$$
\begin{aligned}
&\int_\Omega \left(|\Delta u|^{p(x)-2}\Delta u \Delta v + |\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right) dx \\
&= \lambda \int_\Omega f(x,u)v dx + \mu \int_\Omega g(x,u) v dx
\end{aligned}
$$

for all $v \in X$. Clearly problem $(P_{\lambda,\mu})$ is variational and its weak solutions are nothing else but the critical points of the functional $\mathcal{E}\colon X \to \mathbb{R}$ defined for all $u \in X$ by

$$
\mathcal{E}(u) := \Psi(u) + \lambda J_F(u) + \mu J_G(u),
$$

where
$$\Psi(u) := \int_\Omega \frac{|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + V(x)|u|^{p(x)}}{p(x)}\, dx,$$
$$J_F(u) := -\int_\Omega F(x,u)\, dx, \tag{9}$$
$$J_G(u) := -\int_\Omega G(x,u)\, dx,$$

where, for all $(x,t) \in \Omega \times \mathbb{R}$
$$F(x,t) := \int_0^t f(x,s)\, ds, \quad G(x,t) := \int_0^t g(x,s) ds\,.$$

As already stated in the Introduction, we will prove our main result with the aid of an abstract result (now classical), Theorem 2.5 of [17], recalled below for the reader's convenience.

Theorem 2.4. *Let E be a reflexive real Banach space and let $\Phi, \Psi \colon E \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is strongly continuous and coercive.*

For each $r > \inf_E \Psi$, define $\varphi(r)$ by
$$\varphi(r) := \inf_{x \in \Psi^{-1}(]-\infty, r[)} \frac{\Phi(x) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}_w} \Phi}{r - \Psi(x)},$$

where $\overline{\Psi^{-1}(]-\infty, r[)}_w$ is the closure of $\Psi^{-1}(]-\infty, r[)$ in the weak topology.

Fix $\lambda \in \mathbb{R}$, then

(a) *if $\{r_k\}$ is a real sequence such that $\lim_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \lambda$ for all $k \in \mathbb{N}$, the following alternative holds: either $\Phi + \lambda\Psi$ has a global minimum or there exists a sequence $\{x_k\} \subset E$ of critical points of $\Phi + \lambda\Psi$ such that $\lim_{k\to\infty} \Psi(x_k) = +\infty$;*

(b) *if $\{s_k\}$ is a real sequence such that $\lim_{k\to\infty} s_k = (\inf_E \Psi)^+$ and $\varphi(s_k) < \lambda$ for all $k \in \mathbb{N}$, the following alternative holds: either there exists a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$ or there exists a sequence $\{x_k\} \subset E$ of pairwise distinct critical points of $\Phi + \lambda\Psi$ with $\lim_{k\to\infty} \Psi(x_k) = \inf_E \Psi$ which weakly converges to a global minimum of Ψ.*

3. Our results

The main result of this paper reads as follows:

Theorem 3.1. *Let $V \in C^0(\Omega)$ with $V^- > 0$ and let $f, g \in \mathcal{C}$ satisfy:*

(f_1) *there exist a measurable function $m \colon \Omega \to \mathbb{R}$, with $1 \le m \le p$ in Ω and $m^+ < p^-$ and a function $h_1 \in L^1(\Omega)$, such that*
$$|f(x,t)| \le h_1(x) \left(1 + |t|^{m(x)-1}\right)$$
for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$;

(g_1) *$G(x,t) \geq 0$ for a.e. $x \in \Omega$, for all $t \geq 0$;*

(g_2) *there exist $x_0 \in \Omega$, $\varrho, q_1, q_2 > 0$, such that $B(x_0, \varrho) \subset \Omega$ and*

$$\alpha := \liminf_{t\to+\infty} \frac{\int_\Omega \max_{|\xi|\leq t} G(x,\xi)dx}{t^{q_1}} < +\infty, \quad \beta := \limsup_{t\to+\infty} \frac{\int_{B(x_0,\varrho)} G(x,t)dx}{t^{q_2}} > 0.$$

Then the following facts hold:

(i) *if $q_1 < p^-$ and $q_2 > p^+$, for all $\lambda \in \mathbb{R}$ and for all $\mu > 0$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(ii) *if $q_1 < p^-$ and $q_2 = p^+$, there exists $\mu_1 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu > \mu_1$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(iii) *if $q_1 = p^-$ and $q_2 > p^+$, there exists $\mu_2 > 0$ such that, for all $\lambda \in \mathbb{R}$ and for all $\mu \in (0, \mu_2)$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions;*

(iv) *if $q_1 = p^-$ and $q_2 = p^+$, there exist $\gamma > 1$ and $C_{V,\gamma,\varrho} > 0$ such that, if*

$$C_{V,\gamma,\varrho} < \frac{\beta}{\alpha\omega p^+ c_\infty^{p^-}}, \tag{10}$$

(the previous inequality always being satisfied whether $\alpha = 0$ or $\beta = +\infty$) for all $\lambda \in \mathbb{R}$ and every $\mu \in (\mu_1, \mu_2)$, $(P_{\lambda,\mu})$ admits a sequence of non-zero weak solutions.

Proof. To prove (i), we invoke Theorem 2.4(a), with $E = X$, Ψ as in (9) and $\Phi = J_F + J_G$. It is clear that Ψ is C^1, sequentially weakly lower semicontinuous and, being

$$\Psi(u) \geq \frac{1}{p^+} \|u\|_V^{p^-},$$

coercive as well. The sequential weak lower semicontinuity of Φ follows, by standard arguments, from (f_1), (g_2) and Proposition 2.2. Next, define

$$\varphi(r) = \inf_{\Psi(u)<r} \frac{\sup_{\Psi(w)\leq r} \Phi - \Phi(u)}{r - \Psi(u)}$$

for all $r > 0$. Our next step is to find a sequence $\{r_k\} \subset \mathbb{R}$, diverging to $+\infty$, such that $\varphi(r_k) < 1$ for all $k \in \mathbb{N}$. Thanks to the definition of φ, it amounts to building a sequence $\{u_k\} \subset X$, with $\Psi(u_k) < r_k$ for all $k \in \mathbb{N}$, and satisfying

$$\sup_{\Psi(w)\leq r_k} \left(\lambda \int_\Omega F(x,w)dx + \mu \int_\Omega G(x,w)dx\right) - \lambda \int_\Omega F(x,u_k)dx - \mu \int_\Omega G(x,u_k)dx$$
$$< r_k - \Psi(u_k). \tag{11}$$

We set $u_k = 0$ for any $k \in \mathbb{N}$. Thanks to (g_2), fixed $\tilde{\alpha} > \alpha$, for any $k \in \mathbb{N}$ there exists $\alpha_k \geq k$ such that

$$\int_\Omega \max_{|\xi|\leq\alpha_k} G(x,\xi)dx \leq \tilde{\alpha}\alpha_k^{q_1}.$$

For any $k \in \mathbb{N}$ define $$r_k := \frac{1}{p^+ c_\infty^{p^-}} \alpha_k^{p^-}.$$

It is clear that $r_k \to +\infty$ as $k \to \infty$ and $\Psi(u_k) < r_k$. We check the validity of (11). Observe that one has

$$\|w\|_\infty \le c_\infty \|w\|_V \le c_\infty \max\left\{(p^+\Psi(w))^{\frac{1}{p^+}}, (p^+\Psi(w))^{\frac{1}{p^-}}\right\}$$

for any $w \in X$. Taking this fact into account, if $w \in X$ and $\Psi(w) \le r_k$, one has

$$\|w\|_\infty \le c_\infty (p^+ r_k)^{1/p^-} = \alpha_k$$

and by (f_1), (g_1), (g_2) we obtain, for k large enough,

$$\begin{aligned}
&\lambda \int_\Omega F(x,w)dx + \mu \int_\Omega G(x,w)dx \qquad (12)\\
&\le |\lambda| \int_\Omega |h_1(x)| \left(1 + \frac{|w|^{m(x)}}{m(x)}\right) dx + \mu \int_\Omega \max_{|\xi|\le\alpha_k} G(x,\xi)dx\\
&\le |\lambda| \|h_1\|_{L^1(\Omega)} \left(1 + \frac{\alpha_k^{m^+}}{m^-}\right) + \mu\tilde{\alpha}\alpha_k^{q_1}\\
&\le |\lambda| \|h_1\|_{L^1(\Omega)} + \frac{|\lambda| \|h_1\|_{L^1(\Omega)}}{m^-} c_\infty^{m^+} (p^+)^{\frac{m^+}{p^-}} r_k^{\frac{m^+}{p^-}} + \mu\tilde{\alpha} c_\infty^{q_1} (p^+)^{\frac{q_1}{p^-}} r_k^{\frac{q_1}{p^-}} < r_k.
\end{aligned}$$

According to part (a) of Theorem 2.4, either the functional $\Phi + \Psi$ has a global minimum or there exists a sequence of weak solutions $\{v_k\} \subset X$ such that $\|v_k\|_V \to +\infty$ as $k \to \infty$. Let us show that $\Phi + \Psi$ is unbounded from below.

Owing to (g_2), fixed $0 < \tilde{\beta} < \beta$, for any $k \in \mathbb{N}$ there exists $\beta_k \ge k$ such that

$$\int_{B(x_0,\varrho)} G(x,\beta_k)dx \ge \tilde{\beta}\beta_k^{q_2}.$$

Let $\gamma > 1$ and $\{\eta_k\} \in S(\varrho, \gamma\varrho, \beta_k)$. Similarly to (3), consider the function u_k defined by

$$u_k(x) := \begin{cases} 0 & \text{in } \Omega \setminus B(x_0,\gamma\varrho) \\ \beta_k & \text{in } B(x_0,\varrho) \\ \eta_k(|x-x_0|) & \text{in } B(x_0,\gamma\varrho) \setminus B(x_0,\varrho). \end{cases} \qquad (13)$$

By virtue of (8) one has

$$\begin{aligned}
\varrho_{p(x),V}(u_k) \le \omega\beta_k^{p^+} &\Bigg(2^{p^+-1}\kappa_2^{p^+} \frac{\varrho^{n-2p^+}(\gamma^n - 1)}{(\gamma-1)^{2p^+}} + \\
&+ \left(2^{p^+-1}\frac{(n-1)^{p^+}}{\varrho^{p^+}} + 1\right)\kappa_1^{p^+} \frac{\varrho^{n-p^+}(\gamma^n-1)}{(\gamma-1)^{p^+}} + \left(\max_{B(x_0,\gamma\varrho)} V\right)\gamma^n\varrho^n\Bigg)
\end{aligned}$$

and hence $\Psi(u_k) \le \omega C_{V,\gamma,\varrho}\beta_k^{p^+}$, where

$$C_{V,\gamma,\varrho} := \frac{1}{p^-}\left(2^{p^+-1}\kappa_2^{p^+}\frac{\varrho^{n-2p^+}(\gamma^n-1)}{(\gamma-1)^{2p^+}} + \right. \tag{14}$$
$$\left. + \left(2^{p^+-1}\frac{(n-1)^{p^+}}{\varrho^{p^+}}+1\right)\kappa_1^{p^+}\frac{\varrho^{n-p^+}(\gamma^n-1)}{(\gamma-1)^{p^+}} + \left(\max_{B(x_0,\gamma\varrho)} V\right)\gamma^n\varrho^n\right).$$

So, one has

$$\Psi(u_k)+\Phi(u_k) \le \omega C_{V,\gamma,\varrho}\beta_k^{p^+} + |\lambda|\int_\Omega |h_1(x)|\left(1+\frac{|u_k|^{m(x)}}{m(x)}\right)dx - \mu\tilde{\beta}\beta_k^{q_2}$$
$$\le \omega C_{V,\gamma,\varrho}\beta_k^{p^+} + |\lambda|\,\|h_1\|_{L^1(B(x_0,\varrho))} + \frac{|\lambda|\,\|h_1\|_{L^1(B(x_0,\varrho))}}{m^-}\beta_k^{m^+} - \mu\tilde{\beta}\beta_k^{q_2} \tag{15}$$

and, since $m^+ < p^+ < q_2$ and $\beta_k \to +\infty$ as $k \to \infty$, the functional $\Psi + \Phi$ does not possess any global minimum, as desired. This concludes the proof of (i).

The proof of (ii) and (iii) follows from slight modifications. When $q_1 < p^-$ and $q_2 = p^+$, set $\mu_1 := \dfrac{\omega C_{V,\gamma,\varrho}}{\beta}$; if $\beta = +\infty$, we agree to read, as usual, $\mu_1 = 0$. Then, if $\lambda \in \mathbb{R}$ and $\mu > \mu_1$, choosing $\dfrac{\omega C_{V,\gamma,\varrho}}{\mu} < \tilde{\beta} < \beta$, in the wake of the proof of (i), we obtain

$$\Psi(u_k)+\Phi(u_k) \le \omega C_{V,\gamma,\varrho}\beta_k^{p^+} + |\lambda|\,\|h_1\|_{L^1(B(x_0,\varrho))} + \frac{|\lambda|\,\|h_1\|_{L^1(B(x_0,\varrho))}}{m^-}\beta_k^{m^+} - \mu\tilde{\beta}\beta_k^{p^+}$$

and the lower unboundedness of $\Psi + \Phi$ is guaranteed by the range of $\tilde{\beta}$. On the other hand, in the case $q_1 = p^-$ and $q_2 > p^+$, it suffices to pick $\mu_2 := \dfrac{1}{\alpha p^+ c_\infty^{p^-}}$ (as before, if $\alpha = 0$, we set $\mu_2 = +\infty$). Then, fixing $\lambda \in \mathbb{R}$, $\mu < \mu_2$ and picking $\alpha < \tilde{\alpha} < \dfrac{1}{\mu p^+ c_\infty^{p^-}}$, similarly to (i), we use (g_2) with such an $\tilde{\alpha}$ to get

$$\lambda\int_\Omega F(x,w)dx + \mu\int_\Omega G(x,w)dx$$
$$\le |\lambda|\,\|h_1\|_{L^1(\Omega)} + \frac{|\lambda|\,\|h_1\|_{L^1(\Omega)}}{m^-}c_\infty^{m^+}(p^+)^{\frac{m^+}{p^-}}r_k^{\frac{m^+}{p^-}} + \mu\tilde{\alpha}c_\infty^{p^-}p^+r_k < r_k$$

for k large enough, due to the choice of $\tilde{\alpha}$.

Finally, in the double limit case (iv), assumption (10) ensures that $\mu_1 < \mu_2$. So, in the light of (ii) and (iii), the conclusion is achieved for any $\lambda \in \mathbb{R}$ and $\mu \in (\mu_1, \mu_2)$. □

A direct consequence of Theorem 3.1 is the following one.

Theorem 3.2. *Let* $V \in C^0(\overline{\Omega})$ *with* $V^- > 0$, $\tilde{h}_1 \in L^1(\overline{\Omega})$, $h_2 \in L^1(\Omega) \setminus \{0\}$ *with* $h_2 \geq 0$ *in* Ω, $m\colon \Omega \to \mathbb{R}$ *measurable with* $1 \leq m \leq p$ *in* Ω *and* $m^+ < p^-$. *Let* $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function with* $\int_0^t \tilde{g}(\xi)d\xi \geq 0$ *for all* $t \geq 0$.

Finally assume that there exist two real sequences $\{\alpha_k\}$, $\{\beta_k\}$, *with the property* $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = +\infty$, *and* $q_1, q_2, a, b > 0$, *such that*

$$\max_{|\xi|\leq\alpha_k} \int_0^\xi \tilde{g}(t)dt \leq a\alpha_k^{q_1}, \qquad \int_0^{\beta_k} \tilde{g}(t)dt \geq b\beta_k^{q_2}.$$

Then, considering the problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = \\ \qquad = \lambda\tilde{h}_1(x)|u|^{m(x)-2}u + \mu h_2(x)\tilde{g}(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

the following facts hold:

(i) *if* $q_1 < p^-$ *and* $q_2 > p^+$, *for all* $\lambda \in \mathbb{R}$ *and for all* $\mu > 0$, $(\tilde{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions;*

(ii) *if* $q_1 < p^-$ *and* $q_2 = p^+$, *there exists* $\mu_1 > 0$ *such that, for all* $\lambda \in \mathbb{R}$ *and for all* $\mu > \mu_1$, $(\tilde{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions;*

(iii) *if* $q_1 = p^-$ *and* $q_2 > p^+$, *there exists* $\mu_2 > 0$ *such that, for all* $\lambda \in \mathbb{R}$ *and for all* $\mu \in (0, \mu_2)$, $(\tilde{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions;*

(iv) *if* $q_1 = p^-$ *and* $q_2 = p^+$, *there exist* $x_0 \in \Omega$, $\varrho > 0$, $\gamma > 1$ *and* $C_{V,\gamma,\varrho} > 0$ *such that, if*

$$C_{V,\gamma,\varrho} < \frac{b\,\|h_2\|_{L^1(B(x_0,\varrho))}}{a\,\|h_2\|_{L^1(\Omega)}\,\omega p^+ c_\infty^{p^-}}, \tag{16}$$

for all $\lambda \in \mathbb{R}$ *and every* $\mu \in (\mu_1, \mu_2)$, $(\tilde{P}_{\lambda,\mu})$ *admits a sequence of non-zero weak solutions.*

Proof. The proof plainly follows by applying Theorem 3.1 to the nonlinearities

$$f(x,t) = \tilde{h}_1(x)|t|^{m(x)-2}t, \quad g(x,t) = h_2(x)\tilde{g}(t)$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Assumptions (f_1) and (g_1) are immediate to verify. Since $h_2 \not\equiv 0$, let us choose $x_0 \in \Omega$ and $\varrho > 0$ so that $B(x_0,\varrho) \subset \Omega$ and $h_2 > 0$ in $B(x_0,\varrho)$. One has

$$\begin{aligned} \int_\Omega \max_{|\xi|\leq\alpha_k} G(x,\xi)dx &= \int_\Omega \max_{|\xi|\leq\alpha_k} \left(\int_0^\xi h_2(x)\tilde{g}(t)dt\right)dx \\ &= \|h_2\|_{L^1(\Omega)} \max_{|\xi|\leq\alpha_k} \int_0^\xi \tilde{g}(t)dt \leq a\,\|h_2\|_{L^1(\Omega)}\,\alpha_k^{q_1} \end{aligned}$$

and thus
$$\liminf_{t\to+\infty}\frac{\int_\Omega \max_{|\xi|\le t} G(x,\xi)dx}{t^{q_1}} \le a\,\|h_2\|_{L^1(\Omega)} < +\infty. \tag{17}$$

In a similar fashion,

$$\int_{B(x_0,\varrho)} G(x,\beta_k)dx = \|h_2\|_{L^1(B(x_0,\varrho))}\int_0^{\beta_k}\tilde{g}(t)dt \ge b\,\|h_2\|_{L^1(B(x_0,\varrho))}\,\beta_k^{q_2}$$

and therefore
$$\limsup_{t\to+\infty}\frac{\int_{B(x_0,\varrho)} G(x,t)dx}{t^{q_2}} \ge b\,\|h_2\|_{L^1(B(x_0,\varrho))} > 0. \tag{18}$$

So, the conclusions (i)–(iii) follow directly from Theorem 3.1. As for (iv), inequality (10) is verified by the joint use of (16), (17) and (18). □

In the remainder of the section we supply some examples of functions $\tilde{g}$ allowed by Theorem 3.2. The first one concerns the case (i) and works as a prototype for this kind of nonlinearities.

Example 3.3. Let $h_2 \in L^1(\Omega)\setminus\{0\}$, $h_2 \ge 0$ in Ω. Choose $a, b, q_1, q_2 > 0$ with $q_1 < p^-$ and $q_2 > p^+$, and let $\{\beta_r\}$ be a non-decreasing real sequence such that $\lim_{r\to\infty}\beta_r = +\infty$.

Define a subsequence $\{\beta_{r_k}\}$ of $\{\beta_r\}$ and a new sequence $\{\alpha_k\}$ recursively as follows:

$$\beta_{r_1} > \left(\frac{a}{b}\right)^{\frac{1}{q_2-q_1}}, \quad \beta_{r_k} > \left(\frac{b}{a}\right)^{\frac{1}{q_1}}\beta_{r_{k-1}}^{\frac{q_2}{q_1}} := \alpha_{k-1}, \quad \text{for all } k \ge 2. \tag{19}$$

Now, define $\hat{g}\colon \mathbb{R}\to\mathbb{R}$ by

$$\hat{g}(t) := \begin{cases} 0 & \text{for } t \in (-\infty, 0] \\ At^3 + Bt^2 & \text{for } t \in (0, \beta_{r_1}] \\ b\beta_{r_k}^{q_2} & \text{for } t \in (\beta_{r_k}, \alpha_k], \quad k \ge 1 \\ C_k t^3 + D_k t^2 + E_k t + F_k & \text{for } t \in (\alpha_k, \beta_{r_{k+1}}], \quad k \ge 1 \end{cases} \tag{20}$$

where

$$A := -2b\beta_{r_1}^{q_2-3}, \quad B := 3b\beta_{r_1}^{q_2-2}, \quad C_k := -\frac{2(b\beta_{r_k}^{q_2} - a\alpha_{k-1})}{(\beta_{r_k}-\alpha_{k-1})^3},$$

$$D_k := \frac{3(\beta_{r_k}+\alpha_{k-1})(b\beta_{r_k}^2 - a\alpha_{k-1}^{q_1})}{(\beta_{r_k}-\alpha_{k-1})^3}, \quad E_k := -\frac{6\alpha_{k-1}\beta_{r_k}(b\beta_{r_k}^{q_2} - a\alpha_{k-1}^{q_1})}{(\beta_{r_k}-\alpha_{k-1})^3},$$

$$F_k := \frac{a\beta_{r_k}^2\alpha_{k-1}^{q_1}(\beta_{r_k}-3\alpha_{k-1}) + b\beta_{r_k}^{q_2}\alpha_{k-1}^2(3\beta_{r_k}-\alpha_{k-1})}{(\beta_{r_k}-\alpha_{k-1})^3}.$$

It is straightforward to verify that the sequences $\{\alpha_k\}$, $\{\beta_{r_k}\}$ defined by (19) and the function $\int_0^t \tilde{g}(\xi)d\xi := \hat{g}(t)$ obey all the requirements of Theorem 3.2. Indeed, by construction, one has for all $k \in \mathbb{N}$

$$\max_{|\xi|\le\alpha_k} \hat{g}(\xi) = b\beta_{r_k}^{q_2} = a\alpha_k^{q_1}, \quad \text{and} \quad \hat{g}(\beta_{r_k}) = b\beta_{r_k}^{q_2}.$$

The second example is related to case (iv). In this circumstance, for the sake of concreteness we limit ourselves to the one-dimensional setting, providing an explicit estimate of the constant c_∞ in (16).

Example 3.4. Let $n = 1$, $\Omega = (-1, 1)$, $p(x) = -2x^2 + 4$ and $V(x) = x^2$ for all $x \in (-1, 1)$, $q_1 = p^- = 2$, $q_2 = p^+ = 4$, $h_2 \in L^1((-1, 1)) \setminus \{0\}$, $h_2 \ge 0$ in $(-1, 1)$ and $\int_{-1/2}^{1/2} h_2(x)dx > 0$.

Assume $\{\alpha_k\}$, $\{\beta_{r_k}\}$, $\tilde{g}$ as in Example 3.3.

It is well-known that, for all $u \in W^{2,2}((-1, 1)) \cap W_0^{1,2}((-1, 1))$, one has

$$\max_{x\in(-1,1)} |u(x)| \le \frac{\sqrt{2}}{2} \|u'\|_{L^2((-1,1))} \quad \text{and} \quad \|u'\|_{L^2((-1,1))} \le \frac{2}{\pi} \|u''\|_{L^2((-1,1))},$$

so

$$\max_{x\in(-1,1)} |u(x)| \le \frac{\sqrt{2}}{\pi} \|u''\|_{L^2((-1,1))}.$$

Now, since $L^{p(x)}((-1, 1)) \hookrightarrow L^2((-1, 1))$ and

$$\|u\|_{L^2((-1,1))} \le 2\max\left\{2^{\left(\frac{x^2-1}{2(x^2-2)}\right)^+}, 2^{\left(\frac{x^2-1}{2(x^2-2)}\right)^-}\right\} |u|_{p(x)} = 2\sqrt[4]{2}|u|_{p(x)}$$

(cf. Corollary 3.3.4 in [10]), collecting the previous estimates we finally get

$$\max_{x\in(-1,1)} |u(x)| \le \frac{2\sqrt[4]{8}}{\pi} \|u\|_V.$$

Now choose $\varrho = \frac{1}{2}$, $\gamma = \frac{3}{2}$ and for any $k \in \mathbb{N}$ let $\eta_k \in S(\frac{1}{2}, \frac{3}{4}, \beta_{r_k})$ be the function

$$\eta_k(x) := 64\beta_{r_k} \left(2x^3 - \frac{15}{4}x^2 + \frac{9}{4}x - \frac{27}{64}\right),$$

for any $x \in \left(\frac{1}{2}, \frac{3}{4}\right)$. The computations of the first two derivatives of η_k yield

$$|\eta'(x)| \le 6\beta_{r_k}, \quad |\eta''(x)| \le 96\beta_{r_k},$$

so (2) is satisfied by $\kappa_1 = \frac{3}{2}$ and $\kappa_2 = 6$, respectively.

As a next step, consider the sequence $\{u_k\} \subset W^{2,p(x)}((-1,1)) \cap W_0^{1,p(x)}((-1,1))$ defined by

$$u_k(x) := \begin{cases} 0 & \text{in } (-1,1) \setminus (-\frac{3}{4}, \frac{3}{4}) \\ \beta_{r_k} & \text{in } (-\frac{1}{2}, \frac{1}{2}) \\ \eta_k(|x|) & \text{in } (-\frac{3}{4}, \frac{3}{4}) \setminus (-\frac{1}{2}, \frac{1}{2}). \end{cases} \tag{21}$$

It turns out that

$$C_{V,\gamma,\varrho} = 3^4(2^{20}+2) + \frac{3}{8} \max_{[-\frac{3}{4},\frac{3}{4}]} V < 3^4(2^{20}+2) + \frac{27}{128}.$$

Hence, inequality (10) is fulfilled provided that

$$\frac{b}{a} > \frac{64\sqrt{2}}{\pi^2} \left(3^4(2^{20}+2) + \frac{27}{128} \right) \frac{\|h_2\|_{L^1((-1,1))}}{\|h_2\|_{L^1((-1/2,1/2))}}.$$

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