Minimax Theory and its Applications
Volume 11 (2026), No. 4s, 33 to 44

Solution space characterization and mathematical analysis of complex engineering optimization models

J.K. Shalini[1], K. Sindhuja[2], Kavita Prasad[3], Thiyagarajan Jayaraman[4], Jagan Mohan Rao Saride[5], Vijay Shankar Yadav[6]

[1]Assistant Professor, Department of Computer Science and Engineering, K.S. Rangasamy College of Technology, Namakkal, Tamil Nadu, India. shalinijk@ksrct.ac.in, https://orcid.org/0009-0007-8236-990X

[2]Assistant Professor, Department of Information Technology, J.J. College of Engineering and Technology, Tiruchirappalli, Tamil Nadu, India. sindhujajjcet@gmail.com, https://orcid.org/0009-0001-0495-1749

[3]Assistant Professor, Department of Computer Science and Engineering, Rungta College of Engineering and Technology, Rungta International Skills University, Bhilai, Chhattisgarh, India. kavita@rungta.org, https://orcid.org/0009-0008-8500-8728

[4]Assistant Professor, Department of Mechatronics Engineering, Sona College of Technology, Salem, Tamil Nadu, India. thiyagarajan.mct@sonatech.ac.in

[5]Department of ECE, Ramachandra College of Engineering, Eluru, Andhra Pradesh, India. jaganmohanrs@rcee.ac.in, https://orcid.org/0000-0002-2064-4634

[6]Assistant Professor, Kalinga University, Naya Raipur, Chhattisgarh, India. ku.vijayshankaryadav@kalingauniversity.ac.in, https://orcid.org/0009-0001-2485-3177

Abstract:

Nonlinear relationship, constraints and high dimensionality pose major challenges in identifying solutions and analyzing performance of complex engineering optimization models. In this study, a solution space characterization framework is proposed to investigate characteristics of feasible region, optimization landscape, convergence and stability of complex engineering systems. The framework takes advantage of the fundamental concepts of mathematical optimization such as convex analysis, constraint qualifications, duality, sensitivity and optimality conditions to gain insight into the structural behavior of complex optimization models. The proposed solution space characterization framework considers objective function, design constraints, feasible solution space and optimality conditions to investigate engineering optimization problems in a structured manner. It is demonstrated that appropriate continuity, convexity and regularity conditions guarantee existence and stability of optimal solutions and facilitate the investigation of structural properties of solution space through theoretical studies. In addition, dual relationships and sensitivity analysis are carried out to investigate the effect of parameter variation on optimization performance. Two-bar structural optimization problem considering compliance constraint is solved as a case study in order to illustrate the efficiency of the proposed framework. The obtained results reveal that there is no change in the stability of optimal design against compliance variation between 0.60 and 1.40 whereas the performance of objective is deteriorating from 6.6667 to 2.8571 due to increased design freedom. Stability of optimal design with respect to the parameter variation is verified through sensitivity analysis.

Keywords: Solution space characterization, engineering optimization, mathematical analysis, optimization models, convex optimization, sensitivity analysis, complex systems.

1. Introduction

As the complexity of engineering systems increases, the need for sophisticated models for effective exploration of large solution spaces becomes critical. Modern engineering systems, which include smart manufacturing, energy systems, robotics, structural design, and industrial automation, deal with multiple decision variables, stochastic environments, and multiple objective functions. Conventional optimization approaches are found to pose many difficulties in optimizing such systems because of complex search spaces, computational issues, and the nonlinear nature of the problem. Thus, characterization and mathematical analysis of solution space have become vital in this regard.

The recent progress made in the field of metaheuristic optimization has been shown to have considerable promise for solving difficult engineering problems through enhanced ability to explore and exploit within a vast solution space. Metaheuristics inspired by bionics have been proposed to offer effective means of searching within real-world engineering optimization problems that include complex constraints and objective functions (Xiao et al., 2025). The framework of data analytics-based optimization has stressed on the role of coupling between computational intelligence and industrial decision-making processes (Tang & Meng, 2021). Surrogate modeling has been studied as one of the useful ways of decreasing the computational complexity through simulation approximation while retaining optimization precision (Alizadeh et al., 2020). Human-inspired optimization has also been utilized as a way of solving engineering problems. Optimization techniques based on human intuition have also helped in addressing engineering challenges by offering robust mechanisms to explore complex search spaces without getting trapped at local optima (Matoušová et al., 2023).

Intelligent optimization techniques have had a considerable impact on energy and industrial engineering practices. Highly accurate Runge–Kutta schemes offer efficient numerical solutions to stiff systems of differential equations, providing reliable computational methods for optimization and engineering systems (Reshma et al., 2025).Optimization schemes based on artificial intelligence have been used to enhance the performance of energy systems, resource allocation, and efficiency in smart grids. Optimization models based on digital twins have facilitated real-time configuration analysis and control optimization in smart manufacturing systems using the combination of physical and virtual system models (Liu et al., 2021). Mathematical modeling and advanced optimization methods have greatly improved the resolution of complex engineering problems by the application of analytical modeling with optimization methods (Abhang et al., 2026). Global optimization algorithms that draw inspiration from nature have shown better ability in addressing nonlinear engineering optimization problems (El-Kenawy et al., 2026). The improved bee colony algorithm provides a working example of how intelligent optimization can be used in order to effectively solve complex problems relating to routing and resource allocation.

Multi-objective optimization is another research area that has gained popularity in engineering systems in which there should be simultaneous satisfaction of several performance indicators. Researches on optimization through genetic algorithms have been carried out for obtaining balanced solutions in relation to competing objectives in complicated engineering tasks. Adaptive multi-objective optimization has enhanced decision-making processes for solving engineering planning problems through efficient management of costs, time and performance trade-offs (Ateş et al., 2026). Artificial Intelligence-based engineering modeling and optimization has been comprehensively reviewed emphasizing on increasing role of machine learning, intelligent algorithms and data-driven approach for solving the engineering problems in the future (Li et al., 2026). Global optimization algorithms still undergo development aiming at overcoming issues associated with speed of convergence, computational efficiency and diversity of solutions. Efficient global optimization algorithms with advanced search algorithms have been created for improving reliability of the optimization process in complex problem spaces (Ibrahim et al., 2026). Cooperative optimization has been used in human-robot cooperative systems to achieve balanced task planning and operational efficiency in manufacturing facilities (Chen et al., 2026). Another innovation in engineering optimization field was deep learning-assisted optimization framework that contributed to dynamic layout optimization in flexible manufacturing systems through adaptive decision-making. Integration of physics in artificial intelligence has offered new opportunities for inverse modeling and optimization of complicated environmental systems (Ma et al., 2026).

Parameter estimation and continuous solution space optimization are vital in many engineering applications. Parameter estimation using metaheuristics has been found to be successful in modeling the characteristics of materials in complex engineering models (Sanin-Villa et al., 2023). Improved flow-based optimization algorithms have been used to enhance the performance of engineering optimization problems having complex solution landscapes (Fan et al., 2023). Continuous solution space optimization techniques have been used to enhance manufacturing precision through the optimization of parameters such as part orientation and production configurations (Leirmo & Semeniuta, 2022). Intelligent algorithms for task scheduling and information management optimization have further proved the significance of efficient exploration of the solution space in time-sensitive systems. Combination of evolutionary computation, mathematical modeling, and artificial

intelligence has offered new directions in analyzing complex nonlinear systems. Evolutionary mathematics together with fractional models and artificial intelligence has led to advanced techniques in understanding and optimizing the nonlinear dynamics in complex systems (Karaca & Baleanu, 2022). While considerable advances have been made, there are still difficulties in the current optimization techniques related to the high dimensionality of search space, computational expenses, uncertainty, and lack of interpretability of the results acquired. Most of the methods only aim at acquiring an optimal solution without the provision of mathematical modeling of the solution space and optimization problem.

Thus, the characterization and analysis of the solution space and mathematical model of complex optimization models in engineering is important to establish optimization techniques that are both reliable and computationally efficient. This studyaims to analyze the structure, feasibility, convergence, and mathematical properties of complex engineering models for purposes of enhancing the reliability and efficiency of solutions. This analysis will offer insight into the optimization process and develop methodologies for advanced engineering problems.

Key contribution

- Through a detailed analysis of feasible regions, objective landscapes, interaction of constraints, and optimality requirements, a framework is established for characterizing the solution space of complex engineering optimization models in order to enhance comprehension of optimization performance.

- The mathematical analysis framework used in this research examines the structural features of complex optimization spaces such as convergence features, sensitivity features, solution space diversity, and stability features, facilitating the reliable evaluation of engineering optimization problems.

- The research presents an effective methodology that can help in enhancing the optimization process by examining the challenges in the solution space and developing efficient optimization methodologies for complex engineering optimization scenarios.

The rest of this work is organized as follows: In Section 2, provide the preliminaries and the necessary mathematical notations to be used in this study, that is, some basic definitions regarding feasible sets, constraint conditions, convexity, and optimality conditions. In Section 3, formulate the engineering optimization problem and state the necessary assumptions for theoretical investigation. In Section 4, discuss the space of solutions with respect to existence, compactness, convexity properties, and the structure of optimal solutions. Section 5 discusses the minimax duality and saddle-point characterization to investigate optimality conditions and dual problems. Section 6 discusses the sensitivity and stability behaviors of optimization solutions to parameter changes. Section 7 provides a numerical validation for the theory via a compliance-constrained two-bar structure optimization problem. Section 8 provides some discussions on theoretical implications and practical importance of the results obtained. Finally, Section 10 concludes the study and points out future research directions towards dynamic optimization models, uncertainty-oriented systems, machine learning-assisted explorations, and engineering applications.

2. Preliminaries and notations

Table 1: Notation used throughout the paper.

Symbol	Meaning
$\mathbb{R}^n$	$n-dimensional\ real\ Euclidean\ space$
$x = (x_1, \dots, x_n)^T$	$vector\ of\ design\ (decision)\ variables$
$f: \mathbb{R}^n \to \mathbb{R}$	$objective\ (cost\ /\ weight\ /\ compliance)\ function$
$g_i: \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, m$	$inequality\ constraint\ functions$
$h_j: \mathbb{R}^n \to \mathbb{R}, j = 1, \dots, p$	$equality\ constraint\ functions$
$X \subseteq \mathbb{R}^n$	$abstract\ (ground)\ admissible\ set, e.g.\ box\ constraints$
$S = \{x \in X: g_i(x) \leq 0, h_j(x) = 0\}$	$feasible\ set\ (solution\ space)\ of\ problem\ (P)$
$S* \subseteq S$	$optimal\ solution\ set\ of\ (P)$
$\Phi(x, y)$	$saddle\ (minimax)\ function\ on\ X \times Y$
$\|\cdot\|$	$Euclidean\ norm$

Table 1 collects the notation used in the remainder of the paper; further problem-specific notation is introduced where needed.

3. Problem formulation

A mathematically abstract formulation of the problem (P) encompasses a broad range of engineering design problems by partitioning the design variables with bounds from the functional physics,

$$\text{(P):}\quad \min_{x\in X} f(x) \quad \text{subject to} \quad g_i(x) \le 0 \ (i = 1, \dots, m), \quad h_j(x) = 0 \ (j = 1, \dots, p)$$

Here, design variables $x \in X \subseteq R\hat{}n$ are physical quantities (areas, thicknesses, or gains), constrained by simple domain constraints X (box constraints $l \le x \le u$, or nonnegativity $x \in R_ + \hat{}n$). Objective function f(x) is an index of performance (weight, displacement, or cost), whereas inequalities g_i $(x) \le 0$ correspond to structural and performance safety limits (stresses, displacements, or frequencies), and equalities h_j $(x) = 0$ correspond to fundamental physical balance equations (state equations, static equilibrium equations, or conservation laws).

Feasible set topology & epigraph representation

Let $G(x) = \big(g_1(x), \dots, g_m(x)\big)^T \in \mathbb{R}^m$ and $H(x) = \Big(h_1(x), \dots, h_p(x)\Big)^T \in \mathbb{R}^p$. The feasible set S is expressed as the intersection of inverse images of closed sets:

$$S = X \cap G^{-1}(\mathbb{R}^m_-) \cap H^{-1}(\{\mathbf{0}\})$$

Because G and H are continuous under (A2), and the negative orthant $\mathbb{R}^m_- = (-\infty, 0]^m$ alongside $\{\mathbf{0}\} \subset \mathbb{R}^p$ are closed, the inverse images $G^{-1}(\mathbb{R}^m_-)$ and $H^{-1}(\{\mathbf{0}\})$ are closed. Combined with closed X from (A1), S **is guaranteed to be a closed set in** $\mathbb{R}^n$.

Under (A3), the continuous objective f restricted to the non-empty, closed, and bounded/coercive feasible set S guarantees by the Extreme Value Theorem that the optimal value $v(\text{P}) = \inf_{x\in S} f(x) < +\infty$ is finite and the set of minimizers $S^* = \{x \in S \mid f(x) = v(\text{P})\}$ is non-empty and compact.

4. Characterization of the solution space

4.1. Geometric characterization under convexity

Theorem 4.1 (Face characterization of S^*) establishes the geometric and structural properties of the optimal solution set for convex optimization problems under assumptions (A1), (A2), and (A4), given $S^* \neq \emptyset$:

Theorem Statement

(a) Convexity of S and Constancy of f: The feasible set S is convex, and the objective function $f(x)$ is constant on the optimal solution set S^*, satisfying $f(x) = v(\text{P})$ for all $x \in S^*$.

(b) Exposed Face Structure of S^*: The optimal set S^* is a convex, exposed face of S, defined by the intersection of S with the sublevel set of f at height $v(\text{P})$:

$$S^* = S \cap \{x \in X \mid f(x) \le v(\text{P})\}$$

(c) Uniqueness under Strict Convexity: If f is strictly convex on X, then S^* is a singleton, $S^* = \{x^*\}$.

Combined Proof in equation format

Under assumption (A4), X is convex, each g_i is convex, and each h_j is affine, which implies that their respective sublevel and level sets are convex:

$$S_i^{(g)} = \{x \in X \mid g_i(x) \le 0\} \quad \text{is convex for each } i = 1, \dots, m$$

$$S_j^{(h)} = \{x \in X \mid h_j(x) = 0\} \quad \text{is convex for each } j = 1, \dots, p$$

Consequently, the feasible set S is convex as an intersection of convex sets:

$$S = X \cap \left(\bigcap_{i=1}^{m} S_i^{(g)}\right) \cap \left(\bigcap_{j=1}^{p} S_j^{(h)}\right) \Longrightarrow S \text{ is convex}$$

By definition, $f(x) = v(\mathrm{P})$ for all $x \in S^* = \{x \in S \mid f(x) = v(\mathrm{P})\}$, establishing part (a). For part (b), let $x_1, x_2 \in S^*$ and $\lambda \in [0,1]$, defining the convex combination $x_\lambda = \lambda x_1 + (1-\lambda)x_2 \in S$; by the convexity of f and the definition of optimal value $v(\mathrm{P}) = \inf_{x \in S} f(x)$, obtain:

$$v(\mathrm{P}) \leq f(x_\lambda) \leq \lambda f(x_1) + (1-\lambda) f(x_2) = \lambda v(\mathrm{P}) + (1-\lambda) v(\mathrm{P}) = v(\mathrm{P})$$

This forces $f(x_\lambda) = v(\mathrm{P})$, proving $x_\lambda \in S^*$ and establishing that S^* is convex; furthermore, because $f(x) \geq v(\mathrm{P})$ for all $x \in S$, the sublevel set representation holds:

$$S^* = S \cap \{x \in X \mid f(x) \leq v(\mathrm{P})\} = S \cap H^-$$

Where $H^- = \{x \in X \mid f(x) \leq v(\mathrm{P})\}$ is a closed convex supporting sublevel set of f touching S precisely at height $v(\mathrm{P})$, identifying S^* as an exposed face of S. Finally, for part (c), assume for contradiction that f is strictly convex on X and there exist distinct points $x_1, x_2 \in S^*$ with $x_1 \neq x_2$; setting the midpoint $x_{1/2} = \frac{1}{2}x_1 + \frac{1}{2}x_2 \in S$, strict convexity yields:

$$f\left(x_{1/2}\right) < \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2) = \frac{1}{2} v(\mathrm{P}) + \frac{1}{2} v(\mathrm{P}) = v(\mathrm{P})$$

which directly contradicts $v(\mathrm{P}) = \inf_{x \in S} f(x)$, ensuring S^* must be a singleton $\{x^*\}$.

5. Minimax duality and Saddle point characterization

In this part, the optimality of (P) will be restated as the saddle-point minimax property of the Lagrangian, which guarantees optimality independently of any differential properties of KKT and provides a quantitative estimate of the duality gap if the problem is partially convex, which is typical for engineering applications, where there are only a few non-convex constraints in a convex sizing problem.

5.1. Existence of a saddle point

The formulation introduces the **Lagrangian dual function** $q: \mathbb{R}^m_+ \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty\}$ and the dual optimization problem (D):

$$q(\lambda, \mu) = \inf_{x \in X} \mathcal{L}(x, \lambda, \mu)$$

$$(\mathrm{D}): \quad \max_{\lambda \geq 0, \mu} \; q(\lambda, \mu)$$

By construction, $q(\lambda, \mu)$ is a concave function (as the pointwise infimum of affine functions in (λ, μ)). Weak duality $q(\lambda, \mu) \leq v(\mathrm{P})$ holds universally for all feasible designs $x \in S$ and multipliers $\lambda \geq 0$, since $\sum_{i=1}^m \lambda_i \, g_i(x) \leq 0$ and $\sum_{j=1}^p \mu_j \, h_j(x) = 0$, enforcing $\mathcal{L}(x, \lambda, \mu) \leq f(x)$.

Theorem 5.1 (Existence of a Saddle Point)

Under assumptions (A1), (A2), (A4) (convex case), Slater's condition (A5), $S^* \neq \emptyset$, and bounded sublevel sets $X_0 = \{x \in X \mid f(x) \leq c\}$ for $c > v(\mathrm{P})$, there exist optimal multipliers $(\lambda^*, \mu^*) \in \mathbb{R}^m_+ \times \mathbb{R}^p$ such that (x^*, λ^*, μ^*) is a **saddle point** of $\mathcal{L}$ on $X \times \mathbb{R}^m_+ \times \mathbb{R}^p$ for every optimal design $x^* \in S^*$:

$$\mathcal{L}(x^*, \lambda, \mu) \leq \mathcal{L}(x^*, \lambda^*, \mu^*) \leq \mathcal{L}(x, \lambda^*, \mu^*) \quad \forall x \in X, \; \lambda \geq 0, \; \mu \in \mathbb{R}^p$$

Mathematical proof in step-by-step equation format

Step 1 & 2: Truncation and minimax equality

Let $X_0 = \{x \in X \mid f(x) \leq c\}$ (compact and convex) and $\Lambda = [0, \bar{\lambda}]^m \times [-\bar{M}, \bar{M}]^p$ (compact and convex). Because $\mathcal{L}(x, \lambda, \mu)$ is convex in x, concave in (λ, μ), and continuous, **Sion's Minimax Theorem (Fact 2.9)** applies:

$$\min_{x \in X_0} \max_{(\lambda,\mu) \in \Lambda} \mathcal{L}(x, \lambda, \mu) = \max_{(\lambda,\mu) \in \Lambda} \min_{x \in X_0} \mathcal{L}(x, \lambda, \mu)$$

Step 3: Evaluation of the primal inner maximum

Expanding the left-hand side over X_0:

$$\max_{(\lambda,\mu)\in\Lambda} \mathcal{L}(x,\lambda,\mu) = f(x) + \max_{\lambda\in[0,\bar{\Lambda}]^m} \sum_{i=1}^{m} \lambda_i\, g_i(x) + \max_{\mu\in[-\bar{M},\bar{M}]^p} \sum_{j=1}^{p} \mu_j\, h_j(x)$$

$$\Longrightarrow \max_{(\lambda,\mu)\in\Lambda} \mathcal{L}(x,\lambda,\mu) = \begin{cases} f(x), & \text{if } x \in S \cap X_0 \\ +\infty \text{ (penalized for large } \bar{\Lambda}, \bar{M}), & \text{if } x \in X_0 \setminus S \end{cases}$$

$$\min_{x\in X_0} \max_{(\lambda,\mu)\in\Lambda} \mathcal{L}(x,\lambda,\mu) = \min_{x\in S\cap X_0} f(x) = v(\mathrm{P})$$

Step 4: Removal of truncation via slater's condition

Let $(\lambda^*, \mu^*) \in \Lambda$ achieve the maximum. Slater's condition ensures bounds on (λ^*, μ^*) independent of Λ:

$$q(\lambda^*, \mu^*) = \min_{x\in X} \mathcal{L}(x, \lambda^*, \mu^*) = \min_{x\in X_0} \mathcal{L}(x, \lambda^*, \mu^*) = v(\mathrm{P})$$

Combining with weak duality $q(\lambda^*, \mu^*) \le v(\mathrm{P})$ yields **strong duality**:

$$q(\lambda^*, \mu^*) = v(\mathrm{P}) = \max_{\lambda\ge 0,\mu} q(\lambda, \mu)$$

Step 5: Verification of saddle-point inequalities

For any optimal solution $x^* \in S^*$ and multiplier (λ^*, μ^*):

1. Complementary Slackness: $\lambda_i^* g_i(x^*) = 0\ (\forall i), \quad h_j(x^*) = 0\ (\forall j) \Longrightarrow \mathcal{L}(x^*, \lambda^*, \mu^*) = f(x^*)$

2. Left Inequality: $\mathcal{L}(x^*, \lambda, \mu) = f(x^*) + \sum_{i=1}^{m} \lambda_i\, g_i(x^*) \le f(x^*) = \mathcal{L}(x^*, \lambda^*, \mu^*) \quad (\forall \lambda \ge 0, \mu)$

3. Right Inequality: $\mathcal{L}(x^*, \lambda^*, \mu^*) = v(\mathrm{P}) = q(\lambda^*, \mu^*) = \inf_{x\in X} \mathcal{L}(x, \lambda^*, \mu^*) \le \mathcal{L}(x, \lambda^*, \mu^*) \quad (\forall x \in X)$

$$\therefore \mathcal{L}(x^*, \lambda, \mu) \le \mathcal{L}(x^*, \lambda^*, \mu^*) \le \mathcal{L}(x, \lambda^*, \mu^*)$$

5.2. Strong duality

Theorem 5.2 establishes that under the hypotheses of Theorem 5.1—specifically standing assumptions (A1), (A2), (A4), Slater's condition (A5), non-empty optimal solution set $S^* \neq \emptyset$, and bounded sublevel sets $\{x \in X \mid f(x) \le c\}$—strong duality holds with zero duality gap, meaning the optimal value of the primal problem equals the maximum value of the dual problem, $v(\mathrm{P}) = \max_{(\lambda,\mu)\in\mathbb{R}_+^m\times\mathbb{R}^p} q(\lambda, \mu)$, and the dual optimum is attained. The proof follows directly from Step 4 of Theorem 5.1, which constructed an explicit dual multiplier pair $(\lambda^*, \mu^*) \in \mathbb{R}_+^m \times \mathbb{R}^p$ satisfying $q(\lambda^*, \mu^*) = v(\mathrm{P})$; combining this equality with the universal weak duality inequality $q(\lambda, \mu) \le v(\mathrm{P})$ valid for all feasible $(\lambda, \mu) \in \mathbb{R}_+^m \times \mathbb{R}^p$ yields $\sup_{\lambda\ge 0,\mu} q(\lambda, \mu) \le v(\mathrm{P}) = q(\lambda^*, \mu^*) \le \sup_{\lambda\ge 0,\mu} q(\lambda, \mu)$, which establishes $v(\mathrm{P}) = \max_{(\lambda,\mu)\in\mathbb{R}_+^m\times\mathbb{R}^p} q(\lambda, \mu) = q(\lambda^*, \mu^*)$, proving the dual supremum is attained with zero duality gap.

5.3. A duality gap bound in the nonconvex case

Theorem 5.3 (Duality-gap bound) establishes a quantitative bound on the duality gap for mildly nonconvex optimization problems under assumptions (A1)–(A3) with a convex, compact domain X, where each constraint function admits a DC (difference-of-convex) decomposition $g_i = \hat{g}_i - \rho_i$ with convex $\hat{g}_i$ and convex ρ_i satisfying the quadratic curvature bound $\rho_i(x) \le \frac{\kappa_i}{2}\|x\|^2 + b_i$ ($\kappa_i \ge 0$), and convex f; specifically, the gap between the primal optimal value $v(\mathrm{P})$ and the dual optimal value $\max_{(\lambda,\mu)} q(\lambda, \mu)$ is bounded by $0 \le v(\mathrm{P}) - \max_{(\lambda,\mu)} q(\lambda, \mu) \le \sum_{i=1}^{m} \lambda_i^\circ \cdot \frac{\kappa_i}{2} \cdot \mathrm{diam}(X)^2$, where $\mathrm{diam}(X) = \sup_{x,y\in X} \|x - y\|$ and λ° is any dual-optimal multiplier vector of the convexified relaxation $(\hat{\mathrm{P}})$ formed by replacing g_i with $\hat{g}_i$. The proof proceeds by first applying Theorem 5.2 to the convexified problem $(\hat{\mathrm{P}})$—which has optimal value

$v(\hat{\mathrm{P}}) = \hat{q}(\lambda^\circ, \mu^\circ) = \inf_{x \in X}\left[f(x) + \sum \lambda_i^\circ \hat{g}_i(x) + \sum \mu_j^\circ h_j(x)\right]$ for dual-optimal $(\lambda^\circ, \mu^\circ)$—and evaluating $\hat{q}(\lambda^\circ, \mu^\circ)$ at a primal optimizer $x^* \in S^*$ of (P); using $\hat{g}_i(x^*) = g_i(x^*) + \rho_i(x^*) \leq \rho_i(x^*)$, $\lambda_i^\circ \geq 0$, and $h_j(x^*) = 0$, obtain $\hat{q}(\lambda^\circ, \mu^\circ) \leq L^\circ(x^*, \lambda^\circ, \mu^\circ) \leq f(x^*) + \sum \lambda_i^\circ \rho_i(x^*) = v(\mathrm{P}) + \sum \lambda_i^\circ \rho_i(x^*)$, which after normalizing ρ_i over the diameter yields $v(\hat{\mathrm{P}}) \leq v(\mathrm{P}) + \sum \lambda_i^\circ \frac{\kappa_i}{2} \mathrm{diam}(X)^2$. Second, observing that $q(\lambda, \mu) = \inf_{x \in X}\left[f(x) + \sum \lambda_i \left(\hat{g}_i(x) - \rho_i(x)\right) + \sum \mu_j h_j(x)\right] \leq \inf_{x \in X}\left[f(x) + \sum \lambda_i \hat{g}_i(x) + \sum \mu_j h_j(x)\right] = \hat{q}(\lambda, \mu)$ for normalized non-negative ρ_i, taking the maximum over dual variables gives $\max_{(\lambda,\mu)} q(\lambda, \mu) \leq \max_{(\lambda,\mu)} \hat{q}(\lambda, \mu) = v(\hat{\mathrm{P}})$; combining these bounds yields $\max_{(\lambda,\mu)} q(\lambda, \mu) \leq v(\hat{\mathrm{P}}) \leq v(\mathrm{P}) + \sum \lambda_i^\circ \frac{\kappa_i}{2} \mathrm{diam}(X)^2$, which together with universal weak duality $\max_{(\lambda,\mu)} q(\lambda, \mu) \leq v(\mathrm{P})$ establishes the two-sided inequality. Physically, this demonstrates that the duality gap in mildly nonconvex engineering programs is rigorously controlled by the product of constraint sensitivity multipliers λ_i° and the nonconvex curvature parameters κ_i scaled by the square domain diameter $\mathrm{diam}(X)^2$, formalizing why Lagrangian relaxation remains exceptionally tight for localized or low-curvature physical models.

6. Sensitivity and stability analysis of the solution space

Engineering models are seldom solved only once; rather, there is an ongoing updating of loadings, material characteristics, cost factors, and tolerances, and the designer must know whether the feasible region and optimal solution vary continuously, or even Lipschitz-continuously, as these factors vary. This is done formally by embedding (P) in a parameterized collection of problems.

The structure of the parametric optimization problem involves a family of mathematical programming problems (P_θ), where the objective function $f(x, \theta)$, the inequality constraints $g_i\ (x, \theta) \leq 0$, equality constraints $h_j\ (x, \theta) = 0$, and the set of feasible points $X(\theta)$ vary continuously or differentiably with small perturbations of θ around some nominal base value θ_0. In the light of the standing assumption of non-empty solution sets S(θ)≠∅, the main focus in sensitivity analysis is the study of the continuity and differentiability of the optimal value function $f\hat{}* \ (\theta) = inf_(x \in S(\theta)\ ) f(x, \theta)$ and the upper semi-continuity of the solution sets map θ↦S^* (θ) (by virtue of the Maximum Theorem and Lagrangian multipliers calculus).

6.1. Continuity of the value function and upper semi continuity of the solution map

Theorem 6.1 (Stability via Berge's theorem) proves that if the constraint correspondence $\theta \mapsto S(\theta)$ is continuous (both upper and lower semicontinuous), compact-valued, and non-empty on a neighborhood of θ_0, and the objective function $(x, \theta) \mapsto f(x, \theta)$ is continuous on the product domain $\bigcup_\theta S\ (\theta) \times \{\theta\}$, then the optimal value function $f^*(\theta)$ is continuous at θ_0, while the optimal set mapping $S^*(\theta)$ is non-empty, compact-valued, and upper semicontinuous at θ_0. The proof is a direct application of Berge's Maximum Theorem (Fact 2.10) to the feasible set correspondence $\Gamma(\theta) = S(\theta)$ under the negated payoff function $\phi(x, \theta) = -f(x, \theta)$; converting the primal minimization problem (P_θ) into a payoff maximization yields value function $v(\theta) = \max_{x \in \Gamma(\theta)} \phi(x, \theta) = -f^*(\theta)$ and argmax correspondence $M(\theta) = \arg\max_{x \in \Gamma(\theta)} \phi(x, \theta) = S^*(\theta)$, which immediately inherits continuity for v (and thus for $f^* = -v$) as well as non-emptiness, compactness, and upper semicontinuity for $M = S^*$ directly from Berge's theorem. In practical engineering applications, the essential requirement of joint continuity of $S(\theta)$ is commonly verified by combining joint continuity of constraint functions g_i, h_j with a uniform Slater-type condition (a shared feasible point $\hat{x}$ strictly satisfying constraints across a neighborhood of θ_0) and uniform boundedness of the domain $X(\theta)$.

6.2. Local Lipschitz stability of a unique optimizer

Theorem 6.2 (Local Lipschitz stability)

Under assumptions (i)–(iii), there exist a Lipschitz constant $L < \infty$ and a neighborhood $\Theta_1 \subseteq \Theta_0$ of θ_0 such that:

$$\|x^*(\theta_1) - x^*(\theta_2)\| \leq L \cdot d(\theta_1, \theta_2) \quad \forall \theta_1, \theta_2 \in \Theta_1$$

Mathematical proof in step-by-step equation format

Step 1: Local constancy of active sets

By strict complementarity ($\lambda_i^*(\theta_0) > 0$ for $i \in A$) and Linear Independence Constraint Qualification (LICQ), the

active set $A_0 = A(x^*(\theta_0), \theta_0)$ remains locally constant on a neighborhood Θ_1:

$$A_0 = \{i \in \{1, \dots, m\} \mid g_i(x^*(\theta), \theta) = 0\} = \text{constant} \quad \forall \theta \in \Theta_1$$

Step 2: KKT system reduction

On Θ_1, the parameterized KKT optimality system reduces to a square system of equations $F\left(x, \lambda_{A_0}, \mu, \theta\right) = 0$:

$$F\left(x, \lambda_{A_0}, \mu, \theta\right) = \begin{pmatrix} \nabla_x f(x,\theta) + \sum_{i \in A_0} \lambda_i \nabla_x g_i(x,\theta) + \sum_{j=1}^{p} \mu_j \nabla_x h_j(x,\theta) \\ g_{A_0}(x,\theta) \\ h(x,\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Step 3: Nonsingularity of the KKT Jacobian matrix

The Jacobian J of F with respect to $\left(x, \lambda_{A_0}, \mu\right)$ evaluated at $\left(x^*(\theta_0), \lambda^*_{A_0}(\theta_0), \mu^*(\theta_0), \theta_0\right)$ exhibits the standard bordered structure:

$$J = \begin{pmatrix} \nabla^2_{xx}\mathcal{L} & \nabla_x g^T_{A_0} & \nabla_x h^T \\ \nabla_x g_{A_0} & 0 & 0 \\ \nabla_x h & 0 & 0 \end{pmatrix}$$

$$\text{Strong Convexity (i)} \Longrightarrow d^T(\nabla^2_{xx}\mathcal{L})d \geq \alpha\|d\|^2 > 0 \quad \forall d \in \ker\begin{pmatrix} \nabla_x g_{A_0} \\ \nabla_x h \end{pmatrix} \setminus \{0\}$$

$$\text{LICQ (ii)} \Longrightarrow \operatorname{rank}\begin{pmatrix} \nabla_x g_{A_0} \\ \nabla_x h \end{pmatrix} = |A_0| + p \Longrightarrow \det(J) \neq 0$$

Step 4: Application of the implicit function theorem

Since J is non-singular and F is $\mathcal{C}^1$, there exists a unique $\mathcal{C}^1$ mapping $\theta \mapsto \left(x^*(\theta), \lambda^*_{A_0}(\theta), \mu^*(\theta)\right)$ on Θ_1 satisfying $F\left(x^*(\theta), \lambda^*_{A_0}(\theta), \mu^*(\theta), \theta\right) = 0$. The parameter sensitivity matrix is:

$$\frac{\partial x^*(\theta)}{\partial \theta} = -\left[J^{-1}\frac{\partial F}{\partial \theta}\right]_{\left(x^*(\theta), \lambda^*_{A_0}(\theta), \mu^*(\theta), \theta\right)}$$

Step 5: Bounded derivative and mean value estimate

By assumption (iii), J^{-1} and $\frac{\partial F}{\partial \theta}$ are continuous over the compact neighborhood $\overline{\Theta}_1$:

$$L := \sup_{\theta \in \Theta_1} \left\| \frac{\partial x^*(\theta)}{\partial \theta} \right\| < \infty$$

$$\|x^*(\theta_1) - x^*(\theta_2)\| \leq \left(\sup_{\theta \in \Theta_1} \left\| \frac{\partial x^*(\theta)}{\partial \theta} \right\| \right) d(\theta_1, \theta_2) \leq L \cdot d(\theta_1, \theta_2)$$

Corollary: sensitivity & Danskin's envelope formula

Evaluating the sensitivity of the optimal value function $f^*(\theta) = \mathcal{L}(x^*(\theta), \lambda^*(\theta), \mu^*(\theta), \theta)$ via the chain rule yields:

$$\frac{\partial f^*(\theta)}{\partial \theta} = \frac{\partial \mathcal{L}}{\partial \theta}(x^*(\theta), \lambda^*(\theta), \mu^*(\theta), \theta)$$

$$\Longrightarrow \frac{\partial f^*(\theta)}{\partial \theta} = \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) + \sum_{i \in A_0} \lambda_i^*(\theta) \frac{\partial g_i}{\partial \theta}(x^*(\theta), \theta) + \sum_{j=1}^{p} \mu_j^*(\theta) \frac{\partial h_j}{\partial \theta}(x^*(\theta), \theta)$$

7. Engineering case study

The theory on a compliance-constrained two-bar sizing problem, a simplified but structurally representative model of statically determinate truss sizing under a stiffness (inverse-compliance) requirement. Let $x = (x_1, x_2) \in \mathbb{R}^2_+$ denote the cross-sectional areas of two parallel load-bearing members, and consider

The structural optimization problem (P_0) minimizes total material volume $f(x) = x_1 + x_2$ over design variables $x_1, x_2 > 0$ subject to the active inverse-stiffness compliance constraint $g(x) = \frac{1}{x_1} + \frac{1}{x_2} - C \leq 0$ for $C > 0$. Setting up the Lagrangian $\mathcal{L}(x, \lambda) = x_1 + x_2 + \lambda\left(\frac{1}{x_1} + \frac{1}{x_2} - C\right)$, the first-order stationarity condition $\nabla_x \mathcal{L}(x^*, \lambda^*) = \left(1 - \frac{\lambda^*}{(x_1^*)^2} \quad 1 - \frac{\lambda^*}{(x_2^*)^2}\right)^T = (0 \quad 0)^T$ enforces symmetry $x_1^* = x_2^* = \sqrt{\lambda^*}$; substituting this relation into the active constraint $g(x^*) = \frac{2}{x_1^*} - C = 0$ yields the explicit unique primal optimal design $x_1^* = x_2^* = \frac{2}{C}$, minimum volume $f^*(C) = \frac{4}{C}$, and strict positive dual multiplier $\lambda^* = (x_1^*)^2 = \frac{4}{C^2} > 0$. Furthermore, evaluating the parametric derivative of the optimal volume directly gives $\frac{\mathrm{d}f^*(C)}{\mathrm{d}C} = \frac{\mathrm{d}}{\mathrm{d}C}\left(\frac{4}{C}\right) = -\frac{4}{C^2}$, which precisely matches the sensitivity identity predicted by Danskin's envelope formula $\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial C} = -\lambda^* = -\frac{4}{C^2}$, confirming that the dual multiplier λ^* represents the exact marginal volume penalty associated with tightening the compliance bound C.

7.1. Verification of the structural hypothesis

Take $X = \mathbb{R}^2_+$ (open positive orthant) and, to fit (A1), work on the closure $\bar{X} = [0, \infty)^2$ with the convention $1/0 = +\infty$, so that g is continuous as an extended-real-valued function and $S = \{x \in \bar{X}: g(x) \leq 0\}$ is closed; any feasible point automatically has $x_1, x_2 > 0$ since $g(x) = +\infty$ whenever $x_i = 0$. The objective $f(x) = x_1 + x_2$ is coercive on S ($f(x) \to \infty$ as $\|x\| \to \infty$), so (A3) holds via the coercivity branch, and Theorem 4.1 applies: S^* is nonempty and compact. Since $s \mapsto 1/s$ is convex on $(0, \infty)$, g is convex on the open orthant, X is convex, and f is affine hence convex; thus (A4) holds and Theorem 4.1 applies, so S^* is a convex face of S. Because f is affine rather than strictly convex, Theorem 4.1(c) does not by itself guarantee a unique minimizer; uniqueness here follows instead from the strict convexity of the active constraint g together with the KKT system, as verify next. The point $\hat{x} = (2/C, 2/C)$ scaled slightly, e.g. $\hat{x} = (3/C, 3/C)$, gives $g(\hat{x}) = 2C/3 - C = -C/3 < 0$, so Slater's condition (A5) holds, and Theorem 5.1 and Theorem 5.2 (existence of a saddle point and strong duality) both apply.

7.2. Closed form solution via the KKT/saddle point system

For the Lagrangian $\mathcal{L}(x, \lambda) = x_1 + x_2 + \lambda\left(\frac{1}{x_1} + \frac{1}{x_2} - C\right)$ with $\lambda \geq 0$, the KKT stationarity equations $\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \frac{\lambda}{x_1^2} = 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \frac{\lambda}{x_2^2} = 0$ enforce $x_1^2 = x_2^2 = \lambda$, which yields $x_1 = x_2 = \sqrt{\lambda}$ given $x_1, x_2 > 0$. The compliance constraint must be active $(g(x) = 0)$ at optimality because any strictly feasible design could be perturbed to decrease x_1 or x_2 and strictly lower the objective $f(x) = x_1 + x_2$, ensuring complementary slackness holds with strict positivity $\lambda^* > 0$. Substituting $x_1 = x_2 = \sqrt{\lambda}$ into the active boundary equation $\frac{1}{x_1} + \frac{1}{x_2} = C$ yields $\frac{2}{\sqrt{\lambda}} = C$, giving the unique primal optimal design $x_1^* = x_2^* = \frac{2}{C}$, optimal objective value $f^*(C) = \frac{4}{C}$, and optimal multiplier $\lambda^*(C) = \frac{4}{C^2}$. The global uniqueness of $S^*(C) = \left\{\left(\frac{2}{C}, \frac{2}{C}\right)\right\}$ is independently confirmed via the Arithmetic Mean–Harmonic Mean (AM–HM) inequality $\frac{x_1 + x_2}{2} \geq \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2}{C}$, where equality holds if and only if $x_1 = x_2 = \frac{2}{C}$, matching the global uniqueness characterization established in Theorem 4.1 and Corollary 4.3.

7.3. Saddle point and duality verification

By Theorem 5.1, $(x^*, \lambda^*) = ((2/C, 2/C), 4/C^2)$ is a saddle point of L on $\mathbb{R}^2_+ \times \mathbb{R}_+$. verify this directly.

Direct verification: for the right inequality, $min_\{x > 0\}\, L(x, \lambda*)$ is obtained by the same stationarity computation with λ fixed at $\lambda^* = 4/C^2$, giving the unconstrained minimizer $x_1 = x_2 = \sqrt{(\lambda*)} = 2/C$, which coincides with x^*; hence $min_\{x\}\, L(x, \lambda*) = L(x*, \lambda*) = f(x*) = 4/C$, confirming $q(\lambda*) = f*(C)$, i.e. zero duality gap, as guaranteed by Theorem 5.2. For the left inequality, $L(x*, \lambda) = 4/C + \lambda(g(x*)) = 4/C + \lambda \cdot 0 = 4/C$ for every $\lambda \geq 0$, since x^* lies exactly on the constraint boundary $g(x*) = 0$; hence $L(x*, \lambda) = 4/C = L(x*, \lambda*)$ for all $\lambda \geq 0$, so the left saddle inequality holds with equality throughout — a consequence of complementary slackness $(g(x*) = 0)$ rather than a strict inequality, exactly as predicted by

the general saddle-point construction of Theorem 5.1.

7.4. Sensitivity analysis and comparison with theorem

Table 2 reports x*(C), f*(C) and λ*(C) for several values of the compliance bound C, computed directly from the closed-form expressions above.

Table 2: Optimal design, optimal weight and multiplier as functions of the compliance bound C.

C	$x_1^* = x_2^*$	f*(C)	λ*(C)
0.60	3.3333	6.6667	11.1111
0.80	2.5000	5.0000	6.2500
1.00	2.0000	4.0000	4.0000
1.20	1.6667	3.3333	2.7778
1.40	1.4286	2.8571	2.0408

The hypotheses of Theorem 6.2 hold on any compact interval $[Cmin, Cmax] \subset (0,\infty)$: f is strongly convex in the relevant sense once restricted appropriately (indeed here f is affine, but the reduction to the active manifold has strictly positive curvature, which is the operative condition in the proof of Theorem 6.2 for the reduced KKT system), LICQ holds since $\nabla g(x*) = (-1/{x_1}^2, -1/{x_2}^2) \neq 0$ for x* in the interior of the orthant, and strict complementarity holds since $\lambda*(C) = 4/C^2 > 0$ for every C > 0. Theorem 6.2 therefore predicts that $C \mapsto x*(C)$ is locally Lipschitz on any such interval; this is confirmed directly by differentiating the closed form, $dx_1*/dC = dx_2*/dC = -2/C^2$, which is finite and continuous (hence locally bounded, i.e. locally Lipschitz with local constant L = 2/C_min²) on $[Cmin, Cmax]$.

Finally, the envelope formula of Section 6.2 is verified explicitly: differentiating f*(C) = 4/C gives $df*/dC = -4/C^2 = -\lambda*(C)$, matching $\partial L/\partial C = -\lambda$ (since $L = f(x) + \lambda(g(x) - 0)$ written with the C-dependence entering as $-\lambda C, so\ \partial L/\partial C = -\lambda$) evaluated at the optimal multiplier, exactly as predicted by the envelope theorem stated at the end of Section 6.2. For instance, at C = 1.00 the table gives $f* = 4.0000$ and λ*=4.0000, and the finite-difference estimate $[f*(1.20) - f*(0.80)]/(1.20 - 0.80) = (3.3333 - 5.0000)/0.40 = -4.1667$ brackets the exact derivative −4.0000 at C=1, the discrepancy being the expected O(ΔC) truncation error of a centred difference applied to the convex function f*(C)=4/C.

8. Discussion

First of all, Theorem 4.1 provides us with the proof that the existence of the optimal solution in general is more or less automatic as long as deal with the closed model and coerciveness of the objective function or bounded search domain — both of which will be satisfied for any reasonable engineering model (bounded variables, cost increasing with the number of materials used). So, the existence of the solution does not appear to be much of a problem, but, as the Theorem 4.1 proves, the problem of uniqueness becomes much more complicated – the problem whether there exists one optimum design or there is a whole family of them. This is typical for most engineering problems – e.g., in case when the objective function depends very little on some of the variables (on the active face of constraints).

Secondly, the saddle-point results of Section 5 reformulate optimality as a criterion that is testable even in the absence of the differentiability of the primal problem by identifying a certificate $(\lambda*, \mu*)$ – an idea that has been exploited extensively in methods such as Lagrangian relaxation and Lagrangian decomposition in large-scale engineering problems (e.g., multi-load or multi-scenario structural design, in which the dual variables carry physical meaning as the prices of the physical constraints). Theorem 5.3, in opinion, carries particular practical significance since it quantitatively specifies how much is sacrificed in terms of solution accuracy when the machinery of convex duality, including Lagrangian relaxation and semidefinite relaxation bounds, is applied to an approximate convex problem, as measured by a concrete computable quantity (the product of multiplier norm, curvature and the diameter of the feasible set) instead of a vague abstract disclaimer. This has concrete operational implications. For example, an engineer may reduce the duality gap in a problem by tightening the box constraints X (reducing diam(X)) on the feasible region around a good initial point, a method that has already been employed empirically in nonconvex branch-and-bound but now gains an a priori justification.

Thirdly, the stability results in Section 6 provide rigorous justification for the widely held but often unproven belief that "small changes in loads cause small changes in the optimal design". Theorem 6.1 establishes the validity of this belief in the weak (upper semicontinuous) sense, while Theorem 6.2 establishes the stronger and quantitatively meaningful Lipschitz property only under the strict complementarity and constant active set

conditions that may fail precisely when there is a transition between structures of different configurations (such as a slight increase in the load at which the stress constraint that was inactive before becomes active). This is precisely the point where one would expect the optimal design not to vary smoothly with respect to the parameters and where one would mislead oneself if applying local sensitivity analysis (as opposed to the global results of Section 4).

This specific example of section 7, being simple enough to permit closed-form verification, serves to demonstrate how each of the abstract assumptions (A1)-(A5), LICQ, strict complementarity, Slater's condition can be interpreted as tangible, verifiable criteria within a realistic physical example, and how the abstract result of Theorem 6.2 is seen to boil down to the classical engineering notion of "shadow price" of a design constraint (specifically $\lambda * (C) = 4/C^2$, the marginal price of increasing the constraint of compliance C). It is in this translation from abstract assumption to tangible engineering criteria that see the most fundamental contribution of the paper, rather than any particular theorem in isolation.

9. Conclusion

This study provided a complete framework for the characterization of solution space of complex engineering optimization models through the investigation of feasible region, optimization landscape, convergence and stability properties. The suggested framework made use of mathematical optimization, convex analysis, constraint qualification, duality and sensitivity concepts in order to gain insights into structural properties of engineering optimization problems. The theoretical analysis revealed that proper continuity, convexity and regularity conditions guarantee the existence and stability of optimal solutions and also allow the solution structure characterization. This framework enabled the evaluation of the dual relations and sensitivity behavior so as to understand the effect of parameter variations on the optimization process. A two-bar compliance-constrained structural optimization problem was used as a case study to verify the effectiveness of the methodology in which the solution structure was found to be stable against compliance variations in the range of 0.60 to 1.40 and the performance was improved from 6.6667 to 2.8571 by allowing constraint flexibility. The sensitivity analysis verified the stability of the design against parameter variations proving the validity of the proposed solution space characterization methodology. The developed framework offers a systematic basis for the analysis of nonlinear, Mult objective and large-scale engineering optimization problems. In the future, more work will be done regarding the extension of the framework to dynamic optimization models, large scale industrial applications, uncertainty aware optimization and decision-making systems in real-time. The future studies may involve integration of machine learning-based solution space exploration, adaptive and physics-informed optimization methods in order to increase the efficiency and scalability in the engineering environment.

References

1. Abhang, L. B., Patel, M., & Bansode, V. H. (2026). Advanced Mathematical Modeling AND Optimization Techniques FOR Solving Complex Engineering Problems. *International Journal of Advances in Signal and Image Sciences*, 1437-1444. https://doi.org/10.29284/8k920742

2. Agrawal, P., Kaur, H., & Bhardwaj, D. (2012). Enhanced bee colony algorithm for solving traveling salesperson problem. International Journal of Control Theory and Computer Modeling, 2(4), 37–41.

3. Alizadeh, R., Allen, J. K., & Mistree, F. (2020). Managing computational complexity using surrogate models: a critical review. *Research in Engineering Design*, *31*(3), 275-298. https://doi.org/10.1007/s00163-020-00336-7

4. Ateş, B., Maurya, S., Eirgash, M. A., & Sharma, A. (2026). Modified adaptive weight-based multi-objective Jaya optimization algorithm for construction time-cost trade-off problems. *Asian Journal of Civil Engineering*, *27*(1), 89-98. https://doi.org/10.1007/s42107-025-01491-9

5. Chen, Q., Yao, L., Li, G., Zhang, T., & Zhao, H. (2026). Multi-objective human-robot collaborative task planning for assembly lines balancing based on cooperative optimization. *Expert Systems with Applications*, *306*, 131001. https://doi.org/10.1016/j.eswa.2025.131001

6. El-Kenawy, E. S. M., Khodadadi, N., Mirjalili, S., Zaki, A. M., Ibrahim, A., Alhussan, A. A., ... & Eid, M. M. (2026). Glider snake optimizer (GSO): a nature-inspired metaheuristic algorithm for global and engineering optimization problems. *Artificial Intelligence Review*, *59*(3), 91.

https://doi.org/10.1007/s10462-026-11504-x

7. Fan, Y., Zhang, S., Wang, Y., Xu, D., & Zhang, Q. (2023). An improved flow direction algorithm for engineering optimization problems. *Mathematics*, *11*(9), 2217. https://doi.org/10.3390/math11092217

8. Ibrahim, M. Q., Qaraad, M., Hussein, N. K., Farag, M. A., & Guinovart, D. (2026). Secant Optimization Algorithm for efficient global optimization. *Scientific reports*, *16*(1), 6659. https://doi.org/10.1038/s41598-026-36691-z

9. Karaca, Y., & Baleanu, D. (2022). Evolutionary mathematical science, fractional modeling and artificial intelligence of nonlinear dynamics in complex systems. *Chaos Theory and Applications*, *4*(3), 111-118.

10. Leirmo, T. L., & Semeniuta, O. (2022). Minimizing form errors in additive manufacturing with part build orientation: An optimization method for continuous solution spaces. *Open Engineering*, *12*(1), 227-244. https://doi.org/10.1515/eng-2022-0032

11. Li, J. P., Polovina, N., & Konur, S. (2026). A review of AI-driven engineering modelling and optimization: Methodologies, applications and future directions. *Algorithms*, *19*(2), 93. https://doi.org/10.3390/a19020093

12. Liu, Q., Leng, J., Yan, D., Zhang, D., Wei, L., Yu, A., ... & Chen, X. (2021). Digital twin-based designing of the configuration, motion, control, and optimization model of a flow-type smart manufacturing system. *Journal of Manufacturing Systems*, *58*, 52-64. https://doi.org/10.1016/j.jmsy.2020.04.012

13. Ma, F., Chen, J., Dai, Z., Cai, F., & Hu, Y. (2026). Autonomous inverse modeling of complex groundwater systems via a physics-integrated large language model multi-agent framework. *Water Research*, 125886. https://doi.org/10.1016/j.watres.2026.125886

14. Matoušová, I., Trojovský, P., Dehghani, M., Trojovská, E., & Kostra, J. (2023). Mother optimization algorithm: a new human-based metaheuristic approach for solving engineering optimization. *Scientific Reports*, *13*(1), 10312. https://doi.org/10.1038/s41598-023-37537-8

15. Reshma, Prasad, S., Sharma, C. K., Deshpande, A., Bagal, S. B., & Tejaswi. (2025). High-accuracy Runge–Kutta methods for stiff ODE systems. Journal of Interdisciplinary Mathematics, 28(6), 2217–2225.

16. Sanin-Villa, D., Montoya, O. D., & Grisales-Noreña, L. F. (2023). Material property characterization and parameter estimation of thermoelectric generator by using a master–slave strategy based on metaheuristics techniques. *Mathematics*, *11*(6), 1326. https://doi.org/10.3390/math11061326

17. Tang, L., & Meng, Y. (2021). Data analytics and optimization for smart industry. *Frontiers of Engineering Management*, *8*(2), 157-171. https://doi.org/10.1007/s42524-020-0126-0

18.

19. Xiao, Y., Cui, H., Khurma, R. A., & Castillo, P. A. (2025). Artificial lemming algorithm: a novel bionic meta-heuristic technique for solving real-world engineering optimization problems: Y. Xiao et al. *Artificial Intelligence Review*, *58*(3), 84. https://doi.org/10.1007/s10462-024-11023-7