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Second variation-based optimal design analysis for mechanical systems with structural performance constraints

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Abstract:

Optimization of design of mechanical systems in respect to their structural performance requirements is vital for obtaining lightweight, reliable and efficient engineering solutions. Nevertheless, classical first-order optimization procedures frequently lack ability to adequately account for nonlinear interdependencies, effects of uncertainties and structural responses. The present paper presents second variation-based optimal design procedure that takes into consideration second order sensitivity information for increasing accuracy and reliability of structural optimization. The methodology uses variational formulation, adjoint-based sensitivity analysis and second order optimality conditions to assess impact of variation of design parameters on structural performance. The method accounts for objective functions, such as compliance and material efficiency, while using constraints on stress, displacements, volume and structural stability. A case study on a cantilever beam is presented in order to demonstrate advantages of the approach. In particular, second variation analysis is employed to determine local optimum, curvature and stability of design. It is shown that use of second-order curvature information improves assessment of stability of designs through the analysis of reduced Hessian properties along critical design direction. The results prove that compliance-based structural optimization has strict convexity when there are no constraints, while stress constraints create certain curvature impact on structural optimization. The methodology is able to determine marginal stability cases. Developed approach provides a mathematically rigorous solution to advanced design optimization in mechanical systems, offering increased reliability, convergence and performance in the presence of complex engineering constraints. Extensions of the methodology will be done in order to apply large scale finite element implementation, nonlinear material model, multiple loads optimization and efficient solution to second-order problem.

Keywords: Second variation analysis, structural optimization, mechanical systems design, constrained optimization, reduced hessian analysis, structural performance constraints, optimal design framework.

1. Introduction

With the rise in demand for mechanical systems that are light, reliable, and of high performance, there has been considerable research into advanced methods of structural optimization. Contemporary engineering problems in fields such as automotive, aerospace, robotics, and manufacturing require designs to meet a number of structural constraints, which include considerations of strength, stiffness, reliability, durability, and minimization of weight. Traditional methods of design often use iterative analyses and first order optimization techniques that cannot take into account nonlinear interactions and parameter variations. Thus, there is a need for advanced variation-based optimization approaches.

Design methodologies based on sensitivity have received substantial attention with respect to enhancing the performance of products through the study of the effects of the variables and parameters related to the designs. Design commonalities using sensitivity analysis methods have proved to be able to optimize automotive body designs with consistent performance despite product family variations (Fellini et al., 2004). Product family design methods based on variations have emphasized the significance of taking design variations into consideration for optimization purposes (Nayak et al., 2002). Multi-physics optimization, which is one of the recent advancements in this field, has integrated manufacturing uncertainties and physical interactions into mechanical system designs (Duan et al., 2026).

Structural reliability and robustness become significant issues in mechanical optimization because the structures have to deal with uncertainties in their operating environment and manufacturing process. Tolerance analysis methodologies up to the second order have been developed to enhance the robust design optimization of the structure in terms of variation and uncertainty impacts on structural properties (Lewis & Parkinson, 1994). The optimization planning approach has shown the potential of intelligent algorithms to solve the engineering layout problem with constraints. Improved VDMOS design through parasitic reduction illustrates how optimization at the component level can lead to better electrical performance, an important aspect of optimization of engineering systems in practice (Prasad et al., 2025). The surrogate modeling approach using artificial intelligence was studied in the context of efficient optimal design under uncertainty (Hwang et al., 2023).

In most advanced mechanical systems, complex arrangements are common, and structural parameters play an important role in determining the performance of the system. Various investigations on transmission systems through optimization techniques have confirmed that design analysis based on performance can be beneficial for improving the efficiency and reliability of the complicated mechanical systems (Nie et al., 2025). Optimizing the structure of automotive parts like suspension brackets has shown promise for achieving weight reduction while ensuring mechanical strength and safety (Liu et al., 2023). The use of topology optimization techniques has also been helpful in designing structures that are able to resist crashes by improving energy absorption under various loads (Dang et al., 2025). The structural design approach that is based on failure prevention has considered the importance of optimization for safe operation in loading conditions (Ikechukwu et al., 2019).

The utilization of optimization methodologies has extended to structural systems on large scales as well as in manufacturing processes. Experimental optimization of steel bridges has shown that the performance of structural systems can be improved via parameter and design optimization (Zhang et al., 2024). Methods for reliability-based topology optimization have provided further improvement in structural optimization as uncertainties and performance constraints are considered in the design optimization procedure (Zheng et al., 2023). Optimization of nonlinear mechanical systems such as suspension systems has been made possible via evolutionary optimization methodologies and simulation methodologies (Vahedi & Jamali, 2022). Optimization procedures using neural network methodologies have also been used in structural component design (Ahmed, 2024).

Advancements in additive manufacturing and design computing have presented new possibilities in the creation of high-performance mechanical structures. Multi-objective optimization approaches to lattice filled structures have shown that through control of the parameters of structures and materials used, additively manufactured parts can have better mechanical properties (Khan & Riccio, 2025). Variational-based optimal control strategies have shown that principles of mathematical optimization play an important role in performance improvement of dynamic mechanical systems (Chignoli & Wensing, 2020). The combination of topology optimization with additive manufacturing has opened up another path for creating light and geometrically complex structures with better performance abilities (Meng et al., 2020).

Various intelligent optimization techniques have been used in support of engineering design problems, where efficient search is possible through intelligent optimization methods for dealing with complex constrained optimization problems. Hybrid optimization techniques have proved useful in terms of better prediction and

decision-making capability in engineering applications by integrating various optimization techniques (Swathi & Jyothi, 2026). The chaotic and nature inspired optimization techniques have proved their worth in solving non-linear constrained engineering problems using improved exploration and convergence properties (Yıldız et al., 2022). Aerospace component design through thermo-elastic topology optimization and additive manufacturing is one such example (Guanghui et al., 2020).

Despite the notable achievements made in structural optimization, current techniques are limited in their ability to account for issues concerning nonlinearities, uncertainty treatment, computational burden, and prediction of performance variations. In addition, many traditional techniques are dependent on first order sensitivities and thus do not adequately capture the interaction of design variables and system performance. For this reason, second variation-based optimal design analysis is an efficient method through which improvement in accuracy of optimization can be enhanced using second order response properties. The current study examines development of second variation-based optimization technique for mechanical systems that are subject to structural performance constraints.

Key contribution

- In this study, another optimal design model based on variation theory for mechanical systems is introduced, and sensitivity analysis of higher order is used for obtaining the nonlinear behavior of the interaction between design variables and structure response due to performance constraints.
- With the proposed approach, second-order variation and optimization theory is used for improving the reliability of structural design, designing accuracy, and convergence of optimization in consideration of important issues such as strength, stiffness, and material effectiveness.
- With the proposed study, an efficient approach for structural optimization under variable working condition is achieved by considering uncertainty effects and structural performance constraints.

The rest of this paper is structured in the following way. In section II, preliminaries and notations from mathematics are introduced, such as design variables, state variables, admissible design spaces and variations used in the proposed approach. In section III, constrained mechanical optimization problem is formulated using the objective functions, equilibrium equations and structural performance constraints. In section IV, the optimality conditions are stated including KKT optimality conditions and second variation analysis which can be used to determine the nature of solutions. In section V, the reduced Hessian analysis of compliance-constrained elastic structures is discussed and the significance of this concept for structural stability. The cantilever beam problem is used as an example to illustrate the theoretical formulations and second order design behavior in section VI. The applicability, limitations and computational issues of the proposed method are discussed in section VII. Conclusions and further research on second variation based mechanical optimization are given in section VIII.

2. Preliminaries and notations

The design variables from the true Banach space of permissible designs (such as the space of cross-section area functions, thickness, or, in a density-based approach, the material distribution) and the state variable being the elastic displacement field solving the equilibrium boundary value problem depending on the design. Table 1 is a summary of the notation used in this paper.

Table 1: Notations are used.

Symbol	Meaning
Ω	Reference (fixed) design domain, a bounded open subset of $\mathbb{R}\text{^}d, d \in \{1,2,3\}$
ρ	Design variable (e.g., cross-sectional area, thickness, or density field), $\rho \in X$
X	Banach space of admissible design fields, with admissible set $U_ad \subset X$
$u(\rho)$	State field (displacement) solving the equilibrium equation for design $\rho, u \in V$
V	Hilbert space of admissible displacement fields (state space)
$a(\rho; u, v)$	Bilinear elastic energy form associated with design ρ
$\ell(v)$	Linear load functional
$J(\rho)$	Objective (cost) functional, e.g. compliance or weight
$g_i(\rho)$	Inequality performance constraints, $i = 1, \ldots, m$ (stress, displacement, volume, …)
λ_i	Lagrange multiplier associated with g_i

p	Adjoint (co-state) field
$L(\rho, u, \lambda)$	Lagrangian of the constrained design problem
$\delta J, \delta^2 J$	First and second Fréchet variations of J
h	Admissible design perturbation direction, $h \in X$
$C(\rho*)$	Critical cone of the design problem at $\rho*$

Let $\mathcal{X}$ be a Banach space. A functional $\mathcal{F}: \mathcal{X} \to \mathbb{R}$ is Fréchet differentiable at $\rho \in \mathcal{X}$ if there exists a bounded linear functional $\delta\mathcal{F}(\rho) \in \mathcal{X}^*$ such that $\mathcal{F}(\rho + h) = \mathcal{F}(\rho) + \delta\mathcal{F}(\rho)[h] + o(\|h\|)$ as $\|h\| \to 0$, where $\delta\mathcal{F}(\rho)[h]$ is the first variation in direction h. When $\delta\mathcal{F}$ is itself Fréchet differentiable at ρ, its derivative evaluated twice in direction h defines the second variation $\delta^2\mathcal{F}(\rho)[h,h] = \lim_{t\to 0} \frac{1}{t}(\delta\mathcal{F}(\rho + th)[h] - \delta\mathcal{F}(\rho)[h])$, yielding the second-order expansion $\mathcal{F}(\rho + h) = \mathcal{F}(\rho) + \delta\mathcal{F}(\rho)[h] + \frac{1}{2}\delta^2\mathcal{F}(\rho)[h,h] + o(\|h\|2)$.

Optimization over functional spaces operates over the admissible design set $\mathcal{U}_{\text{ad}} = \{\rho \in \mathcal{X}: g_i(\rho) \le 0,\ i = 1, \dots, m, \text{ and } \rho \in \mathcal{K}\}$, where $\mathcal{K} \subset \mathcal{X}$ is a closed convex set enforcing point-wise bound ($0 < \rho_{\min} \le \rho \le \rho_{\max}$). At a feasible point ρ^*, defining the active constraint index set $\mathcal{I}(\rho^*) = \{i: g_i(\rho^*) = 0\}$, the critical cone is given by $C(\rho^*) = \{h \in T_{\mathcal{K}}(\rho^*): \delta g_i(\rho^*)[h] \le 0\ \forall i \in \mathcal{I}(\rho^*),\ \delta J(\rho^*)[h] \le 0\}$, where $T_{\mathcal{K}}(\rho^*)$ is the tangent cone of $\mathcal{K}$ at ρ^*, representing feasible first-order directions that preserve active constraints without increasing the linearized objective J.

The hypotheses make include that there is a unique state solution $u(\rho) \in V$ to the state equation $a(\rho; u, v) = l(v)$ for all $v \in V$ in the vicinity of the reference design, the map $\rho \mapsto u(\rho)$ is twice continuously Fréchet differentiable, and there exists a constraint qualification (for example, Mangasarian–Fromovitz or linear independence) so that there exist KKT multipliers—the conditions that are satisfied by linear elasticity formulations for bounded density designs.

3. Problem formulation

The constrained structural design optimization problem is formulated over the admissible density design set $\mathcal{U}_{\text{ad}}$ as:

$$\min_{\rho \in \mathcal{U}_{\text{ad}}} J\big(\rho, u(\rho)\big) \quad \text{subject to} \quad a(\rho; u, v) = \ell(v) \quad \forall v \in V, \quad g_i\big(\rho, u(\rho)\big) \le 0, \quad i = 1, \dots, m\big)$$

where $J\big(\rho, u(\rho)\big)$ represents the objective functional—such as structural compliance $\ell\big(u(\rho)\big)$ or total material weight $\int_\Omega \rho\ dx$—the variational equality $a(\rho; u, v) = \ell(v)$ encodes the weak form of mechanical equilibrium for design field ρ, and $g_i\big(\rho, u(\rho)\big)$ aggregates inequality metrics including displacement limits, natural-frequency bounds, local or stress-aggregated (p-norm, Kreisselmeier–Steinhauser) metrics, and volume capacity constraints $\int_\Omega \rho\ dx \le V_{\max}$. Because the state $u(\rho) \in V$ is uniquely determined by ρ through the state equation, problem (4) reduces to an unconstrained design variable formulation over ρ via the reduced objective $\hat{J}(\rho) = J\big(\rho, u(\rho)\big)$ and reduced constraints $\hat{g}_i(\rho) = g_i\big(\rho, u(\rho)\big)$. Adjoining the variational state equilibrium via the adjoint state (co-state) field $p \in V$ and the inequality constraints via non-negative Lagrange multipliers $\lambda_i \ge 0$ yields the full Lagrangian functional:

$$\mathcal{L}(\rho, u, p, \lambda) = J(\rho, u) + a(\rho; u, p) - \ell(p) + \sum_{i=1}^{m} \lambda_i\, g_i(\rho, u) \quad \text{with} \quad \lambda_i \ge 0, \quad \lambda_i g_i\big(\rho, u(\rho)\big) = 0$$

Setting the variation of $\mathcal{L}$ with respect to the state variable u to zero yields the linearized adjoint equation defining the adjoint state field $p(\rho) \in V$:

$$a(\rho; v, p) = -\partial_u J(\rho, u)[v] - \sum_{i=1}^{m} \lambda_i\, \partial_u g_i(\rho, u)[v] \quad \forall v \in V$$

Substituting $u(\rho)$ and $p(\rho)$ into $\mathcal{L}$ forms the reduced Lagrangian $\hat{\ell}(\rho, \lambda) = \mathcal{L}(\rho, u(\rho), p(\rho), \lambda)$. Stationarity of $\hat{\ell}(\rho, \lambda)$ with respect to ρ alone yields the total design sensitivity variations $\delta\hat{\ell}(\rho, \lambda)[h]$ without explicit computation of state derivatives $\partial_\rho u(\rho)$—recovering the classical adjoint (co-state) method of structural sensitivity analysis, whose first and second variations govern the first- and second-order optimality conditions of formulation (4).

4. Optimality theorems

4.1. First order necessary condition

Theorem 1 (First-order KKT stationarity conditions)

Let $\rho^* \in \mathcal{U}_{\text{ad}}$ be a local minimizer of the PDE-constrained structural optimization problem (4) at which a suitable constraint qualification holds. Then there exist non-negative Lagrange multipliers $\lambda_i^* \geq 0$ for $i = 1, \dots, m$ and a unique adjoint state field $p^* = p(\rho^*) \in V$ solving the adjoint state equation such that the first-order stationarity and complementary slackness conditions hold:

$$\delta_\rho \mathcal{L}(\rho^*, u^*, p^*, \lambda^*)[h] \geq 0 \quad \forall h \in T_{\mathcal{K}}(\rho^*), \quad \text{and} \quad \lambda_i^* g_i(\rho^*, u^*) = 0, \quad i = 1, \dots, m$$

Proof

1. **Admissible Direction & State Linearization:** For any direction $h \in T_{\mathcal{K}}(\rho^*)$, define an admissible parametric design path $\rho(t) = \rho^* + th + o(t) \in \mathcal{U}_{\text{ad}}$ for $t \in [0, \varepsilon)$ with $\rho(0) = \rho^*$. Applying the Implicit Function Theorem to the weak state residual $a(\rho; u, v) - \ell(v) = 0$ (enabled by the invertibility and coercivity of the bilinear form $\partial_u a$ at (ρ^*, u^*)) guarantees that $t \mapsto u(t)$ is Fréchet differentiable. Its directional state derivative $\dot{u}(0) = w \in V$ uniquely satisfies the linearized state equation:

$$a(\rho^*; w, v) = -\delta_\rho a(\rho^*; u^*, v)[h] \quad \forall v \in V$$

2. **One-Sided Optimality & Directional Variational Inequality:** Since ρ^* minimizes the design objective locally, the scalar functional $\varphi(t) := J\big(\rho(t), u(t)\big)$ achieves a local minimum at $t = 0$ over feasible directions $h \in C(\rho^*)$, yielding the first-order necessary condition $\varphi'(0) = \delta J(\rho^*)[h] + \partial_u J(\rho^*, u^*)[w] \geq 0$. Evaluating the adjoint equation $a(\rho^*; v, p^*) = -\partial_u J(\rho^*, u^*)[v] - \sum_{i=1}^m \lambda_i^* \, \partial_u g_i(\rho^*, u^*)[v]$ at test direction $v = w$, and substituting the linearized state identity $a(\rho^*; w, p^*) = -\delta_\rho a(\rho^*; u^*, p^*)[h]$ via symmetry of the elastic energy form, eliminates the directional state derivative w:

$$\partial_u J(\rho^*, u^*)[w] = -\delta_\rho a(\rho^*; u^*, p^*)[h] - \sum_{i=1}^m \lambda_i^* \, \partial_u g_i(\rho^*, u^*)[w]$$

3. **Multipliers & Stationarity:** Combining the objective variation with the weighted active constraint variations $\lambda_i^* \geq 0$ and applying the Banach space KKT multiplier theorem under the assumed constraint qualification recovers the stationarity variational inequality $\delta_\rho \mathcal{L}(\rho^*, u^*, p^*, \lambda^*)[h] \geq 0$ for all $h \in T_{\mathcal{K}}(\rho^*)$ alongside complementary slackness $\lambda_i^* g_i(\rho^*, u^*) = 0$.

Remarks & Interpretation

Theorem 1 establishes the infinite-dimensional Banach space analogue of the classical Karush–Kuhn–Tucker (KKT) conditions specialized to PDE-constrained optimization. For continuous topology optimization, this variational inequality directly yields the pointwise optimality criterion $\frac{\partial J}{\partial \rho(x)} + \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial \rho(x)} \geq 0$ (satisfied with strict equality for unbounded interior design variables), which forms the theoretical foundation for heuristic Optimality Criteria (OC) update schemes in structural mechanics.

4.2 Second variation and local optimality

Theorem 2 (Second-Order Necessary Conditions)

Under the hypotheses of Theorem 1, if $\rho^* \in \mathcal{U}_{\text{ad}}$ is a local minimizer of the PDE-constrained optimization problem (4), then the second variation of the reduced Lagrangian $\hat{\ell}(\rho) := \hat{\ell}(\rho, \lambda^*)$ evaluated at optimal multipliers λ^* is non-negative over the critical cone:

$$\delta^2 \hat{\ell}(\rho^*)[h, h] \geq 0 \quad \forall h \in C(\rho^*)$$

Proof

1. **Admissible Curve along Critical Directions:** For any direction $h \in C(\rho^*)$, construct an admissible perturbation path $\rho(t) = \rho^* + th$ for $t \in [0, \varepsilon)$. Because h lies in the critical cone $C(\rho^*)$, all constraints

active at ρ^* remain feasible to first order along $\rho(t)$, so by construction and complementary slackness, the reduced Lagrangian $\hat{\ell}\big(\rho(t)\big) = \hat{J}\big(\rho(t)\big) + \sum_{i\in\mathcal{I}} \lambda_i^* \hat{g}_i\big(\rho(t)\big)$ coincides with $\hat{J}\big(\rho(t)\big)$ up to terms vanishing to first order.

2. **Taylor Expansion & Non-negativity:** Local optimality of ρ^* for the restricted objective curve $\varphi(t) := \hat{\ell}\big(\rho(t)\big)$ implies a local minimum at $t = 0$. Expanding to second order yields $\varphi(t) = \varphi(0) + t\varphi'(0) + \frac{t^2}{2}\delta^2\hat{\ell}(\rho^*)[h,h] + o(t^2)$. Since $\varphi'(0) = \delta\hat{\ell}(\rho^*)[h] \geq 0$ by first-order stationarity, any strictly negative second variation $\delta^2\hat{\ell}(\rho^*)[h,h] < 0$ would force $\varphi(t) < \varphi(0)$ for sufficiently small $t > 0$, violating local minimality and concluding that $\delta^2\hat{\ell}(\rho^*)[h,h] \geq 0$.

Theorem 3 (Second-order sufficient conditions)

Let $\rho^* \in \mathcal{U}_{\text{ad}}$ be a feasible design satisfying first-order stationarity (Theorem 1) for Lagrange multipliers $\lambda^* \geq 0$ obeying complementary slackness. If the second variation of the reduced Lagrangian is coercive over the critical cone—that is, there exists $\alpha > 0$ such that:

$$\delta^2\hat{\ell}(\rho^*)[h,h] \geq \alpha\|h\|_{\mathcal{X}}^2 \quad \forall h \in C(\rho^*)$$

then ρ^* is a strict local minimizer: there exist constants $\varepsilon > 0$ and $\beta > 0$ such that $\hat{J}(\rho) \geq \hat{J}(\rho^*) + \beta\|\rho - \rho^*\|_{\mathcal{X}}^2$ for every feasible ρ with $\|\rho - \rho^*\|_{\mathcal{X}} < \varepsilon$.

Proof

1. **Contradiction Sequence Construction:** Assume for contradiction that no such ε, β exist. Then there exists a sequence of feasible designs $\rho_k \to \rho^*$ with $\rho_k \neq \rho^*$ such that $\hat{J}(\rho_k) < \hat{J}(\rho^*) + \frac{1}{k}\|\rho_k - \rho^*\|_{\mathcal{X}}^2$. Expressing $\rho_k = \rho^* + t_k h_k$ with normalized step magnitude $t_k = \|\rho_k - \rho^*\|_{\mathcal{X}} \to 0$ and $\|h_k\|_{\mathcal{X}} = 1$, extract a weakly convergent subsequence $h_k \rightharpoonup h$ with $\|h\|_{\mathcal{X}} \leq 1$.

2. **Critical Cone Membership:** Feasibility of ρ_k combined with first-order constraint expansions establishes $h \in T_{\mathcal{K}}(\rho^*)$ and $\delta g_i(\rho^*)[h] \leq 0$ for active indices $i \in \mathcal{I}(\rho^*)$. Because $\hat{J}(\rho_k) - \hat{J}(\rho^*) \leq o(t_k)$ rules out a positive first-order variation, $\delta\hat{\ell}(\rho^*)[h] = 0$, confirming $h \in C(\rho^*)$.

3. **Asymptotic Variation & Coercivity Bound:** Expanding the reduced Lagrangian to second order yields $\frac{t_k^2}{2}\delta^2\hat{\ell}(\rho^*)[h_k,h_k] < \frac{1}{k}t_k^2 + o(t_k^2)$, implying $\delta^2\hat{\ell}(\rho^*)[h_k,h_k] < \frac{2}{k} + o(1) \to 0$. Weak lower semicontinuity of the quadratic form gives $\delta^2\hat{\ell}(\rho^*)[h,h] \leq \liminf_{k\to\infty} \delta^2\hat{\ell}(\rho^*)[h_k,h_k] \leq 0$. If $h \neq 0$, this directly contradicts coercivity ($\alpha\|h\|_{\mathcal{X}}^2 > 0$). If $h = 0$, applying coercivity directly to unit vectors h_k yields $\delta^2\hat{\ell}(\rho^*)[h_k,h_k] \geq \alpha - o(1) > \frac{2}{k}$ for large k, contradicting $\delta^2\hat{\ell}(\rho^*)[h_k,h_k] < \frac{2}{k} + o(1)$. Hence ρ^* is a strict local minimizer.

Remarks on second-order diagnostics

The sign of $\delta 2\, l\,(\rho *)$ with respect to the critical cone is the crucial second-order test for structural local optimality; non-negative definiteness is necessary, while coerciveness is sufficient for strict local minimality. The degenerate case where $\delta 2\, l\,(\rho *)[h,h] = 0$ at a non-trivial direction $h \in C(\rho *)$ means marginal stability and determines the optimal topology from high-order terms or numerical approximations.

4.3. Corollary: rank deficiency and bifurcation of optical designs

Corollary 1 (Bifurcation and indeterminacy under second-order degeneracy)

Suppose that $\rho *\in U_"ad"$ is a stationary point according to Theorem 1. Suppose further that $\delta 2\, l\,(\rho *)[h,h]$ vanishes for a non-zero critical direction, $h \in C(\rho *) \cap (-C(\rho *))$; that is, the second variation of the reduced Lagrangian is degenerate. In that case, ρ^* cannot be a locally optimal point of $U"ad"$ based on second-order information alone. Under generic assumptions, $\rho *$ constitutes a bifurcation point for the family of stationary structures: An arbitrary change in the load vector, the constraint vector, or the mesh will split ρ^* into several different local minimizers whose objective function values are distinguished from each other to third order in $\|h\|_X$.

Proof (Sketch)

1. **Third-Order Dominance:** By Theorem 2, $\delta^2\hat{\ell}(\rho^*)[h,h] \geq 0$ is a necessary condition for local minimality. When $\delta^2\hat{\ell}(\rho^*)[h,h] = 0$, the local behavior of $\Delta\hat{\ell}(t) := \hat{\ell}(\rho^* + th) - \hat{\ell}(\rho^*)$ for small step sizes t is governed by the third variation $\frac{t^3}{6}\delta^3\hat{\ell}(\rho^*)[h,h,h]$ (assuming $\hat{\ell}$ is C^3-regular). Since t^3 is an odd function, $\Delta\hat{\ell}(t)$ is generically negative on one side of $t = 0$ unless the third variation vanishes identically.

2. **Singularity unfolding:** The application of the Lyapunov-Schmidt reduction, or Splitting Lemma (as known in Morse-Bott singularity theory), allows one to focus on the single dimension degenerate space corresponding to h. By changing system parameters, e.g. the load or constraints, deform the higher order critical point, splitting the degenerate stationary point ρ* into several minima.

Variational & engineering interpretation

Corollary 1 furnishes an analytic basis for the non-uniqueness property of optimal topologies frequently encountered in computational structural optimization under close to identical values of the parameters. This property is not due to any numerical issues, but rather represents the characteristic nature of structural designs that exist in a degenerate state within the critical cone.

5. Analysis: reduced hessian for compliance-constrained elastic structures

Specializing the general variational framework of Section 4 to structural compliance optimization—where the objective is either the compliance functional $J(\rho,u) = \ell(u)$ or total material weight $J(\rho,u) = \int_\Omega \rho\ dx$ bounded by a compliance constraint—leverages the self-adjoint property of linear elasticity. Because the elastic bilinear form $a(\rho;\cdot,\cdot)$ is symmetric and $\ell(u)$ is linear, the adjoint state field coincides identically with the primary displacement state field, $p(\rho) = u(\rho)$. Differentiating the variational equilibrium relation $a(\rho;u(\rho),v) = \ell(v)$ twice with respect to ρ in direction h and substituting $p = u$ yields the classical reduced Hessian for structural compliance:

$$\delta^2\hat{J}(\rho)[h,h] = \delta^2_{\rho\rho}a(\rho;u,u)[h,h] - 2\delta_\rho a(\rho;u,w)[h] - 2a(\rho;w,w)$$

Where $w \in V$ is the linearized state sensitivity direction satisfying the variational equation $a(\rho;w,v) = -\delta_\rho a(\rho;u,v)[h]$ for all test functions $v \in V$.

The final quadratic term, $-2a(\rho;w,w)$, is non-positive due to the positive definiteness and coercivity of the elastic strain energy form $a(\rho;\cdot,\cdot)$. Consequently, it introduces a destabilizing, non-convex (concave) component into the compliance Hessian $\delta^2\hat{J}(\rho)$. This mathematical structure explains why unconstrained compliance minimization under a global weight bound is ill-posed without additional structural regularization (such as density gradient penalization, perimeter control, or spatial length-scale filtering) to enforce the coercivity condition required by Theorem 3. When a linear volume constraint $\int_\Omega \rho\ dx \leq V_{\max}$ is active, its contribution to the second variation vanishes identically ($\delta^2 g \equiv 0$), leaving the sign of the reduced Hessian governed purely by the elastic energy variations. Conversely, when nonlinear stress constraints (p-norm or Kreisselmeier–Steinhauser aggregations) are active, an additional curvature term $\sum_{i=1}^m \lambda_i^*\,\delta^2\hat{g}_i(\rho^*)[h,h]$ enters the reduced Lagrangian $\hat{\ell}$, which can either restore or further degrade local coercivity depending on constraint curvature.

Equations is the structural optimization analog of the reduced Hessian operator in optimization with constraints given by partial differential equations (PDEs). It is exactly the curvature tensor that is approximated (either implicitly through quasi-Newton secant approximation or explicitly through second order sensitivity analysis) using Newton-like or SQP methods in addition to simple OC updates.

5.1. Discretized form and computational role of the hessian

With respect to finite element formulations where the density variables are represented using $\rho \in R\hat{}n$ such that the state equation becomes $K(\rho)u = f$, the continuous reduced Hessian operator is transformed to an $n \times n$ symmetric matrix $H(\rho)$. The element at the (i,j)-th position of the matrix H(ρ) is obtained using discrete sensitivity analysis of $K(\rho)u - f = 0$.

The continuous coercivity requirement $\delta^2\hat{\ell}(\rho^*)[h,h] \geq \alpha\|h\|_X^2$ corresponds to positive semi-definiteness of $\mathbf{H}(\boldsymbol{\rho})$ restricted to the discrete critical cone:

$$\mathrm{h}^T \mathrm{H}(\rho^*)\mathrm{h} \geq \alpha \|\mathrm{h}\|^2 \quad \forall \mathrm{h} \in \mathcal{C}_{\text{discrete}}(\rho^*)$$

When there is no structural regularization in terms of length-scale filtering or density gradient penalty, the coercivity constant $\alpha(n)$ gets worse as n goes to infinity during mesh refinement.

Discrete Hessian Decomposition for Aggregated Stress Constraints

For an aggregated stress constraint $\hat{g}(\boldsymbol{\rho}) \leq 0$ using a Kreisselmeier–Steinhauser (KS) formulation:

$$\hat{g}(\rho) = \frac{1}{\kappa} \ln\left(\sum_{e=1}^{n} \exp\left(\kappa \frac{\sigma_e(\rho)}{\sigma_{\text{allow}}}\right)\right)$$

The second variation adds a structured matrix contribution $\boldsymbol{\nabla}^2 \hat{g}(\boldsymbol{\rho})$ to the reduced Hessian $\mathbf{H}(\boldsymbol{\rho})$, weighted by the optimal multiplier λ^*:

$$\mathrm{H}_{\text{stress}}(\rho) = \lambda^* \left[\frac{\kappa}{\sigma_{\text{allow}}} \sum_{e=1}^{n} w_e(\rho) \nabla \sigma_e(\rho) \nabla \sigma_e(\rho)^T + \sum_{e=1}^{n} w_e(\rho) \nabla^2 \sigma_e(\rho)\right]$$

where the element stress weights $w_e(\boldsymbol{\rho}) \geq 0$ are given by:

$$w_e(\boldsymbol{\rho}) = \frac{\exp\left(\kappa \frac{\sigma_e(\boldsymbol{\rho})}{\sigma_{\text{allow}}}\right)}{\sum_{k=1}^{n} \exp\left(\kappa \frac{\sigma_k(\boldsymbol{\rho})}{\sigma_{\text{allow}}}\right)}$$

The breakdown illustrates how sharper approximation of the actual maximum stress by the increase in the value of the aggregation sharpness parameter κ improves the constraint accuracy while worsening the second-order well-posedness, a well-known empirical phenomenon of stress-constrained topology optimization explained now in terms of a particular Hessian element.

6. Ilustrative case study: variable cross section cantilever Beam

To make the abstract variational theorems concrete, consider a statically determinate Euler–Bernoulli cantilever beam of length L ($x \in [0, L]$) clamped at $x = 0$ and subjected to a transverse tip load F at $x = L$. The design variable $\rho(x) > 0$ represents the cross-sectional area, with second moment of area given by $I(\rho) = c\rho^2$ for a family of similar cross-sections (where $c > 0$ is a geometric shape constant, e.g., $c = 1/(4\pi)$ for solid circular sections), bounded by simple constraints $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max}$ and a total volume limit $\int_0^L \rho\,(x)\,dx \leq V_{\max}$. Because the bending moment distribution $M(x) = -F(L - x)$ is statically determinate and independent of the design profile $\rho(x)$, the compliance objective $J(\rho) = Fu(L)$ simplifies to the explicit closed-form functional $J(\rho) = \int_0^L \frac{M(x)^2}{EI(\rho(x))}\,dx = \frac{F^2}{Ec}\int_0^L \frac{(L-x)^2}{\rho(x)^2}\,dx$.

Directly computing variations of this explicit functional with $m(x) := (L - x)^2$ yields the first variation $\delta J(\rho)[h] = -\frac{2F^2}{Ec}\int_0^L \frac{m(x)h(x)}{\rho(x)^3}\,dx$ and the second variation $\delta^2 J(\rho)[h, h] = \frac{6F^2}{Ec}\int_0^L \frac{m(x)h(x)^2}{\rho(x)^4}\,dx \geq 0$. The strict positivity of the integrand for any non-zero perturbation $h \neq 0$ over $x \in (0, L)$ demonstrates that the compliance functional $J(\rho)$ is strictly convex on the interior of the design space. Since the second variation of the linear volume restriction is identically zero ($\delta 2\, g \equiv 0$), the KKT point of Theorem 1 is always a coercive point according to (8) of Theorem 3, which means the fully stressed design solution $\rho^*(x) \propto (L - x)^{2/3}$ obtained from $-\delta J(\rho^*)[h] = \mu \int_0^L h\,dx$ is a strict, global minimizer.

However, introducing a local pointwise stress constraint $\sigma(x) \propto \frac{M(x)}{c\rho(x)^{3/2}} \leq \sigma_{\text{allow}}$ alters this well-posedness: because the local bending stress is a concave function of cross-sectional area ρ, its contribution to the second variation of the reduced Lagrangian $\hat{\ell}$ yields a negative (destabilizing) term $\sum_i \lambda_i^* \delta^2 \hat{g}_i(\rho)[h,h] < 0$. This negative contribution can override the positive compliance curvature, eroding coercivity over subset directions of the critical cone and triggering the marginal-stability regime (Corollary 1) where non-uniqueness and topological bifurcations emerge.

6.1. Sensivity of the stability margin to the load and constraint data

Evaluating how the second-order coercivity margin depends on problem data yields key analytical insights into the stability of optimal structural profiles. For the fully stressed cantilever beam design profile $\rho^*(x) = k(L-x)^{2/3}$, where $k > 0$ is fixed by the volume constraint $\int_0^L \rho^*(x)\,dx = V_{\max}$, the quadratic form of the reduced Hessian $Q(h) := \delta^2 J(\rho^*)[h,h]$ simplifies to $Q(h) = \frac{6F^2}{Eck^4}\int_0^L (L-x)^{-2/3} h(x)^2\,dx$. Despite the integrable singularity in the kernel weight $(L-x)^{-2/3}$ as $x \to L$, $Q(h)$ remains strictly positive for every non-zero perturbation $h \in L^2(0,L)$. However, examining the local coercivity parameter $\alpha := \inf_{\|h\|_{L^2}=1} Q(h)$ reveals two critical structural vulnerabilities: first, as the lower bound $\rho_{\min} \to 0$, the local weight $\rho^*(x)^{-4}$ in $Q(h)$ blows up precisely where the beam is thinnest, causing the global coercivity constant α to degrade toward zero and formally demonstrating that slender structural members with low relative stiffness possess the narrowest second-order optimality margins and are exceptionally sensitive to external perturbations, discretization errors, and manufacturing tolerances; second, evaluating the Rayleigh-type ratio $\frac{Q(h)}{\|h\|_{L^2}^2}$ for normalized perturbations h concentrated near a localized point $x_0 \approx L$ shows that the second-order curvature scales like $(L-x_0)^{-2/3}$, which degrades as x_0 approaches the free tip ($x \to L$) and confirms that the most lightly loaded, slender sections of an optimal topology are precisely those operating in the most marginally stable regime of Theorem 3.

7. Discussion

Section 4 theorems, combined with the Hessian reduction formula from Section 5 and the closed form example from Section 6, have a number of practical applications for solving structural optimization problems numerically.

First of all, algorithms that rely solely on first order information (such as steepest descent, optimality criteria) may converge to points that satisfy Theorem 1 without any guarantee whatsoever provided by the algorithm that the points obtained are even local minima in the sense of Theorem 3; to distinguish between a real local minimum and a saddle point type stationary point one must calculate or at least estimate the sign of reduced Hessian (9) restricted to critical cone – and this is exactly what second order methods like Newton-SQP and trust region structural optimization.

Second, the instability-inducing $-2a(\rho; w, w)$ component in (9) provides a variational insight into the known phenomenon of the ill-posedness of unconstrained compliance minimization in continuum (density-based) topology optimization, since this term is always unstable, and without regularization that enforces a characteristic length-scale on the density (density filtering, perimeter penalties, or a discrete design domain), the second variation may not be coercive with respect to arbitrary fine perturbations, just like in the case of unregularized compliance minimization. The current framework indicates that the existing numerical tools employed to handle this problem (filters and projection methods) can be viewed as means to make $\delta^2\ell$ coercive with respect to the discretized critical cone.

Finally, the example of the beam from Section 6 demonstrates that whether a particular performance constraint is favorable or unfavorable for second-order well-posedness depends on the curvature of the dependence of the constraint on the design (linear volume constraints do not affect second-order well-posedness; concave stress constraints might be destabilizing), implying that constraint formulation is not only related to differentiability and conditioning, which is traditionally considered in the literature, but also to second-order structural stability of the formulated optimization problem.

The limitations of the current approach include the requirement that the state map and constraint functionals are sufficiently smooth (twice Fréchet differentiable), thus precluding, without additional work, many important non-smooth formulations including exact (i.e., non-aggregated) local stress constraints with potential degeneracy of the constraint, as well as multi-load or robust formulations with a min-max objective. The generalization of Theorems 2-3 to such settings, as well as to the genuinely non-convex nonlinear elasticity problem, would be interesting lines of future research.

7.1. Outline of a second order structural optimization step

Based on the theory above, one should consider the following curvature-aware update, at each iterate ρ_k of an optimization run, instead of simply taking a first-order update: (a) compute the solution (uk, pk) and the active-set multipliers (λk) at ρk from a first-order KKT solve, as in Theorem 1; (b) construct, or approximate using the quasi-Newton approximation of the limited-memory type, the reduced Hessian matrix Hk of ℓ in the discrete critical cone, according to either the adjoint expression (9) or its discrete variant described in Section 5.1; (c) check whether the minimal eigenvalue of Hk in the discrete critical cone exceeds some small positive tolerance; if it does, compute the Newton step $dk\ as\ Hk\ dk\ =\ -\nabla\ell(\rho k)$ projected into the cone, and if not, use the regularized (trust-region or Levenberg-type) step $Hk\ +\ \tau I,$ where τ is selected to ensure coercivity and declare that the design lies within the marginal stability regime of Corollary 1 for further inspection. In terms of Section 4, the above gives the formal justification of the practices routinely used in various second-order and trust-region structural optimization algorithms, which are essentially doing the above, albeit in a less structured way.

A further diagnostic application of the theory, which does not involve optimization at all, concerns post-optimization verification: after a design ρ* has been output by an arbitrary optimizer, (7)-(8) provide a proof — namely, a necessary condition and, if verifiable, a sufficient proof — of its local optimality, which is entirely algorithm-independent and requires only a small number of additional adjoint evaluations on a basis (or random sketch, if n is large) of the discrete critical cone.

8. Conclusion

This study introduced the second variation-based optimal design concept in terms of mechanical structures subject to performance constraints through considering high order sensitivity information, variational principles, adjoint analysis and the second order optimality condition. The proposed model compensates for the shortcomings of the first order optimization approaches due to their incapability of accounting for nonlinear effects, uncertainties and structural variability in the design process. The developed methodology is based on the compliance optimization, material efficiency, stress limitations, displacements, volumes and structural stability requirements for achieving mechanical design with sufficient reliability. The second variation analysis on the cantilever beam structure confirms the validity of the proposed technique in assessing local optimality, curvatures and structural stability through reduced Hessian properties. In fact, it was found that the compliance optimization problem keeps its strict convexity in absence of stress constraints. The stress constraints are responsible for the curvatures that impact the second order stability and marginal stability situations. The proposed method represents a mathematically sophisticated tool for achieving reliability, convergence and structural performance in complicated mechanical optimization problems. Further research efforts will be devoted to the development of the introduced concepts toward large scale finite element formulations, nonlinear material behaviour, multi-load structural optimization and efficient second order optimization algorithms. In addition, future work can involve real-time structural optimization, uncertainty-based design methods, Hessian matrix approximation through machine learning tools and experimental validation of the second variation-based optimization techniques with help of advanced manufacturing platforms.

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