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Harnack Inequality and Smoothness for some Non Linear Degenerate Elliptic Equations

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We prove Harnack inequality and smoothness for weak solutions of quasilinear degenerate elliptic equation with respect to a system of non commuting vector fields. In addition, the structure assumptions allow quadratic growth in the gradient.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $X = \{X_1, \ldots, X_m\}$ be a family of C^∞ vector fields satisfying Hörmander's condition in a neighborhood of $\overline{\Omega}$.

We consider a quasilinear equations of the following kind

$$-X_j^*(a_{ij}X_iu + d_ju) + \frac{b_0}{\lambda}\omega|Xu|^2 + b_iX_iu + cu = f - X_i^*h_i\,. \tag{1}$$

We will assume that the equation is degenerate elliptic with respect to the system X of the given vector fields i.e.

$$\exists\lambda > 0\;:\;\lambda^{-1}\omega(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda\omega(x)|\xi|^2 \text{ a.e. } x \in \Omega\ \forall\xi \in \mathbb{R}^m\,. \tag{2}$$

Here the degeneracy is given by a Muckenhoupt weight ω in the class A_2 with respect to the Carnot-Carathéodory metric (see Section 2 for precise definition).

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Regarding the lower order coefficients we assume that they belong to a generalization of the Stummel-Kato class with respect to the Carnot-Carathéodory metric. This kind of assumption is very important because, in some cases, it is both necessary and sufficient for the smoothness of the solutions (see [8], [31]).

We will achieve the smoothness of the weak solutions of the equation (1) as consequence of a Harnack inequality for nonnegative bounded weak solutions.

The problem we investigate here has a long history and is motivated by the fact that solutions to certain variational inequalities involving partial differential operators of the form $\mathrm{div}A(x,u,\nabla u) + B(x,u,\nabla u)$ are solutions of the equations

$$\mathrm{div}A(x,u,\nabla u) + B(x,u,\nabla u) = \mu\,,$$

for some non negative Radon measure μ and under several growth assumptions.

It is impossible to report here the contributions of all authors in this field. We will try to draw a short history concerning uniformly elliptic and degenerate cases. The main point here is that we assume conditions different from the classical L^p hypotheses. In this regard, the starting point is a remarkable note by Aizenman and Simon ([1]) where, using probabilistic tools, the authors proved Harnack inequality for nonnegative solutions of the equation

$$-\Delta u = Vu$$

where V is a function in the Stummel-Kato class. After a couple of years their results have been generalized to any linear uniformly elliptic operator by Chiarenza, Fabes and Garofalo in 1986 (see [3]) where analytical proof is given by using a representation formula for weak solutions of the uniformly elliptic equation

$$-(a_{ij}\, u_{x_i})_{x_j} - Vu = 0 \tag{3}$$

V belongs to the Stummel-Kato class. A different proof of Harnack inequality and the continuity of weak solutions based on representation formula is also given by Simader in [29]. There he proved that if the Stummel modulus ϕ (see Section 2 for the definition of ϕ) is Hölder continuous, then the solution of the equation is also Hölder continuous (see also [6], [7], [21]). We remark here that the local regularity properties of weak solutions have also been studied by Trudinger in [30] and by Rakotoson and Ziemer in [27], where different growths have been assumed (see also [10], [14], [18], [32], [33], and [34]).

Turning now our attention to operators that can be degenerate elliptic, our the starting point is the paper [20] where Gutierrez, using a suitable weighted version of the Stummel-Kato class, added a potential to the equation studied in [19] and showed Harnack inequality for the nonnegative weak solutions of (3), where the operator is linear elliptic with degeneracy of A_2 kind. It is worth to recall here that in [35] a Harnack inequality for a more general equation has been obtained, without using representation formula, assuming the lower order terms in suitable weighted Morrey spaces (see Section 2 for the definition) (see also [9], [24], [31]).

Among the various kind of degeneracy, we consider now the operators formed by vector fields we recall the result in [5], where Citti, Garofalo and Lanconelli proved a Harnack inequality and the continuity of the weak solutions for a linear sub-elliptic operator. In [4] the Hölder continuity has been obtained for the same equation assuming the known term in a suitable version of the Morrey class modeled on the level sets of the fundamental solution (see also [16], [17]).

In [23] Lu proved a Harnack inequality for solutions of a special case of equation (1) (there he has $d_j = b_0 = b_i = f = h_i = 0$) where c is a Stummel-Kato function and the leading terms of the equation satisfy the condition (2). His proof relies on representation formula for weak solutions.

In [11] and [12] (see also [13] and [15] for a different kind of degeneration) we extended Lu's results to a more general equation avoiding representation formulas by using a modified version of the Moser iterative process (see [28]). Here we extend our previous results to the equation (1) containing a quadratic dependence in the gradient.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, and $X = \{X_1, \ldots, X_m\}$ be a family of C^∞ vector fields satisfying Hörmander's condition in a neighborhood of $\overline{\Omega}$. Let d be the Carnot-Carathéodory metric associated to the system X. We denote by $B = B_r(x)$ the *metric ball* centered at $x \in \Omega$ of radius r. It is well known that the metric balls satisfy the doubling property locally (see [25]), i.e. for any compact K in Ω there exist R_0 and $c_d > 0$ such that

$$|B_{2r}(x)| \leq c_d |B_r(x)| \quad \forall x \in K$$

for any $0 < r \leq R_0/2$. The number $Q = \log_2 c_d$ is called the local homogeneous dimension of Ω respect to the system X.

We now recall the definition of A_p weights.

Definition 2.1. Let $p > 1$ and ω be a nonnegative locally integrable function. We say that ω belongs to the *Muckenhoupt class* A_p if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int_B [\omega(x)]^{\frac{-1}{p-1}}\, dx \right)^{p-1} \equiv C_0 < +\infty\,,$$

the supremum being over all Carnot-Carathéodory metric balls B in $\mathbb{R}^n$. The number C_0 is called the A_p *constant of* ω.

Theorem 2.2. (Doubling property) *Let $p > 1$ and $\omega \in A_p$. Then, there exists a positive constant $C_d > 1$ such that*

$$\omega(B_{2r}(x_0)) \leq C_d \omega(B_r(x_0))\,, \quad \forall x_0 \in \mathbb{R}^n\,, r > 0\,,$$

where $\omega(B_r(x_0)) = \int_{B_r(x_0)} \omega\, dx$.

Definition 2.3. (Lebesgue and Sobolev spaces) Let $1 \le p \le +\infty$. We set

$$L^p(\Omega,\omega) = \left\{ u : \Omega \to \mathbb{R} \text{ such that } \int_\Omega |u|^p \omega dx < +\infty \right\}, \text{ and}$$
$$W^{1,p}(\Omega,\omega) = \{ u \in L^p(\Omega,\omega) \text{ such that } X_j u \in L^p(\Omega,\omega),\ j = 1, \dots m \},$$

endowed with norms, respectively,

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega dx \right)^{1/p}$$

and
$$\|u\|_{W^{1,p}(\Omega,\omega)} = \|u\|_{L^p(\Omega,\omega)} + \|Xu\|_{L^p(\Omega,\omega)}. \tag{4}$$

We denote by $W_0^{1,p}(\Omega,\omega)$ the *closure* of the smooth and compactly supported functions in $W^{1,p}(\Omega,\omega)$ with respect to the norm (4).

We state the following embedding Theorem for weighted Sobolev spaces (see [22]).

Theorem 2.4. *Let $1 \le p < +\infty$, $\omega \in A_p$, $E \Subset \Omega$. Then there exist constants r_0 and C_1 such that for any metric ball $B = B_r(x)$, $x \in E$, and any $f \in C^\infty(\overline{B})$ the following inequality holds true*

$$\left(\frac{1}{\omega(B)} \int_B |f - f_B|^q \omega dy \right)^{\frac{1}{q}} \le C_1 r \left(\frac{1}{\omega(B)} \int_B |Xf|^p \omega dy \right)^{\frac{1}{p}}$$

provided $0 < r < r_0$, $q = \tau p$, $1 \le \tau < \frac{Q}{Q-1} + \delta_p$, $\delta_p > 0$. Here C_1 depends only on the A_p constant of the weight ω, E and Ω, the constant δ_p depends on p and the A_p constant. Here $f_B = \frac{1}{\omega(B)} \int_B f \omega dy$.
In the case $f \in C_0^\infty(\overline{B})$, f_B can be taken to be zero.

In order to formulate the assumptions on the lower order terms of our equation we need to define some other function spaces.

Definition 2.5. (Stummel classes) Let ω be a A_2 weight and let $B_R(x_0)$ be a metric ball such that $\Omega \subset B_R(x_0)$. If V is a locally integrable function in Ω we set

$$\phi(V;r) = \sup_{x\in\Omega} \int_{\{y\in\Omega\,:\,d(x,y)<r\}} |V(y)| \int_{d(x,y)}^{4R} \frac{s^2}{\omega(B(x,s))} \frac{ds}{s} \omega(y)\, dy.$$

We say that V belongs to the class $\tilde{S}(\Omega,\omega)$ if $\phi(V;r)$ is bounded in a neighborhood of the origin. We say that V belongs to the *Stummel-Kato class* $S(\Omega,\omega)$ if, in addition, $\lim_{r\to 0} \phi(V;r) = 0$. We say that the function V belongs to the *restricted Stummel-Kato class* $S'(\Omega,\omega)$ if there exists $\rho > 0$ such that

$$\int_0^\rho \frac{\phi(V;t)^{1/2}}{t}\, dt < +\infty.$$

Definition 2.6. (Morrey spaces) Let ω be a A_2 weight and let $B_R(x_0)$ be a metric ball such that $\Omega \subset B_R(x_0)$. If V is a locally integrable function in Ω we say that V belongs to the *Morrey space* $M_\sigma(\Omega, \omega)$, for some $\sigma \in \mathbb{R}$, if

$$\|V\|_{M_\sigma(\Omega,\omega)} = \sup_{x\in\Omega} \frac{1}{r^\sigma} \int_{\{y\in\Omega\,:\,d(x,y)<r\}} |V(y)| \int_{d(x,y)}^{4R} \frac{s^2}{\omega(B(x,s))} \frac{ds}{s} \omega(y)\, dy < +\infty\,.$$

Remark 2.7. If $\omega = 1$ the previous definitions give back the classical Stummel-Kato class (see [1]) and Morrey space $L^{1,\lambda}$ for some λ. It is easy to check that $M_\sigma \subset S' \subset S \subset \tilde{S}$, $\sigma > 0$.

Lemma 2.8. *Let $V \in \tilde{S}(\Omega, \omega)$. Then there exists a positive constant $\overline{C}_d$ such that, for all $r > 0$,*

$$\phi(V; r) \le \overline{C}_d \phi\left(V; \frac{r}{2}\right)\,.$$

Proof. See [11]. □

The following results will be useful in the sequel.

Theorem 2.9. *Let ω be a A_2 weight and let $B_R(x_0)$ be a metric ball such that $\Omega \subset B_R(x_0)$. Let V be a locally integrable function such that $\frac{V}{\omega} \in S(\Omega, \omega)$.*

Then for any $\varepsilon > 0$ there exists a positive function

$$K(\varepsilon) \sim \frac{\varepsilon}{\left[\phi^{-1}(\frac{V}{\omega}; \varepsilon)\right]^{Q+2}}$$

such that, for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |V(x)|\, |u(x)|^2\, dx \le \varepsilon \int_\Omega |Xu(x)|^2 \omega\, dx + K(\varepsilon) \int_\Omega |u(x)|^2 \omega\, dx.$$

Proof. See Lemma 4.3 in [23]. □

Now we recall the following two lemmata proved in [26].

Lemma 2.10. *Let $0 < \gamma < 1$, $h\colon\ ]0, +\infty[\ \to\ ]0, +\infty[$ a non decreasing function with $\lim_{t\to 0} h(t) = 0$. Let us assume that $h(t) \le c\, h(t/2)$ for some constant $c > 1$. Moreover, let $\varphi\colon\ ]0, +\infty[\ \to\ ]0, +\infty[$ a non decreasing function such that*

$$\varphi(\rho) \le \gamma\varphi(4\rho) + h(\rho) \qquad \forall \rho < \rho_0 < 1. \tag{5}$$

Then, there exist $\overline{\rho} \le \rho_0$, $0 < \sigma \le 1$ and a positive constant K such that

$$\varphi(\rho) \le K h^\sigma(\rho) \qquad \forall \rho < \overline{\rho}.$$

Lemma 2.11. *Let $\mu(r)$ be a continuous positive increasing function defined in $]0,+\infty[$ such that $\lim_{r\to 0}\mu(r)=0$, $0<\theta<1$. The series*

$$\sum_{i=0}^{+\infty} \theta^i log\, \mu^{-1}\left(\theta^{qi}\right),$$

where $q>0$, is convergent if and only if there exists $\rho>0$ such that

$$\int_0^\rho \frac{\mu^{\frac{1}{q}}(t)}{t}\, dt < +\infty.$$

3. Harnack inequality for variational linear equations

Let Ω be a bounded domain in $\mathbb{R}^n$ and $\omega \in A_2$. We denote by X_j^* the formal adjoint of X_j for all $j=1,2,\dots,m$. Let $\{a_{ij}(x)\}$ be a symmetric matrix of measurable functions in Ω satisfying the following ellipticity condition

$$\exists\lambda>0\,:\lambda^{-1}\omega(x)|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda\omega(x)|\xi|^2 \text{ a.e. } x\in\Omega \quad \forall\xi\in\mathbb{R}^m\,. \tag{6}$$

Consider the following strongly degenerate elliptic equation in divergence form

$$-X_j^*(a_{ij}X_iu+d_ju)+\frac{b_0}{\lambda}\omega|Xu|^2+b_iX_iu+cu=f-X_i^*h_i\,, \tag{7}$$

where
$$b_0\in\mathbb{R}\setminus\{0\}\,,\left(\frac{b_i}{\omega}\right)^2,\frac{c}{\omega},\left(\frac{d_i}{\omega}\right)^2,\frac{f}{\omega},\left(\frac{h_i}{\omega}\right)^2\in S'(\Omega,\omega)\,. \tag{8}$$

Definition 3.1. We say that $w\in W^{1,2}_{loc}(\Omega,\omega)$ is a *local weak supersolution* of (7) if for all $\varphi\in W^{1,2}_0(\Omega,\omega)$ with $\varphi\ge 0$ we have

$$\int_\Omega (a_{ij}X_iw+d_jw)X_j\varphi+\left(\frac{b_0}{\lambda}\omega|Xw|^2+b_iX_iw+cw\right)\varphi dx \ge \int_\Omega (f\varphi+h_iX_i\varphi)\,dx.$$

We say that $w\in W^{1,2}_{loc}(\Omega,\omega)$ is a *local weak subsolution* of (7) if $-w$ is a supersolution. We say that $w\in W^{1,2}_{loc}(\Omega,\omega)$ is a *local weak solution* of (7) if w is a supersolution and a subsolution.

Now we prove the weak Harnack inequality for supersolutions.

Theorem 3.2. *Let w be a weak nonnegative supersolution of equation (7) in a ball $B_{3r}\subset\subset\Omega$. Assume (6) and (8). Let $M>0$ be a constant such that $w\le M$ in B_{3r}. Then there exists C depending on Q, M, λ and the A_2 constant of ω, such that*

$$\omega^{-1}(B_{2r})\int_{B_{2r}} w\,\omega dx \le C\left\{\min_{B_r} w+\phi\left(\frac{f}{\omega};3r\right)+\left(\sum_{i=1}^m \phi\left(\left(\frac{h_i}{\omega}\right)^2;3r\right)\right)^{1/2}\right\}.$$

Proof. Let $k = \phi\Big(\frac{f}{\omega}; 3r\Big) + \Big(\sum_{i=1}^{m} \phi\Big(\Big(\frac{h_i}{\omega}\Big)^2; 3r\Big)\Big)^{1/2}$ and suppose $v = w + k$.

We take $\varphi(x) = \eta^2(x)v^\beta(x)e^{-|b_0|v(x)}$, $\beta < 0$ as test function, where $\eta \in C_0^1(B_{3r})$, $\eta \geq 0$. Since w is supersolution in B_{3r} of (7) we have

$$\int_{B_{3r}} \Big[2\eta(a_{ij}X_iw + d_jw - h_j)X_j\eta v^\beta e^{-|b_0|v} + \\ + (-|\beta|v^{\beta-1} - |b_0|v^\beta)\eta^2 e^{-|b_0|v}(a_{ij}X_iw + d_jw - h_j)X_jv + \\ + \frac{b_0}{\lambda}\omega|Xw|^2\eta^2 v^\beta e^{-|b_0|v} + (b_iX_iw + cw - f)\eta^2 v^\beta e^{-|b_0|v}\Big]dx \geq 0\,,$$

and

$$\int_{B_{3r}} \eta^2 e^{-|b_0|v}(b_0v^\beta + |\beta|v^{\beta-1})|Xv|^2\omega\, dx \leq \tag{9}$$

$$\leq \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|b_0|v^\beta + |\beta|v^{\beta-1})|Xv|^2\omega\, dx \leq \\ \leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|b_0|v^\beta + |\beta|v^{\beta-1})a_{ij}X_ivX_jv\, dx \leq \\ \leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|\beta|v^{\beta-1} + |b_0|v^\beta)(h_j - d_jv)X_jvdx + \\ + 2\lambda \int_{B_{3r}} \eta(a_{ij}X_iv + d_jv - h_j)X_j\eta v^\beta e^{-|b_0|v}dx + \\ + \int_{B_{3r}} b_0\omega|Xv|^2\eta^2 v^\beta e^{-|b_0|v}dx + \lambda \int_{B_{3r}} (b_iX_iv + cv - f)\eta^2 v^\beta e^{-|b_0|v}\, dx\,.$$

From (9) it follows

$$\int_{B_{3r}} \eta^2 e^{-|b_0|v}|\beta|v^{\beta-1}|Xv|^2\omega dx \leq \\ \leq \lambda \int_{B_{3r}} \eta^2 e^{-|b_0|v}(|\beta|v^{\beta-1} + |b_0|v^\beta)(h_j - d_jv)X_jvdx + \\ + 2\lambda \int_{B_{3r}} \eta(a_{ij}X_iv + d_jv - h_j)X_j\eta v^\beta e^{-|b_0|v}dx + \\ + \lambda \int_{B_{3r}} (b_iX_iv + cv - f)\eta^2 v^\beta e^{-|b_0|v}dx\,.$$

Since v is bounded we may drop the exponential to obtain

$$\int_{B_{3r}} \eta^2|\beta|v^{\beta-1}|Xv|^2\omega\, dx \leq \\ \leq C(M, b_0)\Big[2\lambda \int_{B_{3r}} \eta a_{ij}X_ivX_j\eta v^\beta dx + \lambda|\beta| \int_{B_{3r}} |d_j||X_jv|v^\beta\eta^2 dx + \\ + 2\lambda \int_{B_{3r}} |d_j|v^{\beta+1}X_j\eta\eta\, dx + 2\lambda \int_{B_{3r}} |h_j|v^\beta X_j\eta\eta\, dx + \lambda \int_{B_{3r}} |b_i||X_iv\eta^2 v^\beta +$$

$$+\lambda\int_{B_{3r}}|c|\eta^2v^{\beta+1}dx+\lambda\int_{B_{3r}}|f|\eta^2v^{\beta}dx+$$
$$+\lambda|\beta|\int_{B_{3r}}h_jX_jvv^{\beta-1}\eta^2dx+\lambda\int_{B_{3r}}|d_j||v_{x_i}|\eta^2v^{\beta}dx\Big].$$

Now, set $V=\sum_{i=1}^n\frac{|b_i|^2}{\omega}+|c|+\sum_{j=1}^n\frac{|d_j|^2}{\omega}+k^{-1}|f|+k^{-2}\sum_{i=1}^n\frac{|h_i|^2}{\omega}$.

Use of the Young inequality yields

$$\begin{aligned}\int_{B_{3r}}\eta^2v^{\beta-1}|Xv|^2\omega dx\le&\\ \le C(M,b_0,\lambda)&\left[\frac{|\beta|+1}{\beta^2}\int_{B_{3r}}v^{\beta+1}|X\eta|^2\omega\,dx+\left(\frac{|\beta|+1}{\beta}\right)^2\int_{B_{3r}}V\eta^2v^{\beta+1}dx\right]\le\\ \le C(M,b_0,\lambda)&\left(\frac{|\beta|+1}{\beta}\right)^2\left[\int_{B_{3r}}v^{\beta+1}|X\eta|^2\omega\,dx+\int_{B_{3r}}V\eta^2v^{\beta+1}dx\right].\end{aligned}\tag{10}$$

We set $\mathcal{U}(x)=\begin{cases}v^{\frac{\beta+1}{2}}(x) & \text{if } \beta\neq-1\\ \log v(x) & \text{if } \beta=-1.\end{cases}$ By (10) we have

$$\begin{aligned}&\int_{B_{3r}}\eta^2|X\mathcal{U}|^2\omega(x)\,dx\le\\ &\le C(\beta+1)^2\left(\frac{|\beta|+1}{\beta}\right)^2\left\{\int_{B_{3r}}|X\eta|^2\mathcal{U}^2\omega(x)dx+\int_{B_{3r}}V\eta^2\mathcal{U}^2\,dx\right\}\end{aligned}\tag{11}$$

in the case $\beta\neq-1$, while

$$\int_{B_{3r}}\eta^2|X\mathcal{U}|^2\omega\,dx\le C\left\{\int_{B_{3r}}|X\eta|^2\omega\,dx+\int_{B_{3r}}V\eta^2\,dx\right\}\tag{12}$$

if $\beta=-1$. Let us start with the case $\beta=-1$. By Theorem 2.9 we have

$$\int_{B_{3r}}\eta^2|X\mathcal{U}|^2\omega\,dx\le C\left(\int_{B_{3r}}|X\eta|^2\omega\,dx+\int_{B_{3r}}\eta^2\omega\,dx\right).$$

Let B_h be a ball contained in B_{2r}. Choosing $\eta(x)$ so that $\eta(x)=1$ in B_h, $0\le\eta\le1$ in $B_{3r}\setminus B_h$ and $|X\eta|\le\frac{C}{h}$, we get

$$\|X\mathcal{U}\|_{L^2(B_h,\omega)}\le C\frac{\omega(B_h)^{\frac12}}{h}.$$

By Sobolev's Theorem 2.4 and John-Nirenberg's Lemma (see [2]) we obtain $\mathcal{U}=\log v\in BMO$. Then there exist two positive constants p_0 and C, such that

$$\left(\fint_{B_{2r}}e^{p_0\mathcal{U}}\omega\,dx\right)^{\frac{1}{p_0}}\left(\fint_{B_{2r}}e^{-p_0\mathcal{U}}\omega\,dx\right)^{\frac{1}{p_0}}\le C.\tag{13}$$

Let us consider the following family of seminorms

$$\Phi(p,h)=\left(\int_{B_h}|v|^p\omega(x)\,dx\right)^{1/p},\quad p\neq 0\,.$$

By (13) we have

$$\frac{1}{\omega(B_{2r})^{1/p_0}}\Phi(p_0,2r)\le C\omega(B_{2r})^{1/p_0}\Phi(-p_0,2r)\,.$$

In the case (11) by Theorem 2.9 we obtain

$$\int_{B_{3r}}|X\mathcal{U}|^2\eta^2\omega\,dx\le C\Bigg\{\left[\left(\frac{\beta+1}{2}\right)^2+1\right]\left(1+\frac{1}{|\beta|}\right)^2\int_{B_{3r}}|X\eta|^2\mathcal{U}^2\omega\,dx+$$
$$+\left[\frac{1}{\phi^{-1}\left(\frac{V}{\omega};\left(\frac{\beta+1}{2}\right)^{-2}\left(1+\frac{1}{|\beta|}\right)^{-2}\right)}\right]^{Q+2}\int_{B_{3r}}\eta^2\mathcal{U}^2\omega\,dx\Bigg\}\,. \tag{14}$$

By Sobolev's inequality we have

$$\left(\int_{B_{3r}}|\eta\mathcal{U}|^{\tau p}\omega\,dx\right)^{\frac{1}{\tau}}\le$$
$$\le C\,\omega(B_{3r})^{\frac{1}{\tau}-1}\Bigg\{\left[\left(\frac{\beta+1}{2}\right)^2+2\right]\left(1+\frac{1}{|\beta|}\right)^2\int_{B_{3r}}|X\eta|^2\mathcal{U}^2\omega\,dx+$$
$$+\left[\frac{1}{\phi^{-1}\left(\frac{V}{\omega};\left(\frac{\beta+1}{2}\right)^{-2}\left(1+\frac{1}{|\beta|}\right)^{-2}\right)}\right]^{Q+2}\int_{B_{3r}}\eta^2\mathcal{U}^2\omega\,dx\Bigg\} \tag{15}$$

where c is a positive constant independent of ω.

Now we choose the function η. Let r_1 and r_2 be real numbers such that $r\le r_1<r_2\le 2r$ and let the function η be chosen so that $\eta(x)=1$ in B_{r_1}, $0\le\eta(x)\le 1$ in B_{r_2}, $\eta(x)=0$ outside B_{r_2}, $|X\eta|\le\frac{c}{r_2-r_1}$ for some fixed constant c. We have

$$\left(\int_{B_{r_1}}\mathcal{U}^{2\tau}\omega(x)\,dx\right)^{\frac{1}{\tau}}\le C\,\omega(B_{3r})^{\frac{1}{\tau}-1}\frac{1}{(r_2-r_1)^2}\left[\left(\frac{\beta+1}{2}\right)^2+2\right]\cdot$$
$$\cdot\left(1+\frac{1}{|\beta|}\right)^2\left[\frac{1}{\phi^{-1}\left(\frac{V}{\omega};\left(\frac{\beta+1}{2}\right)^{-2}\left(1+\frac{1}{|\beta|}\right)^{-2}\right)}\right]^{Q+2}\int_{B_{r_2}}\mathcal{U}^2\omega(x)\,dx\,.$$

Setting $\gamma = \beta + 1$ and recalling that $\mathcal{U}(x) = v^{\frac{\beta+1}{2}}(x)$, we get for negative γ

$$\Phi(\tau\gamma, r_1) \geq C^{\frac{1}{\gamma}} \omega(B_{3r})^{\frac{1}{\gamma}(\frac{1}{\tau}-1)} \left[\left(\frac{\beta+1}{2} \right)^2 + 2 \right]^{\frac{1}{\gamma}} \cdot$$

$$\cdot \left[\frac{1}{\phi^{-1}\left(\frac{V}{\omega}; \left(\frac{\beta+1}{2}\right)^{-2} \right)} \right]^{\frac{Q+2}{\gamma}} \frac{1}{(r_2 - r_1)^{\frac{2}{\gamma}}} \Phi(\gamma, r_2) \,. \tag{16}$$

This is the inequality we are going to iterate. If $\gamma_i = \tau^i p_0$ and $r_i = r + \frac{r}{2^i}$, $i = 1, 2, \dots$, iteration of (16) and use of Lemma 2.11 yields

$$\Phi(-\infty, r) \geq C(\phi_{\frac{V}{\omega}}, \operatorname{diam}\Omega) \omega(B_{3r})^{\frac{1}{p_0}} \Phi(-p_0, 2r) \,.$$

Therefore by Hölder inequality,

$$\Phi(p_0', 2r) \leq \Phi(p_0, 2r) \omega(B_{3r})^{\frac{1}{p_0'} - \frac{1}{p_0}} \,, \quad p_0' \leq p_0 \,.$$

So we obtain $\omega^{-1}(B_{2r})\Phi(1, 2r) \leq C\,\Phi(-\infty, r)$ and the result follows. □

We state the following weak Harnack inequality for subsolutions.

Theorem 3.3. *Let w be a weak nonnegative subsolution of (7) in $B_{3r} \subset\subset \Omega$. Assume (6) and (8). Let $M > 0$ be a constant such that $w \leq M$ in B_{3r}. Then there exists C depending on Q, M, λ and the A_2 constant of ω, such that*

$$\max_{B_r} w \leq C \left\{ \omega^{-1}(B_{2r}) \int_{B_{2r}} w\, \omega dx + \phi\left(\frac{f}{\omega}; 3r \right) + \left(\sum_{i=1}^m \phi\left(\left(\frac{h_i}{\omega}\right)^2; 3r \right) \right)^{1/2} \right\}.$$

From previous results we obtain

Theorem 3.4. *Let w be a weak nonnegative solution of (7) in $B_{3r} \subset\subset \Omega$. Assume (6) and (8). Let M be a constant such that $w \leq M$ in B_{3r}. Then there exists C depending on Q, M, λ and the A_2 constant of ω such that*

$$\max_{B_r} w \leq C \left\{ \min_{B_r} w + \phi\left(\frac{f}{\omega}; 3r \right) + \left(\sum_{i=1}^m \phi\left(\left(\frac{h_i}{\omega}\right)^2; 3r \right) \right)^{1/2} \right\}.$$

Now we prove that weak solutions of (7) are continuous with respect to the Carnot-Carathéodory metric.

Theorem 3.5. *Let $u \in W^{1,2}(\Omega, \omega)$ be a weak solution of (7) such that (6) and (8) hold true and $\sup_\Omega |u| = L < +\infty$. Then u is continuous in Ω.*

Proof. Let B_r be an arbitrary ball contained in Ω and the functions

$$M = M(r) = \max_{B_r} u \quad \text{and} \quad m = m(r) = \min_{B_r} u\,.$$

If we set $\overline{u} = M(r) - u$, $\overline{u}$ is a non negative weak solution of the equation

$$-X_j^*(a_{ij}X_i u - d_j u) + \frac{b_0}{\lambda}\omega|Xu|^2 + b_i X_i u + cu = (Mc - f) - X_i^*(Md_i - h_i)$$

in B_r, with

$$\frac{Mc - f}{\omega}, \left(\frac{Md_i - h_i}{\omega}\right)^2 \in S'(\Omega, \omega)\,.$$

By Harnack's inequality we obtain

$$\max_{B_{\frac{r}{3}}} \overline{u} \leq C\left(\min_{B_{\frac{r}{3}}} \overline{u} + \overline{h}\right)$$

and $$\overline{h} = \overline{h}(r) = \phi\left(\frac{\overline{f}}{\omega}; r\right) + \left(\sum_{i=1}^{m} \phi\left(\left(\frac{\overline{h}_i}{\omega}\right)^2; r\right)\right)^{1/2},$$

where $\overline{f} = Mc - f$ and $\overline{h}_i = Md_i - h_i$. We remark that $\overline{h}$ is a positive non decreasing function such that $\lim_{r\to 0} \overline{h}(r) = 0$ and (see Lemma 2.8)

$$\overline{h}(r) \leq \frac{1}{K}\overline{h}\left(\frac{r}{2}\right), \quad 0 < K < 1\,. \tag{17}$$

Then $$M(r) - m\left(\frac{r}{3}\right) \leq C\left\{M(r) - M\left(\frac{r}{3}\right) + \overline{h}(r)\right\}. \tag{18}$$

In the same way, setting $\overline{\overline{u}} = u - m(r)$ we obtain

$$M\left(\frac{r}{3}\right) - m(r) \leq C\left\{m\left(\frac{r}{3}\right) - m(r) + \overline{h}(r)\right\}. \tag{19}$$

Adding (18) and (19) we get

$$M\left(\frac{r}{3}\right) - m\left(\frac{r}{3}\right) \leq \frac{C-1}{C+1}[M(r) - m(r)] + \frac{2C}{C+1}K^2\overline{h}\left(\frac{r}{4}\right).$$

Set, for $\rho > 0$, $\varphi(\rho) = M(\rho) - m(\rho)$, and $\theta = \frac{C-1}{C+1}$, $h(r) = \frac{2C}{C+1}K^2\overline{h}(r)$. We have

$$\varphi\left(\frac{r}{4}\right) \leq \varphi\left(\frac{r}{3}\right) \leq \theta\varphi(r) + h\left(\frac{r}{4}\right)$$

and the conclusion follows by Lemma 2.10. □

The next result is a natural consequence of the previous one if we assume the lower order terms to belong to the Morrey classes M_σ. Namely

Theorem 3.6. *Let $u \in W^{1,2}(\Omega, \omega)$ be a weak solution of* (7) *such that* (6) *holds true,*

$$\left(\frac{b_i}{\omega}\right)^2, \frac{c}{\omega}, \left(\frac{d_i}{\omega}\right)^2, \frac{f}{\omega}, \left(\frac{h_i}{\omega}\right)^2 \in M_\sigma(\Omega, \omega)\,,$$

and $\sup_\Omega |u| = L < +\infty$. *Then u is locally Hölder continuous in* Ω.

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