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Optimal transport-based mathematical models for resource flow regulation in networked engineering systems

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Abstract:

In this paper, develop a new mathematical formulation using the concept of optimal transport to regulate resource flows in interconnected engineering networks. Specifically, this formulation represents supply and demand distributions defined on the connected nodes and computes a feasible transport plan based on the network structure, the cost of using the shortest path, flow conservation, and capacity constraints. Kantorovich duality theory is used to obtain the optimal regulation cost and interpret the dual variables as decision-making signals at the network level. A Wasserstein distance-based approach is used to analyze the stability of the transport cost with respect to the demand perturbation. An entropic regularization method and Sinkhorn algorithm are considered to reduce the computational complexity of the optimal transport problem for large and sparse networks. In a numerical experiment conducted on a 100-node radial distribution network with 30 supply nodes and 70 demand nodes, the proposed Sinkhorn algorithm can achieve a transport cost of 0.40% deviation from the exact optimal solution computed by linear programming solver in 74 iterations and 0.18 s of computing time. For the same problem, the exact network simplex method takes 2.90 s of computing time which corresponds to an approximately 16-times improvement of the computational time compared with the exact optimal solution. Under randomized ±10% demand perturbations in 500 runs, the theoretical stability bound stays within a factor of 1.4 from the worst cost change of all demand perturbations. It is further shown that the optimal gap decreases from 0.92% to 0.15% when decreasing the regularization coefficient from 0.10 to 0.02, but the iteration number grows from 51 to 146.

Keywords: Optimal transport, resource flow regulation, networked engineering systems, Wasserstein distance, Sinkhorn algorithm, network optimization.

1. Introduction

Efficient resource flow management is a key requirement in the contemporary engineering systems, which feature

networked components that constantly trade energy, materials, information, communication resources, and transportation capabilities under varying operating conditions. The recent advancements show that optimal transport models based on geometry and structure can identify inefficiencies in complex supply chain networks by taking into account network topology and resource properties (Seyedi et al., 2026). In the same way, optimal transport models have been utilized for joint allocation of communication and localization resources in sensing and communications networks, showcasing their ability to allocate multiple types of resources within a network (Zheng et al., 2025). The optimal transport models have also been combined with federated reinforcement learning in order to optimize adaptive resource allocation in cloud-edge collaborative IoT systems (Gan et al., 2023). In transportation systems, the optimal transport routing has helped to design a consistent infrastructure across multiple layers while enhancing connectivity and flow efficiency (Lonardi et al., 2025). The self-optimizing cognitive methods also reveal the significance of intelligent approaches in terms of computational and communication resource management in 6G edge environments (Kollinal & Cheeran, 2025). The interplay of network similarity, economies of scale, and optimal transport-based infrastructure design also prove the significance of optimal transport theory in terms of understanding the resource distribution patterns (Leite & De Bacco, 2024). Finally, digital logistics networks have been used to assess sustainability and efficiency of the energy resource transportation through interconnected supply chains (Barykin et al., 2024).

The mathematical modeling of network flow is fundamental for modeling the flow, distribution, and control of resource within engineering infrastructures (Zaitseva et al., 2020). Optimizations and heuristic techniques have also been developed for optimizing the scheduling and use of resources within industrial systems (Talwar & Shah, 2023). The multi-commodity formulation of network flow models extends this scope and is useful in capturing multiple resource types and their interrelations within the context of dynamic energy management problems (Manfren, 2012). Mathematical optimization is also proven to be useful in controlling the operation of complex engineering infrastructures including sewer networks, where the movement of resource should obey certain physical and operational constraints (Ermolin, 1999). The recent development of the combination of artificial intelligence and knowledge engineering, as well as mathematical models, has been applied for intelligent operations decision making and incident management within enterprise systems (Peralta et al., 2025). Stochastic optimization models are also developed for increasing the robustness and optimizing of interconnected industrial Internet of Things and supply chain systems in uncertain environments (Selwin Mich Priyadharson et al., 2025). Mathematical modeling of advanced wireless networks is another example of importance of having rigorous modeling for capturing the characteristics of resource behavior and connectivity in highly complex communication systems (Moltchanov et al., 2022).

Energy Systems In the area of energy systems, mathematical power flow models provide structured descriptions for analysis of the distribution of resources and operating conditions in electrical systems, whereas the optimization model formulations facilitate the effective control and management of power flows in distribution systems (Fobes et al., 2020; Yang et al., 2023). Moreover, the application of multi-agent systems and complex network theory has allowed an increase in modeling capabilities of the decentralized behavior and decision making of interacting interconnected engineering systems (Herrera et al., 2020). In addition, reinforcement learning models have been developed for dynamic association and resource allocation in heterogeneous cellular networks (Reddy & Kamalanathan, 2026). Learning algorithms based on graphs have been applied to predict urban traffic flows and make resource-aware decisions in distributed computing environments (Ali et al., 2024). Finally, mathematical control models of frequency and power control in energy systems illustrate the need for a continuous balance between the generation, demand and renewable energy sources in dynamic power networks (Denysov, 2025).

Nevertheless, current studies are still concentrating more on application fields like supply chains, wireless communications, transportation, Internet-of-Things, traffic networks, or electrical power systems (Ganesan, 2026). Developing a generalized mathematical model that can regulate resource flows within different networked engineering systems and at the same time consider factors such as transport cost, network topology, capacity constraint, flow conservation, and dynamic resource imbalances is still an open research question. Hence, this study suggests a mathematical model for resource flow regulation in networked engineering systems based on optimal transport theory. Through the proposed model, redistribution of resources is modeled as an optimal transport problem under certain constraints to facilitate intelligent resource movement between supply and demand nodes while minimizing transport cost and ensuring the feasibility of the network.

Key Contribution

- The research creates a mathematical model that views resource allocation and reallocation as an optimal transport problem considering the following factors: network structure, supply and demand distribution, conservation of flow, cost of transport, and capacities of channels.

- The created model uses constraints that are specific to engineering applications and control the resource flow under conditions of variable supply, demand, congestion, and operation of the network.

- The research defines a mathematical formulation that can be tailored for various networked engineering applications: smart grids, transportation networks, telecommunications networks, logistics systems, and industries.

The rest of the paper is structured in the following way. Section II contains the preliminaries and the mathematical notation to be used for the optimal transport-based resource flow regulation problem formulation. Section III deals with the theoretical background of the framework such as the concept of Kantorovich duality, Wasserstein stability analysis, and convergence and complexity issues of the entropic regularization technique. Section IV provides the numerical evaluation of the proposed method under the conditions of network resources regulation. The conclusions regarding the results obtained and their possible implications along with the trade-off between the efficiency and accuracy of the optimization procedure are discussed in Section V. Finally, Section VI conclude this study.

2. Preliminaries and notations

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph with $|\mathcal{V}| = n$, representing an engineering network with edge costs $c: \mathcal{E} \to \mathbb{R}_{>0}$ reflecting per-unit congestion or transmission costs, and let $d(u, v)$ denote the shortest-path (graph-geodesic) distance induced by c. Defining source and sink loads as normalized supply and demand probability measures $\mu, \nu \in \Delta^{n-1}$ on $\mathcal{V}$, a coupling matrix $\pi \in \Pi(\mu, \nu)$ represents a joint distribution on $\mathcal{V} \times \mathcal{V}$ belonging to the polyhedral feasible set $\Pi(\mu, \nu) = \{\pi \in \mathbb{R}_+^{n \times n} \mid \pi\mathbf{1} = \mu,\ \pi^\top \mathbf{1} = \nu\}$.

2.1. Notation table

The following notation used in this research referred as table1.

Table 1: Notation used for this research.

Symbol	Meaning
$G = (V, E), n = \lvert V \rvert$	Network graph and node count
$c(e), d(u, v)$	Edge cost and induced shortest-path cost
$\mu, \nu \in \Delta\{n-1\}$	Supply and demand measures on nodes
$\Pi(\mu, \nu)$	Set of couplings (transport plans) with marginals μ, ν
$\pi \in \Pi(\mu, \nu)$	A feasible resource-flow coupling
$Wc(\mu, \nu)$	Optimal regulation (transport) cost between μ, ν
$\varphi, \psi : V \to R$	Kantorovich dual potentials
$\varepsilon > 0$	Entropic regularization parameter
$H(\pi)$	Shannon entropy of coupling π

2.2. Problem statement

The resource flow regulation problem is formulated as the discrete optimal transport program:

$$\text{(RT)} \quad W_c(\mu, \nu) = \min_{\pi \in \Pi(\mu, \nu)} \sum_{u, v \in \mathcal{V}} d\,(u, v)\, \pi(u, v)$$

under **Assumption 2.1**, where $d: \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geq 0}$ is a proper metric on $\mathcal{V}$ (satisfying symmetry, non-negativity, and the triangle inequality) induced by non-negative edge costs c, and the supply-demand distributions $\mu, \nu \in \Delta^{n-1}$ have full support on $\mathcal{V}$ (i.e., $\mu(u) > 0$ and $\nu(v) > 0$ for all $u, v \in \mathcal{V}$).

3. Theoretical results

3.1. Kantorovich duality

Theorem 3.1 (Duality for Regulated Resource Flow). Under Assumption 2.1, the regulated-flow problem (RT) admits the dual representation $W_c(\mu,\nu) = \max\{\sum_{u\in\mathcal{V}} \phi(u)\mu(u) + \sum_{v\in\mathcal{V}} \psi(v)\nu(v) \mid \phi(u) + \psi(v) \le d(u,v)\ \forall u,v \in \mathcal{V}\}$, where the maximum is attained at an optimal pair (ϕ^*,ψ^*) satisfying $\psi^* = \phi^{*,d}$ (the d-transform/conjugate); moreover, ϕ^* can be chosen to be 1-d-Lipschitz on $\mathcal{V}$.

Proof of Theorem 3.1. The primal optimization problem (RT) is a finite linear program that minimizes a linear function of π over the transportation polytope $\Pi(\mu,\nu)$, defined by $2n$ linear equality constraints (with one redundant constraint) under non-negativity bounds $\pi \ge 0$. Assigning Lagrange multipliers $\phi(u)$ to the supply constraints $(\pi\mathbf{1})(u) = \mu(u)$ and $\psi(v)$ to the demand constraints $(\pi^\top\mathbf{1})(v) = \nu(v)$ yields the LP dual:

$$\max_{\phi,\psi}\{\textstyle\sum_{u\in\mathcal{V}} \phi(u)\mu(u) + \sum_{v\in\mathcal{V}} \psi(v)\nu(v) \mid \phi(u) + \psi(v) \le d(u,v) \quad \forall u,v \in \mathcal{V}\}$$

Which matches the claimed dual formulation. Because $\mu \otimes \nu \in \Pi(\mu,\nu)$, the feasible polytope $\Pi(\mu,\nu)$ is non-empty and bounded, guaranteeing that the primal LP attains a finite optimal value. By LP strong duality, the dual program is feasible and attains the same optimal value $W_c(\mu,\nu)$, proving equality and dual attainment.

Given any optimal dual pair (ϕ,ψ), define the d-conjugate $\phi^*(u) = \inf_{v\in\mathcal{V}}[d(u,v) - \psi(v)]$. Feasibility requires $\psi(v) \le d(u,v) - \phi(u)$ for all u,v, which implies $\phi^*(u) \ge \phi(u)$ everywhere on $\mathcal{V}$. Thus, (ϕ^*,ψ) is dual-feasible and yields a dual objective value at least as large as (ϕ,ψ). Optimality of (ϕ,ψ) forces the objective values to coincide, establishing that ϕ^* is also an optimal potential. Moreover, for any $u,u' \in \mathcal{V}$, the triangle inequality for d implies:

$$\phi^*(u) - \phi^*(u') = \inf_{v\in\mathcal{V}}[d(u,v) - \psi(v)] - \inf_{v\in\mathcal{V}}[d(u',v) - \psi(v)] \le \sup_{v\in\mathcal{V}}[d(u,v) - d(u',v)] \le d(u,u')$$

confirming that ϕ^* is 1-d-Lipschitz. Constructing $\psi^* = \phi^{*,d}(v) = \inf_{u\in\mathcal{V}}[d(u,v) - \phi^*(u)]$ symmetrically completes the optimal conjugate pair (ϕ^*,ψ^*).

3.2. Wasserstein Stability

Theorem 3.2 (Stability of Regulation Cost). Under Assumption 2.1, for any two demand measures $\nu_1,\nu_2 \in \Delta^{n-1}$ under a fixed supply measure μ, the difference in optimal transport regulation cost satisfies the total-variation bound $|W_c(\mu,\nu_1) - W_c(\mu,\nu_2)| \le \mathrm{diam}_d(\mathcal{V}) \cdot \|\nu_1 - \nu_2\|_{\mathrm{TV}}$, and, more sharply, $|W_c(\mu,\nu_1) - W_c(\mu,\nu_2)| \le W_c(\nu_1,\nu_2)$.

Proof of Theorem 3.2. Because d is a metric on $\mathcal{V}$, the 1-Wasserstein distance W_c defines a metric on the probability simplex Δ^{n-1} (a direct consequence of the Kantorovich LP-duality representation established in Theorem 3.1). Applying the metric triangle inequalities $W_c(\mu,\nu_1) \le W_c(\mu,\nu_2) + W_c(\nu_2,\nu_1)$ and $W_c(\mu,\nu_2) \le W_c(\mu,\nu_1) + W_c(\nu_1,\nu_2)$ yields the sharp stability estimate:

$$|W_c(\mu,\nu_1) - W_c(\mu,\nu_2)| \le W_c(\nu_1,\nu_2)$$

To establish the total-variation bound, evaluate the difference in optimal transport values using the dual potential pair (ϕ^*,ψ^*) associated with $W_c(\mu,\nu_1)$ from Theorem 3.1, where ϕ^* is 1-d-Lipschitz:

$$\begin{aligned} W_c(\mu,\nu_1) - W_c(\mu,\nu_2) &\le \sum_{v\in\mathcal{V}} \psi^*(v)\nu_1(v) - \sum_{v\in\mathcal{V}} \psi^*(v)\nu_2(v) \\ &= \sum_{v\in\mathcal{V}} \psi^*(v)\big(\nu_1(v) - \nu_2(v)\big) \\ &\le \mathrm{osc}(\psi^*) \cdot \|\nu_1 - \nu_2\|_{\mathrm{TV}} \\ &\le \mathrm{diam}_d(\mathcal{V}) \cdot \|\nu_1 - \nu_2\|_{\mathrm{TV}} \end{aligned}$$

Where $\mathrm{osc}(\psi^*) = \max_v \psi^*(v) - \min_v \psi^*(v) \le \mathrm{diam}_d(\mathcal{V}) = \max_{u,v\in\mathcal{V}} d(u,v)$ follows from the 1-d-Lipschitz property of conjugate dual potentials. Exchanging ν_1 and ν_2 symmetrically establishes the upper and lower bounds, completing the proof.

Theorem 3.2 provides a quantitative measure of the robustness of the regulation cost with respect to errors in the demand prediction, which states that a small perturbation of the demand distribution in total variation norm, or even Wasserstein distance, can change the optimum regulation cost by at most a constant time the perturbation size, $\mathrm{diam}_d(V)$.

3.3. Convergence, complexity and analysis

To address the computational complexity of solving the discrete optimal transport program (RT)—which scales as $\mathcal{O}(n^3 \log n)$ under standard network-simplex or interior-point methods—consider the entropic-regularized surrogate:

$$(\mathrm{RT}_\varepsilon) \quad W_{c,\varepsilon}(\mu,\nu) = \min_{\pi\in\Pi(\mu,\nu)} \sum_{u,v\in\mathcal{V}} d\,(u,v)\,\pi(u,v) - \varepsilon H(\pi)$$

Where $H(\pi) = -\sum_{u,v} \pi\,(u,v) \ln \pi\,(u,v)$ is the Shannon entropy. Program $(\mathrm{RT}_\varepsilon)$ is efficiently solved via Sinkhorn's matrix-scaling algorithm, generating factorized iterates $\pi^{(k)} = \mathrm{diag}\big(a^{(k)}\big) K \mathrm{diag}\big(b^{(k)}\big)$ with the Gibbs kernel matrix $K = \exp(-d/\varepsilon)$. The dynamic scaling vectors $a, b \in \mathbb{R}^n_{>0}$ are updated via alternating marginal normalization:

$$a^{(k+1)} = \mu \oslash \left(K b^{(k)}\right), \quad b^{(k+1)} = \nu \oslash \left(K^\top a^{(k+1)}\right)$$

By Franklin's matrix-scaling theorem, the Sinkhorn iterates converge globally and linearly in the Hilbert projective metric. The contraction rate is bounded by $\kappa(K) = \frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}$, where $\eta = \max_{i,j,k,l} \frac{K_{ik}K_{jl}}{K_{il}K_{jk}}$, while the regularized value $W_{c,\varepsilon}(\mu,\nu)$ converges to the unregularized distance $W_c(\mu,\nu)$ at rate $\mathcal{O}(\varepsilon \log n)$ as $\varepsilon \to 0$. By exploiting physical graph sparsity to truncate the effective support of K, the per-iteration computational complexity drops from $\mathcal{O}(n^2)$ to $\mathcal{O}\big(n\bar{k}\big)$, where $\bar{k}$ denotes the average number of cost-significant node pairs per subregion. Achieving an accuracy threshold δ requires $\mathcal{O}\left(\frac{\log(1/\delta)}{\log(1/\kappa)}\right)$ iterations at $\mathcal{O}\big(n\bar{k}\big)$ work per iteration, offering a decisive scaling advantage over exact LP solvers for large sparse infrastructure networks.

The contraction factor $\kappa(K)$ exhibits a direct dependence on the regularization parameter ε through $K = \exp(-d/\varepsilon)$: smaller values of ε concentrate K sharply along shortest-path routes, increasing η and pushing $\kappa \to 1^-$, which slows convergence speed. Conversely, larger values of ε smooth K, accelerating linear convergence at the cost of a higher entropic bias $\mathcal{O}(\varepsilon \log n)$. This structural tradeoff between iterations and the degree of approximation is used to calibrate empirically the value of ε in the process of large-scale flow regulation. Also, since each iteration involves the multiplication of vectors by elements of the scarified kernel matrix K, the algorithm is highly scalable and can be parallelized using node clusters.

Theorem 4.1 (Monotonicity and Convexity of Entropic Optimal Transport). Let $W_{c,\varepsilon}(\mu,\nu)$ be the optimal value of the entropic-regularized resource flow problem $(\mathrm{RT}_\varepsilon)$. Under Assumption 2.1, the map $\varepsilon \mapsto W_{c,\varepsilon}(\mu,\nu)$ is monotonically decreasing and strictly concave on $(0,\infty)$. Furthermore, for any fixed distributions $\mu,\nu \in \Delta^{n-1}$, $W_{c,\varepsilon}(\mu,\nu)$ is jointly convex in (μ,ν).

Proof of Theorem 4.1. Let π^*_ε denote the unique optimal coupling for $(\mathrm{RT}_\varepsilon)$ at regularization parameter $\varepsilon > 0$. By definition of $W_{c,\varepsilon}(\mu,\nu) = \langle D, \pi^*_\varepsilon\rangle - \varepsilon H(\pi^*_\varepsilon)$, where $D_{uv} = d(u,v)$, the derivative of the value function with respect to ε evaluated via the Envelope Theorem yields:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} W_{c,\varepsilon}(\mu,\nu) = -H(\pi^*_\varepsilon)$$

Since $\pi^*_\varepsilon \in \Pi(\mu,\nu)$ is a valid joint probability distribution on $\mathcal{V}\times\mathcal{V}$, its Shannon entropy satisfies $H(\pi^*_\varepsilon) = -\sum_{u,v} \pi^*_\varepsilon\,(u,v) \ln \pi^*_\varepsilon\,(u,v) \geq 0$. Thus, $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} W_{c,\varepsilon}(\mu,\nu) \leq 0$, proving that $\varepsilon \mapsto W_{c,\varepsilon}(\mu,\nu)$ is monotonically decreasing. Taking the second derivative yields:

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} W_{c,\varepsilon}(\mu,\nu) = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} H(\pi^*_\varepsilon) = -\mathrm{Var}_{\pi^*_\varepsilon}(\ln \pi^*_\varepsilon) < 0$$

Which establishes strict concavity on $(0,\infty)$.

To establish joint convexity in (μ, ν), consider two supply-demand pairs (μ_0, ν_0) and (μ_1, ν_1) with corresponding optimal couplings π_0^* and π_1^*. For any $\alpha \in [0,1]$, define the convex combination $\mu_\alpha = (1-\alpha)\mu_0 + \alpha\mu_1$ and $\nu_\alpha = (1-\alpha)\nu_0 + \alpha\nu_1$. The linear combination $\pi_\alpha = (1-\alpha)\pi_0^* + \alpha\pi_1^*$ is feasible for $\Pi(\mu_\alpha, \nu_\alpha)$. By linearity of the inner product $\langle D, \cdot \rangle$ and strict concavity of the entropy function $H(\cdot)$:

$$\begin{aligned} W_{c,\varepsilon}(\mu_\alpha, \nu_\alpha) &\leq \langle D, \pi_\alpha \rangle - \varepsilon H(\pi_\alpha) \\ &= (1-\alpha)\langle D, \pi_0^* \rangle + \alpha \langle D, \pi_1^* \rangle - \varepsilon H\big((1-\alpha)\pi_0^* + \alpha\pi_1^*\big) \\ &\leq (1-\alpha)[\langle D, \pi_0^* \rangle - \varepsilon H(\pi_0^*)] + \alpha[\langle D, \pi_1^* \rangle - \varepsilon H(\pi_1^*)] \\ &= (1-\alpha) W_{c,\varepsilon}(\mu_0, \nu_0) + \alpha W_{c,\varepsilon}(\mu_1, \nu_1) \end{aligned}$$

Which proves joint convexity of $W_{c,\varepsilon}(\mu, \nu)$ in (μ, ν).

Theorem 4.2 (Exponential Convergence of Dual Sinkhorn Potentials). Let $\big(\alpha^{(k)}, \beta^{(k)}\big) \in \mathbb{R}^n \times \mathbb{R}^n$ be the dual potentials defined by $\alpha^{(k)} = \varepsilon \ln a^{(k)}$ and $\beta^{(k)} = \varepsilon \ln b^{(k)}$ at iteration k of the Sinkhorn algorithm. Under Assumption 2.1, the sequence of dual potentials $\big(\alpha^{(k)}, \beta^{(k)}\big)$ converges linearly in the Hilbert projective metric d_H to the unique optimal dual pair (α^*, β^*) of $(\mathrm{RT}_\varepsilon)$ satisfying the centering condition $\sum_u \alpha^*(u) = 0$:

$$\|\alpha^{(k)} - \alpha^*\|_\infty + \|\beta^{(k)} - \beta^*\|_\infty \leq M \cdot [\kappa(K)]^k$$

Where $K = \exp(-D/\varepsilon)$, $\kappa(K) = \frac{\sqrt{\eta}-1}{\sqrt{\eta}+1} < 1$ is the Birkhoff contraction coefficient of the linear operator induced by K, and $M > 0$ is a constant depending on the initial scaling vectors $a^{(0)}, b^{(0)}$ and ε.

Proof of Theorem 4.2. The Sinkhorn updates $a^{(k+1)} = \mu \oslash \big(Kb^{(k)}\big)$ and $b^{(k+1)} = \nu \oslash \big(K^\top a^{(k+1)}\big)$ can be rephrased as alternating projection operators on the positive cone $\mathbb{R}^n_{>0}$. Define the nonlinear operator $\mathcal{T}: \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ by $\mathcal{T}(b) = \mu \oslash \Big(K\big(\nu \oslash (K^\top b)\big)\Big)$. The Hilbert projective metric on $\mathbb{R}^n_{>0}$ is defined for any $x, y \in \mathbb{R}^n_{>0}$ by:

$$\mathrm{d}_H(x, y) = \ln\left(\max_{i,j} \frac{x_i y_j}{x_j y_i}\right)$$

By Birkhoff's Theorem, a positive linear operator $K: \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ is a strict contraction in the Hilbert metric d_H, with contractivity factor $\kappa(K) = \frac{\sqrt{\eta}-1}{\sqrt{\eta}+1} < 1$, where $\eta = \max_{i,j,k,l} \frac{K_{ik}K_{jl}}{K_{il}K_{jk}} < \infty$ under full-support Assumption 2.1. Since entrywise inversion and vector scaling by fixed positive vectors (μ, ν) are isometric self-maps under d_H, the composition operator $\mathcal{T}$ satisfies:

$$\mathrm{d}_H\big(\mathcal{T}(b_1), \mathcal{T}(b_2)\big) \leq [\kappa(K)]^2 \mathrm{d}_H(b_1, b_2)$$

By the Banach Fixed-Point Theorem applied to the complete metric space of normalized positive vectors $(\mathbb{R}^n_{>0}/\mathbb{R}_{>0}, \mathrm{d}_H)$, $b^{(k)}$ converges linearly to a unique fixed point $b^* \in \mathbb{R}^n_{>0}$ with rate $\kappa(K)^k$:

$$\mathrm{d}_H\big(b^{(k)}, b^*\big) \leq \frac{[\kappa(K)]^k}{1-\kappa(K)} \mathrm{d}_H\big(b^{(1)}, b^{(0)}\big)$$

Applying the logarithmic transformation $\alpha^{(k)} = \varepsilon \ln a^{(k)}$ and $\beta^{(k)} = \varepsilon \ln b^{(k)}$ converts the multiplicative error bound under d_H into an additive norm bound in ℓ_∞ space. Using the standard inequality $\| \ln x - \ln y \|_\infty \leq \mathrm{d}_H(x, y)$ for scale-centered vectors yields $\|\beta^{(k)} - \beta^*\|_\infty \leq M_1[\kappa(K)]^k$. Passing through the continuous updating relation $\alpha^{(k+1)} = \varepsilon \ln \mu - \varepsilon \ln\big(K \exp\big(\beta^{(k)}/\varepsilon\big)\big)$ yields a matching bound for $\alpha^{(k)}$, completing the proof with $M = M_1(1 + \|K\|\infty/\varepsilon)$.

4. Numerical results

On a benchmark problem instance for a radial distribution network with 100 nodes, including 30 sources and 70 sinks, the entropic Sinkhorn approach to regularization (ε = 0.05, scarified cost matrix) achieved a cost for transportation 0.4% off from the optimal one (LP) after 74 iterations in 0.18 s compared to 2.9 s of the exact network simplex procedure — a speed-up factor of 16 — while the theoretical stability bound of Theorem 3.2 (diamd(V)=8.6) matched cost variations due to the ±10% random demand perturbation within a factor of 1.4 from the worst case, over 500 runs.

Table 2: Results of various distribution networks.

Method	Iterations	Time (s)	Gap to LP optimum (%)
Exact network-simplex LP	—	2.90	0.00 (reference)
Sinkhorn, ε = 0.10	51	0.13	0.92
Sinkhorn, ε = 0.05	74	0.18	0.40
Sinkhorn, ε = 0.02	146	0.34	0.15

A reduction in ε leads to smaller gap towards the optimal LP solution but increases the number of iterations, which is expected from the Ω(ε log n) bound on the bias and the Hilbert-metric contraction analysis mentioned in Section 4; ε = 0.05 was the right choice for this instance, where the gap was below 0.5% and speed-up was 16 times.

5. Discussion

The OT-based approach reinterprets flow-regulation shadow prices as Kantorovich potentials (Theorem 3.1), thereby providing a solid theoretical basis for congestion pricing mechanisms that are currently applied in the power grid and transportation networks, whereas the stability theorem (Theorem 3.2) offers a robustness guarantee to the planner before solving the regulation problem for any scenario. The entropic-Sinkhorn method uses a bias, which can be controlled by parameter ε, for the sake of dramatic acceleration of computations on sparse graphs.

Two considerations arise from the discussion. First, the selection of ε should be based on the trade-off between precision and speed illustrated in table 2 and the O (ε log n) bias bound derived in Section 4: problems where auditable shadow prices close to the exact values (such as regulatory tariff setting) require either small values of ε or resort to the exact LP problem of Theorem 3.1, while problems involving real-time or scenario-based control allow one to benefit from relatively large ε in order to get significant gains in speed. Second, since the stability bound provided by Theorem 3.2 is a function of diam_d(V) and not of n, it still makes sense in cases of extremely sparse large graphs, unlike other bounds that tend to deteriorate as n increases. This feature makes the OT approach especially applicable to large distribution-level infrastructure networks which have many nodes but relatively low transport distances. Convex formulations of the transport problem with time-dependence, capacity constraints and stochastic edge costs are important topics for further research.

6. Conclusion

The paper presented a mathematical formulation to regulate the resource flows of networked engineering systems using supply-demand distribution, network structure, shortest path transport cost, flow conservation and operational constraint conditions in an integrated optimization formulation via the optimal transport theory. In theoretical analysis, the dual formulation of the resource flow regulation problem was obtained, and the stability of optimal transport costs was analyzed with respect to demand perturbation. To enhance the computation speed, the entropic regularization scheme with Sinkhorn algorithm was considered for large and sparse networks. With the numerical evaluation on 100 nodes radial distribution network, the proposed Sinkhorn method with ε = 0.05 gave a solution that was 0.40% close to the exact linear programming optimization solution in 0.18 s, while the exact network simplex solution required 2.90 s. Hence, there is a computation speed-up factor of 16. The stability analysis was also practical even with randomized demand variation by ±10% through 500 trial runs. In addition, the results also confirmed the presence of the trade-off between computational efficiency and solution accuracy depending on the value of the regularization parameter, which suggests that the proposed framework is an efficient balance between the required properties of such problems. The further research can extend the static model into a dynamic optimal transport problem capable of considering the changes in supply, demand, and transportation network. Moreover, it is possible to consider the capacity-constrained and unbalanced transport that will be able to consider the inequality between supply and demand, generation and loss of resources, and resource storages. The consideration of stochastic demand, uncertain cost of edges, renewable resources, and unexpected network disruptions will improve the robustness of the framework. The adaptive mechanism of determining the regularization parameter can maximize the trade-off between computational efficiency and solution accuracy. Also, the validation using the large real-world datasets from the smart grid networks, transportation networks, logistic systems, communication network infrastructure, and industrial networks will provide proof of concept for the applicability of the framework.

References

1. Ali, A., Ullah, I., Shabaz, M., Sharafian, A., Khan, M. A., Bai, X., & Qiu, L. (2024). A resource-aware multi-graph neural network for urban traffic flow prediction in multi-access edge computing systems. *IEEE Transactions on Consumer Electronics*, *70*(4), 7252-7265. https://doi.org/10.1109/TCE.2024.3439719

2. Barykin, S. E., Sergeev, S. M., Provotorov, V. V., Lavskaya, K. K., Shidlovskaya, K. A., Dedyukhina, N., ... & Dzhamaludinova, M. Y. (2024). Sustainability analysis of energy resources transport based on a digital nd logistics network. *Engineered science*, *29*, 1093. http://dx.doi.org/10.30919/es1093

3. Denysov, V. (2025). Mathematical models for controlling active emergency frequency and power regulators in power systems with potential integration of wind and solar power plants. Priority directions. *System Research in Energy*, (3 (83)), 56-64. https://doi.org/10.15407/srenergy2025.03.056

4. Ermolin, Y. A. (1999). Mathematical modelling for optimized control of Moscow's sewer network. *Applied Mathematical Modelling*, *23*(7), 543-556. https://doi.org/10.1016/S0307-904X(98)10098-7

5. Fobes, D. M., Claeys, S., Geth, F., & Coffrin, C. (2020). PowerModelsDistribution. jl: An open-source framework for exploring distribution power flow formulations. *Electric Power Systems Research*, *189*, 106664. https://doi.org/10.1016/j.epsr.2020.106664

6. Gan, D., Ge, X., & Li, Q. (2023). An optimal transport-based federated reinforcement learning approach for resource allocation in cloud–edge collaborative IoT. *IEEE Internet of Things Journal*, *11*(2), 2407-2419. https://doi.org/10.1109/JIOT.2023.3292368

7. Herrera, M., Pérez-Hernández, M., Kumar Parlikad, A., & Izquierdo, J. (2020). Multi-agent systems and complex networks: Review and applications in systems engineering. *Processes*, *8*(3), 312. https://doi.org/10.3390/pr8030312

8. Kollinal, R. K., & Cheeran, M. T. (2025). Neuromorphic-Inspired Hybrid Cognitive Model for Self-Optimizing Resource Management In 6g Edge Networks. *Archives for Technical Sciences/Arhiv za Tehnicke Nauke*, (34), 393. https://doi.org/10.70102/afts.2025.1834.393

9. Leite, D., & De Bacco, C. (2024). Similarity and economy of scale in urban transportation networks and optimal transport-based infrastructures. *Nature Communications*, *15*(1), 7981. https://doi.org/10.1038/s41467-024-52313-6

10. Lonardi, A., Szell, M., & De Bacco, C. (2025). Cohesive urban bicycle infrastructure design through optimal transport routing in multilayer networks. *Journal of the Royal Society Interface*, *22*(223), 20240532. https://doi.org/10.1098/rsif.2024.0532

11. Manfren, M. (2012). Multi-commodity network flow models for dynamic energy management–Mathematical formulation. *Energy Procedia*, *14*, 1380-1385. https://doi.org/10.1016/j.egypro.2011.12.1105

12. Moltchanov, D., Sopin, E., Begishev, V., Samuylov, A., Koucheryavy, Y., & Samouylov, K. (2022). A tutorial on mathematical modeling of 5G/6G millimeter wave and terahertz cellular systems. *IEEE Communications Surveys & Tutorials*, *24*(2), 1072-1116. https://doi.org/10.1109/COMST.2022.3156207

13. Peralta, A., Olivas, J. A., Romero, F. P., & Navarro-Illana, P. (2025). Intelligent incident management leveraging artificial intelligence, knowledge engineering, and mathematical models in enterprise operations. *Mathematics*, *13*(7), 1055. https://doi.org/10.3390/math13071055

14. Reddy, P. P. K., & Kamalanathan, C. (2026). Multi-Agent Reinforcement Learning for Joint User Association and Resource Allocation in Heterogeneous Cellular Networks. *Indian Journal of Information Sources and Services*, *16*(2), 215–232. https://doi.org/10.51983/ijiss-2026.16.2.23

15. Selwin Mich Priyadharson, A., Ugli, M. N. N., Gopalakrishnan, R., & Bawankar, S. (2025). A Multidimensional Stochastic Framework for Optimizing 6G-Enabled Industrial IoT and Supply Chain Resilience. *International Academic Journal of Science and Engineering*, *12*(4), 373-384. https://doi.org/10.71086/IAJSE/V12I4/IAJSE12105

16. Seyedi, I., Candelieri, A., & Archetti, F. (2026). Geometric Optimal Transport for Sustainable Closed-Loop Supply Chain: A Fused Gromov–Wasserstein Framework for Structural and Attribute Inefficiency Diagnosis. *Sustainability*, *18*(13), 6906. https://doi.org/10.3390/su18136906

17. Talwar, A., & Shah, R. B. (2023). Optimizing Production Schedules Using Advanced Heuristic Algorithms. *International Academic Journal of Innovative Research*, *10*(3), 16-19. https://doi.org/10.71086/IAJIR/V10I3/IAJIR1020

18. Yang, C., Sun, Y., Zou, Y., Zheng, F., Liu, S., Zhao, B., ... & Cui, H. (2023). Optimal power flow in distribution network: A review on problem formulation and optimization methods. *Energies*, *16*(16), 5974. https://doi.org/10.3390/en16165974

19. Zaitseva, I. V., Malafeyev, O. A., Zakharov, V. V., Smirnova, T. E., & Novozhilova, L. M. (2020, June). Mathematical model of network flow control. In *IOP Conference Series: Materials Science and Engineering* (Vol. 873, No. 1, p. 012036). IOP Publishing. https://doi.org/10.1088/1757-899X/873/1/012036

20. Zheng, Y., Li, L., Lin, W., Liang, W., Du, Q., & Han, Z. (2025). Optimal transport framework for ISAC in low-altitude networks: Joint resource allocation for cooperative communication and non-cooperative localization. *IEEE Transactions on Communications*, *74*, 1984-2000. https://doi.org/10.1109/TCOMM.2025.3642742

21. Ganesan, A. (2026). Sustainable fishery management balancing economic development and ecosystem conservation. *Aquatic Ecosystems and Environmental Frontiers, 4*(1), 68-76.