Minimax Theory and its Applications
Volume 4 (2019), No. 1, 101–112

Multiple Entire Solutions for Schrödinger-Hardy Systems Involving Two Fractional Operators

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Received: September 25, 2018
Accepted: November 21, 2018

The paper is devoted to the study of the following fractional Schrödinger-Hardy system in $\mathbb{R}^n$

$$\begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu \dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v), \end{cases}$$

where μ and σ are real parameters, dimension $n > ps$, with $s \in (0,1)$, $1 < m \leq p < m^*_s = mn/(n-ms)$, a and b are positive potentials, while H_u and H_v are derivatives of a suitable continuous function H. The main feature of the paper is the combination of two possibly different fractional operators and different Hardy terms with a nonlinearity H which does not necessarily satisfy the Ambrosetti-Rabinowitz condition. By using the symmetric mountain pass theorem, we provide the existence of an unbounded sequence of nonnegative entire solutions. For this, we complete the picture of the existence result stated in Theorem 1.1 by the author, P. Pucci and S. Saldi in [*"Existence of entire solutions for Schrödinger-Hardy systems involving the fractional p-Laplacian"*, Nonlinear Anal. 158 (2017) 109–131].

Keywords: Schrödinger-Hardy systems, existence of entire solutions, fractional p-Laplacian operator.

2010 Mathematics Subject Classification: Primary: 35J47, 35B08, 35B09, 35R11; secondary: 35Q55, 35B33, 47G20.

1. Introduction

Jointly with Pucci and Saldi, we recently studied in our work [10] the following fractional Schrödinger-Hardy system in $\mathbb{R}^n$

$$\begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu \dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v), \end{cases} \tag{1}$$

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

where μ and σ are real parameters, $n > ps$, with $s \in (0,1)$, $1 < m \leq p < m_s^* = mn/(n-ms)$ and $(-\Delta)_\wp^s$ is the fractional $\wp$-Laplacian operator, $\wp > 1$, which, up to normalization factors, is defined for $x \in \mathbb{R}^n$ by

$$(-\Delta)_\wp^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{\wp-2}(\varphi(x) - \varphi(y))}{|x-y|^{n+\wp s}}\, dy$$

along any $\varphi \in C_0^\infty(\mathbb{R}^n)$; see [7] and the references therein.

The weight functions a and b are of class $\mathscr{V}(\mathbb{R}^n)$. The family $\mathscr{V}(\mathbb{R}^n)$ consists of all functions $V \in C(\mathbb{R}^n)$ satisfying

(V_1) *V is bounded from below by a positive constant*;

(V_2) *there exists $\kappa > 0$ such that* $\lim_{|y|\to\infty} \operatorname{meas}(\{x \in B_\kappa(y) : V(x) \leq c\}) = 0$ *for any* $c > 0$,

where $B_\kappa(y)$ denotes any open ball of $\mathbb{R}^n$ centered at y and of radius $\kappa > 0$. Condition (V_2), which is weaker than the coercivity assumption, $V(x) \to \infty$ as $|x| \to \infty$, was originally discussed by Bartsch and Wang in [3] to overcome the lack of compactness.

The nonlinearities H_u and H_v in (1) denote the partial derivatives of H with respect to the second variable and the third variable, respectively. In the main result [10, Theorem 1.1], we provide the existence of an entire solution of (1), whenever the nonlinearity H satisfies

(H_1) *$H\colon \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and admits partial derivatives H_u and H_v of class $C(\mathbb{R}^n \times \mathbb{R}^2)$, $H \geq 0$ in $\mathbb{R}^n \times \mathbb{R}^2$, $H(x,0,0) = 0$ in $\mathbb{R}^n$ and $H_u(x,u,v) = 0$ if $x \in \mathbb{R}^n$ and $u \leq 0$, $v \in \mathbb{R}$, while $H_v(x,u,v) = 0$ if $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $v \leq 0$;*

(H_2) *There are an exponent $q \in (p, m_s^*)$ and a number $\lambda \in [0, \lambda_p)$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|H_z(x,z)| \leq (\lambda + \varepsilon)|z|^{p-1} + C_\varepsilon |z|^{q-1}, \quad z = (u,v),\ |z| = \sqrt{u^2 + v^2},$$

holds for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^2$, where λ_p is introduced in (4) *and $H_z = (H_u, H_v)$;*

(H_3) $\displaystyle\lim_{\substack{|z|\to\infty \\ u>0 \vee v>0}} \frac{H(x,z)}{|z|^p} = \infty$, *uniformly in* $\mathbb{R}^n$;

(H_4) *There exist a nonnegative function g of class $L^1(\mathbb{R}^n)$ and a constant $C_F \geq 1$ such that $F(x,tu,tv) \leq C_F F(x,u,v) + g(x)$ for a.e. $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}_0^+$, $v \in \mathbb{R}_0^+$ and $t \in (0,1)$, where*

$$F(x,z) = H_z(x,z) \cdot z - pH(x,z).$$

By assuming $(H_1)-(H_4)$, [10, Theorem 1.1] generalizes the existence result of [16, Theorem 1.3], where system (1) is considered with $\mu = \sigma = 0$, namely without Hardy terms.

Besides [10, 16], there have been made very few attempts to study systems driven by two possibly different $\wp$-fractional operators. We refer to [12], where the authors provide existence and nonexistence results for a system with two coupled subcritical terms. Also, we recall [6] for a multiplicity result in a similar context, with a nonlinearity H satisfying the Ambrosetti-Rabinowitz condition. While, for the study of a system with equal fractional operators and involving Hardy terms, we mention [11], where two Kirchhoff coefficients are also considered.

Motivated by the above works, in the present paper we provide the multiplicity of entire solutions of (1). For this, we require the following symmetric condition for H

(H_5) $H(x,-u,-v) = H(x,u,v)$ *for a.e.* $x \in \mathbb{R}^n$, *for all* $u \in \mathbb{R}$ *and* $v \in \mathbb{R}$,

and we assume a weaker condition than (H_3), given by

(H_3') $\lim\limits_{|z|\to\infty} \dfrac{H(x,z)}{|z|^p} = \infty$, *uniformly in* $\mathbb{R}^n$.

For models and general comments on nonlinearity H satisfying $(H_1)-(H_5)$ and (H_3'), we refer to the Introduction of [10].

From this setting, we can introduce the functional space and some related constants, which play a crucial role on the resolution of (1). Let $1 < \wp < \infty$. By Theorems 1 and 2 of [14], we know that

$$\begin{aligned} &\|u\|^{\wp}_{\wp^*_s} \leq c_{n,\wp} \frac{s(1-s)}{(n-\wp s)^{\wp-1}} [u]^{\wp}_{s,\wp}, \quad \wp^*_s = \frac{\wp n}{n-\wp s}, \quad n > \wp s, \\ &\|u\|^{\wp}_{H_\wp} \leq c_{n,\wp} \frac{s(1-s)}{(n-\wp s)^{\wp}} [u]^{\wp}_{s,\wp}, \qquad \|u\|^{\wp}_{H_\wp} = \int_{\mathbb{R}^n} |u(x)|^{\wp} \frac{dx}{|x|^{\wp s}} \end{aligned} \tag{2}$$

for all $u \in D^{s,\wp}(\mathbb{R}^n)$, where the positive constant $c_{n,\wp}$ depends only on n and $\wp$, and $D^{s,\wp}(\mathbb{R}^n)$ is the fractional Beppo-Levi space, that is the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm $[\,\cdot\,]_{s,\wp}$ set as

$$[\varphi]_{s,\wp} = \left(\iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)|^{\wp}}{|x-y|^{n+\wp s}} dxdy \right)^{1/\wp},$$

well defined along any test function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Thus, we can set the best fractional Hardy-Sobolev constant, which for all $\wp > 1$ is denoted by $\mathcal{H}_\wp = \mathcal{H}(\wp, n, s)$, and is given by

$$\mathcal{H}_\wp = \inf_{\substack{u \in D^{s,\wp}(\mathbb{R}^n) \\ u \neq 0}} \frac{[u]^{\wp}_{s,\wp}}{\|u\|^{\wp}_{H_\wp}}, \tag{3}$$

where $\|\cdot\|_{H_\wp}$ is defined in (2). Clearly, $\mathcal{H}_\wp > 0$ thanks to (2).

The natural solution space for system (1) is the real Banach space $W = E_{m,a} \times E_{p,b}$, endowed with the norm $\|(u,v)\| = \|u\|_{E_{m,a}} + \|v\|_{E_{p,b}}$, where

$$E_{m,a} = \left\{ u \in D^{s,m}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a(x)|u(x)|^m dx < \infty \right\},$$

$$E_{p,b} = \left\{ v \in D^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} b(x)|v(x)|^p dx < \infty \right\},$$

$$\|u\|_{E_{m,a}} = \left([u]_{s,m}^m + \|u\|_{m,a}^m\right)^{1/m}, \quad \|v\|_{E_{p,b}} = \left([v]_{s,p}^p + \|v\|_{p,b}^p\right)^{1/p},$$

and $\|\varphi\|_{\wp,V} = \left(\int_{\mathbb{R}^n} V(x)|\varphi|^{\wp} dx\right)^{1/\wp}$ for all $\wp > 1$, $V \in \mathscr{V}(\mathbb{R}^n)$ and $\varphi \in L^{\wp}(\mathbb{R}^n, V)$. As noted in Lemma 2.2, see [5, Lemma 4.1] for a proof, under the solely condition (V_1), the embeddings $W \hookrightarrow W^{s,m}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{\nu}(\mathbb{R}^n) \times L^{\nu}(\mathbb{R}^n)$ are certainly continuous for all $\nu \in [p, m_s^*]$, being $1 < m \le p < m_s^*$. Thus, the numbers

$$\lambda_\nu = \inf\left\{ \|u\|_{E_{m,a}}^{\nu} + \|v\|_{E_{p,b}}^{\nu} : \int_{\mathbb{R}^n} |(u,v)|^{\nu} dx = 1 \right\} \tag{4}$$

are well defined and strictly positive.

Now, we are ready to introduce the multiplicity result of the paper.

Theorem 1.1. *Let $s \in (0,1)$, $n > ps$ and $1 < m \le p < m_s^*$. Assume that (H_1), (H_2), (H_3'), (H_4), (H_5) hold true, and a, b satisfy (V_1)–(V_2).*

Then, system (1) admits an unbounded sequence of nonnegative entire solutions in W, for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ such that

$$1 - \frac{\delta_{m,p}}{m}\left(1 + \frac{|\mu^-|}{\mathcal{H}_m}\right) - \frac{1}{p}\left(1 + \frac{|\sigma^-|}{\mathcal{H}_p}\right) > 0, \tag{5}$$

where $\delta_{m,p}$ denotes the Kronecker delta of m and p, and

$$1 - \frac{\mu^+}{\mathcal{H}_m} - \frac{\sigma^+}{\mathcal{H}_p} - 2^{p-1}\frac{\lambda}{\lambda_p} > 0, \tag{6}$$

being $\lambda \in [0, \lambda_p)$ given in (H_2).

The proof of Theorem 1.1 is mainly variational, based on the application of the symmetric mountain pass theorem due by Ambrosetti and Rabinowitz; see [13, Theorem 11.5]. For this, we first need a technical lemma for subspaces of W which helps us in verifying the mountain pass geometry. This lemma is the vectorial counterpart of a result given in [8], which forces (5). A similar restriction to (5) already appeared in [8], where it is well explained why (5) is a consequence of the setting of (1) in the whole space $\mathbb{R}^n$. While, (6) is crucial also for the existence result stated in [10, Theorem 1.1]. Hence, Theorem 1.1 somehow completes the picture of [10, Theorem 1.1] and [16, Theorem 1.3].

This article is organized as follows. In Section 2, we construct the weak formulation of (1) and present some preliminary results. In Section 3, using the symmetric mountain pass Theorem 2.1, we prove Theorem 1.1. Finally, in Section 4 we extend Theorem 1.1 when the fractional $\wp$-Laplacian operator is replaced by a more general elliptic nonlocal integro-differential operator, generated by a singular kernel K, satisfying the natural assumptions described by Caffarelli, e.g., in [4]. See also [9] and the references therein.

2. Preliminaries

From now on, we assume that $n > ps$, $s \in (0,1)$, $1 < m \le p < m_s^*$, (H_1), (H_2), (H_3'), (H_4) and (H_5) hold true, a and b satisfy (V_1) – (V_2), without further mentioning. As a matter of notations, we denote with $t^+ = \max\{t,0\}$ and $t^- = \min\{t,0\}$ respectively the positive and negative part of a number $t \in \mathbb{R}$. Also, we indicate with $\|(\cdot,\cdot)\|_\nu$ the norm of the product Lebesgue space $L^\nu(\mathbb{R}^n)\times L^\nu(\mathbb{R}^n)$, for any $\nu \in [1,\infty]$.

We start this section by defining the variational setting of equation (1). We say that the couple $(u,v) \in W$ is an *entire* (*weak*) *solution* of (1) if

$$\begin{aligned}\langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{H_m} - \sigma\langle v,\Psi\rangle_{H_p} \\ = \int_{\mathbb{R}^n}[H_u(x,u,v)\Phi(x) + H_v(x,u,v)\Psi(x)]dx \end{aligned} \tag{7}$$

for any $(\Phi,\Psi) \in W$, where

$$\langle u,\Phi\rangle_{s,\wp} = \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{\wp-2}(u(x)-u(y))(\Phi(x)-\Phi(y))}{|x-y|^{n+\wp s}}dxdy,$$

$$\langle u,\Phi\rangle_{E_{m,a}} = \langle u,\Phi\rangle_{s,m} + \int_{\mathbb{R}^n} a(x)|u(x)|^{m-2}u(x)\Phi(x)dx,$$

$$\langle v,\Psi\rangle_{E_{p,b}} = \langle v,\Psi\rangle_{s,p} + \int_{\mathbb{R}^n} b(x)|v(x)|^{p-2}v(x)\Psi(x)dx,$$

$$\langle u,\Phi\rangle_{H_m} = \int_{\mathbb{R}^n} |u(x)|^{m-2}u(x)\Phi(x)\frac{dx}{|x|^{ms}},\ \ \langle v,\Psi\rangle_{H_p} = \int_{\mathbb{R}^n} |v(x)|^{p-2}v(x)\Psi(x)\frac{dx}{|x|^{ps}}.$$

Clearly, the entire (weak) solutions of (1) are exactly the critical points of the Euler-Lagrange functional associated with (1), that is of

$$I_{\mu,\sigma}(u,v) = \frac{1}{m}\|u\|^m_{E_{m,a}} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\mu}{m}\|u\|^m_{H_m} - \frac{\sigma}{p}\|v\|^p_{H_p} - \int_{\mathbb{R}^n} H(x,u,v)dx.$$

The functional $I_{\mu,\sigma}$ is well defined in W, of class $C^1(W)$ under the conditions (H_1) – (H_2), and for any fixed $(u,v) \in W$ and all $(\Phi,\Psi) \in W$ it satisfies

$$\begin{aligned}\langle I'_{\mu,\sigma}(u,v),(\Phi,\Psi)\rangle = \langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{H_m} - \sigma\langle v,\Psi\rangle_{H_p} \\ - \int_{\mathbb{R}^n}[H_u(x,u,v)\Phi + H_v(x,u,v)\Psi]dx\,.\end{aligned}$$

The idea is to apply the following symmetric mountain pass theorem of Ambrosetti and Rabinowitz, given in [13, Theorem 11.5], to the functional $I_{\mu,\sigma}$.

Theorem 2.1. *Let X be infinite dimensional Banach space and $J \in C^1(X)$ a functional satisfying the (PS) condition as well as the following three properties:*

(i) $J(0) = 0$ *and there exist* r, α *such that* $J|_{\partial B_r} \geq \alpha$;

(ii) J *is even;*

(iii) *for all finite dimensional subspaces* $\widetilde{X} \subset X$ *there exists* $R = R(\widetilde{X}) > 0$ *with*

$$J(u) \leq 0 \text{ for any } u \in X \setminus B_R(\widetilde{X})$$

where $B_R(\widetilde{X}) = \{u \in \widetilde{X} : \ \|u\| < R\}$.

Then, J possesses an unbounded sequence of critical values characterized by a minimax argument.

We end the section by recalling some basic facts on the fractional Sobolev space W. By [15, Lemma 10], we have that $E_{m,a} = (E_{m,a}, \|\cdot\|_{E_{m,a}})$ and $E_{p,b} = (E_{p,b}, \|\cdot\|_{E_{p,b}})$ are two separable, reflexive Banach spaces. Hence, $W = (W, \|\cdot\|)$ is a separable and reflexive Banach space by [1, Theorem 1.12]. Furthermore, combining the results of [5, Lemma 4.1] and [15, Lemma 2.1], we get the following embedding lemma.

Lemma 2.2. *Let (V_1) hold. Then the embeddings*

$$W \hookrightarrow W^{s,m}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

are continuous if $\nu \in [p, m_s^*]$, and

$$\|(u,v)\|_\nu \leq \|u\|_\nu + \|v\|_\nu \leq C_\nu \|(u,v)\| \quad \text{for all } (u,v) \in W, \tag{8}$$

where C_ν depends on ν, n, s, m and p. If in addition also (V_2) holds, then the embedding $W \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$ is compact when $\nu \in [p, m_s^*)$.

The number C_ν in (8) is related to λ_ν defined in (4) by $C_\nu = \lambda_\nu^{-1/\nu}$.

3. Multiple solutions of (1)

In order to prove Theorem 1.1, we shall apply Theorem 2.1 to the functional $I_{\mu,\sigma}$ introduced in Section 2.

Let us recall that a functional $J\colon X \to \mathbb{R}$ of class $C^1(X)$, on a real Banach space $X = (X, \|\cdot\|)$, with its dual space X', is said to satisfy the *Cerami condition* (C) if any *Cerami sequence* associated with J has a strongly convergent subsequence in X. A sequence $(u_k)_k$ in X is called a *Cerami sequence*, if $(J(u_k))_k$ is bounded and $(1 + \|u_k\|) \cdot \|J'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

Then, by [10, Lemma 3.2], with (H_3') instead of (H_3), we still have the following compactness result for $I_{\mu,\sigma}$.

Lemma 3.1. *The functional $I_{\mu,\sigma}$ satisfies the Cerami condition (C) in W for all $\mu < \mathcal{H}_m$ and for all $\sigma < \mathcal{H}_p$.*

We introduce the next technical lemma, to prove part (iii) of Theorem 2.1.

Lemma 3.2. *For any finite dimensional subspace $\widetilde{W} \subset W$, there exists $\delta > 0$ such that*

$$|\{x \in \mathbb{R}^n : |(u(x), v(x))| \geq \delta \|(u, v)\|\}| \geq \delta, \quad \text{for any } (u, v) \in \widetilde{W} \setminus \{(0, 0)\}.$$

Proof. Let us argue by contradiction and suppose that for any $k \in \mathbb{N}$ there exists a nontrivial $(u_k, v_k) \in \widetilde{W}$ such that

$$\left| \left\{ x \in \mathbb{R}^n : |(u_k(x), v_k(x))| \geq \frac{1}{k} \|(u_k, v_k)\| \right\} \right| < \frac{1}{k}, \quad \text{for any } k \in \mathbb{N}.$$

Let $(U_k, V_k) = (u_k, v_k)/\|(u_k, v_k)\| \in \widetilde{W}$ so that $\|(U_k, V_k)\| = 1$ and

$$\left| \left\{ x \in \mathbb{R}^n : |(U_k(x), V_k(x))| \geq \frac{1}{k} \right\} \right| < \frac{1}{k}, \quad \text{for any } k \in \mathbb{N}. \tag{9}$$

Since $\{(U_k, V_k)\}_k$ is bounded in $\widetilde{W}$ finite dimensional, up to a subsequence, still denoted by $\{(U_k, V_k)\}_k$, there exists $(U, V) \in \widetilde{W}$ such that $(U_k, V_k) \to (U, V)$ in $\widetilde{W}$ as $k \to \infty$. While, by Lemma 2.2 and considering all norms are equivalent in $\widetilde{W}$, we also have $(U_k, V_k) \to (U, V)$ in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ as $k \to \infty$. Hence, considering $\|(U, V)\| = 1$ and since (U, V) is nontrivial, there exists $\delta_0 > 0$ with

$$|\{x \in \mathbb{R}^n : |(U(x), V(x))| \geq \delta_0\}| \geq \delta_0. \tag{10}$$

Let us set $\qquad \Lambda_0 = \{x \in \mathbb{R}^n : |(U(x), V(x))| \geq \delta_0\}$

and for every $k \in \mathbb{N}$ $\qquad \Lambda_k = \{x \in \mathbb{R}^n : |(U_k(x), V_k(x))| \geq 1/k\}$.

By (9) and (10), we get $|\Lambda_k \cap \Lambda_0| \geq |\Lambda_0| - |\Lambda_k^c| > \delta_0 - \dfrac{1}{k} \geq \dfrac{\delta_0}{2}$, for k sufficiently large. Therefore, we obtain

$$\|(U_k, V_k) - (U, V)\|_p^p \geq \int_{\Lambda_k \cap \Lambda_0} |(U_k, V_k) - (U, V)|^p dx$$

$$\geq \int_{\Lambda_k \cap \Lambda_0} (|(U, V)|^p - |(U_k, V_k)|^p)\, dx \geq \left(\delta_0 - \frac{1}{k} \right)^p |\Lambda_k \cap \Lambda_0| > \left(\frac{\delta_0}{2} \right)^{p+1} > 0,$$

which is in contradiction to the fact that $(U_k, V_k) \to (U, V)$ in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ as $k \to \infty$. □

Now, we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $\mu \in (-\infty, \mathcal{H}_m)$ and $\sigma \in (-\infty, \mathcal{H}_p)$. We want to apply Theorem 2.1 to functional $I_{\mu,\sigma}$. For this, by Lemma 3.1, functional $I_{\mu,\sigma}$ satisfies the Cerami condition and so the Palais-Smale condition (PS). Also, by (H_5) we have that $I_{\mu,\sigma}(-u,-v) = I_{\mu,\sigma}(u,v)$ for any $(u,v) \in W$, proving (ii).

Let us show that $I_{\mu,\sigma}$ satisfies condition (i). Take $\varepsilon > 0$, with $2^p\varepsilon/\lambda_p = \kappa - 2^{p-1}\lambda/\lambda_p$, where $\kappa = \min\{1-\mu^+/\mathcal{H}_m,\, 1-\sigma^+/\mathcal{H}_p\} > 0$. This is possible thanks to the restriction (6). By (4), (8) and (H_2), we have for all $(u,v) \in W$, with $\|(u,v)\| \le 1$,

$$
\begin{aligned}
I_{\mu,\sigma}(u,v) &\ge \frac{1}{m}\left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|u\|^m_{E_{m,a}} + \frac{1}{p}\left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\|v\|^p_{E_{p,b}} \\
&\qquad - \frac{1}{p}\int_{\mathbb{R}^n}\big(\lambda+\varepsilon\big)|(u,v)|^p dx - C_\varepsilon\int_{\mathbb{R}^n}|(u,v)|^q dx \\
&\ge \frac{\kappa}{p}\big(\|u\|^p_{E_{m,a}} + \|v\|^p_{E_{p,b}}\big) - \frac{1}{p}(\lambda+\varepsilon)\|(u,v)\|^p_p - C_\varepsilon C_q^q\|(u,v)\|^q \qquad (11) \\
&\ge \frac{1}{2^{p-1}p}\left(\kappa - 2^{p-1}\frac{\lambda+\varepsilon}{\lambda_p}\right)\|(u,v)\|^p - C_\varepsilon C_q^q\|(u,v)\|^q \\
&= \frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p\, C_\varepsilon C_q^q\|(u,v)\|^{q-p}\right)\|(u,v)\|^p.
\end{aligned}
$$

Now fix $r \in (0,1)$ so small that $\kappa - 2^{p-1}\lambda/\lambda_p - 2^p p\, C_\varepsilon C_q^q\, r^{q-p} > 0$. This can be done by (6). Therefore, for all $(u,v) \in W$, with $\|(u,v)\| = r$, we have

$$
I_{\mu,\sigma}(u,v) \ge \frac{r^p}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p\, C_\varepsilon C_q^q\, r^{q-p}\right) = \alpha > 0
$$

proving (i) of Theorem 2.1.

In order to conclude, we must prove that the functional $I_{\mu,\sigma}$ satisfies condition (iii) of Theorem 2.1. Indeed, for any finite dimensional subspace $\widetilde{W} \subset W$, by Lemma 3.2 there exists $\delta > 0$ such that

$$
|\{x \in \mathbb{R}^n : |(u(x),v(x))| \ge \delta\|(u,v)\|\}| \ge \delta, \quad \text{for any } (u,v) \in \widetilde{W} \setminus \{(0,0)\}. \qquad (12)
$$

Thus, by (H_3), there exists $\rho > 0$ such that

$$
H(x,u,v) \ge \frac{|(u,v)|^p}{\delta^{p+1}} \quad \text{for any } x \in \mathbb{R}^n \text{ and any } |(u,v)| \ge \rho. \qquad (13)
$$

For any $(u,v) \in \widetilde{W} \setminus \{(0,0)\}$, let us set $\Omega = \{x \in \mathbb{R}^n : |(u(x),v(x))| \ge \delta\|(u,v)\|\}$, with $|\Omega| \ge \delta$ by (12). Then, for any $(u,v) \in \widetilde{W}$ with $\|(u,v)\| \ge \rho/\delta$, by (3) and (13) we have

$$
\begin{aligned}
I_{\mu,\sigma}(u,v) &= \frac{1}{m}\|u\|^m_{E_{m,a}} - \frac{\mu}{m}\|u\|^m_{H_m} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\sigma}{p}\|v\|^p_{H_p} - \int_{\mathbb{R}^n} H(x,u,v)dx \\
&\le \frac{1}{m}\left(1+\frac{|\mu^-|}{\mathcal{H}_m}\right)\|u\|^m_{E_{m,a}} + \frac{1}{p}\left(1+\frac{|\sigma^-|}{\mathcal{H}_p}\right)\|v\|^p_{E_{p,b}} - \int_\Omega \frac{|(u,v)|^p}{\delta^{p+1}}dx \\
&\le \frac{1}{m}\left(1+\frac{|\mu^-|}{\mathcal{H}_m}\right)\|(u,v)\|^m + \frac{1}{p}\left(1+\frac{|\sigma^-|}{\mathcal{H}_p}\right)\|(u,v)\|^p - \|(u,v)\|^p.
\end{aligned}
\tag{14}
$$

From this, let us distinguish two situations.

If $m < p$, by (14) we obtain

$$
I_{\mu,\sigma}(u,v) \le \left(D_1 - D_2\|(u,v)\|^{p-m}\right)\|(u,v)\|^m, \tag{15}
$$

with $$D_1 = \frac{1}{m}\left(1+\frac{|\mu^-|}{\mathcal{H}_m}\right) > 0 \quad \text{and} \quad D_2 = 1 - \frac{1}{p}\left(1+\frac{|\sigma^-|}{\mathcal{H}_p}\right) > 0,$$

thanks to (5). Therefore, choosing $R = R(\widetilde{W})$ such that

$$
R > \max\{\rho/\delta, (D_1/D_2)^{1/(p-m)}\},
$$

by (15) we have $I_{\mu,\sigma}(u,v) < 0$ for any $(u,v) \in \widetilde{W}$ with $\|(u,v)\| \ge R$, proving (iii).

If $m = p$, by (14) we get

$$
I_{\mu,\sigma}(u,v) \le -\left[1 - \frac{1}{p}\left(2 + \frac{|\mu^-| + |\sigma^-|}{\mathcal{H}_p}\right)\right]\|(u,v)\|^p,
$$

from which, also by (5), it is enough to consider $R = R(\widetilde{W})$ with $R > \rho/\delta$ to show that $I_{\mu,\sigma}(u,v) < 0$ for any $(u,v) \in \widetilde{W}$ with $\|(u,v)\| \ge R$, proving again (iii).

Thus, by applying Theorem 1.1 we prove the existence of an unbounded sequence of entire solutions of (1). Now, let $(u,v) \in W \setminus \{(0,0)\}$ be an entire solution of (1). Taking $\Phi = u^- = \min\{0,u\}$ and $\Psi = v^- = \min\{0,v\}$ in (7), and considering the following elementary inequality valid for all $\wp > 1$

$$
|\xi^- - \eta^-|^\wp \le |\xi - \eta|^{\wp-2}(\xi - \eta)(\xi^- - \eta^-) \quad \text{for } \xi, \eta \in \mathbb{R},
$$

we have by (H_1) and (6)

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n} \left[H_u(x,u,v)u^- + H_v(x,u,v)v^-\right]dx \\
&\ge \left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u^-\|^m_{E_{m,a}} + \left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v^-\|^p_{E_{p,b}} \ge 0.
\end{aligned}
$$

In conclusion, $u^- = 0$ and $v^- = 0$ a.e. in $\mathbb{R}^n$, that is, $u \ge 0$ and $v \ge 0$ a.e. in $\mathbb{R}^n$. This completes the proof. □

4. Multiple solutions of (16)

In this section, we extend Theorem 1.1 when the fractional $\wp$-Laplacian operators are replaced by more general elliptic nonlocal integro-differential operators.

Hence, we consider in $\mathbb{R}^n$ problem

$$\begin{cases} -\mathcal{L}_{K_m}u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v), \\ -\mathcal{L}_{K_p}v + b(x)|v|^{p-2}v - \sigma\dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v), \end{cases} \tag{16}$$

governed by the operator $-\mathcal{L}_{K_\wp}$, which up to a multiplicative constant depending only on n, s and $\wp$ is defined for all $x \in \mathbb{R}^n$ by

$$-\mathcal{L}_{K_\wp}\varphi(x) = \int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^{\wp-2}[\varphi(x) - \varphi(y)]K_\wp(x-y)dy,$$

along any function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

The *weight* $K_\wp\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$, $\wp > 1$, satisfies the natural restrictions

(K_1) *there exists a number* $K_0 > 0$ *with* $K_\wp(x)|x|^{n+\wp s} \geq K_0$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$;

(K_2) $\bar{m}K_\wp \in L^1(\mathbb{R}^n)$, *where* $\bar{m}(x) = \min\{1, |x|^\wp\}$.

Without loss of generality we are allowed to suppose that $K_\wp$ is even, since the odd part of $K_\wp$ does not give any contribution in the integral above. Clearly, when $K_\wp(x) = |x|^{-(n+\wp s)}$, the operator $-\mathcal{L}_{K_\wp}$ reduces to the more familiar fractional $\wp$-Laplacian operator $(-\Delta)^s_\wp$.

Let us denote by $D^{s,\wp}_{K_\wp}(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to

$$[u]_{s,K_\wp} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^\wp K_\wp(x-y)dxdy\right)^{1/\wp},$$

which is well-defined by (K_2). The embedding $D^{s,\wp}_{K_\wp}(\mathbb{R}^n) \hookrightarrow D^{s,\wp}(\mathbb{R}^n)$ is continuous, since $[u]_{s,\wp} \leq K_0^{-1/\wp}[u]_{s,K_\wp}$ for all $u \in D^{s,\wp}_{K_\wp}(\mathbb{R}^n)$ by (K_1). Thus (2) is still valid.

The natural solution space for (16) is $W_K = E_{K_m,a} \times E_{K_p,b}$, where

$$E_{K_m,a} = \left\{u \in D^{s,m}_{K_m}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a(x)|u(x)|^m dx < \infty\right\},$$

$$E_{K_p,b} = \left\{v \in D^{s,p}_{K_p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} b(x)|v(x)|^p dx < \infty\right\},$$

endowed with the norm $\|(u,v)\| = \|u\|_{E_{K_m,a}} + \|v\|_{E_{K_p,b}}$, with

$$\|u\|_{E_{K_m,a}} = \left([u]^m_{s,K_m} + \|u\|^m_{m,a}\right)^{1/m}, \quad \|v\|_{E_{K_p,b}} = \left([v]^p_{s,K_p} + \|v\|^p_{p,b}\right)^{1/p}.$$

Assuming only condition (V_1), the embeddings

$$W_K \hookrightarrow W^{s,m}_{K_m}(\mathbb{R}^n) \times W^{s,p}_{K_p}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

are certainly continuous for all $\nu \in [p, m^*_s]$, since $1 < m \leq p < m^*_s$ and, again, by Lemma 2.2. Thus, the numbers

$$\lambda_\nu = \inf \left\{ \|u\|^\nu_{E_{K_m,a}} + \|v\|^\nu_{E_{K_p,b}} : \int_{\mathbb{R}^n} |(u,v)|^\nu dx = 1 \right\} \tag{17}$$

are well defined and strictly positive.

From this setting, the proof in Section 3 can proceed in the same way, up to the replacement of the appropriate norms. Thus, we obtain the following result, recalling that λ_p in (H_2) is now defined by (17).

Theorem 4.1. *Let* $s \in (0,1)$, $n > ps$ *and* $1 < m \leq p < m^*_s$. *Under the assumptions* (K_1) – (K_2) *on* K_m *and* K_p, (H_1), (H_2), (H'_3), (H_4), (H_5) *and* (V_1) – (V_2) *on a and b, system* (16) *admits an unbounded sequence of nonnegative entire solutions in* W_K, *for any* $\mu \in (-\infty, \mathcal{H}_m)$ *and for any* $\sigma \in (-\infty, \mathcal{H}_p)$ *verifying* (5) *and* (6).

Theorem 4.1 somehow completes the picture of [10, Theorem 5.1].

Acknowledgements. The author realized the manuscript within the auspices of the FAPESP Project entitled *"Fractional problems with lack of compactness"* (2017/19752-3) and of the CNPq Project entitled *"Variational methods for singular fractional problems"* (3787749185990982).

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