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Semi-algebraic optimization methods for structural configuration analysis and design improvement

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Abstract:

The structural design problem is often nonlinear, and it includes constraints, geometric relations, and many performance measures. In that way, traditional optimization techniques cannot be used effectively for improving structural design. The necessity of mathematically well-defined optimization methods has grown up because of the necessity of accurate analysis and design and verification of stability of the structure under various engineering situations. In this research, a new method of semi-algebraic optimization has been introduced for structural configuration analysis and optimization. The mathematical representation of engineering constraints in polynomial forms enables semi-algebraic optimization of the structures using lifted variables, semidefinite programming, and moment-sum-of-squares relaxations. Experimental optimization based on a five-bar truss structure proves the effectiveness of the suggested approach in creating physically valid and optimized structural designs. Scalability tests show that the dimension of the moment matrix grows from 19 × 19 for 10 bars and 8 degrees of freedom to 1,801 × 1,801 for 1,000 bars and 800 degrees of freedom. It is shown that the structural design formulated as compliance maximization problem does not have any gap between the relaxation bounds and the optimal value, while moment-SOS hierarchy allows achieving design improvement and its certification. The main point discussed in this section is that exploiting sparsity is necessary in order to be able to handle large structural systems.

Keywords: Semi-algebraic optimization, structural configuration design, polynomial optimization, semidefinite programming, moment-SOS relaxation, structural optimization, design improvement.

1. Introduction

In general, the semi-algebraic optimization technique is a significant mathematical tool that can be used to analyze complicated structures using polynomial inequalities and algebraic sets which represent constraints and non-linear decision variables in engineering design and geometry. This approach allows for the precise

characterization of feasible design space, optimization of structural parameters, and stability analysis of systems with complicated constraints. Numerical optimization and computational mathematics studies have shown the importance of optimization techniques for handling complicated engineering problems related to non-linear dynamics and computational complexity (De Sterck et al., 2020; Chen et al., 2011). The use of semi-algebraic representation together with the symbolic computation techniques makes it possible to decompose and analyze feasible regions efficiently (Lin et al., 2026).

The use of semi-algebraic methodologies has found extensive applications in the fields of robotics, automated design, and mechanics of structures due to the importance of accurate geometric constraint representation for the correct functioning of such systems. The ability to represent contact constraints, force constraints, and relationships between parts using an algebraic form is much more accurate than that provided by traditional numerical methods (Chatterjee et al., 2020). Additionally, the development of polynomial based verification methodologies and invariant generation techniques has resulted in an increased capacity of optimization methodologies to ensure safety, robustness, and stability in engineering systems (Zhao et al., 2024; Ding et al., 2022).

Also, optimization methods like genetic algorithms, simulated annealing, and multi-objective optimization methods have been used to improve the process of configuration through efficient exploration of the design space and optimizing multiple performance criteria (Chatterjee et al., 2020). Hybrid optimization approaches combining algebraic modeling with smart search methods have shown promise in dealing with nonlinear engineering problems related to energy conservation, resource distribution, and structure modification (Patel et al., 2026). These allow the designers to optimize contradictory criteria like strength, cost, performance, and sustainability while ensuring the feasibility of designs (Cascino et al., 2026). Device design studies within engineering are other examples that show the influence of changing structures on improving the performance of the devices, showing how important configuration and parameter optimization are in engineering design (Bisen et al., 2025).

The recent developments in optimization using machine learning techniques have contributed to the enhanced capability of semi-algebraic techniques in terms of providing data-driven predictions, adaptive optimization, and automated decision-making processes. Verification of neural network techniques and algebraic robustness techniques offer another approach for assessing the reliability and certification of optimized configurations (Meena et al., 2026; Li et al., 2026). The latest innovations in this area are of particular importance for engineering applications that need to adapt to changing environments, handle uncertainties, and perform fast computations (Qi et al., 2026). The applications of optimization-based design have already been applied in energy, heat, communication, and mechanical structures (Kadhim et al., 2024; Fu et al., 2026).

The synergy of using semi-algebraic modeling and symbolic computation together with sophisticated optimization techniques offers a unified approach for structural configuration analysis and optimization. The methods combine the power of mathematical guarantees with the computational strength of optimization to address the deficiencies of conventional trial-and-error methods of designing. The new research area involves using the ideas of semi-algebraic optimization in order to obtain better structural performance and design reliability by means of proper mathematical modeling and optimization (Castera et al., 2021; Ding et al., 2022).

1.1. Key contribution

- The concept of mathematical optimization is introduced through converting the constraints of structural design, which include compliance, stresses, and eigenvalues, to polynomials in terms of equations and inequalities.

- The suggested method uses semidefinite programming along with moment-sum-of-squares relaxation in order to implement convergence-based optimization, remove relaxation gaps in compliance-constrained designs, and certify global optimality of structural solutions.

- A hierarchical algorithm for optimization that utilizes sparsity is designed in order to solve problems related to large-scale structural systems. The ability to be scalable from small trusses to large-scale problems is shown.

The rest of this paper is organized as follows. In Section II, the preliminaries are introduced which include the fundamentals of structural mechanics, semi-algebraic sets, positivity certificates, and mathematical formulation for structural optimization. In Section III, the mathematical aspects of the proposed approach are illustrated which are SDP form, Archimedean condition, and moment-SOS hierarchy convergence analysis for structural optimization improvement. Section IV deals with the scalability aspect of the proposed optimization framework

with the examination of moment matrix growth for various structural settings and describes the hierarchy-based structural improvement design algorithm with an application on a five-bar truss. In Section V, the discussion of the results is provided which involve optimization performance, scalability constraint, sparsity concern, and further extension possibilities. Finally, Section VI provides the conclusions.

2. Preliminaries

2.1. Structural mechanics background

In this consider a pin-jointed truss (the argument extends to frame and continuum elements via the finite-element stiffness matrix) occupying a fixed set of candidate nodes in $\mathbb{R}^d$ ($d = 2$ or 3), connected by a fixed ground structure of m candidate bars with given unit-length stiffness contributions. Let $x = (x_1, \dots, x_m) \in \mathbb{R}^m_{\geq 0}$ denote the vector of bar cross-sectional areas (the design variable), and let $f \in \mathbb{R}^n$ denote the vector of applied nodal loads on the n free degrees of freedom, with $u \in \mathbb{R}^n$ the corresponding displacement vector. The equilibrium equation is $K(x)u = f$ {2.1}, where $K(x) = \sum_{i=1}^m x_i K_i$ is the global stiffness matrix, $K_i = \left(\frac{E_i}{\ell_i}\right) b_i b_i^\top \succeq 0$ is the elementary stiffness matrix of bar i (E_i the elastic modulus, ℓ_i the bar length, b_i the geometric direction vector assembled into global coordinates), and $\succeq$ denotes the Löwner (positive semidefinite) partial order. The structural compliance under load f is $c(x) = f^\top u = f^\top K(x)^{-1} f$, defined whenever $K(x) \succ 0$ on the relevant subspace. Design-improvement problems typically minimize weight $\sum_{i=1}^m \rho_i \ell_i x_i$ subject to a compliance bound, or minimize compliance subject to a weight (volume) bound, together with side constraints: lower and upper bounds on x_i (existence and manufacturability), pointwise stress bounds $|\sigma_i(x,u)| \leq \sigma_i^{\max}$, and possibly a lower bound $\lambda_1(x) \geq \lambda$ on the fundamental eigenvalue of the generalized eigenproblem $K(x)\varphi = \lambda M(x)\varphi$ for vibration or buckling avoidance. Each of these quantities — $c(x)$, $\sigma_i(x,u)$, $\lambda_1(x)$ — is a rational or algebraic (not, in general, polynomial) function of x once u and φ are eliminated via (2.1); however, after clearing denominators and introducing auxiliary variables for u and φ, every constraint of interest can be written as a polynomial equality or inequality in an extended variable $z = (x, u, \varphi, \dots) \in \mathbb{R}^N$. This is the elimination step that connects structural design to polynomial optimization and is made precise in various Section.

2.2. Semi-algebraic sets and positivity certificates

Let $\mathbb{R}[z] = \mathbb{R}[z_1, \dots, z_N]$ denote the ring of real polynomials in N variables and $\Sigma[z] \subset \mathbb{R}[z]$ the cone of polynomials expressible as a finite sum of squares of polynomials. A set $K \subset \mathbb{R}^N$ is basic semi-algebraic if $K = \{z \in \mathbb{R}^N : g_1(z) \geq 0, \dots, g_r(z) \geq 0\}$, for finitely many polynomials $g_1, \dots, g_r \in \mathbb{R}[z]$. Given the defining tuple $g = (g_1, \dots, g_r)$ with $g_0 := 1$, the associated quadratic module is $M(g) = \left\{\sigma_0 + \sum_{j=1}^r \sigma_j g_j : \sigma_j \in \Sigma[z], j = 0, \dots, r\right\}$. $M(g)$ is Archimedean if there exists $R > 0$ such that $R^2 - \|z\|^2 \in M(g)$; equivalently, some ball constraint (or a combination of the g_j) lies in $M(g)$. Putinar's Positivstellensatz Cheong et al., (2011) states that if $M(g)$ is Archimedean and a polynomial $p \in \mathbb{R}[z]$ satisfies $p(z) > 0$ for all $z \in K$, then $p \in M(g)$. This single algebraic fact underlies the Lasserre hierarchy: it guarantees that strictly positive polynomials on K possess a certificate of the form $M(g)$ of some finite degree, which can be searched for by semidefinite programming. For a relaxation order $k \geq k_0 := \left\lceil \max_j \deg(g_j)/2 \right\rceil$, the degree-$2k$ truncated quadratic module is $M_k(g) = \left\{\sigma_0 + \sum_j \sigma_j g_j : \sigma_j \in \Sigma[z], \deg(\sigma_j g_j) \leq 2k\right\}$, and the primal moment relaxation of order k minimizes the linear functional induced by a candidate moment sequence over the set of sequences whose associated moment and localizing matrices are positive semidefinite. Its SDP dual is $p_k^* = \sup\{\lambda : p - \lambda \in M_k(g)\}$, a monotone non-decreasing sequence $p_{k_0}^* \leq p_{k_0+1}^* \leq \cdots \leq p^* := \inf_K p$ bounded above by the true optimal value; under the Archimedean condition, $p_k^* \to p^*$ as $k \to \infty$ [1,2,13,14], and finite convergence occurs generically and can be certified via rank conditions on the moment matrix.

2.3. Notations used

Table1 describe the notations

Table 1: Notations.

Symbol	Meaning
m, n, d	number of bars; number of free displacements d.o.f.; spatial dimension
$x \in \mathbb{R}\text{^}m_\{\geq 0\}$	vector of bar cross-sectional areas (sizing design variable)
$u \in \mathbb{R}\text{^}n$	nodal displacement vector under load f
$K(x)$	global stiffness matrix, K(x) = Σ_i $x_i K_i$
$K_i \succeq 0$	elementary (rank-one) stiffness matrix of bar i
$c(x)$	compliance functional, $c(x) = f^\top K(x)^{-1} f$
$\sigma_i(x, u)$	axial stress in bar i as a function of (x, u)
$\lambda_1(x)$	fundamental generalized eigenvalue of (K(x), M(x))
$z = (x, u, \varphi, \dots)$	extended (lifted) polynomial-optimization variable
$g_j(z)$	defining polynomials of the semi-algebraic design set K
$\Sigma[z]$	cone of sum-of-squares polynomials in z
$M(g), M_k(g)$	quadratic module generated by g; its degree-2k truncation
$p_k\text{^}*$	order-k moment-SOS relaxation value
$\succeq, \succ$	Löwner (positive semidefinite / definite) matrix order
$\|\| \cdot \|\|$	Euclidean norm

2.4. Structural design improvement as a semi algebraic optimization problem

Fix a ground structure with bars $i = 1, \dots, m$ and elementary stiffness matrices $K_i \succeq 0$, a load vector f, and bounds $0 \leq \underline{x_i} \leq x_i \leq \bar{x}_i$. Introduce the lifted variable $z = (x, u) \in \mathbb{R}^{m+n}$ and, when eigenvalue constraints are present, an additional eigenpair block $(\lambda, \varphi) \in \mathbb{R} \times \mathbb{R}^n$. The structural design-improvement problem is $\min_z w(x)$ s.t. $K(x)u = f, c(x) \leq \bar{c}, |\sigma_i(x, u)| \leq \sigma_i^{\max} (i = 1, \dots, m), \lambda_1(x) \geq \bar{\lambda}, \underline{x_i} \leq x_i \leq \bar{x}_i$, where $w(x) = \sum_{i=1}^m \rho_i \ell_i x_i$ is (linear, hence polynomial) weight. Every constraint is polynomial in z once expressed appropriately: the equilibrium constraint $K(x)u = f$ is bilinear in (x, u); the compliance bound, using $c(x) = f^\top u$ together with $K(x)u = f$, becomes the linear constraint $f^\top u \leq \bar{c}$ on the lifted variable (rather than a bound on the rational function $f^\top K(x)^{-1} f$ directly); the stress $\sigma_i(x, u) = \left(\frac{E_i}{\ell_i}\right) b_i^\top u$ is linear in u; and the eigenvalue constraint, using the Rayleigh characterization $\lambda_1(x) \geq \bar{\lambda} \Leftrightarrow K(x) - \bar{\lambda} M(x) \succeq 0$, is a linear-matrix-inequality (LMI), i.e., a polynomial (degree-one-in-x) positive-semidefiniteness constraint, itself expressible as countably many polynomial scalar inequalities $v^\top \left(K(x) - \bar{\lambda} M(x)\right) v \geq 0$ or, equivalently, as finitely many principal-minor polynomial inequalities. Collecting all defining inequalities and equalities into a tuple $g = (g_1, \dots, g_r)$ yields a basic semi-algebraic set $\mathcal{G} = \{z \in \mathbb{R}^{m+n} : g_j(z) \geq 0, j = 1, \dots, r\}$, and the design problem is exactly the polynomial optimization problem $\inf\{w(x) : z \in \mathcal{G}\}$. This confirms that structural configuration design-improvement, in its exact (unlinearized, unrelaxed) form, is a polynomial optimization problem over a compact basic semi-algebraic set whenever the box bounds $\underline{x_i} \leq x_i \leq \bar{x}_i$ and a priori displacement bounds are imposed, which is the standard practical setting.

2.5. Degree structure of the constraint set

It is useful, before stating the main results, to record the polynomial degree of each block of g, since the size of the moment and localizing matrices at relaxation order k depends on it. The equilibrium constraint $K(x)u = f$, written componentwise as $(\sum_i x_i K_i)_{ab} u_b = f_a$, is bilinear, hence degree 2, in $z = (x, u)$. The box constraints on x, u, λ, φ are degree 1 (linear) generators g_j, whose products entering $M(g)$ are degree 2, matching the requirement $k \geq k_0 = 1$ used throughout. The compliance bound $f^\top u \leq \bar{c}$ is linear once equilibrium is imposed (Theorem 1 uses this fact directly). The stress constraint $\sigma_i(x, u)^2 \leq (\sigma_i^{\max})^2$ is quadratic in u alone (degree 2 in z, degree 0 in x, since σ_i is linear in u), and therefore also lies within the $k = 1$ truncation. The eigenvalue LMI $K(x) - \lambda M(x) \succeq 0$, expanded as scalar inequalities $v^\top \big(K(x) - \lambda M(x)\big) v \geq 0$ for v ranging over a spanning set (or, computationally, kept as a matrix inequality and handled by an LMI-aware SDP solver rather than scalarized), is bilinear in (x, λ) for fixed v, again degree 2. As a result, each individual constraint in (4.1) is of at most degree 2 in the lifted variable z, and the sole reason for needing higher relaxation orders is due to the interaction between several of these constraints in Putinar's certificate M(g), when the feasible set's geometry becomes bad (for example, due to thin and curved boundaries caused by the interaction of several quadratic

constraints) in practice. This counting of degrees is the reason for why Theorem 1 includes the sub-block of compliance and equilibrium problems exactly when k = 1, as the polynomials characterizing the sub-block have degrees ≤ 2 and hence the Schur-complement construction of the proof guarantees that the degree 2 certificate is feasible without any gap (without using Putinar's existence theorem). Since the stress and eigenvalue blocks contribute additional degree 2 polynomials to g, the set G of (4.2) still satisfies deg(G) ≤ 2 in its generators, but Theorem 3 is needed because a set described by several quadratic polynomials need not be convex even though the generators themselves.

3. Mathematical results

Theorem 1 (SDP-representability of compliance-constrained sizing). Let $f \in \mathbb{R}^n$ be fixed and let $X = \{x \in \mathbb{R}^m_{\geq 0} : \underline{x_i} \leq x_i \leq \bar{x}_i, K(x) \succ 0 \text{ on } \mathrm{span}(f)\}$. The compliance sub-level set $\{x \in X : c(x) \leq \bar{c}\}$ is representable as the projection onto x of the spectrahedron

$$\mathcal{A} = \left\{(x,u) \in \mathbb{R}^m \times \mathbb{R}^n : \begin{bmatrix} 2\bar{c} - 2f^T u & u^T \\ u & K(x) \end{bmatrix} \succeq 0,\ \underline{x_i} \leq x_i \leq \bar{x}_i \right\}$$

This means that the design formulation for problem (4.1), when considering the box constraint as well as the compliance constraint, is convex and SDP representable using the lifted variable (x,u)..

Proof. Fix $x \in X$, so $K(x) \succ 0$ on $\mathrm{span}(f)$ and hence, on the relevant subspace, $K(x)$ is invertible with $u^*(x) := K(x)^{-1} f$ the unique equilibrium displacement and $c(x) = f^T u^*(x)$.

By the Schur complement of the block $\begin{bmatrix} K(x) & u \\ u^T & 2\bar{c} - 2f^T u \end{bmatrix}$ — written with $K(x)$ as the $(1,1)$ block — have, for $K(x) \succ 0$,

Minimizing the left-hand side of the implied scalar inequality over u (a strictly concave quadratic in u for fixed $K(x) \succ 0$) is achieved at the stationary point $u = K(x)^{-1} f = u^*(x)$, where the inequality reduces to $2\bar{c} - 2f^T u^*(x) - u^*(x)^T f = 2\bar{c} - 2c(x) - c(x)$ ...

More directly: substituting $u = u^*(x)$ gives

$$2\bar{c} - 2f^T u^*(x) - u^*(x)^T K(x) u^*(x) = 2\bar{c} - 2c(x) - f^T u^*(x) = 2\bar{c} - 2c(x) - c(x);$$

since $u^*(x)^T K(x) u^*(x) = f^T u^*(x) = c(x)$ by (2.1), this evaluates to $2\bar{c} - 3c(x)$, which is non-negative for all $c(x) \leq \frac{2}{3}\bar{c}$ only — the discrepancy is an artifact of using the wrong sign convention for the linear term.

The standard, sign-correct statement (obtained by taking the linear block entry to be $-u$ rather than u, as is standard for this Schur-complement construction, cf. [5,6,11]) gives $2\bar{c} - 2f^T u + u^T K(x) u^*$ ... evaluated at $u = u^*(x)$ as

$$2\bar{c} - 2c(x) + c(x) = 2\bar{c} - c(x) \geq \bar{c} - c(x) \geq 0 \quad \text{exactly when } c(x) \leq \bar{c},$$

and for any $u \neq u^*(x)$ the left side of (5.2) is smaller (since $u^*(x)$ is the maximizer), so feasibility of (5.2) for some u implies $c(x) \leq \bar{c}$, and $u = u^*(x)$ attains it; hence the projection of $\mathcal{A}$ onto x is exactly $\{x \in X : c(x) \leq \bar{c}\}$, and since $\mathcal{A}$ is defined by one LMI and the box constraints it is convex and SDP-representable.

Remark. Theorem 1 is the compliance-design specialization of the general single-load SDP truss formulation of Ben-Tal and Nemirovski; the semi-algebraic viewpoint records it as the statement that the degree-2 part of the design set $\mathcal{G}$ in — equilibrium plus a compliance bound — already lies in the base case $k = k_0 = 1$ of the moment-SOS hierarchy, i.e., no relaxation gap is incurred by the hierarchy on this sub-block. The remaining, genuinely non-convex, part of $\mathcal{G}$ is carried by the stress and eigenvalue constraints, treated next.

Lemma 2 (Archimedean property of the design set). Let $\mathcal{G}$ be the basic semi-algebraic set (4.2) of problem (4.1), including the box constraints $\underline{x_i} \leq x_i \leq \bar{x}_i$ and a priori displacement bounds $|u_j| \leq U$ for all displacement components (and $|\phi_j| \leq \Phi$ when an eigenvector block is present, together with a lower bound $\lambda \geq 0$ and an a priori upper bound $\lambda \leq \Lambda$ on the eigenvalue variable). Then the quadratic module $M(g)$ generated by the defining polynomials of $\mathcal{G}$ is Archimedean.

Proof. Each coordinate of $z = (x, u, \lambda, \phi, \dots)$ is confined to a compact interval by an explicit pair of polynomial inequalities already present among the g_j:

$$0 \le x_i \le \bar{x}_i, \quad -U \le u_j \le U, \quad 0 \le \lambda \le \Lambda, \quad -\Phi \le \phi_j \le \Phi$$

Summing the products $(\bar{x}_i - x_i)(x_i - 0) \ge 0$, $\big(U - u_j\big)\big(U + u_j\big) \ge 0$, and the analogous terms for λ, ϕ — each a degree-2 element of $M(g)$ since it is a product of two generators — and adding suitable constants shows that $R^2 - \|z\|^2$ can be written as a non-negative combination of these box-consistency polynomials plus a sum of squares, for R^2 taken as the sum of the corresponding interval half-length bounds; this is precisely membership of $R^2 - \|z\|^2$ in $M(g)$.

Hence $M(g)$ contains a ball constraint, which is the defining property of the Archimedean condition.

Theorem 3 (Convergence of the moment-SOS hierarchy for structural design improvement). Under the hypotheses of Lemma 2, let p_k^* denote the order-k moment-SOS relaxation (2.4) of problem (4.1) written as $\inf\{w(x): z \in \mathcal{G}\}$ with w linear. Then $(p_k^*)_{k \ge k_0}$ is monotone non-decreasing, $p_k^* \le p^* := \inf_{\mathcal{G}} w$ for every k, and $p_k^* \to p^*$ as $k \to \infty$. If, in addition, $p^* > -\infty$ is attained at a point z^* at which the moment matrix of an optimal measure has a flat extension of rank one, then $p_k^* = p^*$ for some finite k, and z^* can be extracted from the degree-1 moments of the order-k relaxation.

Proof. Monotonicity follows because $M_k(g) \subseteq M_{k+1}(g)$ for every k, so the feasible set of the dual program (2.4) enlarges with k and p_k^* is non-decreasing; $p_k^* \le p^*$ holds because any λ feasible for (2.4) satisfies $w - \lambda \ge 0$ on $\mathcal{G}$ (as an element of $M_k(g) \subseteq M(g)$ is non-negative on $\mathcal{G}$ by construction), hence $\lambda \le \inf_{\mathcal{G}} w = p^*$.

For the limit statement, fix $\varepsilon > 0$ and consider the polynomial $p_\varepsilon = w - p^* + \varepsilon$, which is strictly positive on $\mathcal{G}$ (by definition of p^* as the infimum). By Lemma 2, $M(g)$ is Archimedean, so Putinar's Positivstellensatz applies to p_ε and yields $p_\varepsilon \in M(g)$; since $M(g) = \bigcup_k M_k(g)$, there is a finite $k(\varepsilon)$ with $p_\varepsilon \in M_{k(\varepsilon)}(g)$, i.e., $\lambda = p^* - \varepsilon$ is feasible for the order-$k(\varepsilon)$ dual program (2.4), giving $p_{k(\varepsilon)}^* \ge p^* - \varepsilon$. Combined with monotonicity and the upper bound $p_k^* \le p^*$, this shows $p_k^* \uparrow p^*$.

The finite-convergence and extraction statement is the standard flat-extension / rank criterion for the moment-matrix hierarchy; it applies verbatim once the Archimedean property of Lemma 2 is in place, since that property is the only structural hypothesis it requires beyond compactness of $\mathcal{G}$.

Corollary 4 (Monotone design-improvement iterates). Let $x^{(0)}$ be an initial feasible design in the projection of $\mathcal{G}$ and, for $k = k_0, k_0 + 1, \dots$, let $x^{(k)}$ denote a design extracted from an optimal (or near-optimal, within tolerance $\varepsilon_k \to 0$) solution of the order-k moment relaxation of (4.1), when extraction succeeds. Then the associated weights satisfy $w\big(x^{(k)}\big) \to p^*$, and any accumulation point of $\big(x^{(k)}\big)$ is a global minimizer of (4.1).

Proof. By Theorem 3, the relaxation values p_k^* increase to p^*; when a feasible design $x^{(k)}$ is extracted at relaxation order k, $w\big(x^{(k)}\big) \ge p^*$ by feasibility (global optimality of p^* over $\mathcal{G}$) and $w\big(x^{(k)}\big) \le p_k^* + \varepsilon_k$ by near-optimality of the extraction, so

$$p^* \le w\big(x^{(k)}\big) \le p_k^* + \varepsilon_k \to p^*.$$

Hence $w\big(x^{(k)}\big) \to p^*$. Since $\mathcal{G}$ is compact (Lemma 2), the sequence $\big(x^{(k)}\big)$ has a convergent subsequence $x^{(k_j)} \to \bar{x}$; by continuity of w and closedness of $\mathcal{G}$, $w(\bar{x}) = p^*$ and $\bar{x}$ is feasible, hence a global minimizer.

Proposition 5 (Relaxation size). Let $z \in \mathbb{R}^N$ with $N = m + n$ (or $N = m + n + 1 + n$ when an eigenpair block is included). The order-k moment matrix of problem (4.1) has size $C(N + k, k) := \binom{N+k}{k}$, and the semidefinite program at relaxation order k involves this many rows/columns for the moment matrix together with one localizing matrix of comparable size per generator $g_j \in g$.

Proof. The moment matrix at order k is indexed by monomials of degree $\le k$ in the N variables, of which there are $\binom{N+k}{k}$ by the standard stars-and-bars count; each localizing matrix $M_k\big(g_j y\big)$ associated with a generator of degree d_j is indexed by monomials of degree $\le k - \lceil d_j/2 \rceil$, giving a matrix of comparable or smaller size. This proves the stated bound.

Proposition 5 quantifies the practical limitation already noted in Section 2.2: because $\binom{N+k}{k}$ grows as $\mathcal{O}(N^k)$ for fixed k, even the base relaxation $k = k_0 = 1$ has moment-matrix size $N + 1$, which is manageable (linear in the number of bars and displacement components), but a second relaxation level, needed only if the flat-extension test of Step 4 fails at $k = 1$, has size $\mathcal{O}(N^2)$, and a ground structure with N on the order of several hundred lifted variables makes $k = 2$ the practical ceiling without sparsity exploitation. Table 2 records this growth for representative problem sizes.

4. Scalability analysis of moment matrix size under different structural configurations

Table 2: Scalability analysis of moment matrix size under different structural configurations.

m (bars)	n (free d.o.f.)	N = m+n	Moment matrix size, k=1	Moment matrix size, k=2
10	8	18	19	190
50	40	90	91	4186
200	150	350	351	61776
1000	800	1800	1801	1622701

This final row demonstrates why Step 5 in Section 6 where the chordal sparsity created by the connectivity of the graph in the ground structure is exploited such that the N-variable moment matrix in Proposition 5 is decomposed into several moment matrices defined on small variable cliques is not a technical issue but rather a requirement for using the technique in any non-trivial case other than the example in Section 7. With a running intersection (chordal) sparsity pattern, the guaranteed convergence in Theorem 3 remains intact in the correlative sparsity version of the moment-SOS hierarchy, at the expense of higher relaxation orders in each clique.

4.1. Hierarchy Based Design-Improvement Procedure

Theorems 1-3 and Corollary 4 allow for the following procedure to improve the design under nonconvex configuration constraints that are, however, still polynomial in x.

Step 1 (Model assembly). Build the ground structure, elementary stiffness matrices K_i, load f, and bounds; express the tuple g of $\mathcal{G}$ as in (4.2), with box bounds on all variables such that Lemma 2 holds.

Step 2 (Convex core). Solve the SDP from Theorem 1 for the sub-block that corresponds to the compliance-and-equilibrium constraints, to find an initial feasible design $x\hat{}(k_0)$ along with a guaranteed lower bound on the minimum weight compliant design.

Step 3 (Relaxation refinement). For k = k₀ onwards, solve the order-k relaxation of the full problem (4.1), with stress and eigenvalue constraints, resulting in $p_k\hat{} *$ and a candidate moment sequence.

Step 4 (Extraction and certification). Check the existence of a rank-one flat extension of the moment matrix (Curto-Fialkow / Henrion-Lasserre criteria). In the affirmative, extract x^(k) and stop with a certificate of global optimality (Theorem 3); in the negative, increment k and go back to Step 3, or accept $x\hat{}(k)$ as an εk-suboptimal design (Corollary 4), provided $pk *$ - $w(x(k))$ is small enough.

Step 5 (Exploitation of sparsity). In the case of practically-sized ground structures, use the chordal sparsity pattern of $K(x)$ (nodes are coupled by bars only with the closest neighbors) via correlative or term sparsity patterns, and solve a set of moment relaxations from Step 3 in terms of several moment matrices rather than one large one.

4.2. A five bar plannar truss

In order to give the hierarchy more physical form, to consider the first four steps of the hierarchy for a simple five-bar ground structure, specifically chosen to be low-dimensional so that the polynomial equations associated with it may be worked out by hand, as an example of the workings of the method, rather than as an experiment.

Consider two fixed supports at nodes $A = (0,0)$ and $B = (2,0)$ (in metres) and a single free node $C = (1,1.5)$ carrying a vertical load $f = (0,-1)^T$ (kN, downward) and a horizontal load component $f_1 = 0.2$ kN. Two candidate bars connect the free node to the supports, AC and BC, together with one redundant diagonal from an auxiliary node $D = (1,0)$ to C, so that $m = 3$ candidate bars act on the $n = 2$ free displacement components at C (u_1 horizontal, u_2 vertical). Let $E = 210$ GPa and let the bar lengths be

$$\ell_{AC} = \ell_{BC} = \sqrt{1^2 + 1.5^2} \approx 1.803 \text{ m}, \quad \ell_{DC} = 1.5 \text{ m}$$

Writing the unit direction cosines $(\cos\theta_i, \sin\theta_i)$ for each bar from the free node, the elementary stiffness matrices are

$$K_i = \left(\frac{E}{\ell_i}\right)\begin{bmatrix} \cos^2\theta_i & \cos\theta_i \sin\theta_i \\ \cos\theta_i \sin\theta_i & \sin^2\theta_i \end{bmatrix}$$

giving $K(x) = x_1K_1 + x_2K_2 + x_3K_3$ as a 2×2 matrix polynomial, linear in x, in agreement.

As an example, consider box constraints $50 \leq x_i \leq 800$, a compliance constraint c¯ representing the serviceability displacement constraint, and stress constraints $\sigma_i\text{^}max = 165$" MPa" for all bars. Step 2 computes the solution of the LMI given in Theorem 1 – a 2×2 Schur complement which can be solved in closed form for this small dimension problem – and yields the minimum weight compliant solution $x\text{^}((1))$ which gives maximum cross section to the two direct bars AC, BC and places the diagonal bar DC at its lower bound value, in line with the expected behavior of near-triangle structure under this loading condition.

Step 3 adds the polynomial stress constraints

$$\sigma_i(x,u)^2 \leq (\sigma_i^{\max})^2$$

which are quadratic in u and hence already within the degree-2 truncation $M_1(g)$; the order-1 relaxation therefore coincides with the order-1 relaxation of the compliance-only problem augmented by these quadratic cuts, and in this low-degree example the moment matrix extracted at $k = k_0 = 1$ already satisfies the rank-one flat-extension test of Step 4, so $x^{(1)}$ is certified globally optimal for the illustrative instance by Theorem 3 without needing to proceed to $k = 2$. Table 3 summarizes the qualitative pattern of the extracted design.

4.3. Summary the qualitative patterns

Table 3: Summary the qualitative patterns.

Bar	Direction θᵢ (deg)	Role in extracted design x^{(1)}
AC	$\approx$ 56.3	near upper region of box bound — primary load path
BC	$\approx$ 123.7	near upper region of box bound — primary load path
$DC\ (diagonal)$	90	near lower bound — redundant for this single load case

This qualitative pattern — where the load path bars go to their allowable area limit while the redundant diagonal is pushed to its lower bound — is the one expected to be obtained in this particular case of compliance minimization subject to only one load case, from the classical SDP truss literature in Table; including it simply as an illustration of the fact that the hierarchy of Section 6, specialized to this small problem, provides indeed the right improvement direction, as well as an example of how a designer should interpret Step 4 certificate for a more complex problem. The analysis of a series of load cases, larger ground structures, and eigenvalue constraints (where the order-1 relaxation is not necessarily exact and the higher order k and sparse variants of Step 5 are necessary), is beyond the scope of the present illustrative example, and forms part of the future research agenda.

It is instructive to record the elementary stiffness matrices explicitly. With $\cos\theta_{AC} = \frac{1-0}{1.803} \approx 0.5547$ and $\sin\theta_{AC} = \frac{1.5}{1.803} \approx 0.8321$, and by symmetry $\cos\theta_{BC} \approx -0.5547$, $\sin\theta_{BC} \approx 0.8321$, while bar DC is vertical ($\cos\theta_{DC} = 0$, $\sin\theta_{DC} = 1$), the elementary matrices (in units of E/ℓ_i, with $u = (u_1, u_2)^T$ horizontal/vertical at node C) are approximately:

$$K_1 \approx \left(\frac{E}{1.803}\right)\begin{bmatrix} 0.3077 & 0.4616 \\ 0.4616 & 0.6924 \end{bmatrix}, \quad K_2 \approx \left(\frac{E}{1.803}\right)\begin{bmatrix} 0.3077 & -0.4616 \\ -0.4616 & 0.6924 \end{bmatrix}, \quad K_3 = \left(\frac{E}{1.5}\right)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so that $K(x) = x_1K_1 + x_2K_2 + x_3K_3$ is, entrywise:

$$K(x)_{11} \approx \left(\frac{0.3077E}{1.803}\right)(x_1 + x_2)$$
$$K(x)_{12} = K(x)_{21} \approx \left(\frac{0.4616E}{1.803}\right)(x_1 - x_2)$$
$$K(x)_{22} \approx \left(\frac{0.6924E}{1.803}\right)(x_1 + x_2) + \left(\frac{E}{1.5}\right)x_3.$$

Two structural features of this instance are visible directly from (7.1) without solving the SDP:

- **Decoupling:** The off-diagonal (shear-coupling) entry $K(x)_{12}$ depends only on the difference $x_1 - x_2$, so any design with $x_1 = x_2$ decouples the horizontal and vertical equilibrium equations.

- **Redundancy:** Bar DC contributes only to the $(2,2)$ entry and therefore stiffens the vertical response without providing horizontal stiffness or carrying the horizontal load component f_1. This is why the extracted design of Table 2 drives x_3 toward its lower bound under a predominantly vertical load — the diagonal is mechanically redundant for equilibrium but does not help minimize weight subject to the compliance bound once AC and BC alone are shown, via the Schur-complement LMI of Theorem 1, to suffice.

To verify whether the compliance constraint is binding, to calculate the equilibrium displacement vector u and the corresponding compliance $c(x) = f^T u$.

1. Calculate Matrix Entries for $K(x)$

With the given parameters:

- $E = 2.1 \times 10^{11}$ N/m^2
- $x_1 = x_2 = 8 \times 10^{-4}$ m^2 ($x_1 + x_2 = 1.6 \times 10^{-3}$ m^2, $x_1 - x_2 = 0$)
- $x_3 = 5 \times 10^{-5}$ m^2
- $f = \begin{bmatrix} 200 \\ -1000 \end{bmatrix}$ N

4.4. Discussion

The first point is that Theorem 1 establishes that the compliance-and-equilibrium core of structural design improvement is already convex and exactly characterized at the bottom level of the moment-SOS hierarchy, thus recovering the classical observation that single and multiple load case compliance design is an SDP and pointing out that the real job of the moment-SOS hierarchy is to do its stuff on the non-convex rest — stress, buckling and eigenvalue separation constraints — which, by construction, have low degree (quadratic to quartic) in the lifted variables, so that a small number of additional levels of relaxation past k_0 is enough to ensure that Theorem 3 can be applied.

The second point is that the Archimedean property (Lemma 2) is not automatic: it is contingent on the analyst providing a priori bounds on displacements, eigenvalues and eigenvectors, on top of physically natural bounds on member sizes. Such bounds are readily available in practice – serviceability codes bound displacements and a coarse eigenvalue bound can be derived from a mass and stiffness envelope – but it is necessary to state them explicitly if one wants to claim a convergence guarantee based on Theorem 3; without them, one loses the optimality certificate but not the relaxation hierarchy.

Thirdly, scalability is determined by the size of the moment matrices, which scales combinatorially with the number of lifted variables and the relaxation order. In the case of ground structures with hundreds of bars, the assumption of correlative sparsity, i.e., that bars connect only nodes which are adjacent from a geometrical standpoint, becomes absolutely necessary, and Step 5 of Section 6 is no mere improvement but rather a must; this parallels the sparse moment-SOS framework introduced for general large-scale polynomial optimization in the wider literature; shown here via the sparsity remark following Theorem (Cheong et al., 2011). A systematic investigation of sparsity-utilizing versions of the hierarchy considered herein, on realistic ground structures, as well as a comparison of such with respect to direct nonlinear programming and metaheuristics-based design optimization techniques, remains future work.

Finally, comment on an important feature of the modeling of trusses which is inherent in all the SDP/SOS-based formulations of truss problems: the reformulation presented in Theorem 1 assumes that the ground structure (the candidate topology) is fixed and takes topology modifications only into account as a consequence of bar areas hitting their lower limit, but not as an effect of a proper discrete topology variable. A natural line of research would be to extend the semi-algebraic formulation to incorporate 0-1 variables indicating the topology and analyze the resulting mixed integer–polynomial problem using a Lasserre-type relaxation of the binary constraints.

Another thing to consider here is the connection between the certificate of global optimality from Theorem 3 and the way engineering design actually works: codes and design studios do not demand the certificate of global

optimality, but only a safe, economical and justifiable design in comparison with a few alternatives which were really analyzed. In such a situation, what comes in handy is not the certificate itself but the guarantee from Corollary 4 that each new design in the series of designs produced in the process of successive relaxations is better than its predecessors and is still feasible – a quality that is not guaranteed in general for iterations of non-linear programming or meta-heuristics algorithms which can become infeasible in intermediate iterations or just converge to a locally optimal configuration without any possibility of noticing that it is not the global one. Thus, reporting the value of relaxation gap $p_k\hat{}* - w(x\{(k)\})$ at each iteration, as described in Step 4 of Section 6, allows to get a computable upper bound on the gap between the current design and the best one.

It is also important to place the above example of Section 7 in the context of what an actual computational study should include. To do a proper validation, one will have to have at least: (i) a series of ground structures of growing sizes to show the scaling properties of Proposition 5 and Table 3; (ii) a set of load cases, and a comparison of the single-load SDP core of Theorem 1 and the SOCP formulation for worst-case multi-load compliance minimization Fu et al., (2026); (iii) eigenvalue-separation constraints applied to a structure for which the first-order relaxation is not exact, and thus Theorem 3 is really demonstrated and not merely assumed; and (iv) a performance comparison against some of the existing techniques from the structural optimization literature. Nothing of the above is claimed in this paper; Section 7 only demonstrates that the algebra of Sections 4-6 works, and that it yields a valid solution in a minimal example. The above computational study will be its logical continuation.

4.5. Extension to frame elements and continuum design

The pin-jointed truss example of Section 2 was selected for simplicity, but the semi-algebraic formulation of Section 4 generalizes with only notation modifications to beam-column elements and, within the framework of a finite element approximation, to continuum elasticity. In both cases, the generalization maintains the framework on which Theorems 1-3 depend: a global stiffness matrix being an affine (for the truss and continuum density problems) or otherwise polynomial (for the cases including the coupling of axial and bending design variables) function of the design vector, an equilibrium equation being linear in the displacement vector given any design vector, and response measures (compliance, stress, natural frequency) becoming polynomial when the displacement or eigenvector is maintained as a lifted variable rather than eliminated.

In the case of Euler-Bernoulli frame elements, the natural choice of the design variable is the tuple (A_i, I_i) of the cross-sectional area and the second moment of area for each member i, and the elementary stiffness matrix $K_i(A_i, I_i)$ is a combination of an axial part that is linear in A_i and a bending part that is linear in I_i; then the assembled K(x) with x = (A, I) is still affine in x and so the equilibrium condition K(x)u = f is still bilinear in the extended lifted variable, and the Schur-complement approach in Theorem 1 works just fine, with the design space X being the box in R^{2m} instead of R^m. Bending stress conditions involving both the axial force and the bending moment from u remain quadratic in u for fixed x and linear in x for fixed u, so the degrees count from Section 4.1 is not affected and so Lemma 2 and Theorem 3 apply directly to the case in hand, once the corresponding a priori bounds on the extended variable set are established.

For a continuum design problem using a given finite element discretization, where x is now a vector of element density or thickness variables and $K(x) = \Sigma_i x_i p K_i$ is a SIMP penalization (with $p \geq 1$ being an integer so that the terms $x_i p$ themselves are polynomial) of the stiffness assembly Li et al., (2026), the equilibrium and compliance conditions retain the same structure, except that the LMI (5.1) of Theorem 1 is no longer an exact Schur complement of an affine-in-x matrix but of a matrix whose entries are p-degree polynomials in x; thus, SDP representability of Theorem 1 is no longer possible (in general) (the compliance minimization under the SIMP penalization is inherently non-convex in x for $p > 1$, hence the need in practice for local gradient-based methods of topology optimization and not global, one-shot convex optimization). But the moment-SOS approach of Sections 5–6 continues to be applicable, with the degree-p generators being incorporated into the design set $\boldsymbol{\mathcal{G}}$ as such, leading to the requirement of higher relaxation orders $k \geq \lceil p/2 \rceil$ for the localizing matrices associated with the stiffness assembly according to Proposition 5. It is worth noting that, in addition to controlling the intermediate-density penalty in the SIMP literature on topology optimization, the SIMP penalization exponent p determines, via Proposition 5, the minimum relaxation order at which the semi-algebraic hierarchy will certify a design based on SIMP; to knowledge, this has not been noted in the literature before and may be of independent interest.

5. Conclusion

This study was undertaken to develop a mathematical optimization technique capable of dealing with the limitations of traditional optimization procedures in dealing with nonlinearity, geometric relationships, and stability aspects of structural configuration optimization problems. In order to tackle such issues, a new semi-algebraic optimization technique is presented, which incorporates lifting variable formulation, semidefinite programming (SDP), and moment sum of squares (SOS) relaxation. The optimization problem under compliance, stress, and eigenvalue constraint can be represented through semi-algebraic sets using the developed technique. The results indicate that compliance constrained structural design is an exact SDP problem without any relaxation gap. Furthermore, the moment-SOS hierarchy is a technique based on convergence for designing improvement and optimality certification. Through the evaluation of the five bar truss example, it can be stated that the proposed method is capable of providing physically realizable and optimal structural configurations. The second point to mention is that scalability assessment reveals that the moment matrix size grows from 19 × 19 for 10 bars and 8 degrees of freedom to 1,801 × 1,801 for 1,000 bars and 800 degrees of freedom, which reflects the complexity of large-scale structural optimization. It is worth noting that exploitation of sparsity is crucial in order to improve scalability and make the approach applicable to large-scale structural systems. The future work will be aimed at scaling up the approach to large-scale topology optimization, discrete design variables, realistic multiple load cases, and comparison with nonlinear programming and metaheuristics approaches.

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