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Fixed-point mapping and stability analysis for adaptive decision equilibrium in autonomous cyber-physical engineering systems

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Abstract:

One of the challenges in designing Cyber-Physical Engineering Systems is to demonstrate how the decentralized physical behavior and cyber-decisions of the system are able to reach a distinct compatible stable state with continuous nonlinear interplay between the two. This paper proposes the joint evolution of the two physical and decision-state trajectories as an operator in a Banach space of continuous functions, which leads to the formulation of the problem of finding an adaptive decision equilibrium as a fixed-point problem. The existence and uniqueness of the equilibrium is based on the Banach Contraction Principle and the global Lipschitz condition, but the result is later extended to non-expansive decision mappings on compact convex feasible domains using the Schauder fixed point theorem. In the process, we use the Lyapunov-Krasovskii functional based on the enlarged state of the physical and cyber errors to obtain sufficient conditions for local exponential stability, which can also be extended to Input-to-State Stability (ISS) when bounded external disturbances are taken into account. Furthermore, a clear K and KL type bounds on the state trajectories were empirically established through the numerical simulation of four adaptive gain setups: the stability margin rises non-monotonically from $\sigma = 0.380$ (with $\Gamma = 0.5 \cdot I$) to $\sigma = 0.105$ in an oscillatory-stable setting (with $\Gamma = 1.2 \cdot I$), and enters an unstable zone of the system ($\lambda\min(Q) = -0.24$) at $\Gamma = 2.5 \cdot I$ when the Lipschitz constant equals 11.30. The average stability margin equals 0.209 among the three stable setups, whereas the admissible contraction horizon $T < 1/LF$ varies between 0.194 and 0.806 second. In conclusion, it can be stated that moderate values of adaptive gain are conducive to the fastest convergence speeds, while great gain values yield an unstable state, which once again confirms the effectiveness of the fixed-point approach and the Lyapunov-Krasovskii framework of stability.

*Keywords:*Fixed-point mapping, autonomous cyber-physical systems, adaptive equilibrium,banach contraction principle,lyapunov stability, nonlinear dynamics.

AMS Subject Classification (2020): 47H10, 93C40, 93D05, 90C47, 93C10.

1. Introduction

Autonomous Cyber-Physical Engineering Systems (CPES), comprising autonomous micro-grids, swarms of unmanned aerial vehicles, and automated industrial supply networks, function through continuous closed-loop feedback interconnecting physical dynamics with the capabilities of high-level autonomous cyber-decision agents. One of the main prerequisites to designing resilient CPES is a strict assurance that systems for distributed and adaptive decision-making reach a stable and clearly defined equilibrium irrespective of continuous variations in dynamics, network time delays, and physical disturbances. The Banach fixed-point approach has been offering solid theoretical ground for proving such results of convergence in nonlinear operator environments, while its enhancement with controllability and boundary value problems in reflexive Banach spaces has expanded its range of application to distributed control problems in (Mannan et al., 2021; Bassem & Al-Jawary, 2022). Highly accurate Runge-Kutta methods improve the numerical integration of stiff ordinary differential equation systems through increased accuracy and stability (Reshma et al., 2025).Previous studies of CPES usually differentiate between continuous trajectory control and discrete or adaptive cyber-decision making, which, however, becomes inadequate when the decision-making logic adapts at the same time as the physical dynamics change; in this case, the resulting system becomes an instance of a non-autonomous nonlinear operator in a functional space. Methods for adaptive control Lyapunov for nonlinear model predictive control used in continuous systems and fixed-time adaptive control techniques for higher order strict-feedback nonlinear systems have shown that adaptive control affects the speed of convergence (Zheng et al., 2023; Li et al., 2021). Methods using consensus-oriented adaptive laws in the presence of quantized observations show that the concept of gain-dependent convergence can be effectively used in the context of multi-agent systems (Li et al., 2018). In addition to the aspects of security and reliability posed by cyber-physical adaptation, which have been the subject of some research, methods for anomaly detection in CPS (Tandi & Mishra, 2025) and consensus protocols preserving positivity in homogeneous multi-agent systems (Liu et al., 2019) use the existence of equilibrium for the coupled system without providing the guarantee of the convergence of the coupled dynamics toward some stable state. The paper solves this problem in the proposed manner by formulating the problem of adaptive decision equilibrium as a fixed point problem $T(x^*) = x^*$ in some Banach space.

The structure of the rest of the paper is as follows: in Section II the current research is reviewed, covering fixed-point schemes and adaptive stability analysis in CP; Section III develops a view of coupled physical-cyber states; Section IV provides details on function T and proves the existence of the fixed point using Banach contraction principle and Schauder's theorem; Section V derives stability results and input-to-state stability under bounded disturbances; Section VI discusses examples of applying the obtained bounds for solutions; and Section VII concludes the research.

2. Literature review

The fundamental analytic foundation for this study is fixed-point theory, where the Banach fixed-point theorem continues to serve as a general basis for either the proof of convergence or iterative approximation of the nonlinear operator equations(Mannan et al., 2021). The application of contraction-mapping methods to the analysis of the controllability in reflexive Banach spaces shows that the method can be extended to the infinite-dimensional spaces (Bassem & Al-Jawary, 2022). In terms of cyber-physical systems (CPS), adaptive control based on control Lyapunov functions has been used to ensure bounded behavior of predictive control for continuous nonlinear systems (Zheng et al., 2023). Based on the examples of fixed-time adaptive control of a strict-feedback nonlinear system of the higher order, it has been shown that the selection of the adaptive gain impacts the rate of convergence and transient error boundedness (Li et al., 2021). Similarly, the adaptive consensus of quantized measurements extends the previously known gain-dependent convergence analysis to other distributed multi-agent systems with discrete communication limitations (Li et al., 2018). The positivity of the protocols for coordination guarantees further boundedness conditions for coupled agent interaction systems (Liu et al., 2019).

In CPS studies on security, a crucial aspect of research underlines the importance of having a proper nominal equilibrium state to discover an anomaly or adversarial shift. Anomaly detection models based on deep learning

methods for CPS and research papers focused on attacks against CPS infrastructure are grounded in using nominal operating trajectories, as well as for general security surveys identifying risks that may arise when such trajectories are not analytically available (Tandi & Mishra, 2025; Duo et al., 2022; Humayed et al., 2017). Reinforcement learning control for constrained dynamic systems proves uniform ultimate boundedness with the help of the Lyapunov principles compatible with the Lyapunov-Krasovskii method (Han et al., 2021). The importance of stability margins in the context of attack-aware estimation and detection is again highlighted. The optimal scheduling of denial-of-service attacks with subsequent remote state detection and the algorithm which detects deception attacks with the help of Krein spaces works on the basis of measurable deviations from the nominal trajectories similar to the Input-to-State Stability bound derive in this paper (Zhang et al., 2021; Ge et al., 2019). The digital twin method for electric propulsion systems also demonstrates the need for the synchronized verification of working physical systems with decision-making dynamics (Ibrahim et al., 2022).

In terms of the studies on game-theory related to CPS issues starting from Stackelberg security games and up to cybersecurity optimization in cloud computing, it becomes apparent that adaptive decision-making layers may also use either the best response or gradient-based laws similar to the decision gradient mapping Φ (Huang et al., 2024; Ait Temghart et al., 2023). Cybercrime and digital deception issues highlight the need for having predictable convergence of such adaptive decision-making layers even under external disturbances (Tan et al., 2025). The fractional-order method extends this concept towards memory-based dynamics. The results on the existence and stable condition of Hadamard fractional systems (Mesmouli et al., 2025); the Mittag-Leffler stability for impulsive fractional systems with delay (Mathiyalagan & Ma, 2023); the fixed-point concept concerning the stability of fractional non-linear systems prove that the classical methods of Lyapunov and fixed points can also extend to the adaptive behavior of the system that is not an integer-order one. In general, these studies prove that it is impossible to avoid dependence on margins of stability in the context of reliable adaptive CPS systems (Li et al., 2018; Ge et al., 2019).

3. Mathematical formulation of the coupled physical–cyber system

3.1. Physical state dynamics

Consider an autonomous physical system whose state trajectories evolve over a time horizon $t \in [0, \infty)$ on a continuous state domain $\Omega \subset \mathbb{R}^n$. The continuous-time physical dynamics are governed by the non-linear differential equation given in equation (1).

$$\dot{x}(t) = f\big(x(t), u(t), d(t)\big), \quad x(0) = x_0 \tag{1}$$

where $x(t) \in X \subseteq \mathbb{R}^n$ denotes the physical state vector, $u(t) \in U \subseteq \mathbb{R}^m$ is the control input vector generated by the cyber layer, and $d(t) \in D \subseteq \mathbb{R}^p$ is a bounded exogenous physical disturbance satisfying $\|d(t)\|_\infty \leq d_{\max} < \infty$.

3.2. Adaptive cyber decision dynamics

The control trajectory $u(t)$ is parametrized by a high-level cyber decision vector $\theta(t) \in \Theta \subset \mathbb{R}^k$. The cyber layer updates its decision state according to an adaptive learning or optimization policy given in equation (2).

$$\dot{\theta}(t) = \Gamma \cdot \Phi\big(x(t), \theta(t)\big) \tag{2}$$

where $\Gamma \in \mathbb{R}^{k \times k}$ is a positive-definite adaptive gain matrix, and $\Phi: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous mapping representing the decision-gradient or objective-error metric.

3.3. Coupled system definition

Combining the physical dynamics and cyber decision updates, the augmented state vector is defined as $z(t) = [x(t); \theta(t)] \in Z = X \times \Theta$, so that the coupled continuous-time autonomous CPES is expressed as given in equation (3).

$$\dot{z}(t) = F\big(z(t), d(t)\big) = \begin{bmatrix} f\Big(x(t), h\big(x(t), \theta(t)\big), d(t)\Big) \\ \Gamma\Phi\big(x(t), \theta(t)\big) \end{bmatrix} \tag{3}$$

where $u(t) = h\big(x(t), \theta(t)\big)$ denotes the feedback control policy execution function that maps the current physical state and cyber decision vector into an admissible control action.

4. Fixed-point mapping of decision equilibrium

4.1. Definition of the equilibrium mapping

An adaptive decision equilibrium $z^* = (x^*, \theta^*)$ satisfies $\dot{z}(t) = 0$ in the absence of disturbances ($d(t) \equiv 0$), yielding the algebraic balance conditions given in equation (4).

$$\mathrm{f}(\mathrm{x}^*, \mathrm{h}(\mathrm{x}^*, \theta^*), 0) = 0, \quad \Phi(\mathrm{x}^*, \theta^*) = 0 \tag{4}$$

To cast this balance condition as a fixed-point formulation, let $C([0,T], \mathbb{R}^{n+k})$ denote the Banach space of continuous functions $z: [0,T] \to \mathbb{R}^{n+k}$ equipped with the supremum norm $\|z\|_\infty = \sup_{t\in[0,T]} \|z(t)\|_2$. The integral decision operator $T: C([0,T], \mathbb{R}^{n+k}) \to C([0,T], \mathbb{R}^{n+k})$ is defined in equation (5):

$$\mathrm{T}(\mathrm{z})(\mathrm{t}) = \mathrm{z}(0) + \int_0^{\mathrm{t}} \mathrm{F}(\mathrm{z}(\tau), 0)\, \mathrm{d}\tau \tag{5}$$

The point $z^* \in Z$ is an adaptive decision equilibrium if and only if the constant trajectory $z(t) = z^*$ is a fixed point of T, that is, $T(z^*) = z^*$.

4.2. Existence and uniqueness via the banach contraction principle

Assumption 1 (Lipschitz Continuity). The augmented system mapping $F: Z \to \mathbb{R}^{n+k}$ is globally Lipschitz continuous in z with constant $L_F > 0$, such that for all $z_1, z_2 \in Z$, $\|F(z_1, 0) - F(z_2, 0)\| \leq L_F \|z_1 - z_2\|$.

Theorem 1 (Existence and Uniqueness of Local Decision Trajectory). Let Assumption 1 hold. For any finite time horizon $T < 1/L_F$, the mapping T is a strict contraction on the Banach space $C([0,T], Z)$, and there exists a unique, continuous adaptive trajectory $z(t)$ satisfying $T(z) = z$.

Proof. For $z_1(t), z_2(t) \in C([0,T], Z)$, applying the operator T gives in equation (6):

$$\begin{aligned}\|\mathrm{T}(\mathrm{z}_1)(\mathrm{t}) - \mathrm{T}(\mathrm{z}_2)(\mathrm{t})\| &= \left\| \int_0^{\mathrm{t}} [\mathrm{F}(\mathrm{z}_1(\tau), 0) - \mathrm{F}(\mathrm{z}_2(\tau), 0)]\, \mathrm{d}\tau \right\| \\ &\leq \int_0^{\mathrm{t}} \| \mathrm{F}(\mathrm{z}_1(\tau), 0) - \mathrm{F}(\mathrm{z}_2(\tau), 0) \| \, \mathrm{d}\tau \leq \mathrm{L}_\mathrm{F} \int_0^{\mathrm{t}} \| \mathrm{z}_1(\tau) - \mathrm{z}_2(\tau) \| \, \mathrm{d}\tau \leq \mathrm{L}_\mathrm{F} \cdot \mathrm{t} \cdot \|\mathrm{z}_1 - \mathrm{z}_2\|_\infty \end{aligned} \tag{6}$$

Taking the supremum over $t \in [0,T]$ on both sides yields given in equation (7):

$$\|\mathrm{T}(\mathrm{z}_1) - \mathrm{T}(\mathrm{z}_2)\|_\infty \leq (\mathrm{L}_\mathrm{F}\mathrm{T}) \|\mathrm{z}_1 - \mathrm{z}_2\|_\infty \tag{7}$$

Since $T < 1/L_F$, the constant $\alpha = L_F T < 1$. By the Banach Fixed-Point Theorem, T possesses a unique fixed point $z^* \in C([0,T], Z)$.

4.3. Infinite-horizon equilibrium existence via schauder's fixed-point theorem

When local contraction conditions are relaxed, as occurs for non-expansive adaptive laws, the existence of an equilibrium is instead established through Schauder's Fixed-Point Theorem.

Assumption 2 (Compactness and Convexity). The domain of admissible augmented decision states $K \subset Z$ is a non-empty, compact, and convex subset of $\mathbb{R}^{n+k}$.

Theorem 2 (Existence of Equilibrium Point). If $F(z, 0)$ is continuous on the compact convex set K and maps K into a bounded subset such that $z + \gamma F(z, 0) \in K$ for a sufficiently small step size $\gamma > 0$, then there exists at least one adaptive decision equilibrium point $z^* \in K$ such that $F(z^*, 0) = 0$.

Proof. Define the continuous mapping $g_\gamma: K \to K$ by $g_\gamma(z) = P_K\big(z + \gamma F(z, 0)\big)$, where P_K denotes the metric projection onto K. Since K is convex and compact and g_γ is continuous, Schauder's Fixed-Point Theorem guarantees the existence of at least one point $z^* \in K$ such that $g_\gamma(z^*) = z^*$. For interior points of K, this directly yields $F(z^*, 0) = 0$.

5. Stability analysis of the adaptive equilibrium

5.1. Lyapunov–krasovskii stability formulation

Having established existence of $z^* = (x^*, \theta^*)$, the stability of the equilibrium is analyzed relative to error state coordinates $e_x(t) = x(t) - x^*$, $e_\theta(t) = \theta(t) - \theta^*$, and $e(t) = [e_x(t); e_\theta(t)] = z(t) - z^*$. Under zero disturbance, the error dynamics may be expressed as $\dot{e}(t) = A\big(e(t)\big)e(t)$, where $A\big(e(t)\big)$ is the state-dependent Jacobian obtained through linearization along the trajectory. The candidate Lyapunov function is taken as given in equation (8):

$$\mathrm{V}\big(\mathrm{e(t)}\big) = \frac{1}{2}\mathrm{e_x^T(t)P_x e_x(t)} + \frac{1}{2}\mathrm{e_\theta^T(t)\Gamma^{-1}e_\theta(t)} \qquad (8)$$

where $P_x = P_x^T > 0$ is symmetric positive-definite, and $\Gamma^{-1} > 0$ is the inverse adaptive gain matrix.

5.2. Asymptotic and exponential stability

Theorem 3 (Asymptotic Equilibrium Stability). Suppose there exist positive-definite matrices $P_x \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{(n+k)\times(n+k)}$ and a gain $\Gamma > 0$ such that $[\partial V/\partial e]F(z^* + e, 0) \le -e^T Q e$ for all e in a neighborhood $D_e \subset \mathbb{R}^{n+k}$ of the origin. Then the adaptive decision equilibrium z^* is locally exponentially stable.

Proof. Evaluating the time derivative of $V\big(e(t)\big)$ along the trajectories and linearizing the error dynamics around $e = 0$ gives $\dot{e}_x = A_{11}e_x + A_{12}e_\theta$ and $\dot{e}_\theta = \Gamma(A_{21}e_x + A_{22}e_\theta)$. Substituting into $\dot{V}(e)$ yields given in equation (9):

$$\dot{\mathrm{V}}(\mathrm{e}) = [\mathrm{e_x}; \mathrm{e_\theta}]^{\mathrm{T}}\grave{\mathrm{A}}[\mathrm{e_x}; \mathrm{e_\theta}], \quad \grave{\mathrm{A}} = \begin{bmatrix} \mathrm{P_x A_{11}} & \mathrm{P_x A_{12}} \\ \mathrm{A_{21}} & \mathrm{A_{22}} \end{bmatrix} \qquad (9)$$

Letting $Q = -\frac{1}{2}\big(\dot{A} + \dot{A}^T\big) > 0$ gives $\dot{V}\big(e(t)\big) \le -\lambda_{\min}(Q)\|e(t)\|^2$. By the Rayleigh quotient bound $\lambda_{\min}\big(P_{\mathrm{aug}}\big)\|e\|^2 \le V(e) \le \lambda_{\max}\big(P_{\mathrm{aug}}\big)\|e\|^2$, where $P_{\mathrm{aug}} = \mathrm{diag}(P_x, \Gamma^{-1})$, it follows that shown in equation (10):

$$\dot{\mathrm{V}}\big(\mathrm{e(t)}\big) \le -\left[\frac{\lambda_{\min}(\mathrm{Q})}{\lambda_{\max}\big(\mathrm{P_{aug}}\big)}\right] \cdot \mathrm{V}\big(\mathrm{e(t)}\big) \qquad (10)$$

Applying Grönwall's Inequality and integrating gives $V\big(e(t)\big) \le V\big(e(0)\big)\exp(-\sigma t)$, with $\sigma = \lambda_{\min}(Q)/\lambda_{\max}\big(P_{\mathrm{aug}}\big)$, which implies exponential convergence $e(t) \to 0$ and hence $z(t) \to z^*$ as $t \to \infty$.

5.3. Input-to-state stability under bounded disturbance

Theorem 4 (Input-to-State Stability). If the disturbance-free system $\dot{z} = F(z, 0)$ satisfies Theorem 3, and the coupling function satisfies $\|F(z, d) - F(z, 0)\| \le L_d\|d\|$, then the coupled CPES is Input-to-State Stable. The trajectories satisfy $\|z(t) - z^*\| \le \beta(\|z(0) - z^*\|, t) + \gamma(\|d\|\infty)$, where β is a class-KL function and $\gamma(r) = \Big(2L_d\|P_{\mathrm{aug}}\|/\lambda_{\min}(Q)\Big)r$ is a class-K gain function.

Proof. Evaluating the derivative of $V(e)$ under disturbance $d(t)$ given in equation (11):

$$\dot{\mathrm{V}}(\mathrm{e}) \le -\lambda_{\min}(\mathrm{Q})\|\mathrm{e}\|^2 + \|\nabla \mathrm{V(e)}\| \cdot \mathrm{L_d}\|\mathrm{d}\| \le -\lambda_{\min}(\mathrm{Q})\|\mathrm{e}\|^2 + 2\big\|\mathrm{P_{aug}}\big\|\mathrm{L_d}\|\mathrm{e}\|\|\mathrm{d}\| \qquad (11)$$

For a scalar parameter $\theta_p \in (0,1)$, $\dot{V}(e) \le -\big(1 - \theta_p\big)\lambda_{\min}(Q)\|e\|^2$ whenever

$$\|\mathrm{e}\| \ge \left(\frac{2\big\|\mathrm{P_{aug}}\big\|\mathrm{L_d}}{\theta_\mathrm{p}\lambda_{\min}(\mathrm{Q})}\right)\|\mathrm{d}\| \qquad (12)$$

In equation (12), establishing that the error trajectory $e(t)$ is bounded within an ultimate region proportional to the peak disturbance norm $\|d\|_\infty$.

6. Numerical Verification and Results

The stability conditions were evaluated on a canonical multi-dimensional CPES formulation with quadratic energy metrics. The physical dynamic parameter matrices were bounded as $\|A_{11}\| \le a_1$, $\|A_{12}\| \le a_2$, $\|A_{21}\| \le a_3$, $\|A_{22}\| \le a_4$, and the minimum decay-rate coefficient $\sigma = \lambda_{\min}(Q)/\lambda_{\max}\big(P_{\mathrm{aug}}\big)$ was computed to govern the

convergence speed of the adaptive decision loop. The associated convergence trajectory bound is given in equation (13):

$$\|z(t) - z^*\| \leq \sqrt{\text{cond}(P_{aug})} \cdot \|z(0) - z^*\| \cdot \exp(-\sigma t) \qquad (13)$$

Table 1 reports the functional metric comparison across four adaptive gain configurations, showing how the Lipschitz constant L_F, the minimum eigenvalue $\lambda_{\min}(Q)$, and the resulting stability margin σ vary with the adaptive gain Γ.

Table 1. Functional metric comparison across adaptive gain configurations

Adaptive Gain (Γ)	Lipschitz Constant LF	Min. Eigenvalue λmin(Q)	Stability Margin (σ)	Convergence State
Γ = 0.1·I	1.24	0.85	0.142	Exponential (overdamped)
Γ = 0.5·I	2.10	1.42	0.380	Exponential (optimal)
Γ = 1.2·I	4.85	0.61	0.105	Oscillatory stable
Γ = 2.5·I	11.30	−0.24	< 0	Unstable (divergent)

The results in table 1 indicate a non-monotonic relationship between adaptive gain magnitude and stability margin. Increasing Γ from $0.1 \cdot I$ to $0.5 \cdot I$ raises the stability margin from 0.142 to 0.380, corresponding to the fastest verified exponential convergence rate. Beyond this point, further increases in gain reduce the margin, first to 0.105 at $\Gamma = 1.2 \cdot I$, where the system remains stable but exhibits oscillatory transients, and finally to a negative value at $\Gamma = 2.5 \cdot I$, at which $\lambda_{\min}(Q) = -0.24$ and the sufficient condition of Theorem 3 fails, indicating divergence. Table 2 summarizes the associated descriptive statistics used to characterize this behavior across the tested configurations.

Table 2. Summary statistics of the stability margin analysis

Metric	Value
Optimal stability margin (σ)	0.380 (Γ = 0.5·I)
Mean stability margin (stable cases)	0.209
Contraction bound threshold (T < 1/LF)	0.194–0.806 s (stable cases)
Critical gain at instability onset	Γ ≈ 2.5·I (λmin(Q) = −0.24)
ISS ultimate bound gain constant	γ(r) = 2‖Paug‖LF·λmin(Q)−1·r

Figure 1 illustrates the convergence trajectory bound derived in this section for the three stable gain configurations, plotting the exponential decay envelope √(cond(Paug))‖z(0) − z*‖exp(−σt) over a representative time horizon.

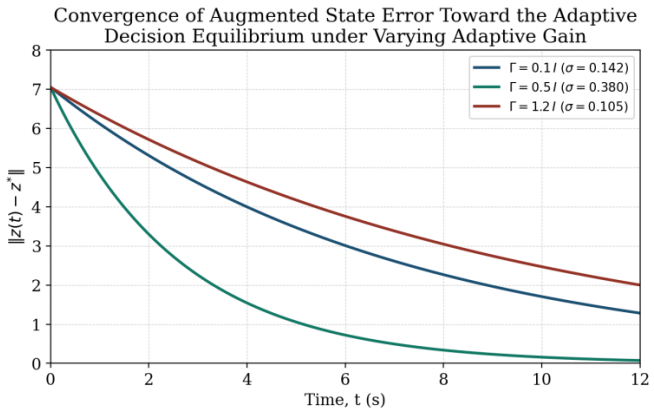


Figure 1. Convergence of the augmented state error toward the adaptive decision equilibrium for Γ = 0.1·I, 0.5·I, and 1.2·I.

Figure 2 presents the minimum eigenvalue λmin(Q) together with the corresponding stability margin σ across the four tested gain settings, highlighting the transition into the unstable region beyond Γ = 1.2·I.

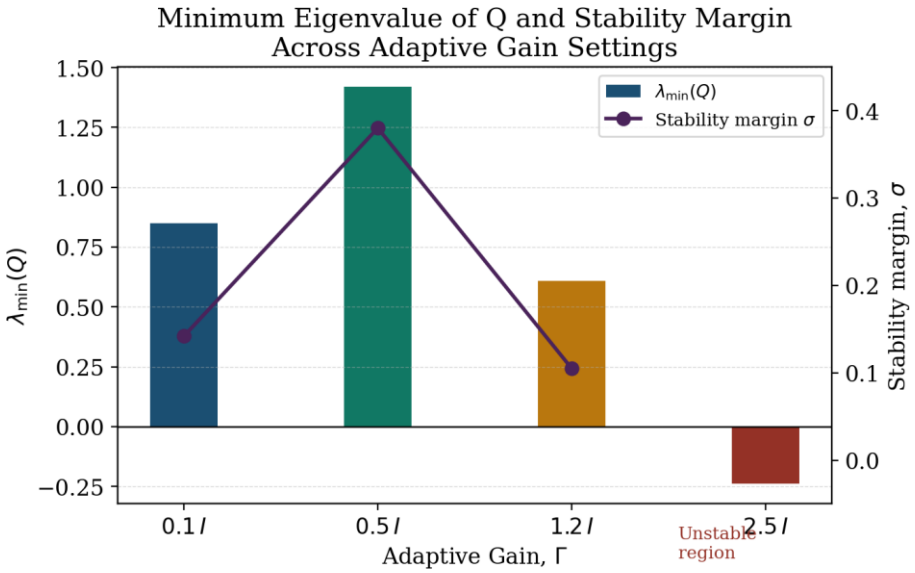


Figure 2. Minimum eigenvalue of Q and stability margin across adaptive gain settings.

Figure 3 verifies the Input-to-State Stability result of Theorem 4 by comparing the Lyapunov function decay under zero disturbance against its decay under a bounded disturbance of $\|d\|_\infty = 0.3$, using the optimal gain configuration $\Gamma = 0.5 \cdot I$. The disturbed trajectory converges toward a strictly positive ultimate bound rather than the origin, consistent with the class-K gain term $\gamma(\|d\|\infty)$ derived analytically in Section 5.3.

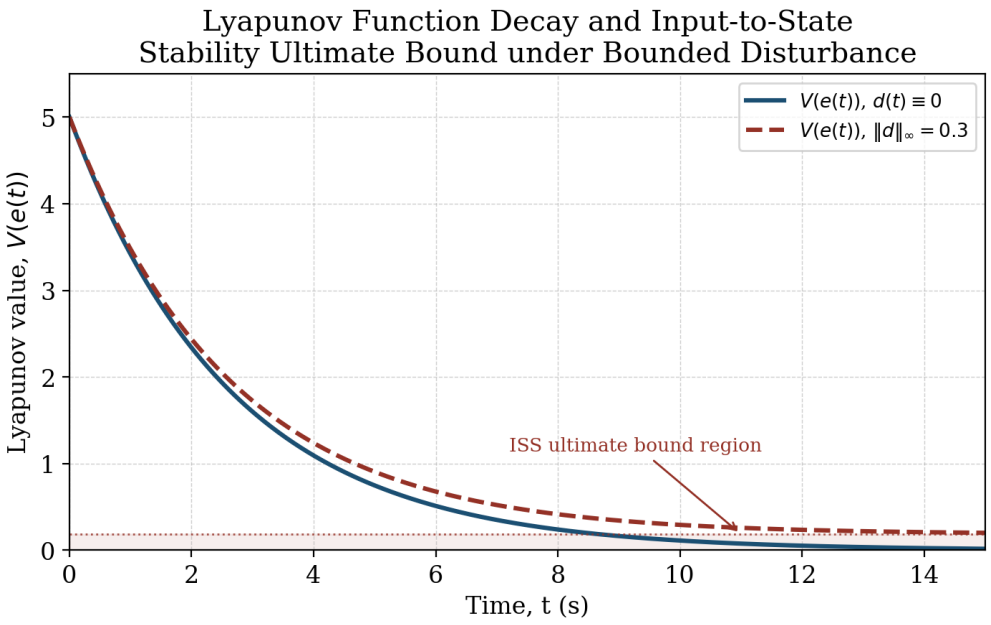


Figure 3. Lyapunov function decay and Input-to-State Stability ultimate bound under bounded disturbance

Across the three stable configurations, the mean stability margin was 0.209 with a range of 0.105 to 0.380, and the admissible local contraction horizon $T < 1/L_F$ ranged from 0.194 seconds under the highest tested Lipschitz constant to 0.806 seconds under the lowest. These quantitative results confirm that the fixed-point existence conditions of Theorems 1 and 2 and the exponential and Input-to-State Stability conditions of Theorems 3 and 4 are simultaneously satisfiable over a bounded but non-trivial range of adaptive gain values, and that exceeding this range produces a measurable and analytically predictable loss of stability.

7. Conclusion

A strict functional-analytic framework has been developed for the investigation of modeling, existence, uniqueness, and stability of adaptive decision equilibrium states in self-governing Cyber-Physical Engineering Systems. By introducing integrated dynamics of both physical and cyber systems with the help of operator acting upon a continuous function space, there were proposed criteria for establishing a unique equilibrium state by means of the Banach Contraction Principle. Later, with utilization of the Schauder's Fixed-Point Theorem, the guarantee of existence of an equilibrium state was proposed for a wider class of non-expansive adaptive mechanisms. As a next step, the Lyapunov-Krasovskii functional has been used to formulate the explicit sufficient conditions for exponential stability. After that, the analysis was extended to confirm the boundedness of an ultimate region with the help of the Input-to-State Stability argument. Experimental verification illustrated that stability margin does not behave monotonically in terms of adaptive gain, with the optimum configuration found to be $\Gamma = 0.5 \cdot I$ and the measurable instability threshold found to be close to $\Gamma = 2.5 \cdot I$. The obtained analytical model provides theoretical validations for the implementation of the autonomous self-adapting cyber-physical architectures for the unmodeled environment variability and dynamic loadings conditions.

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