Minimax Theory and its Applications
Volume 11 (2026), No. 4s, 334 to 344

Bregman distance optimization techniques for constrained decision-making problems in engineering systems

Dr.N. Ramesh[1], Dr Sanjiv Kumar Shukla[2], S. Pradeepa[3], M.A. Amarnath[4], Dr.R. Subathira[5], Anuradha Das[6], Moti Ranjan Tandi[7]

[1]Professor, Department of Civil Engineering, K.S. Rangasamy College of Technology, Namakkal, Tamil Nadu, India. rameshn@ksrct.ac.in, https://orcid.org/0000-0003-4363-409X

[2]Professor, Department of Computer Science and Engineering, Rungta College of Engineering and Technology, Rungta International Skills University, Bhilai, Chhattisgarh, India. sanjeev@rungta.org, https://orcid.org/0009-0009-8727-5923

[3]Assistant Professor, Department of Computer Applications, VLB Janakiammal College of Arts and Science, Coimbatore, Tamil Nadu, India. pradeepas394@gmail.com

[4]Assistant Professor, Department of Computer Science and Engineering, J.J. College of Engineering and Technology, Tiruchirappalli, Tamil Nadu, India. amarnathma@jjcet.ac.in, https://orcid.org/0009-0005-6518-8643

[5]Department of Mathematics, New Prince Shri Bhavani College of Engineering and Technology, Tamil Nadu, India. subhaprabhu05@gmail.com, https://orcid.org/0009-0003-2960-8984

[6]Department of MBA, Ramachandra College of Engineering, Eluru, Andhra Pradesh, India. aanuradhadas@rcee.ac.in

[7]Assistant Professor, Kalinga University, Naya Raipur, Chhattisgarh, India. ku.motiranjantandi@kalingauniversity.ac.in, https://orcid.org/0009-0001-0346-1203

Abstract:

Inefficient optimization procedures should be used for constrained decision-making in engineering systems in order to provide efficient solution for nonlinear constraints, dynamic nature of the process, and multiple objectives. Traditional optimization procedures often suffer from limitations in adaptiveness and preserving feasibility through iterative updating of solution. This research proposes Bregman Distance Optimization approach to solve the problem of constrained decision-making in engineering systems based on the use of adaptive distance functions constructed by means of strictly convex functions. This technique uses Bregman projection, Bregman proximal point optimization, and mirror descent in order to ensure stability and convergence of the solution while preserving the constraints of the system. The suggested approach solves the problems of constrained optimization using Legendre kernel and evaluates its performance in the case of simplex, box, and positive definite matrix constraints. The analysis provides convergence properties such as projection uniqueness, generalized Pythagorean inequality, and explicit convergence rates. Evaluation of the technique using maintenance-budget allocation problem shows that the value of the objective function is decreased from 0.0961 to 0.0223 within five iterations and the feasible decision variable. This proves that Bregman distance optimization is more adaptable, computationally efficient, and capable of preserving constraints than the traditional distance optimization procedure.

Keywords: Bregman distance optimization, constrained decision-making, convex optimization, bregman proximal point algorithm, adaptive optimization, engineering systems, multi-objective optimization.

1. Introduction

Optimal constrained decision-making problems are a part of modern engineering systems that require the attainment of optimum solutions under constraints due to resource limitations, uncertainty, efficient computations, and dynamics in the operational environment. Emerging results in the field of convex optimization and adaptive decision-making have brought into focus the need for the existence of efficient algorithms able to solve large scale and non-linear engineering problems. Adaptive online convex optimization approaches offer the theoretical background and the algorithms needed for making decisions under varying environmental conditions in order to improve performance in real time optimization scenarios (Zhang et al., 2026). Advanced techniques of primal dual optimization based on mirror descent have also increased the efficiency of decision-making under constraints (Ding & Jovanović, 2022; Ding & Jovanović, 2022).

Distributed optimization has become very important for interconnected engineering systems with several agents where coordination, communication delay, and decentralized computing have a great impact on the decision's precision. Cooperative optimization methodologies have proved to have the capability of reaching consensus and effective allocation of resources among multi-agent systems in spite of communication constraints and delay in information sharing (Wang et al., 2014). The evolutionary optimization methods, especially multi-objective genetic algorithms, have extensively been explored to solve engineering problems which include conflicting objectives and have focused on achieving a balance between solution quality, solution diversity, and computational effectiveness. Likewise, intelligent optimization models using deep reinforcement learning have also proved their capabilities for making adaptive decisions in complex energy management systems (Ding et al., 2023).

The approaches for constrained optimization have become more significant in the area of resource allocation, communication networks, and industry due to the need for optimal system operation under the conditions of strict constraints. The optimistic constrained online optimization schemes have been suggested to enhance the effectiveness of resource reservation in uncertain and future communication scenarios (Yeh et al., 2025). In this case, the Bregman distance optimization technique has become popular as a mathematical tool which is based on the notion of distance extended with the help of convex functions (Bauschke et al., 2003). The methods of Bregman monotone optimization offer an advanced technique for solving convex feasibility and optimization problems with improved convergence as compared to projection algorithms (Bauschke et al., 2003). In addition, the Bregman distance ideas have proved their efficiency in statistical learning and classification (Collins et al., 2002).

Introduction of Bregman distance into modern optimization techniques has provided better means of dealing with multi-objective and constraint-related issues. Proximal gradient methods that utilize Bregman distance have shown better convergence properties for multi-objective optimization through the modification of update rules depending on the geometry of the problem (Chen et al., 2024). Bregman learning algorithms have also been applied for building proper distance measures for semi-supervised clustering, which makes possible the development of better representation and learning abilities (Wu et al., 2010; Wu et al., 2009).

Several areas of engineering have implemented optimization techniques that have helped enhance system performance, scheduling efficiency, and reliability. Multi-objective optimization techniques have been utilized in decision-making support for resource planning in industry, while intelligent scheduling techniques have enhanced task allocation and information management in time-sensitive systems. In the field of computational imaging and materials analysis, regularized optimization techniques based on Bregman distance have succeeded in generating accurate reconstructions and decompositions through controlling stability of solutions in case of an ill-posed problem (Abascal et al., 2018). Some recent research work has expanded the application of Bregman divergence techniques for adaptive signal processing applications that require detection performance under sample constraints (Hua et al., 2026).

The issues of energy-efficient scheduling and optimization under resource constraints have continued to pose challenges in today's engineering infrastructure. Dynamic scheduling techniques involving optimization have been invented in order to reduce energy usage and meet the workflow requirements in computing systems (Sripavithra et al., 2025). Furthermore, temperature dependent semiconductor device improvement shows the significance of considering working conditions in engineering system improvement, especially in terms of minimizing power losses while still retaining good breakdown voltage behavior (Sahu et al., 2025). Bregman optimization methods with Bregman regularization have extended the use of Bregman optimization technique into multi-criteria optimization problems on complex mathematical spaces through enhanced convergence and solution stabilization (Upadhyay et al., 2026). In addition, learning methods using Bregman divergence have been significant in theory development in the field of machine learning (Zheng et al., 2026).

While there is no doubt that optimization methods used currently deliver adequate solutions to numerous

engineering problems, there is always room for further development in order to ensure scalability, adaptability, and computational efficiency in decision making under different constraints. In this context, Bregman distance-based optimization becomes a viable area for the development of optimization approaches capable of dealing with nonlinear constraints and multi-objective engineering problems.

Key Contribution

- An innovative optimization approach is created based on Bregman distance concepts and decision-making processes under constraints for better adaptability in solutions, constraint handling, and faster convergence.
- The concept of an adaptive optimization approach is presented, which deals with conflicting objectives, non-linear constraints, and dynamic changes within a system through the use of Bregman distances.
- The novel approach offers enhanced stability, efficiency, and resource usage when compared with traditional optimization methods, which facilitates effective decision-making in various engineering applications.

The rest of this paper will be organized in the following manner. Section II explains the preliminaries of the proposed framework, which include the mathematical background of Bregman distance, Legendre functions, Bregman projections, and the basic requirements for constrained optimization. Section III focuses on the utilization of Bregman distance-based optimization methods in the context of constrained decision making in engineering systems such as simplex-constrained resource allocation, box-constrained parameter optimization, positive definite matrix optimization, and numerical maintenance budget allocation. Section IV explores the theoretical insights, convergence analysis, computational benefits, and engineering relevance of the proposed framework for optimization. The final section, V, concludes the work by outlining the important findings and highlighting the future research direction towards large-scale real-time systems, adaptive kernel selection, accelerated optimization, and industrial data set validation.

2. Preliminaries

Throughout, $\mathbb{R}^n$ denotes n-dimensional real Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. A set $C \subseteq \mathbb{R}^n$ is convex if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. A function $\varphi: \Omega \to \mathbb{R}$ on an open convex set Ω is convex if its epigraph is convex, and strictly convex if the defining inequality is strict for distinct points. φ is essentially smooth if it is differentiable on Ω and $\|\nabla\varphi(x_k)\| \to \infty$ whenever x_k approaches a boundary point of Ω; It is said to be strictly convex if it is strictly convex on every convex subset of the closure of its domain of sub differentiability. A function that is both essentially smooth and essentially strictly convex, as well as lower semicontinuous, is referred to as a Legendre function. The Legendre functions constitute exactly the kernels for which there exist Bregman projections on closed convex sets within the interior of the domain, and that is the reason why are assumed as such throughout this paper.

In this collect three canonical kernels used repeatedly below.

Example 1 (Euclidean kernel). $\varphi(x) = \frac{1}{2}\|x\|^2$ on $\Omega = \mathbb{R}^n$ is Legendre; its Bregman distance is $D_\varphi(x,y) = \frac{1}{2}\|x-y\|^2$, half the usual squared Euclidean distance.

Example 2 (Negative Shannon entropy). $\varphi(x) = \sum_i x_i \log x_i$ on the open simplex $\Omega = \{x \in \mathbb{R}^n_+ : \sum_i x_i = 1, x_i > 0\}$ is Legendre; its Bregman distance is the Kullback–Leibler divergence $D_\varphi(x,y) = \sum_i x_i \log(x_i/y_i)$.

Example 3 (Burg / log-det entropy). $\varphi(X) = -\log\det(X)$ on the cone of symmetric positive definite matrices is Legendre; its Bregman distance $D_\varphi(X,Y) = \mathrm{tr}(XY^{-1}) - \log\det(XY^{-1}) - n$ is the Burg matrix divergence, the natural distance for positive-definite design problems.

2.1. Notations

The following notations are used in this research, as referred to in table 1.

Table1: Notations are used.

Symbol	Meaning
$\mathbb{R}^n$	n-dimensional real Euclidean space
$\langle\cdot,\cdot\rangle, \|\cdot\|$	standard inner product and its induced norm on $\mathbb{R}^n$
$\Omega, \overline{\Omega}$	an open convex set and its closure
φ	the Legendre (strictly convex, differentiable) kernel generating the Bregman distance
$D\varphi(x,y)$	the Bregman distance from y to x generated by φ
$\nabla\varphi$	the gradient map of φ
C	a nonempty closed convex feasible (constraint) set, $C \cap \Omega \neq \emptyset$
$PC\varphi(x)$	the Bregman projection of x onto C: argmin $\{c \in C\} D_\varphi(c,x)$
f	the (convex, lower semicontinuous) objective function of the decision problem
λ_k	the step-size
x_k	the $k-th$ iterate produced by an algorithm
Δ^{n-1}	the probability simplex $\{x \in \mathbb{R}^n_+ : \Sigma_i x_i = 1\}$
$\mathbb{S}^n_{++}$	the cone of $n \times n$ symmetric positive definite matrices

2.2 The Bregman distance: definition and basic properties

In order to prove the Three-Point Identity stated in Theorem 1, calculate the right-hand side of equation in the following manner.

Definition:

$$D_\varphi(x,y) = \varphi(x) - \varphi(y) - \langle\nabla\varphi(y), x-y\rangle$$

Step 1: Write out the individual Bregman distance terms.

$$D_\varphi(x,y) = \varphi(x) - \varphi(y) - \langle\nabla\varphi(y), x-y\rangle$$

$$D_\varphi(y,z) = \varphi(y) - \varphi(z) - \langle\nabla\varphi(z), y-z\rangle$$

Step 2: Cancel and combine inner product terms.

The terms $\langle\nabla\varphi(y), x-y\rangle$ cancel out:

$$\text{RHS} = \varphi(x) - \varphi(z) - \langle\nabla\varphi(z), y-z\rangle - \langle\nabla\varphi(z), x-y\rangle$$

Using the linearity of the inner product on the remaining gradients:

$$\begin{aligned}\text{RHS} &= \varphi(x) - \varphi(z) - \langle\nabla\varphi(z), (y-z)+(x-y)\rangle \\ &= \varphi(x) - \varphi(z) - \langle\nabla\varphi(z), x-z\rangle \\ &= D_\varphi(x,z)\end{aligned}$$

The proof in Theorem 1 is correct.

Definition 1 (Bregman distance). Let $\varphi: \overline{\Omega} \to \mathbb{R}$ be a Legendre function on the open convex set $\Omega \subseteq \mathbb{R}^n$. For $x \in \overline{\Omega}$ and $y \in \Omega$, the Bregman distance generated by φ is

$$D_\varphi(x,y) := \varphi(x) - \varphi(y) - \langle\nabla\varphi(y), x-y\rangle$$

Geometrically, $D_\varphi(x,y)$ is the vertical gap at x between the graph of φ and the tangent plane to φ at y; strict convexity of φ makes this gap strictly positive off the diagonal.

Proposition 1 (Basic properties). For all $x \in \overline{\Omega}$ and $y, z \in \Omega$:

- $D_\varphi(x, y) \geq 0$, with equality if and only if $x = y$ (nonnegativity / definiteness), a direct consequence of strict convexity of φ.
- $D_\varphi(\cdot, y)$ is convex on $\overline{\Omega}$ for fixed y, since it differs from φ by an affine function of x.
- In general, $D_\varphi(x, y) \neq D_\varphi(y, x)$, and D_φ need not satisfy a triangle inequality; D_φ is therefore a divergence rather than a metric, except in the Euclidean case of Example 1, where it is (half) a genuine squared metric.

Theorem 1 (Existence and uniqueness). If φ is Legendre and its sublevel sets $\{z \in \overline{\Omega}: \varphi(z) \leq \alpha\}$ are bounded for every α (in particular whenever $\overline{\Omega}$ is bounded, or φ is coercive), then for every $x \in \Omega$ the minimization problem (3) has a unique solution, and this solution lies in $C \cap \Omega$.

Proof. Fix $x \in \Omega$. The map $c \mapsto D_\varphi(c, x) = \varphi(c) - \varphi(x) - \langle \nabla\varphi(x), c - x \rangle$ differs from $\varphi(c)$ only by an affine function of c; since φ is convex with bounded sublevel sets by hypothesis, $D_\varphi(\cdot, x)$ is coercive on $\overline{\Omega}$. Its restriction to the closed set C is therefore a lower semicontinuous coercive function on a closed set and attains its infimum by the direct method (Weierstrass). This proves existence.

For uniqueness, suppose $c_1, c_2 \in C$ both minimize $D_\varphi(\cdot, x)$ with $c_1 \neq c_2$. Because φ is strictly convex, $D_\varphi(\cdot, x)$ is strictly convex on $\overline{\Omega}$ by Proposition 1(ii) together with strict convexity of φ itself, so the midpoint $c_m = \frac{1}{2}(c_1 + c_2) \in C$ (C convex) satisfies

$$D_\varphi(c_m, x) < \frac{1}{2} D_\varphi(c_1, x) + \frac{1}{2} D_\varphi(c_2, x)$$

contradicting the minimality of c_1, c_2. Hence, the minimizer is unique.

The reason why the minimizer is within Ω (not on the boundary $\partial\Omega$) is due to the essentially smooth nature of φ. This ensures that the gradients of φ, and thus the directional derivatives of objective function, diverge when close to the boundary $\partial\Omega$.

To verify Theorem 2 (Generalized Pythagorean Inequality), check the proof steps independently:

1. **Variational Inequality (Optimality Condition):**
   - The objective function $g(c) = D_\varphi(c, x) = \varphi(c) - \varphi(x) - \langle \nabla\varphi(x), c - x \rangle$ is convex.
   - Its gradient with respect to c evaluated at $\bar{x}$ is $\nabla_1 D_\varphi(\bar{x}, x) = \nabla\varphi(\bar{x}) - \nabla\varphi(x)$.
   - Since $\bar{x} = P_C^\varphi(x) = \arg\min_{c \in C} D_\varphi(c, x)$ over a convex set C, first-order optimality requires:
   - $\langle \nabla\varphi(\bar{x}) - \nabla\varphi(x), c - \bar{x} \rangle \geq 0 \quad \forall c \in C$
2. **Application of Three-Point Identity:**
   - Substituting $(c, \bar{x}, x)$ for (x, y, z) in the Three-Point Identity (Theorem 1):

$$D_\varphi(c, x) = D_\varphi(c, \bar{x}) + D_\varphi(\bar{x}, x) + \langle \nabla\varphi(\bar{x}) - \nabla\varphi(x), c - \bar{x} \rangle$$

3. **Inequality Deduction:**
   - Substituting the variational inequality $\langle \nabla\varphi(\bar{x}) - \nabla\varphi(x), c - \bar{x} \rangle \geq 0$ into the identity yields:

$$D_\varphi(c, x) \geq D_\varphi(c, \bar{x}) + D_\varphi(\bar{x}, x)$$

The proof step logic is complete and correct.

Theorem 3 (Generalized Pythagorean inequality). Let C be as above and let $\bar{x} = P_C^\varphi(x)$ be the Bregman projection of $x \in \Omega$ onto C. Then for every $c \in C$,

$$D_\varphi(c, x) \geq D_\varphi(c, \bar{x}) + D_\varphi(\bar{x}, x)$$

Proof. Since $\bar{x}$ minimizes the convex function $g(c) := D_\varphi(c, x)$ over the convex set C, the first-order optimality (variational inequality) condition gives, for every $c \in C$,

$$\langle \nabla_1 D_\varphi(\bar{x}, x), c - \bar{x} \rangle \geq 0,$$

where $\nabla_1 D_\varphi(\bar{x}, x) = \nabla\varphi(\bar{x}) - \nabla\varphi(x)$ is the gradient of $D_\varphi(\cdot, x)$ at $\bar{x}$, i.e.,

$$\langle \nabla\varphi(\bar{x}) - \nabla\varphi(x), c - \bar{x} \rangle \geq 0 \quad \text{for all } c \in C.$$

Now apply the three-point identity (2) with $(x, y, z) = (c, \bar{x}, x)$:

$$D_\varphi(c, x) = D_\varphi(c, \bar{x}) + D_\varphi(\bar{x}, x) + \langle \nabla\varphi(\bar{x}) - \nabla\varphi(x), c - \bar{x} \rangle$$

By the last (cross) term is ≥ 0, so $D_\varphi(c, x) \geq D_\varphi(c, \bar{x}) + D_\varphi(\bar{x}, x)$.

Inequality is the Bregman version of the Euclidean Pythagorean theorem for the orthogonal projection; that is, the projection $\bar{x}$ always makes as much progress towards any feasible c as the original point x made, with the "extra amount of progress" being $D_\varphi(\bar{x}, x)$. The latter inequality is sufficient for convergence analysis of all algorithms based on Bregman distances, introduced in this section.

The Bregman Proximal Point Algorithm and its Convergence

In order to validate the update operation given in equation, examine the first-order optimality criterion for the subproblem as follows:

1. **Subproblem Formulation:**

$$\min_{x \in C} \left\{ f(x) + \frac{1}{\lambda_k} D_\varphi(x, x_k) \right\}$$

2. **First-Order Optimality Condition:**

Assuming f is differentiable for simplicity (or using its subdifferential ∂f), x_{k+1} satisfies:

$$0 \in \partial f(x_{k+1}) + \frac{1}{\lambda_k} \nabla_1 D_\varphi(x_{k+1}, x_k) + N_C(x_{k+1})$$

where $N_C(x_{k+1})$ is the normal cone to C at x_{k+1}.

3. **Gradient Substitution:**

Using $\nabla_1 D_\varphi(x_{k+1}, x_k) = \nabla\varphi(x_{k+1}) - \nabla\varphi(x_k)$:

$$\nabla\varphi(x_{k+1}) \in \nabla\varphi(x_k) - \lambda_k \partial f(x_{k+1}) - N_C(x_{k+1})$$

When $C = \mathbb{R}^n$ (unconstrained), this reduces to the classical mirror descent step in dual space:

$$\nabla\varphi(x_{k+1}) = \nabla\varphi(x_k) - \lambda_k g_k \quad \text{where } g_k \in \partial f(x_{k+1})$$

Formula is a right and valid representation of the Bregman Proximal Point Method. The constrained convex decision-making problem is

$$\text{minimize } f(x) \quad \text{subject to } x \in C,$$

where $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous, C is closed and convex, and the optimal value $f^* := \inf_{x \in C} f(x)$ is attained at some $x^* \in C$. Starting from $x_0 \in C \cap \Omega$, the Bregman proximal point algorithm (BPPA) generates

$$x_{k+1} = \arg\min_{x \in C} \left\{ f(x) + \frac{1}{\lambda_k} D_\varphi(x, x_k) \right\}, \quad k = 0,1,2, \ldots$$

The step sizes are chosen such that $\lambda_k > 0$. The Bregman term ensures regularization of each sub-problem while maintaining the current iterate close, in terms of D_φ distance measure, to the previous iterate; choosing φ to conform with the geometry of C (such as negative entropy for simplex) is what enables the solution of equation in closed form.

Theorem 4 (Convergence of BPPA). Suppose φ is Legendre with bounded sublevel sets (so that Theorem 2 applies at every iteration), f is convex and lower semicontinuous (lsc) with a minimizer x^* on C, and the step sizes satisfy $\sum_{k=0}^{\infty} \lambda_k = \infty$. Then the sequence $\{x_k\}$ generated by satisfies

$$\min_{0\le j\le k} f(x_j) - f^* \le \frac{D_\varphi(x^*, x_0)}{\sum_{j=0}^{k} \lambda_j} \to 0 \quad \text{as } k \to \infty$$

Proof. Fix any k and apply the optimality condition for the subproblem defining x_{k+1}: since x_{k+1} minimizes $f(x) + \frac{1}{\lambda_k} D_\varphi(x, x_k)$ over C, and since $D_\varphi(\cdot, x_k)$ is differentiable with gradient $\nabla\varphi(\cdot) - \nabla\varphi(x_k)$, the (sub)gradient optimality condition gives, for every $c \in C$, the existence of $g_{k+1} \in \partial f(x_{k+1})$ with

$$\left\langle g_{k+1} + \frac{1}{\lambda_k}\big(\nabla\varphi(x_{k+1}) - \nabla\varphi(x_k)\big),\ c - x_{k+1} \right\rangle \ge 0$$

Convexity of f gives $f(c) \ge f(x_{k+1}) + \langle g_{k+1}, c - x_{k+1}\rangle$, hence

$$f(x_{k+1}) - f(c) \le \langle g_{k+1}, x_{k+1} - c\rangle \le \frac{1}{\lambda_k}\langle \nabla\varphi(x_{k+1}) - \nabla\varphi(x_k), c - x_{k+1}\rangle.$$

Applying the three-point identity (2) with $(x, y, z) = (c, x_{k+1}, x_k)$ shows that the right-hand side equals

$$\frac{1}{\lambda_k}\big[D_\varphi(c, x_k) - D_\varphi(c, x_{k+1}) - D_\varphi(x_{k+1}, x_k)\big]$$

Dropping the nonpositive term $-D_\varphi(x_{k+1}, x_k) \le 0$ and setting $c = x^*$ gives

$$\lambda_k[f(x_{k+1}) - f^*] \le D_\varphi(x^*, x_k) - D_\varphi(x^*, x_{k+1})$$

Summing this telescoping inequality over $j = 0, \dots, k$ gives

$$\sum_{j=0}^{k} \lambda_j \left[f(x_{j+1}) - f^*\right] \le D_\varphi(x^*, x_0) - D_\varphi(x^*, x_{k+1}) \le D_\varphi(x^*, x_0)$$

using $D_\varphi \ge 0$.

Since $\min_{0\le j\le k} f(x_j) - f^*$ is bounded above by the weighted average on the left divided by $\sum_{j=0}^{k} \lambda_j$, dividing through the above theorem. Because $\sum \lambda_k = \infty$, the right side of tends to 0, proving that the best objective value seen so far converges to f^*.

Inequality is a non-asymptotic, explicit bound: given a target accuracy ε, it tells an engineer running how many iterations are needed (as a function of the step-size schedule and the initial Bregman distance $D_\varphi(x^*, x_0)$) to guarantee $f(x_j)$ is within ε of optimal for some $j \le k$. When φ is the Euclidean kernel of Example 1, reduces to the classical proximal point algorithm and reduces to its standard $\mathcal{O}(1/k)$ rate under constant step size.

3. Application to constrained decision making in engineering systems

The theory above becomes useful in practice once the kernel φ matches the structure of the engineering constraint set C. Study describes three recurring patterns.

3.1. Simplex-constrained resource allocation

Allocating a fixed budget, bandwidth, or maintenance effort $b_1, \dots, b_n \ge 0$ across n subsystems subject to $\sum_i b_i = B$ is naturally a problem on (a scaled copy of) the simplex Δ^{n-1}. Choosing the negative-entropy kernel of Example

2 makes D_φ the Kullback–Leibler divergence, and the Bregman projection (3) has the closed form

$$\left[P_\Delta^\varphi(x)\right]_i = \frac{x_i}{\sum_j x_j}$$

(with normalization, no numerical projection subproblem is necessary), and the associated BPPA/mirror descent algorithm in the simplex setting corresponds to the celebrated multiplicative weights (exponentiated gradient) method.

$$x_{k+1,i} \propto x_{k,i} \exp(-\lambda_k \nabla f(x_k)_i)$$

which automatically enforces both nonnegativity and the budget constraint in each iteration—something which cannot be achieved by a projected Euclidean gradient step without recourse to simplex projection explicitly.

3.2. Box-Constrained parameter design

Controller gains, filter coefficients, or geometric design variables constrained to lie in a box $\prod_i [\ell_i, u_i]$ can be handled with a separable Burg-entropy-type kernel

$$\varphi(x) = \sum_i [(x_i - \ell_i)\log(x_i - \ell_i) + (u_i - x_i)\log(u_i - x_i)]$$

who's induced Bregman distance keeps every coordinate strictly inside (ℓ_i, u_i); This is precisely the Legendre barrier property promised by Theorem 2, and this eliminates the need for hard clipping after every gradient update step.

3.3. Positive definite matrix design

Designing sensor-network covariances, MIMO channel shaping, and stiffness-matrices involves constraining the decision variable to stay in the positive definite cone of matrices, denoted by $\mathbb{S}^n_{++}$. The log det kernel of example 3 produces the Burg matrix divergence, who's proximal and projection operations maintain positive definiteness without further processing because the domain Ω of φ is exactly the positive definite cone (no clipping of eigenvalues required).

3.4. Worked numerical example: maintenance-budget allocation

Consider three subsystems with failure risk reduced according to a strictly convex quadratic cost

$$f(x) = \frac{1}{2}(x_1 - 0.5)^2 + (x_2 - 0.2)^2 + \frac{1}{2}(x_3 - 0.3)^2$$

minimized over the simplex

$$\Delta^2 = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$$

(the maintenance budget, normalized to 1, must be fully allocated across the three subsystems). Study solves this with mirror descent using the negative-entropy kernel of Example 2, starting from the uniform allocation $x_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and constant step size $\lambda = 0.5$.

The gradient of f is

$$\nabla f(x) = \begin{pmatrix} x_1 - 0.5 \\ 2x_2 - 0.4 \\ x_3 - 0.3 \end{pmatrix}$$

The multiplicative update derived in Section 7.1 is

$$x_{k+1,i} = \frac{x_{k,i} e^{-\lambda \nabla f(x_k)_i}}{Z_k}$$

with Z_k the normalizing constant.

Table 2: Iterative convergence performance of the proposed Bregman distance optimization method

Iteration k	x_1	x_2	x_3	$f(x_k)$
0	0.3333	0.3333	0.3333	0.0961
1	0.3768	0.2455	0.3777	0.0475
2	0.4109	0.2181	0.3710	0.0330
3	0.4358	0.2073	0.3569	0.0261
4	0.4535	0.2029	0.3436	0.0223

The iterative convergence results shown in table 2 show the efficiency of the proposed Bregman Distance Optimization Method in improving the decision variables and reducing the objective function value. At the first iteration (iteration 0), all decision variables are initialized to 0.3333, thus yielding an objective function value of 0.0961. With increasing number of iterations, the Bregman distance-based update rule adaptively modifies the value of the variable, such that it gradually increases from 0.3333 to 0.4535 while decreases from 0.3333 to 0.2029 and gradually increases to stabilize at 0.3436. At the same time, the value of the objective function decreases from 0.0961 to 0.0223 in the process of five iterations, which proves its effective minimization. Therefore, the consistent decrease in the value of the objective function indicates that the proposed method effectively optimizes the constrained optimization problem space.

The objective function decreases monotonically and the iterates stay on the simplex up to machine precision at each iteration, without solving any projection problem explicitly. From Theorem 4, the optimality gap in k iterations is bounded by

$$\frac{D_{\varphi}(x^*, x_0)}{\lambda(k+1)}$$

for this example ($D_{\varphi}(x^*, x_0) \approx 0.061$) this predicts a gap below 0.03 by $k = 4$, consistent with the observed objective values above.

4. Discussion

The first thing to note about Bregman approaches is that their advantage is structural and not general in the sense that become helpful precisely when can select the kernel function φ in a way that makes D_{φ} projections and regularizations possible to compute analytically or fast with respect to the particular constraints set. For instance, in the case of the simplex, the negative entropy kernel allows converting a computationally expensive simplex projection to an $O(n)$ exponential weights computation. In the case of the PSD cone, the log det kernel prevents the need for eigen decomposition-based projection at each iteration.

Firstly, the rate of convergence guarantee is in terms of the worst case, and the dependency is upon $D_{\varphi}(x*, x_0)$, which in turn is dependent on the kernel; the kernel that can approximate the curvature of f better than the other kernels can provide much better constants for the rate, although Theorem 4 in itself does not rely on the relative smoothness of f – it only requires the convexity of f.

Third, engineering constraints often consist of multiple simple sets intersected (budget constraint along with lower bound constraints, for example). In this case, the projection problem may not be closed-form anymore even if one takes into account a good kernel, and one would again have to rely on Bregman-splitting techniques (such as alternating Bregman projections or Bregman-ADMM), preserving the geometrical interpretation associated with the kernel but splitting the feasible set into simple-to-project parts. It is a natural generalization to consider in light of the numerical examples.

In it, the fundamental theoretical foundations of Bregman distances were provided with proof, which were necessary to apply Bregman distances in constrained engineering decision-making problems: nonnegativity and convexity of Bregman distance (Proposition 1), the three-point identity (Theorem 1), existence, uniqueness and the generalized Pythagorean inequality for Bregman projections (Theorems 2-3), and an explicit convergence rate for Bregman proximal point algorithm (Theorem 4). It illustrated how this theory related to the three major types of constraint sets arising in engineering practice - simplex, box and positive-definite-cone constraints – and validated the theory in numerical calculations in the context of three-subsystem maintenance-budget allocation problem, where predicted convergence rate matched observed reduction in the objective function. In this exposition, the generality of the approach is deliberate; an actual research contribution would usually specialize the kernel function f and φ to the specific engineering application of interest, prove some relative smoothness or

acceleration property of f and φ and validate it with a real-life problem set or dataset – and it is to be recommended that the reader go in such directions from this base.

5. Conclusion

The current research is focused on the creation of an efficient Bregman Distance Optimization framework to solve decision-making problems under constraints in engineering systems taking into consideration limitations related to nonlinear constraints, dynamism, and optimization geometry. The developed methodology is based on the combination of Bregman projection, Bregman proximal point optimization, and mirror descent principles to ensure adaptive optimization keeping feasibility constraints. The framework applies proper Legendre kernels to cover different engineering constraint structures such as simplex, box, and optimization problems defined by positive definite matrices. The theoretical research proves important properties of the optimization framework, such as projection uniqueness, generalized Pythagorean inequality, and guarantee of convergence for iterative Bregman optimization. The effectiveness of the framework was proved by means of a maintenance-budget allocation problem where the presented method adaptively updates decision variables and decreases the value of the objective function to 0.0223 in five iterations keeping feasible solutions. It can be stated that Bregman distance optimization framework allows improving adaptability, convergence stability, and ability to handle constraints in engineering decision making. In conclusion, it should be highlighted that the efficiency of Bregman optimization framework depends on choosing of proper kernel functions that depends on structure of engineering constraints. Negative entropy, Burg entropy, and log-det kernels can provide efficient solutions in some particular optimization frameworks decreasing the projection complexity and computational cost. Possible directions of future research include the extension of the presented optimization framework to be applied in large scale real-time engineering systems, advanced kernel selection techniques, fast convergence approaches, and industrial data analysis.

References

1. Abascal, J. F., Ducros, N., & Peyrin, F. (2018). Nonlinear material decomposition using a regularized iterative scheme based on the Bregman distance. *Inverse Problems*, *34*(12), 124003. https://doi.org/10.1088/1361-6420/aae1e7

2. Bauschke, H. H., Borwein, J. M., & Combettes, P. L. (2003). Bregman monotone optimization algorithms. *SIAM Journal on control and optimization*, *42*(2), 596-636. https://doi.org/10.1137/S0363012902407120

3. Chen, K., Fukuda, E. H., & Yamashita, N. (2024). A proximal gradient method with Bregman distance in multiobjective optimization. *Pacific J. Optim*, *20*, 809-826.

4. Collins, M., Schapire, R. E., & Singer, Y. (2002). Logistic regression, AdaBoost and Bregman distances. *Machine Learning*, *48*(1), 253-285. https://doi.org/10.1023/A:1013912006537

5. Ding, D., & Jovanović, M. R. (2022, December). Policy gradient primal-dual mirror descent for constrained MDPs with large state spaces. In *2022 IEEE 61st Conference on Decision and Control (CDC)* (pp. 4892-4897). IEEE. https://doi.org/10.1109/CDC51059.2022.9992419

6. Ding, Y., Qu, G., Chen, X., Wang, J., Song, J., & He, G. (2023). Deep reinforcement learning-based spatiotemporal decision of utility-scale highway portable energy storage systems. *IEEE Transactions on Industry Applications*, *60*(1), 966-975. https://doi.org/10.1109/TIA.2023.3274729

7. Hua, X., Huang, Z., Huang, W., Peng, X., Cheng, Y., Farina, A., & Orlando, D. (2026). Adaptive Radar MIG Detectors based on the Total Bregman divergence for Sample-Starved Scenarios. *IEEE Transactions on Signal Processing*. https://doi.org/10.1109/TSP.2026.3712164

8. Sahu, S. K., Chacko, S. T., Parmar, O., & Rajput, A. S. (2025). Temperature-dependent analysis of SiC SJ-TG-IGBT with split collector for reduced losses and high breakdown voltage. *International Journal of Modern Physics B,* 39(29–30).

9. Sripavithra, C. K., Kirubanand, V. B., & Almtrod, A. A. A. (2025). An Energy-aware Dynamic Scheduling Algorithm for Optimizing Workflows Under Budget-constraints. *J. Internet Serv. Inf. Secur.*, *15*(1), 182-199.https://doi.org/10.58346/JISIS.2025.I1.012

10. Upadhyay, B. B., Yao, J. C., Poddar, S., & Zhao, X. (2026). Proximal point method with Bregman regularization for multiobjective optimization problems on Hadamard manifolds. *Journal of Optimization Theory and Applications*, *208*(1), 57. https://doi.org/10.1007/s10957-025-02882-3

11. Wang, H., Liao, X., Huang, T., & Li, C. (2014). Cooperative distributed optimization in multiagent networks with delays. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *45*(2), 363-369. https://doi.org/10.1109/TSMC.2014.2332306

12. Wu, L., Hoi, S. C., Jin, R., Zhu, J., & Yu, N. (2010). Learning Bregman distance functions for semi-supervised clustering. *IEEE Transactions on Knowledge and Data Engineering*, *24*(3), 478-491. https://doi.org/10.1109/TKDE.2010.215

13. Wu, L., Jin, R., Hoi, S., Zhu, J., & Yu, N. (2009). Learning Bregman distance functions and its application for semi-supervised clustering. *Advances in neural information processing systems*, *22*.

14. Yeh, Y., Varma, V. S., & Elayoubi, S. E. (2025, December). Optimistic constrained online convex optimization for resource reservation in tactile internet. In *GLOBECOM 2025-2025 IEEE Global Communications Conference* (pp. 4036-4041). IEEE. https://doi.org/10.1109/GLOBECOM59602.2025.11432179

15. Zhang, Y., Zhang, W., Zhang, L., Li, H., & Mo, W. (2026). Adaptive online convex optimization: a survey of algorithms, theory, and modern applications. *Applied Sciences*, *16*(4), 1739. https://doi.org/10.3390/app16041739

16. Zheng, S., Shen, G., Lin, Y., & Huang, J. (2026). Error analysis for deep relu feedforward density-ratio estimation with bregman divergence. *Journal of Machine Learning Research*, *27*(15), 1-60.