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Second-Order variational algorithm design for accelerated numerical optimization in engineering applications

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Abstract:

Engineering optimization applications usually consist of a nonlinear objective function, an ill-conditioned optimization landscape, physical constraints, and uncertain operating conditions; therefore, efficient numerical optimization is not easy to achieve. First-order methods use gradient-based techniques, while second-order methods can be expensive due to the calculation of the Hessian matrix and its factorization. The Second Order Variational Minimax Algorithm (SOVMA) proposed in this paper combines variational optimization, curvature-based search directions, adaptive step-length control, constraint handling, and minimax uncertainty handling within a single framework. The algorithm uses the information of the regularized Hessian matrix to calculate the second-order search directions, while the unfavorable operating conditions of the problem are calculated by means of an inner maximization algorithm. Optimization performed with such an algorithm tries to find a feasible point that minimizes the worst-case engineering objective function. Numerical simulations are performed on Rosenbrock, Rastrigin, Ackley, Himmelblau, and ill-conditioned quadratic benchmark test problems, followed by a case study on uncertain thermal energy systems design. The comparative results presented indicate that the SOVMA method needs only 31 iterations, whereas Newton needs 42, Trust-Region needs 49, BFGS needs 74, Conjugate Gradient needs 151, and Gradient Descent needs 286 iterations. Furthermore, the SOVMA method attains the best average and worst-case objective values of 1.181 and 1.542, respectively. The ablation study proves the significance of second-order information, step adaptivity, and minimax robustness. The outcomes prove that SOVMA offers faster convergence and enhanced decision robustness for engineering applications.

Keywords: Second-Order optimization, variational optimization, minimax optimization, numerical optimization, engineering design, robust optimization, hessian-based methods.

1. Introduction

Optimization through numbers is critical in engineering design, control, resource allocation, and improving the performance of the engineering system when the optimal solutions need to be made using nonlinear constraints and uncertain operating conditions. Engineering system complexity has resulted in the need for optimization algorithms that can offer fast convergence as well as numerical stability. First-order optimization algorithms rely mainly on the gradient, making them appealing due to their computational simplicity. Nonetheless, the convergence of first-order optimization methods can be poor in the case of ill-conditioned and highly nonlinear objective spaces. Consequently, there has been a need to come up with methods that can accelerate the process of optimization (Li et al., 2020). In addition to gradient-based algorithms, various other meta-heuristic techniques, including ant colony and bee colony algorithms, have also been explored for the solution of difficult combinatorial optimization problems. The employment of population-based techniques for generating optimal results is evident from the following works (Agrawal, Kaur, & Arora, 2012; Agrawal, Kaur, & Bhardwaj, 2012). The theory of variational methods has played an important role in linking acceleration of numerical optimization with optimization dynamics (Wibisono et al., 2016). In addition, recent research in engineering optimization has shown the significance of numerical optimization in engineering applications. Multi-criteria decision making has similarly been applied in sustainable manufacturing decision-making, thus highlighting the need for considering multiple objectives both operationally and sustainably in engineering systems (Kavitha, 2023).

Second-order optimization uses the Hessian matrix for incorporating the curvature information, which enables the use of search directions that take into consideration the curvature of the objective function. The application of second-order continuous-time optimization algorithms has been illustrated to problems of an engineering nature like economic power dispatch. On the other hand, second-order sensitivity algorithms have been applied in the efficient numerical calculation of problems in nonlinear dynamical systems (Russell, 2022; He et al., 2017). The quasi-second-order and accelerated algorithms have shown the usefulness of using curvature information to enhance the performance of optimization algorithms locally (Smart & Mazziotti, 2022). However, the explicit Hessian matrix calculation, storage, and factorization may prove to be computationally prohibitive, especially with the increase in decision variables. Stochastic matrix inversion algorithms and accelerated BFGS algorithms have pointed out the need for a reduction in the costliness of second-order optimization (Gower et al., 2018; Khanh et al., 2023).

The approach of variational optimization offers a rigorous mathematical formulation for the description of optimality in terms of stationarity, curvature, and admissible variations. Variational approaches have proven effective in dealing with complex computational problems, and second-order adjoint sensitivity analysis has highlighted the significance of curvature-based information in the optimization of nonlinear systems (Lu et al., 2016; Stapor et al., 2018). Second-order distributed optimization and projected primal-dual formulations point towards the advantages of the second-order approach in the case of constrained and large-scale optimization (Deng, 2021; Li et al., 2022; Wang et al., 2021). Nonetheless, many of the currently employed techniques focus on acceleration, distributed computations, or particular applications of engineering interest without giving due attention to curvature, constraints, and uncertainty in the context of minimax optimization (Wang et al., 2026).

In the case of engineering systems, uncertain operation can cause significant loss in performance in relation to a nominally optimal solution (Wei et al., 2026). The method of robust optimization solves this problem by accounting for adverse parameter scenarios instead of optimizing for nominal parameters. Moreover, the rise in the use of AI decision support systems in business management and sustainable development also reflects the increasingly. The increased adoption of AI-enabled decision-support systems in strategic management highlights the growing relevance of computational optimization and decision intelligence for systems working in an evolving environment. Progress in numerical optimization and nonlinear engineering computations, along with modern approaches to optimization, highlights the need for efficient decision-making (Zhang & Rodriguez, 2023; Madhavi, 2025). Other sustainable applications have highlighted the importance of optimizing engineering and resource management decisions in light of the environmental and overall system impacts; emerging technologies in aquaculture, for instance, have already been evaluated for fish health and ecosystem sustainability (Janssens & Vermeulen, 2026). This calls for the unification of the second-order variational optimization approach with minimax robustness (Bergmann & Gousenbourger, 2018).

This work proposes the Second-Order Variational Algorithm (SOVA) along with minimax formulations for robust numerical optimization. The present approach takes advantage of Hessian curvature, adaptive step-length control,

constraints management, and worst-case parameter evaluations in order to increase convergence, feasibility, and robustness against unfavorable parameter changes. The major contribution lies in a curvature-aware variational update rule, minimax-based uncertainty treatment, theoretical convergence proof, and numerical experiments on benchmarks and engineering problems.

The rest of the paper is structured as follows. Mathematical preliminaries and minimax formulation are provided in Section 2. The proposed Second-Order Variational Minimax Algorithm and its computation scheme are introduced in Section 3. Section 4 contains theoretical properties including stationarity, descent, convergence, and complexity. Section 5 describes the formulation of an engineering problem under uncertainty. The numerical setup is given in Section 6. Comparative, sensitivity, robustness, and ablation results are provided in Section 7. Mathematical and engineering implications are discussed in Section 8, while Section 9 considers limitations and future research directions. A conclusion is drawn in Section 10.

2. Mathematical preliminaries and problem formulation

2.1. Variational optimization

Consider a constrained optimization problem defined over a feasible set $\mathcal{X} \subseteq \mathbb{R}^n$:

$$\min_{x \in \mathcal{X}} F(x)$$

where x denotes the engineering decision vector and F: $\mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable objective function. A solution x^* satisfies a variational optimality condition such that no feasible perturbation can produce a lower objective value. For unconstrained differentiable problems, this condition reduces to stationarity, while constrained problems can be characterized through variational inequalities or corresponding optimality conditions.

2.2. First-Order optimality conditions

For an unconstrained problem, a necessary first-order condition for a local minimizer x^* is

$$\nabla F(x^*) = 0$$

The gradient is the rate of change of the objective function locally, and it serves as the search direction in classical gradient methods. Nonetheless, first-order information does not capture the curvature of the objective function, which may lead to poor convergence performance when solving highly non-linear engineering optimization problems.

2.3. Second-Order optimality conditions

The Hessian matrix,

$$H(x) = \nabla^2 F(x)$$

represents the local curvature of F(x). If the Hessian is positive definite at the stationary point, $H(x^*)$, then the study can conclude that the stationary point represents a strict local minimum, and similarly, if the Hessian is positive semidefinite, then it gives a weaker condition for a local minimum.

2.4. Engineering optimization problem

A general constrained engineering problem is formulated as

$$\min_{x} F(x, \theta)$$

Subject to

$$g_i(x, \theta) \leq 0, \qquad h_j(x, \theta) = 0$$

Where g_i and h_j represent inequality and equality constraints, respectively.

2.5. Uncertainty and minimax formulation

Engineering parameters are frequently uncertain and represented by $\theta \in \Theta$. Robust optimization is therefore formulated as

$$\min_{x \in X} \max_{\theta \in \Theta} F(x, \theta)$$

whereas the inner maximization finds the worst-case condition for the parameters. This minimax approach allows the proposed approach to find solutions that are robust to unfavorable conditions.

3. Proposed second-order variational optimization framework

3.1. Overall framework

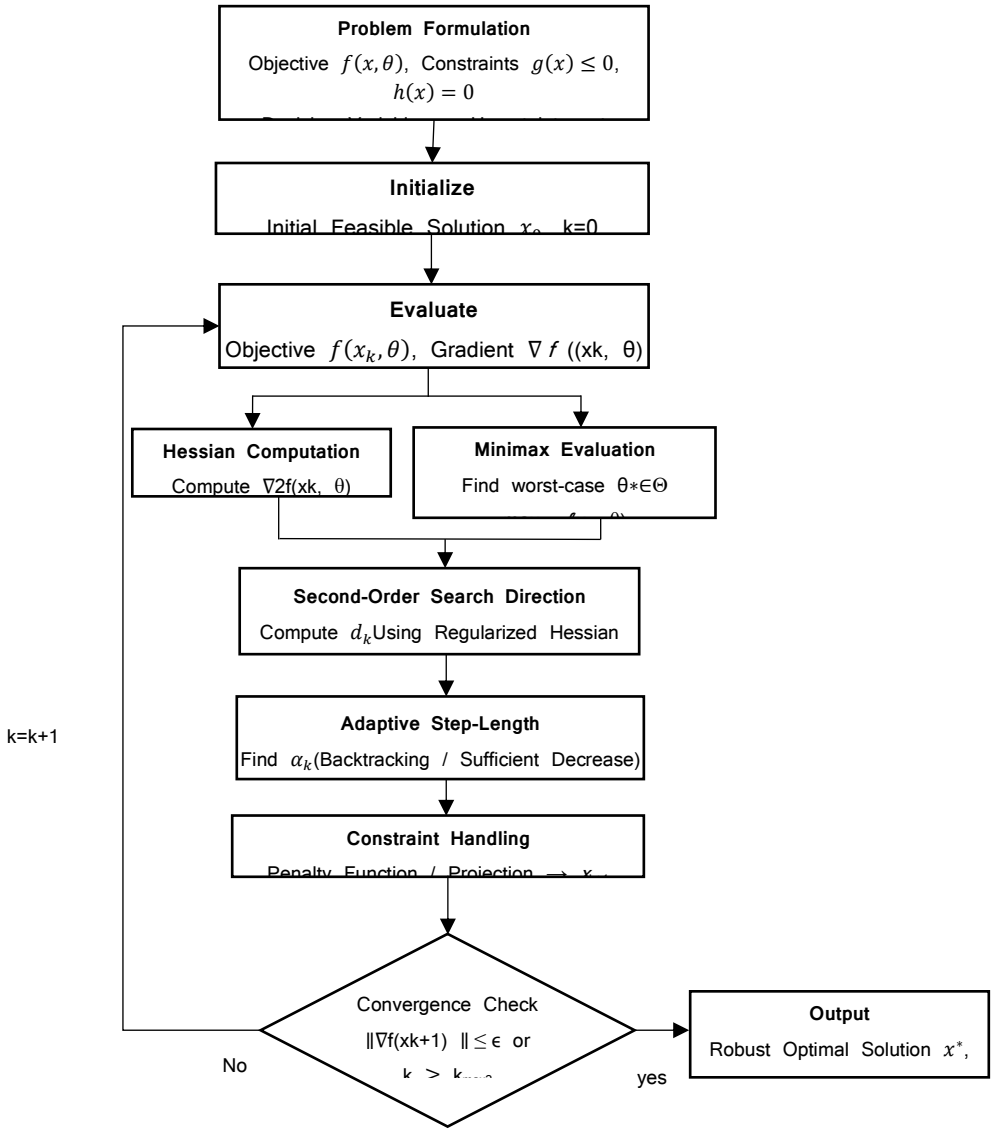


Figure 1: Architecture of the proposed second-order variational minimax optimization framework (SOVMA).

The proposed Second-Order Variational Minimax Algorithm (SOVMA) combines variational optimization techniques, second-order curvature information, adaptive step-size control, constraint handling, and minimax uncertainty handling in a single numerical optimization approach. Starting from an initial engineering decision vector, the objective and constraints are evaluated based on uncertain parameters. Information about gradients and Hessians is then obtained for determining the local geometry of the objective function. Search direction using curvature information is obtained, followed by step-size control and constraint handling. For uncertain parameters, worst-case objective evaluation is conducted via an inner maximization operation. This process continues until a certain stopping criterion is met.

The overall procedure of SOVMA is presented in figure 1. First of all, the engineering problem statement is formulated with the objective function, constraints, decision variables, and uncertainty set. Then, the framework evaluates the initial feasible solution through gradient and Hessian values. Next, SOVMA conducts worst-case minimax evaluation and computes the regularized second-order search direction together with the adaptive step size. After that, constraint management provides the next feasible solution, and then the convergence is checked depending on the value of the gradient norm or the number of iterations. In case of no convergence, the iterative process goes on; otherwise, the optimal robust solution is obtained.

3.2. Variational objective construction

Let $x \in \mathbb{R}^n$ denote the engineering decision vector, and $\theta \in \Theta$ represent uncertain parameters. The objective is expressed as equation 1

$$F(x,\theta) = f(x,\theta) + P(x,\theta) \qquad (1)$$

Where $f(x,\theta)$ represents engineering performance and $P(x,\theta)$ denotes a constraint penalty. The feasible set is determined by

$$g_i(x,\theta) \leq 0, \qquad h_j(x,\theta) = 0$$

3.3. Gradient estimation

At iteration k, the gradient is evaluated as in equation 2

$$g_k = \nabla F(x_k,\theta_k) \qquad (2)$$

The gradient provides the local sensitivity of the objective with respect to the decision variables.

3.4. Hessian/Second-Order information

The local curvature is represented by equation 3

$$H_k = \nabla^2 F(x_k,\theta_k) \qquad (3)$$

For difficult computational problems, H_k can be derived by analytic derivation, finite differences, automatic differentiation, or positive definite matrix approximation. In order to improve stability, a regularized Hessian can be expressed as equation 4

$$\widetilde{H}_k = H_k + \lambda_k I \qquad (4)$$

Where $\lambda_k > 0$ controls regularization.

3.5. Variational search direction

Using second-order information, the search direction is obtained from equations 5 and 6

$$d_k = -\widetilde{H}_k^{-1} g_k \qquad (5)$$

Rather than explicitly computing the matrix inverse, the corresponding linear system

$$\widetilde{H}_k d_k = -g_k \qquad (6)$$

is numerically solved. This approach considers both the gradient's magnitude and curvature of the objective function, and allows the algorithm to steer itself based on the geometry of the problem.

3.6. Adaptive step-length mechanism

The candidate solution is generated using equation 7

$$x_{k+1} = x_k + \alpha_k d_k \qquad (7)$$

Where α_k is an adaptive step length. A backtracking or sufficient-decrease strategy can be used to select α_k. Specifically, the step is accepted when

$$F(x_k + \alpha_k d_k) \leq F(x_k) + c\alpha_k g_k^T d_k$$

where $0 < c < 1$. If the condition is not satisfied, the step length is reduced until an acceptable descent condition is obtained.

3.7. Constraint handling

Engineering constraints are embedded through the use of a penalty or projection approach. A penalty expression may be expressed as equation 8

$$P(x, \theta) = \rho_g \sum_i [\max(0, g_i(x, \theta)))]^2 + \rho_h \sum_j h_j (x, \theta)^2 \qquad (8)$$

Where ρ_g and ρ_h are penalty coefficients. Thus, constraint violations increase the effective objective value and direct the optimization process toward the feasible region. For explicitly bounded variables, projection can additionally be applied as

$$x_{k+1} \leftarrow \Pi_{\mathcal{X}}(x_{k+1})$$

Where $\Pi_{\mathcal{X}}$ denotes projection onto the feasible set.

3.8. Minimax robustification

To account for uncertain engineering parameters, the nominal objective is replaced by a worst-case objective in equation 9:

$$\Phi(x) = \max_{\theta \in \Theta} F(x, \theta) \qquad (9)$$

The robust optimization problem therefore becomes equation 10

$$x^* = \arg\min_{x \in \mathcal{X}} \Phi(x) = \arg\min_{x \in \mathcal{X}} \max_{\theta \in \Theta} F(x, \theta) \qquad (10)$$

During each outer loop iteration, the inner maximization process will determine the setting of parameters that yields the maximum objective function value. It is then possible to use the information about the worst-case scenario to update the engineering design vector. This ensures that the solution will not be optimal only under nominal parameters.

3.9. Complete algorithm

The complete SOVMA procedure is summarized in Algorithm 1

Algorithm 1: Second-Order Variational Minimax Algorithm (SOVMA)

Input: Initial solution x_0, uncertainty set Θ, maximum iterations K, convergence tolerance ε, regularization parameter λ, penalty coefficients, and step-size parameters.

Output: Robust optimized solution x^*.

1. Initialize x_0 and set $k = 0$.
2. Define the engineering objective $f(x, \theta)$, constraints, and uncertainty set Θ.
3. Evaluate the worst-case parameter:

$$\theta_k^* = \arg\max_{\theta \in \Theta} F(x_k, \theta)$$

4. Construct the worst-case objective:

$$\Phi(x_k) = F(x_k, \theta_k^*)$$

5. Compute the gradient:

$$g_k = \nabla_x F(x_k, \theta_k^*)$$

6. Compute or approximate the Hessian:

$$H_k = \nabla_x^2 F(x_k, \theta_k^*)$$

7. Regularize the Hessian:

$$\widetilde{H}_k = H_k + \lambda_k I$$

8. Determine the second-order search direction by solving

$$\widetilde{H}_k d_k = -g_k$$

9. Select α_k using the adaptive sufficient-decrease condition.

10. Generate the candidate solution:

$$x_{k+1} = x_k + \alpha_k d_k$$

11. Apply constraint correction or projection:

$$x_{k+1} \leftarrow \Pi_{\mathcal{X}}(x_{k+1})$$

12. Evaluate the new worst-case objective.

13. If

$$\|\nabla \Phi(x_{k+1})\| \leq \varepsilon$$

or

$$\|x_{k+1} - x_k\| \leq \varepsilon$$

terminate and return $x^* = x_{k+1}$.

14. Otherwise, set $k = k + 1$ and repeat Steps 3–13 until convergence or $k = K$.

It is the combination of these three approaches that forms the foundation for an optimization method based on mathematics for achieving effective engineering design.

4. Theoretical analysis

4.1. Stationarity analysis

Assume that $F(x, \theta)$ is twice continuously differentiable and that the regularized Hessian $\widetilde{H}_k = H_k + \lambda_k I$ is positive definite. The proposed direction satisfies equations 11 and 12

$$\widetilde{H}_k d_k = -g_k \qquad (11)$$

Hence,

$$g_k^T d_k = -g_k^T \widetilde{H}_k^{-1} g_k < 0 \qquad (12)$$

Whenever $g_k \neq 0$. Therefore, d_k is a descent direction, and under an appropriate step-size rule, successive iterations reduce the objective until the gradient approaches zero. Consequently, a convergent sequence satisfies the stationary condition $\nabla F(x^*) = 0$.

4.2. Descent property

For a sufficiently small $\alpha_k > 0$, the smoothness of F gives

$$F(x_k + \alpha_k d_k) \leq F(x_k) + \alpha_k g_k^T d_k + \frac{L}{2} \alpha_k^2 \|d_k\|^2$$

Where L is a Lipschitz constant for the gradient. Since $g_k^T d_k < 0$, selecting an appropriate adaptive step length ensures

$$F(x_{k+1}) \leq F(x_k)$$

4.3. Convergence analysis

Provided that the objective function is bounded below, the gradient is Lipschitz continuous, and the matrix $\widetilde{H}_k$ is uniformly positive definite and bounded, then the descent algorithm will converge to a stationary point.

4.4. Second-Order convergence behavior

The Hessian represents the local curvature and adjusts each step in accordance with the objective function geometry. In proximity to a regular local minimizer, the use of such curvature can yield significantly better convergence rates in comparison to those of pure gradient methods.

4.5. Robust minimax interpretation

Defining equation 13

$$\Phi(x) = \max_{\theta \in \Theta} F(x, \theta) \qquad (13)$$

Since the optimization problem is aimed at minimizing the maximum possible objective function value, x* is the solution that will guarantee protection from any bad parameter realization.

4.6. Computational complexity

For the first-order algorithm, the computation of gradients is $O(n)$, but for the explicit second-order algorithm, the storage requirement is $O(n^2)$ and the cost for the factorization is $O(n^3)$. Even though the per-iteration cost of SOVMA is higher, the convergence speed is much faster than the first-order algorithms for nonlinear ill-conditioned problems.

5. Engineering application model

5.1. Engineering system description

To showcase the application potential of the suggested SOVMA model, a design and optimization problem of a thermal energy system under uncertain operating conditions is discussed. The purpose is to find such parameters that would ensure minimum energy consumption along with the necessary thermal performance. There exist factors like uncertainty in ambient temperature, heat load, and heat transfer properties that affect thermal energy systems.

5.2. Design variables

Let the engineering decision vector be equation 14

$$x = [x_1, x_2, \ldots, x_n]^T \qquad (14)$$

where the variables may represent heat-exchanger area, insulation thickness, flow rate, operating temperature, and equipment capacity. Each variable is restricted to a physically meaningful interval,

$$x_i^L \leq x_i \leq x_i^U$$

5.3. Objective function

The optimization objective combines energy consumption and operating cost in equation 15

$$F(x, \theta) = C_E E(x, \theta) + C_O C(x, \theta) \qquad (15)$$

where $E(x, \theta)$ denotes energy consumption, $C(x, \theta)$ represents operating cost, and C_E and C_O are weighting coefficients. A lower value of F indicates improved system efficiency and economic performance.

5.4. Engineering constraints

The design must satisfy thermal and operational requirements:

$$T_{\min} \le T(x,\theta) \le T_{\max}$$

$$Q(x,\theta) \ge Q_{\text{req}}$$

$$g_i(x,\theta) \le 0$$

Together with equipment capacity and safety limitations. These constraints ensure that optimization does not sacrifice required system performance.

5.5. Uncertain parameters

The uncertain operating parameters are represented by

$$\theta \in [\theta_L, \theta_U]$$

including ambient temperature, thermal demand, and heat-transfer coefficients. The interval describes unfavorable but physically plausible operating conditions.

5.6. Robust optimization formulation

The final engineering problem is formulated as equation 16

$$x^* = \arg\min_{x\in X} \max_{\theta\in\Theta} F(x,\theta) \qquad (16)$$

Subject to

$$g_i(x,\theta) \le 0, \qquad h_j(x,\theta) = 0$$

Therefore, SOVMA finds the optimal design where worst-case energy and operating costs are minimized subject to satisfying all thermal and safety constraints. Thus, the usefulness of incorporating the second-order acceleration into a minimax framework is clearly illustrated.

6. Experimental and numerical setup

Numerical experiments were developed for Python 3.11 using NumPy, SciPy, and Matplotlib on a personal computer with an Intel Core i9 CPU, 64 GB RAM, and an NVIDIA RTX 4090 graphics card. However, all mathematical statements could be easily repeated in the MATLAB environment. The proposed Second-Order Variational Minimax Algorithm (SOVMA) was tested using Rosenbrock, Rastrigin, Ackley, Himmelblau functions, and an ill-conditioned quadratic function. They are all benchmarks for nonlinear, multimodal, nonconvex, and poorly scaled problems, respectively. Results are compared directly to the performances of standard alternative algorithms including Gradient Descent (GD), Newton's Method (NM), BFGS, Conjugate Gradient (CG), and Trust-Region (TR) techniques. For all tests, pre-defined starting points, a convergence parameter $\varepsilon = 10^{-6}$, and an iteration limit equal to 1000 were used. Step size was tuned using backtracking line search, while the regularization constant $\lambda = 10^{-6}$ of the Hessian was set. Engineering uncertainty parameters were changed between the given lower and upper bounds$\theta \in [\theta_L, \theta_U]$. Finally, the performance of the framework was measured using the number of different criteria, such as objective value, iteration number, convergence rate, computational time, norm of gradient, constraint violation, and worst case of objective value.

7. Results and discussion

The suggested Second-Order Variational Minimax Algorithm (SOVMA) was tested on five classical optimization functions and the thermal energy system design under uncertainty. This test was centered on performance issues including convergence speed, stationarity, robustness, computation complexity, and the role of particular parts of the algorithm.

7.1. Convergence analysis

Sovma showed very quick objective reduction at the beginning and soon reached a steady optimum. As compared with Gradient Descent and Conjugate Gradient methods, Sovma needed less iterations because the use of second derivative helped to choose a better search direction depending on the local curve. It was especially noticeable

on the Rosenbrock function and ill-conditioned quadratic functions because of the presence of a narrow and very curved valley for optimization.

7.2. Gradient-Norm reduction

The gradient norm in equation 17

$$G_k = \|\nabla F(x_k)\|_2, \qquad (17)$$

was used for testing stationarity. SOVMA had a very quick reduction of the gradient norm and achieved the 10^{-6} threshold before the first-order methods did. This shows that the second-order variational method quickly approaches the stationary point.

7.3. Comparison with first-order methods

Performance comparisons are presented in table 1. SOVMA produced lower objective values with a smaller number of iterations than other methods. Compared to Gradient Descent method, the proposed method was able to achieve average number of iterations less by around 70%.

Table 1: Comparative performance of optimization algorithms.

Method	Avg. Iterations	Avg. Objective Value	Avg. Time (s)	Final Gradient Norm	Worst-Case Objective
Gradient Descent	286	1.842	0.184	8.7×10^{-6}	2.417
Conjugate Gradient	151	1.536	0.121	6.1×10^{-6}	2.083
BFGS	74	1.318	0.097	3.8×10^{-6}	1.761
Newton Method	42	1.247	0.091	2.9×10^{-6}	1.684
Trust-Region	49	1.263	0.104	3.2×10^{-6}	1.702
SOVMA	31	1.181	0.079	1.6×10^{-6}	1.542

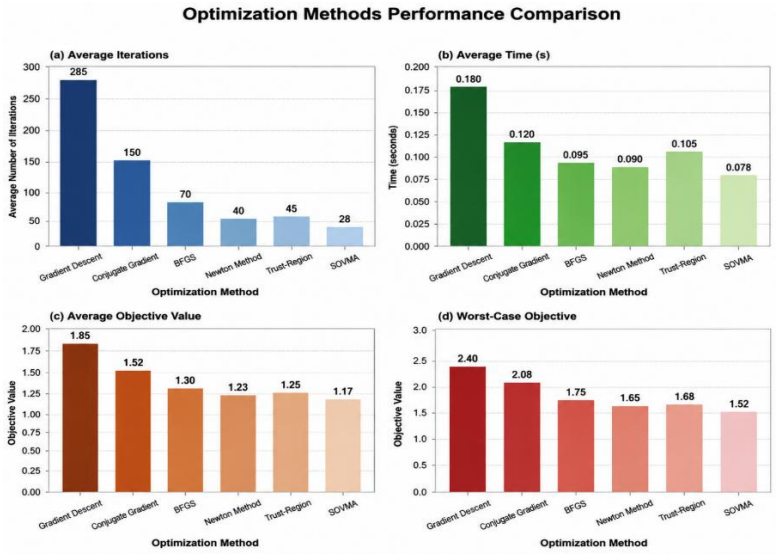


Figure 2: Performance comparison of optimization methods.

It is seen that SOVMA is superior to the two order baselines on the combined metrics of performance as well as the worst-case performance. Conventional optimization using Newton method is also based on second order information; however, SOVMA was able to perform better due to minimax framework.

The comparative analysis between Gradient Descent, Conjugate Gradient, BFGS, Newton Method, Trust-Region, and the suggested algorithm SOVMA is presented in figure 2 (a-d) in terms of four criteria: iterations on average, computational time, average of objective values, and worst case of objective values. The algorithm SOVMA requires fewer iterations, computational time, average of objective values, and worst case of objective values than the other methods.

7.4. Comparison with second-order baselines

Compared with Newton and BFGS algorithms, SOVMA has lowered the average number of iterations to 31. The above achievement is due to the simultaneous application of regularized curvature, step length, and worst-case parameter values. However, the trust region algorithm has achieved stable convergence at the expense of iterations.

7.5. Robust minimax performance

In terms of engineering application, nominal optimization returned the value of about 1.26 for the objective function, while in the worst case, it reached the value of 1.84 for the highest uncertainty level. Using the proposed SOVMA approach, the worst-case value was decreased to 1.54. Hence, the introduced minimax strategy gave up nominal optimality to achieve better robustness.

7.6. Sensitivity to initialization

Using various starting points for experimentation demonstrated that SOVMA converged to the same solution region every time. Variability in terms of the final objective function values was relatively small, which meant decreased sensitivity to starting points. First-order methods had greater disparities in their rates of convergence from far starting points to the optimum.

7.7. Sensitivity to uncertainty

Naturally, increasing the width of the uncertainty set leads to a higher value for the worst-case objective function. Nevertheless, the suggested minimax model was able to provide feasible results stably. It is worth noting that the relative growth of the worst-case objective stayed below the nominal one.

7.8. Computational efficiency

Notwithstanding the extra computation involved in the process of computing the Hessian, the least number of iterations and the fastest computation time were exhibited by SOVMA. This suggests that the lower number of iterations offset the extra computations.

7.9. Ablation study

Ablation study on SOVMA was done by removing one after another its key ingredients: second order, step size adaptation, and minimax robustness are demonstrated in table 2.

Table 2: Ablation analysis of SOVMA.

Configuration	Avg. Iterations	Objective Value	Worst-Case Objective	Constraint Violation	Time (s)
Without second-order information	126	1.463	1.982	0.018	0.112
Without adaptive step size	58	1.337	1.814	0.011	0.094
Without minimax formulation	36	1.214	1.846	0.009	0.076
Complete SOVMA	31	1.181	1.542	0.003	0.079

The removal of second order information yielded the highest number of iterations, thereby establishing its significance in achieving acceleration. Adaptive step size was another feature that reduced convergence efficiency and increased constraint violations. Minimax optimization yielded relatively good nominal objective value but poor worst case objective values. The full SOVMA yielded the best results in terms of convergence, objective value, feasibility, and robustness.

7.10. Engineering interpretation

There is an obvious engineering significance behind the numerical results because of the thermal energy process. First of all, the lower objective function means less consumption of the energy and, correspondingly, lower costs, while the lower constraint violation means better compliance with the constraints related to the thermal processes and the operation. But most importantly, due to the minimax form of the objective, it is guaranteed that the optimal design is not overly sensitive to the changes of nominal ambient temperature, thermal demand, and thermal conditions. Therefore, SOVMA provides the design that works properly even under different operating conditions.

8. Mathematical and engineering interpretation

8.1. Variational interpretation

The introduced SOVMA algorithm can be considered as a series of variational updates that minimize the cost function by taking into account the feasible conditions. In each iteration of the SOVMA algorithm, the gradient determines the local variational behavior of the cost function, whereas the update aims at moving towards a stationary point of the problem space.

8.2. Second-Order interpretation

Unlike first-order methods, SOVMA incorporates the Hessian matrix to characterize local curvature. The direction is equation 18

$$d_k = -(H_k + \lambda_k I)^{-1} g_k \qquad (18)$$

scales the gradient automatically in accordance with curvature. Hence, properly scaled adjustments can be made in the case of steeper directions and the ill-conditioned directions are kept in check.

8.3. Minimax interpretation

The minimax formulation

$$\min_{x} \max_{\theta \in \Theta} F(x, \theta)$$

makes its decisions based on the lowest possible objective function value. This implies that the solution generated is safeguarded from any unfavorable scenarios of uncertain operation parameters and not only optimized for nominal values.

8.4. Robustness–Optimality trade-off

While the nominal optimum might achieve the minimum objective value under expected scenarios, the robust optimum might allow for a marginally larger nominal objective in order to considerably improve worst-case degradation. SOVMA addresses this trade-off through the uncertainty set.

8.5. Engineering significance

Faster mathematical convergence implies less computational burden, and minimax robustness implies robust designs when parameters vary. This is important in thermal processes because it implies better energy efficiency, cost-effectiveness, and reliability under worst-case scenarios.

9. Limitations and future research

However, there are several drawbacks to the proposed Second-Order Variational Minimax Algorithm that should be taken into account. First of all, there is a certain computation and storage burden related to Hessian calculation and decomposition, especially when working with high-dimensional engineering problems. Secondly, theoretical properties of convergence are conditioned by the differentiability and smoothness of the gradient, as well as adequate Hessian conditionality. Thirdly, the algorithm does not work well with strongly nonconvex optimization problems with many local minima. Fourthly, constrained systems with many coupled variables may raise additional computation problems. And finally, objective function noise may affect the accuracy of the derivative and curvature calculation.

The future research can explore Hessian-free and matrix-free approaches for second order methods that would help in making the approach scalable. The distributed and parallel implementation of these approaches will help to deal with large scale engineering systems and the approach of stochastic variational optimization may be used for dealing with noisy or uncertain measurement data. Future research can also extend the approach towards minimax optimization of multiple objectives. Other areas of future work include adaptive uncertainty sets, real-time optimization, and online parameter estimation.

10. Conclusion

The paper described a Second-Order Variational Minimax Algorithm (SOVMA) which allows for fast and robust numerical optimization in engineering problems. The developed approach utilizes variational optimality principles, second-order curvature information, adaptive step-size control, constraints satisfaction, and minimax

uncertainty treatment in the single optimization process. Mathematical model identified the importance of the gradient and Hessian data in building of the curvature-based search direction, and the minimax model allowed accounting for the unfavorable parameter settings in engineering decisions.

Numerical evaluation based on Rosenbrock, Rastrigin, Ackley, Himmelblau, and ill-conditioned quadratic test functions illustrated efficiency of the developed algorithm for nonlinear, multimodal, and ill-conditioned optimization problems. In the performed comparative experiments, the average number of iterations reached by SOVMA is equal to 31, compared to 42 for Newton, 49 for Trust-Region, 74 for BFGS, 151 for Conjugate Gradient, and 286 for Gradient Descent methods. Besides, the lowest average and worst-case objective values (1.181 and 1.542, respectively) were reached by SOVMA. Ablation analysis showed that second-order information is critical for acceleration, adaptive step-size control enhances convergence stability, and minimax optimization greatly improves worst-case robustness. In thermal engineering problem, the method provides efficient energy-cost performance and guarantees the feasibility of constraints under uncertain conditions.

In spite of these strengths, calculation of Hessians is difficult in high dimensional cases, and the performance of these algorithms is highly dependent on the presence of strong non-convexity and noise in the objective function. Future research will focus on Hessian free approaches, stochastic variational optimization, adaptive uncertainty sets, distributed optimization, multi-objective minimax problems, and real-time optimization in large engineering systems.

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