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Dual of the Class of HK$_r$ Integrable Functions

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We define for $1 \leq r < \infty$ a norm for the class of functions which are Henstock-Kurzweil integrable in the L^r sense. We then establish that the dual in this norm is isometrically isomorphic to $L^{r'}$ and is therefore a Banach space, and in the case $r = 2$, a Hilbert space. Finally, we give results pertaining to convergence and weak convergence in this space.

Keywords: L^r-Henstock-Kurzweil integral, HK_r-dual, HK_r-norm.

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1. History and aim

Many variants on the classical derivative have been formulated. The idea of integration as a means for recovering a function from its derivative goes back to the earliest days of calculus. Denjoy (1912) and Perron (1914) developed nonabsolute integration methods that recover a function from its classical derivative. Later, Burkill (1931) developed a Perron-type integration process for recovering a function from its approximate derivative. Working independently, Kurzweil and Henstock developed an integration process equivalent to those of Denjoy and Perron, but which maintains the sense of Riemann integration by defining the integral as the limit of Riemann sums, subject to a pointwise-defined positive gauge function. This integration process and some his generalization has found to be suitable for many applications, for example [15], [16], [18], in Haar, Walsh and Vilekin series [19] [20], in Riesz Space [2] [3], in zero-dimensional groups [12], [13], [14], [17], and in the multidimensional case [21].

In [1], Alexiewicz defined a norm on the space of Denjoy integrable functions f. He showed that the dual of this space is isomorphic to the space of functions of bounded variation on $[a, b]$. In 1961, Calderon and Zygmund [5] described the L^r derivative, which is preserved at individual points under operations such as

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fractional integration and singular integral transformations. In 1967, Gordon [6] described the P_r integral, a Perron type integral that recovers a function from its L^r derivative. Bongiorno and Panchapagesan [4] and Talvila [23] extended Alexiewicz's work by describing the completion of the space of Denjoy integrable functions with respect to the Alexiewicz norm. In 2004 Musial and Sagher [8] developed a Henstock-Kurzweil type integral, (HK_r), that also recovers a function from its L^r derivative. Indeed, the HK_r integral extends the P_r integral, though it is not known whether the P_r integral integrates all HK_r-integrable functions. In 2015, Musial and Tulone [9] gave an integration by parts formula for the HK_r integral.

In this paper, the authors describe a norm on the space of HK_r-integrable functions, as well as the dual and completion of this space.

2. Introduction

Definition 2.1. [8] A real-valued function $f \in HK_r\,[a,b]$ is called *L^r Henstock-Kurzweil integrable* if there exists a function $F \in L^r\,[a,b]$ so that for any $\varepsilon > 0$ there exists a gauge function $\delta\,(x) > 0$ so that, whenever $\mathcal{P} = \left\{(x_i,[c_i,d_i])_{1\le i\le q}\right\}$ is a δ-fine tagged partition of $[a,b]$, we have

$$\sum_{i=1}^{q}\left(\frac{1}{d_i-c_i}\,(L)\int_{c_i}^{d_i}\left|F\,(y)-F\,(x_i)-f\,(x_i)\,(y-x_i)\right|^r dy\right)^{\frac{1}{r}} < \varepsilon. \tag{1}$$

We then say that F is the *indefinite HK_r primitive* of f. □

In this paper, if an integral is not specified, it is a Lebesgue integral. We will say that a gauge δ is *HK_r-appropriate* for ε and for f if (1) holds for any δ-fine tagged partition $\mathcal{P}$. If f is HK_r-integrable on $[a,b]$, the following function is well-defined for all $x \in [a,b]$:

$$F\,(x) = (HK_r)\int_a^x f\,(t)\;dt \tag{2}$$

In 2015 we gave an integration by parts formula for the HK_r integral.

Theorem 2.2. [9] *Suppose that f is HK_r-integrable on $[a,b]$, and G is absolutely continuous on $[a,b]$ with $G' \in L^{r'}([a,b])$, where $1 \le r < \infty, r' = r/(r-1)$. Then fG is HK_r-integrable on $[a,b]$ and if $F\,(x)$ is defined as in (2) then*

$$(HK_r)\int_a^b f\,(t)\,G\,(t)\;dt = F\,(b)\,G\,(b) - \int_a^b F\,(t)\,G'\,(t)\;dt.$$

3. HK_r Norm

For every $f \in HK_r\,([a,b])$ there is a unique function $F \in ACG_r\,([a,b])$ such that $F_r^{'} = f$ a.e. [8]. It is shown in [9] that the space $HK_r\,([a,b])$ is linear. We now equip it with a norm:

Definition 3.1. Let $f \in HK_r([a,b])$. We define the HK_r norm of f as follows:

$$\|f\|_{HK_r} := \|F\|_r \,,$$

where F is the indefinite HK_r integral of f as defined in (2). □

This is a natural extension of the norm for the classical case $r = \infty$ first proposed in 1948 by Alexiewicz [1].

Fix $r \geq 1$. Let us show that $HK_r([a,b])$ is separable under this norm. Consider $\mathcal{P}$, the collection of polynomials having rational coefficients. This is a countable subset of $HK_r([a,b])$ which is dense in L^r and therefore dense in the collection of $HK_r([a,b])$-primitives. Since the collection of derivatives of functions in $\mathcal{P}$ is again $\mathcal{P}$, we have that $\mathcal{P}$ is dense in $HK_r([a,b])$.

Suppose $\{f_n\}$ is a sequence of functions in $HK_r([a,b])$ having indefinite HK_r primitives $\{F_n\}$. We will say $\{f_n\}$ *converges* (strongly) in HK_r sense to $f \in HK_r([a,b])$ if $\{F_n\} \to F$ in L^r, where F_n and F are the HK_r primitives of f_n and f respectively.

We identify a function $f \in HK_r$ with the equivalence class of real-valued functions that are equal to f a.e. Thus we have established an isometric one-to-one correspondence between the space HK_r and the space of HK_r primitives.

It is clear that if a sequence $\{f_n\} \in L^r([a,b])$ converges in the L^r norm, then $\{f_n\}$ converges in HK_r. However, the converse does not hold. Take e.g.

$$f_n(x) = n^{1/r}\chi_{[1-1/n,n]}(x).$$

These functions converge to zero in HK_r but do not converge in L^r.

4. HK_r Dual

We now characterize the linear functionals on HK_r.

Theorem 4.1. *Let* $1 \leq r < \infty$. ϕ *is a linear functional on* $\left(HK_r\left([a,b]\right), \|\cdot\|_{HK_r}\right)$ *if and only if there exists an absolutely continuous function* $\widehat{G}$ *defined on* $[a,b]$ *so that* $\widehat{G}(b) = 0$ *and* $\left(\widehat{G}\right)' \in L^{r'}([a,b])$, *where* $r' = r/(r-1)$, *so that for every* $f \in HK_r([a,b])$ *we have*

$$\phi(f) = (HK_r)\int_a^b f(x)\,\widehat{G}(x)\,dx = -\int_a^b F(x)\left(\widehat{G}\right)'(x) \tag{3}$$

where F *is given by* (2). *In this case,*

$$\|\phi\|_{HK_r} = \left\|\left(\widehat{G}\right)'\right\|_{L^{r'}}.$$

Proof. First we consider $1 < r < \infty$. Suppose ϕ is a linear functional on HK_r. There is a corresponding linear functional $\widehat{\phi}$ defined on the space of HK_r primitives on $[a,b]$ so that for every $f \in HK_r\left([a,b]\right)$ we have

$$\phi\left(f\right) = \widehat{\phi}\left(F\right), \quad \text{and therefore:} \quad \|\phi\| = \left\|\widehat{\phi}\right\|.$$

By the Hahn-Banach Theorem, the functional $\widehat{\phi}$ can be extended to all of L^r without altering the norm. Thus by the Riesz Representation Theorem, there exists a unique function $G \in L^{r'}$ so that

$$\widehat{\phi}(F) = \int_a^b F(x)G(x)dx$$

for every $F \in L^r$. Define $\widehat{G}\left(x\right) := -\int_x^b G\left(t\right)\, dt.$

Hence we have $\widehat{G}\left(b\right) = 0$. We use Theorem 2.2 to write for every $f \in HK_r\left([a,b]\right)$

$$\phi\left(f\right) = -\left(HK_r\right)\int_a^b f\left(x\right)\widehat{G}\left(x\right)\, dx.$$

Thus the first part of the theorem is proved.

Next, suppose $\widehat{G}$ satisfies the hypotheses of the theorem. Let $G = \left(\widehat{G}\right)'$ and let ϕ be defined as in (3) for functions $f \in HK_r\left([a,b]\right)$. Define F as in equation (2). Then by Theorem 2.2 we have

$$\phi\left(f\right) = -\int_a^b F\left(x\right)G(x)\, dx.$$

By Hölder's inequality we have

$$\left|\phi\left(f\right)\right| \le \|F\|_r \|G\|_{r'} = \|f\|_{HK_r} \|G\|_{r'}$$

Therefore ϕ is a linear functional on HK_r with $\|\phi\|_{HK_r} \le \|G\|_{r'}$.

To show the reverse inequality, let $G \in L^{r'}$ and define as in [11]

$$G^* = \|G\|_{r'}^{1-r'} \cdot \operatorname{sgn}(G) \cdot |G|^{r'-1}$$

so that $G^* \in L^r([a,b]), \|G^*\|_r = 1$ and

$$\int_a^b G \cdot G^* = \|G\|_{r'}.$$

Let $\varepsilon > 0$. Since $AC([a,b])$ is dense in $L^r([a,b])$, we may choose $F \in AC([a,b])$ so that $\|F - G^*\|_r < \varepsilon$. Let $f = F'$ so that F is the HK_r primitive of f. Then

$$\begin{aligned}|\phi(f)| &= \left|\int_a^b FG\right| = \left|\int_a^b G^* \cdot G + \int_a^b (F - G^*) \cdot G\right| \\ &\ge (1-\varepsilon)\|G\|_{r'} \ge (1-\varepsilon)(\|F\|_r - \varepsilon)\|G\|_{r'} = (1-\varepsilon)(\|f\|_{HK_r} - \varepsilon)\|G\|_{r'}.\end{aligned}$$

Since ε is an arbitrary positive number, we have

$$|\phi(f)| \geq \|f\|_{HK_r} \|G\|_{r'}.$$

Now we consider $r = 1$. Since the dual to $L^1([a,b])$ is $L^\infty([a,b])$, the dual to HK_1 is the linear space of absolutely continuous functions $\widehat{G}$ having bounded derivative a.e. (that is, Lipschitz functions of order 1) and such that $\widehat{G}(b) = 0$. The rest of the proof follows as in the case $r > 1$ except that for $G \in L^\infty$ we define $G^* = \operatorname{sgn}(G)$.

Thus Theorem 4.1 is proved. □

For $r \geq 1$ we have thus established that the dual of the space of HK_r-integrable functions is isometrically isomorphic to $L^{r'}$. That the dual is complete follows from the completeness of $L^{r'}$, so the dual is a Banach space, and in the case of $n = 2$ a Hilbert space.

It is clear from the preceding proof that the following modification of Hölder's inequality is true.

Corollary 4.2. *Let* $f \in HK_r([a,b])$, $g \in L^{r'}$ *and* $G(x) := \int_x^b g$.

Then $fG \in HK_r([a,b])$ *and* $\left|(HK_r)\int_x^b fG\right| \leq \|f\|_{HK_r} \cdot \|g\|_{r'}$.

We use Theorem 4.1 to characterize weak convergence in the space HK_r.

Corollary 4.3. *A sequence of functions* $\{f_n\}$ *in* HK_r *converges weakly to a function* $f \in HK_r$ *if for every absolutely continuous function* $\widehat{G}$ *defined on* $[a,b]$ *such that* $\widehat{G}(b) = 0$ *and* $\left(\widehat{G}\right)' \in L^{r'}([a,b])$*,where* $r' = r/(r-1)$*, we have*

$$(HK_r)\int_a^b f_n(x)\,\widehat{G}(x)\,dx \to (HK_r)\int_a^b f(x)\,\widehat{G}(x)\,dx$$

or, equivalently,

$$\int_a^b F_n(x)\left(\widehat{G}\right)'(x)\,dx \to \int_a^b F(x)\left(\widehat{G}\right)'(x)\,dx$$

where F_n *and* F *are defined as in equation* (2).

In other words, the sequence $\{f_n\}$ converges weakly to f in HK_r if and only if the sequence $\{F_n\}$ of HK_r primitives of $\{f_n\}$ converges weakly in L^r to F, the HK_r primitive of f.

For $1 \leq r < \infty$, a sequence in $HK_r([a,b])$ can converge weakly to at most one function in HK_r. This follows from the uniqueness of weak limits in L^r.

Remark 4.4. The results proven thus far allow us to extend such classical theorems as those of Radon-Riesz and Banach-Saks to the to the HK_r setting. See [11], chapter 8.

We now show that for $1 \leq r < \infty$, $HK_r([a,b])$ fails to be complete when paired with the $\|\cdot\|_{HK_r}$ norm. Fix $1 \leq r < \infty$, let $[a,b] = [0,2]$ and for $k \geq 1$ let $f_k = k\chi_{[1-1/k,1]}$ and let F_k be the indefinite HK_r primitive of $\{f_k\}$. $\{F_k\}$ converges in $\|\cdot\|_r$ norm to $F(x) = \chi_{[1,2]}(x)$. Let $[a,b] = \cup_{n=1}^{\infty} E_n$. Let n be such that $1 \in E_n$. Let $G = F$ a.e. in $[a,b]$. We will show that G cannot be in $AC_r(E_n)$. For any $\gamma \in (0,1)$ we have

$$\left(\frac{1}{2\gamma}\int_{1-\gamma}^{1+\gamma} |G(y) - G(1)|^r \, dy\right)^{\frac{1}{r}} \geq 1/2.$$

Thus we have that $G \notin AC_r(E_n)$ and therefore $G \notin ACG_r([a,b])$ so G cannot be an HK_r primitive. If the sequence $\{f_k\}$ were to converge in HK_r norm then $\{F_k\}$ would need to converge in L^r norm to an ACG_r function, and we have shown that that cannot happen.

We now characterize the completion of $HK_r([a,b])$. For $h \in HK_r([a,b])$ let $\Phi_0(h)$ denote the HK_r primitive of h, i.e.,

$$\Phi_0(h)(x) = (HK_r)\int_a^x h$$

so that $\Phi_0(h)(a) = 0$. We will say that two sequences, $\{f_n\}$ and $\{g_n\}$ of HK_r integrable functions are equivalent if their HK_r primitives $\{\Phi_0(f_n)\}$ and $\{\Phi_0(g_n)\}$ converge in L^r to the same function F; this function is unique up to a set of measure zero. We will denote by $\mathcal{H}$ the collection of equivalence classes induced by this relation. $\mathcal{H}$ is the completion of $HK_r([a,b])$, and we will denote elements of $\mathcal{H}$ in boldface type, e.g., $\mathbf{h}$. If $\{h_n\}$ converges to $\mathbf{h}$ in HK_r we denote the function F defined above by $\widehat{\mathbf{h}}$ and define

$$\|\mathbf{h}\|_{HK_r} = \|\widehat{\mathbf{h}}\|_{L^r} = \lim_n \|h_n\|_{HK_r}.$$

Since HK_r is dense in $\mathcal{H}$, the following characterizes the dual of $\mathcal{H}$.

Theorem 4.5. *Let $1 \leq r < \infty$. ϕ is a linear functional on $\mathcal{H}$ if and only if there exists a function $G \in L^{r'}([a,b])$ so that*

$$\phi(\mathbf{h}) = \int_a^b \widehat{\mathbf{h}}(x)\, G(x)\, dx. \tag{4}$$

In this case, $\|\phi\|_{HK_r} = \|G\|_{L^{r'}}$.

A function ϕ is said to be a *test function* if ϕ is infinitely differentiable and has support in (a,b). A distribution is a continuous linear functional on the space of test functions. We say that two distributions are equal if they agree on the space of test functions.

Theorem 4.6. *Let the mapping* $\Phi : \mathcal{H} \to L^r$ *be given by* $\Phi(\mathbf{h}) = \widehat{\mathbf{h}}$. *Then* $\widehat{\mathbf{h}}$ *is unique up to a set of measure zero and* Φ *is a bijection extending* Φ_0. *Thus* Φ *is the unique isometric extension of* Φ_0 *to* $\mathcal{H}$.

Proof. Given $\mathbf{h}$, that $\widehat{\mathbf{h}}$ exists and is unique up to a set of measure zero, and that Φ is an injection, are clear from the definition of Φ. Let us show that Φ is a surjection. Let $F \in L^r([a,b])$. There exists a sequence of test functions $\{F_n\}$ which converges to F in L^r. Each of the F_n is the HK_r primitive of $F_n' \in HK_r([a,b])$. Thus we have

$$\|F_n' - F_m'\|_{HK_r} = \|F_n - F_m\|_{L^r} \to 0 \text{ as } n, m \to \infty.$$

Hence there is some $\mathbf{h} \in \mathcal{H}$ so that $F_n' \to \mathbf{h}$ in HK_r and so $F = \widehat{\mathbf{h}}$. □

Given a function $F \in L^r([a,b])$ we will denote its *distributional derivative* by

$$D_F(\phi) = -\int_a^b F\phi'\,dt.$$

for any test function ϕ.

If $\{h_n\} \to \mathbf{h}$ we define the following on the space of test functions ϕ:

$$\langle \mathbf{h}, \phi\rangle = \lim_n \langle h_n, \phi\rangle = \lim_n\,(HK_r)\int_a^b h_n\phi\,dt.$$

This is clearly a continuous functional on the space of test functions, and so is a distribution. It is well-defined because it is independent of the choice of sequence converging to $\mathbf{h}$.

Theorem 4.7. *The following assertions hold:*

(1) $\mathbf{h} \in \mathcal{H}$ *if and only if there exists some* $F \in L^r([a,b])$ *so that* $\mathbf{h} = D_F$ *(as distributions).*

(2) *For each* $\mathbf{h} \in \mathcal{H}$ *there exists* $F \in L^r([a,b])$ *such that* $\mathbf{h} = D_F$; *in particular,* $\mathbf{h} = D_{\widehat{\mathbf{h}}}$. *Moreover, if* $D_F = D_G$ *then* $F - G$ *is a constant almost everywhere.*

Proof. (1) Let $\mathbf{h} \in \mathcal{H}$, let $\{h_n\}$ be a sequence in $HK_r([a,b])$ converging to $\mathbf{h}$ and let $F_n = \Phi_0(h_n)$. Since $\|F_n - F_m\|_{L^r} = \|h_n - h_m\|_{HK_r} \to 0$, the sequence $\{F_n\}$ converges in L^r to a function $F = \widehat{\mathbf{h}}$. Let ϕ be a test function. Using integration by parts we have

$$\begin{aligned}\langle \mathbf{h}, \phi\rangle &= \lim_n \langle h_n, \phi\rangle = \lim_n\,(HK_r)\int_a^b h_n\phi\,dt = \lim_n\,[F_n\phi]_a^b - \lim_n\,(HK_r)\int_a^b F_n\phi'\,dt\\ &= -\int_a^b F\phi'\,dt = D_F(\phi).\end{aligned}$$

Therefore we have $\mathbf{h} = D_F$ and $F = \widehat{\mathbf{h}}$.

Conversely, let $F \in L^r([a,b])$. There exists a sequence of test functions $\{F_n\}$ which converges to F in L^r. Each of the F_n is the HK_r primitive of some $h_n \in HK_r([a,b])$. Therefore $\|h_n - h_m\|_{HK_r} = \|F_n - F_m\|_{L^r} \to 0$. Hence there is some $\mathbf{h} \in \mathcal{H}$ so that $h_n \to \mathbf{h}$ in HK_r. Let ϕ be a test function. Then we have

$$\begin{aligned} D_F(\phi) &= -\int_a^b F\phi'\,dt = -\lim_n \int_a^b F_n\,\phi'\,dt \\ &= \lim_n [F_n\phi]_a^b - \lim_n \int_a^b F_n\,\phi'\,dt = \lim_n (HK_r)\int_a^b h_n\,\phi\,dt = \langle \mathbf{h}, \phi\rangle . \end{aligned}$$

Thus $D_F(\phi) = \langle \phi, \mathbf{h}\rangle$ and $F = \widehat{\mathbf{h}}$.

(2) The first part of the assertion follows immediately from part (1). We have that the distributional derivative $\mathbf{h} = D_F$ has a primitive which is a distribution whose action on any test function ϕ is given by

$$\langle F, \phi\rangle = \int_a^b F\phi\,dt.$$

A distributional derivative has infinitely many primitives, any two of which differ by a constant (see [22]). Suppose we have $D_F(\phi) = D_G(\phi)$ for all test functions ϕ. Then F and G, when considered as distributions, differ by a constant. That is, there exists C so that for all test functions ϕ

$$\langle F - G - C, \phi\rangle = \int_a^b (F(t) - G(t) - C)\phi\,dt = 0. \tag{5}$$

It can easily be shown that if for $F, G \in L^r([a,b])$, we have (5) holds for all test functions ϕ then $F - G = C$ a.e. on $[a,b]$ where C is given above.

Suppose the contrary and let $J = F - G - C$. Then $\int_a^b |J|\,dt > 0$ and

$$\max\left(\int_a^b J^+\,dt, \int_a^b J^-\,dt\right) > 0;$$

we will assume that $\int_a^b J^+\,dt = \eta > 0$. Let $E = \{x \in [a,b] : J(x) > 0\}$. Let δ be such that $\int_A J\,dt < \eta/3$ if $|A| < \delta$. Choose $S = \bigcup_{k=1}^q I_k$ a disjoint union of open intervals such that set $D = \{x : \chi_E(x) \neq \chi_S(x)\}$ has measure less than δ. Choose also a test function ϕ so that $0 \le \phi \le 1$ and the support of $\phi - \chi_S$ has measure less than δ. We then have

$$\begin{aligned} \int_a^b J\phi\,dt &= \int_a^b J\chi_E\,dt + \int_a^b J(\chi_S - \chi_E)\,dt + \int_a^b J(\phi - \chi_S) \\ &\ge \int_a^b J^+\,dt - 2\int_D |J|\,dt \ge \frac{\eta}{3} \end{aligned}$$

contradicting the claim that such an integral must equal zero. □

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