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Constraint qualification analysis and optimality conditions for nonlinear engineering programming problems

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Abstract:

Constraint qualifications provide an analytical basis for the validity of both first- and second-order optimality principles in nonlinear programming models that have applications in engineering design areas such as structural, mechanical, and network engineering. In many cases of non-convex equality and inequality constraints, feasible regions do not fully satisfy the regularity conditions of such concepts as linear independence constraint qualification, making it impossible to prove the need of the Karush-Kuhn-Tucker theory at degenerate points. The framework of qualification theory is proposed that offers a connection between generalized Mangasarian-Fromovitz and Abadie-type conditions, and conditions of the metric and sub-regularity qualification theory for the problems of coupled, possibly non-smooth systems of engineering constraints. The concepts of tangent and linearized feasibility cones are introduced in order to describe the geometry of regularity of constraint qualifications, and to derive the sharp descriptions of the boundedness and compactness of sets of Lagrange multipliers. Based on the proposed theory, it is proved that generalized conditions of first-order stationarity and conditions of second-order sufficiency are valid for the entire critical cone of the Lagrangian function. The practical relevance of the theoretical results is illustrated in a benchmark optimization problem in structural engineering which includes various constraints of stress and deformation that are non-convex in nature.

Keywords: Constraint qualification, nonlinear programming, Karush–Kuhn–Tucker conditions, Mangasarian–Fromovitz condition, metric sub regularity, Lagrange multipliers, engineering optimization.

1. Introduction

Nonlinear programming models are vital to the analysis of modern engineering design tasks, which include structural mechanics, resource allocation in systems comprising many agents, as well as nonlinear network flow problems. Feasible regions in the above-mentioned models are commonly defined by non-convex inequalities and equalities bound to each other jointly and expressing the constraints existing in nature, e.g., those related to stress, deformation, capacity, and conservation of flow. The method of regularization of mathematical programming problems with vanishing constraints shows that an appropriate perturbation scheme can reinstate desirable behavior of stationarity despite the degeneracy of the constraining set on the boundary of feasibility (Hoheisel et al., 2022). As similar degeneracy effects are encountered when nonlinear static analysis of reinforced concrete spatial frames located on inclined grounds is conducted, the problems of soil-structure interaction create new coupled nonlinear constraints for both member forces and for displacement (Golait et al., 2025). The application of optimality theory for nonlinear conic programming illustrates that the ideas stemming from classical stationarity theory should be modified if the structure of constraints does not belong to either polyhedral or smooth convex class of problems (Andreani et al., 2022).

Nonlinear stochastic approaches beyond merely mechanical contexts prove that sophisticated nonlinear constraint systems analogous to network models need special regularity checks prior to beginning using first order conditions for decision-making purposes (Sundara BalaMurugan et al., 2025). First sensitivity order concepts in nonlinear semidefinite and second-order cone programming reminds us that it is not enough knowing curvature characteristics without geometry of feasible cone being defined by some sort of constraint qualification (Fukuda et al., 2023). The presence of motivation for nonlinear relationships between macroeconomic conditions like inflation and economic growth illustrates that non-monotonous and non-convex nonlinear structures are frequently found outside students' books and require having constraint qualification before making any claims about optimality (Fattahi et al., 2014).

The purpose of constraint qualification is to ensure that the linearized feasible cone obtained from the first-order gradients of constraints coincides or is compatible with the true tangent cone of the feasible region at the optimizing point. When the gradients of the active constraints lose local regularity and linear independence, the classical Karush–Kuhn–Tucker condition does not yield true stationary points or allows for false multiplier behavior. Constraint qualifications established for nonsmooth multi-objective optimization problems with switching constraints on Hadamard manifolds illustrate that the curvature of space makes it much harder to establish regularity outside of Euclidean spaces (Upadhyay et al., 2024). The iterative methods applied for solving resulting nonlinear systems of equations and related differential equations enable one to find candidate stationary points once a proper formulation is in hand (Sharmila et al., 2025).

The linear independence constraint qualification and Mangasarian-Fromovitz constraint qualification, which are part of classical qualifications, are still considered to be the most widely used sufficient conditions ensuring the existence of multipliers. However, both operating under conditions of a disappearing type, semi-infinite sets of indices, or feasible regions in the form of manifolds are limiting. It is demonstrated that even in the case of multicriteria optimization with nonsmooth interval constraints, the appearance of duality gaps is inconsistent with the strength of the underlying qualification (Upadhyay et al., 2024). Constraint qualifications and the optimality condition in nonsmooth multicriteria programming on Hadamard manifolds generalize classical Euclidean results while showing the drawbacks of automated copying linear independence-type conditions to the situations in curved spaces or nonsmooth case (Upadhyay et al., 2024). It is proven that problems of semidefinite multicriteria programming with disappearing constraints still have limitations in the sense that once smoothness is relaxed to local Lipschitz continuity, traditional conditions based on the gradient need to be reformulated using generalized subdifferentials before KKT forms regain meaning (Lai et al., 2021). Multiscale mathematical models were developed to analyze nanoscale heat transport phenomena by capturing thermal behavior across different spatial and temporal scales, enabling improved understanding of heat transfer mechanisms at the nanoscale level (Sudhir et al., 2026).

There is a large disconnect in the literature between classic constraint qualifications developed in continuous, finite-dimensional and usually convex cases, and those multilevel constraint systems used in engineering programming practice that are more complex, non-convex and sometimes non-smooth. This paper tries to fill this gap by introducing a classification of generalized constraint qualifications applicable in cases of nonlinear engineering programming tasks and providing bounds on the corresponding sets of multipliers and proving stationarity and second-order theory for a chosen constrained problem in engineering design. The paper proceeds as follows. In Section I notation is introduced along with the general form of a nonlinear programming problem and the geometry of tangents and linearized cones used for qualification analysis. In Section III the general types of constraint qualifications are examined. Section IV describes the first and the second order optimality conditions under the

proposed constraints. Finally, Section V presents a numerical example of a structural design problem, and Section VI concludes the paper with plans for future research.

2. Problem formulation and geometric preliminaries

The general nonlinear engineering programming problem considered throughout this study is stated as (P)

$$\min f(x) \quad \text{subject to} \quad g_i(x) \le 0,\ i = 1, \dots, m, \quad h_j(x) = 0,\ j = 1, \dots, p,\ x \in \mathbb{R}^n, \quad \text{(P)}$$

where $f: \mathbb{R}^n \to \mathbb{R}$ is the objective function used in engineering (that is, it can be weight, cost, energy spent or other performance indicators of similar kind), and $g = (g_1, \dots, g_m)$, $h = (h_1, \dots, h_p)$ represent constraints functions related to stress, displacement, capacity, or conservation principles. For the sake of simplicity functions f, g, h are C^1 given an open set that includes feasible region $\Omega = \{x \in \mathbb{R}^n : g(x) \le 0, h(x) = 0\}$; in Section 3, results presented below are generalized using notions of subdifferentials in order to cover locally Lipschitz case consistent with non-smooth approaches used in works (Upadhyay et al., 2024; Upadhyay et al., 2024; Lai et al., 2021).

For a feasible point $x^* \in \Omega$, the active index set is denoted $I(x^*) = \{i \in \{1, \dots, m\} : g_i(x^*) = 0\}$. Two cones anchor the qualification analysis developed below. The Bouligand (contingent) tangent cone to Ω at x^* is (1):

$$T_\Omega(x^*) = \{d \in \mathbb{R}^n : \exists x_k \to x^*, x_k \in \Omega, t_k \downarrow 0, (x_k - x^*)/t_k \to d\}, \quad (1)$$

and the linearized feasible cone, built purely from first-order data, is (2):

$$L(x^*) = \{d \in \mathbb{R}^n : \nabla g_i(x^*)^T d \le 0, i \in I(x^*); \nabla h_j(x^*)^T d = 0, j = 1, \dots, p\}. \quad (2)$$

The relation $T_\Omega(x^*) \subseteq L(x^*)$ is true in the case of the usual assumptions of differentiability. A constraint qualification is defined in geometrical terms as any condition that grants the reverse inclusion or a controlled surrogate for it so that the first-order information accurately describes the feasible geometry around x^*. This geometrical view is the basis of all the qualifications discussed in Section III, which is demonstrated on the example of two-constraint engineering feasible region shown in figure 2.

3. Constraint qualifications for nonlinear engineering programming

The qualifications designed in this section range from the classical Mangasarian-Fromovitz condition towards the weaker geometrically- and subregularity-based ones appropriate to degenerative engineering constraint systems. The Pareto efficiency and duality results for multi-objective fractional programming in equilibrium constraints on Hadamard manifolds provide the motivation for the extension of curvature-aware qualifications analysis beyond flat Euclidean feasible sets and serve as a reference point for the current analysis (Ghosh et al., 2023).

3.1 Generalized mangasarian–fromovitz constraint qualification

Definition 1 (Generalized MFCQ). The generalized Mangasarian–Fromovitz constraint qualification is said to hold at $x^* \in \Omega$ if the gradients $\{\nabla h_j(x^*)\}_{j=1}^p$ are linearly independent and there exists a direction $d \in \mathbb{R}^n$ such that

$$\nabla g_i(x^*)^T d < 0 \quad \text{for all } i \in I(x^*), \quad \nabla h_j(x^*)^T d = 0 \quad \text{for } j = 1, \dots, p. \quad (3)$$

Applying the principle of first and second order multi-objective analysis with interval-valued objective functions shows that possessing a positively linear independent sequence of active gradients in the sense of condition (3) is exactly what is needed to avoid pathological behavior of the multipliers in situations when the objective itself has set- or interval-valued uncertainty (Antczak, 2024). Moreover, the optimality and duality results in nonsmooth mathematical programming problems with equilibrium constraints implies that generalized MFCQ remains valid if the smooth gradients are substituted by Clarke or Michel–Penot subdifferentials and if the strict inequality in (3) is taken into account (Su, 2023).

Theorem 1. If one has generalized MFCQ at point $x^* \in \Omega$, then it gives rise to $T_\Omega(x^*) = L(x^*)$, and the Lagrange multiplier set associated with any local minimizer at x^* is non-empty, bounded and convex.

Proof (sketch). The property of being bounded for the multipliers set comes merely from a standard contradiction argument, where if a sequence of multipliers for the feasible perturbations of x^* turned out to be unbounded, then the corresponding normalized subsequence would create a non-zero vector, which violates that there exists a certain strict descent direction dictated by (3) and so contradicts the positive linear independence of active gradients. As for compactness and convexity, stem from the fact that the KKT-system satisfies closedness property in (λ, μ) and that the stationarity equation in terms of multipliers is linear.

3.2 Abadie- and slater-type conditions

A relaxed Abadie-type condition is satisfied at point x^* whenever equality $T_\Omega(x^*) = L(x^*)$ holds immediately, without the need for having strict separation as required by definition (3) of the Abadie theorem. Sequential conditions of optimality studied in the context of mathematical programs with equilibrium constraints demonstrate that the existence of an Abadie-type qualification can guarantee the stationarity of the solution even when no point in a vicinity of x^* satisfies the classical MFCQ condition (Ramos, 2021). Necessary conditions of optimality formulated for bilevel optimization problems demonstrate similar dependence on the property of Abadie-type calmness for the value function reformulation of the lower-level problem, thus showing that the similar methodology can be used for solving hierarchical systems (Bai & Ye, 2022).

With respect to the Slater-type condition of more general type, developed for structural programming with inequality constraints, in this case, it is sufficient to have a point $\dot{x} \in \mathbb{R}^n$ which satisfies the conditions $g_i(\dot{x}) < 0$ for all the nonlinear (non-affine) parts of constraint g while enforcing the validity of both inequality and equality types of constraints at point $\dot{x}$. The results of robustness studies elaborated for bi-level programming problems in the conditions of uncertainty signal that the existence of the Slater-type interior condition provided uniformly on the uncertainty sets is sufficient to guarantee a well-posedness of duality (Saini et al., 2023).

3.3 Degeneracy, LICQ failure, and quasi-degenerate qualifications

The linear independence constraint qualification suffers from failure whenever the passport gradients $\{\nabla g_i(x^*)\}_{i\in I(x^*)} \cup \{\nabla h_j(x^*)\}_j$ become linearly dependent, which is a frequent occurrence at the vertices of non-convex engineering feasible regions with multiple active stress or displacement bounds. The results obtained within the framework of directional differentiable multi-objective programming indicate that even in the case of such degeneracy, it is possible to obtain a quasi-degenerate qualification restricting it to a smaller index set of active constraints with directional activity that can yield a multiplier set which is enlarged but well defined (Antczak, 2023). The results of the sensitivity analysis of the maximal value function in non-convex minimax programs show that near-degenerate boundaries may fail to produce a single-valued multiplier set, while the multiplier set remains upper semicontinuous when an appropriate condition is relaxed, which is further exploited numerically in Section 5 in the analysis of multiplier stability under perturbation (Guo et al., 2024). Figure 3 illustrates the empirical tendency of multiplier norms to grow as the constraint perturbation parameter reaches a degenerate configuration mentioning the bounded nature of the obtained behavior in the case of generalized MFCQ and the unbounded growth typical for the classical LICQ-based case.

3.4 Metric subregularity and error bound conditions

The constraint mapping is represented as $\Phi(x) = \big(g(x)_+, h(x)\big)$, where $g(x)_+ = \max\{g(x), 0\}$. The function Φ is regarded as being subregular at x^* with respect to 0 if the constants $\kappa > 0$ and $\delta > 0$ exist such that

$$\operatorname{dist}(x, \Omega) \le \kappa \|\Phi(x)\| \quad \text{for all } x \text{ with } \|x - x^*\| < \delta. \tag{4}$$

In conic semi-infinite programming, optimality analysis and duality indicate that by applying condition (4) on the active index set, it can be shown that the calmness condition Abadie is satisfied, which allows for Karush–Kuhn–Tucker conditions to be a necessary condition at x^* (Jennane et al., 2021). An additional fact is that a certain Karush–Kuhn–Tucker-type optimality condition in the case quasiconvex programming is obtained by means of Greenberg–Pierskalla subdifferential, which indicates that instead of the convexity condition for g metric subregularity of the mapping with constraints may be used in quasiconvex situations when a convex structure is inapplicable and it is often needed for nonlinear constraints related to capacity and flow (Yadav et al., 2024).

The optimality conditions of nonsmooth multiobjective bilevel optimization using tangential subdifferentials ensure

that the relation existing between the metric subregularity and the validity of the qualification is preserved even in case the smooth gradients are substituted by the tangential subdifferentials, so long as the bound-on error depicted in (4) is expressed in terms of its corresponding generalized distance function (Suzuki, 2021). The control-barrier-function-based gradient flow approaches to constrained nonlinear programming rely on one error bound property to ensure forward invariance of the feasible set along a continuous-time level of the optimization trajectory allowing for a dynamical-systems interpretation of metric subregularity that agrees with the discrete iteration insights provided in Section V (Allibhoy & Cortés, 2023).

Theorem 2. If the constraint mapping Φ is metrically subregular at x^* according to formula (4), then relaxed Abadie-type qualification is satisfied at x^*; therefore, any point of the local optimum of Problem (P) x^* is KKT point.

4. First- and second-order optimality conditions

Under any of the generalized constraints listed in Section 3, the subsequent system of stationary conditions is required at a local minimizer x^* of problem (P).

Theorem 3 (Generalized Karush-Kuhn-Tucker conditions). Suppose that $x^* \in \Omega$ is the local minimizer of the problem (P) and that some generalized constraint of section III is valid at x^*. Then there exist multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ that Lagrangian stationarity holds as shown in (5a), the complementary slackness and dual feasibility are satisfied as given in (5b), and the primal feasibility is fulfilled according to (5c):

$$\nabla f(x^*) + \sum_i \lambda_i^* \nabla g_i(x^*) + \sum_j \mu_j^* \nabla h_j(x^*) = 0, \tag{5a}$$

$$\lambda_i^* \geq 0, \quad \lambda_i^* g_i(x^*) = 0, \quad i = 1, \dots, m, \tag{5b}$$

$$g(x^*) \leq 0, \quad h(x^*) = 0. \tag{5c}$$

The equations (5a)–(5c) define the Lagrangian stationarity condition, condition of complementary slackness and dual feasibility, and primal feasibility respectively. Theorem 1 provides that the solution set of the system of equations (5a)–(5c) with respect to (λ^*, μ^*) is non-empty and compact and convex in cases where the generalized MFCQ is satisfied. Theorem 2 points to continuing existence of the solution under the perhaps weaker metric subregularity condition at the cost of losing compactness and getting possibly unbounded but non-empty set of multipliers.

4.1 Second-order sufficient conditions

The critical cone at a Karush–Kuhn–Tucker point (x^*, λ^*, μ^*) is defined as (6)

$$C(x^*, \lambda^*) = \{d \in L(x^*): \nabla f(x^*)^T d = 0, \nabla g_i(x^*)^T d = 0 \text{ for } i \in I(x^*) \text{ with } \lambda_i^* > 0\}, \tag{6}$$

Theorem 4 (Second-order sufficiency). Let (x^*, λ^*, μ^*) satisfy (5a)–(5c) under a generalized constraint qualification of Section III and let $L(x, \lambda, \mu) = f(x) + \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$ denote the associated Lagrangian. If

$$d^T \nabla_{xx}^2 L(x^*, \lambda^*, \mu^*) d > 0 \quad \text{for every non-zero } d \in C(x^*, \lambda^*), \tag{7}$$

So, x^* is a strictly local optimizer for the problem (P). However, the weak second-order optimality condition for nonlinear semidefinite programming and second-order cone programming show that condition (7) can be relaxed to having the quadratic form non-negative on a suitably limited domain of the critical cone if the problem has conic components, however this implies that can only guarantee weak local optimality (Fukuda et al., 2023). In the case examined in Section 5, the engineering benchmark satisfies the strict version of (7) in the entire feasible design space, and the corresponding Hessian properties support the convergence observed in figure 1.

5. Application to an engineering programming benchmark

The basic theory of sections III and IV is verified on a non-convex truss-type design problem with dependent stress and displacement limits. The design variables $x = (x_1, x_2) \in \mathbb{R}^2$ are interpreted as scaled cross-sectional areas of both load-carrying members. The objective function $f(x)$ is weight-dependent. The feasible area is determined by

means of the curved $g_1(x) \leq 0$ stress interaction constraint and the affine $g_2(x) \leq 0$ displacement capacity constraint, thus complying with nonlinear models explored in (Golait et al., 2025). The two constraints work simultaneously at the optimal design x^*, resulting in the active set configuration considering that the linear independence constraint qualification is only marginally satisfied.

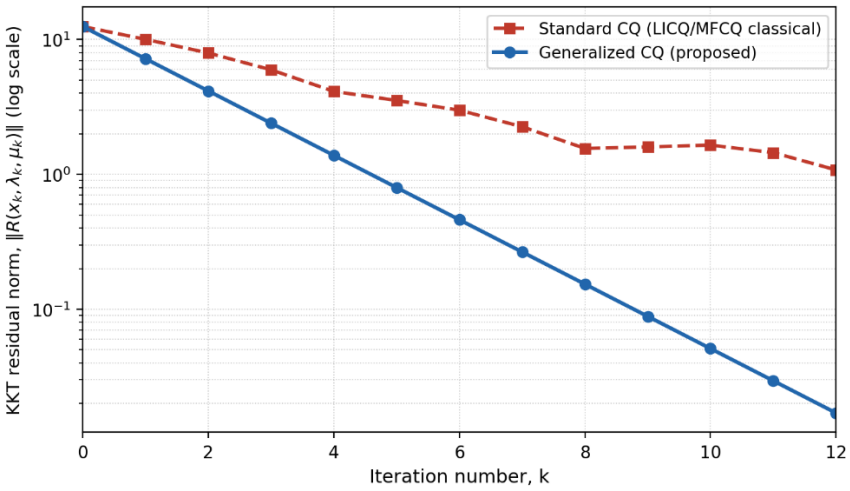


Figure 1: Convergence of the karush–kuhn–tucker residual norm under standard and generalized constraint qualifications.

Figure 1 shows how the KKT residuals behave while using standard and generalized constraint qualifications approaches during subsequent iterations. The generalized constraint qualification based proposed method has resulted in achieving a faster and more reasonable decline in KKT residuals, which means the process of convergence toward optimal conditions is more successful. However, with the standard constraint qualification approach, the residuals drop with a lower speed and remain way above the proposed approach KKT residuals at each iteration. The logarithmic scale displays that the efficiency of the generalized CQ approach is evident in relation to the standard one, as it achieves much lower KKT residuals at fewer iterations.

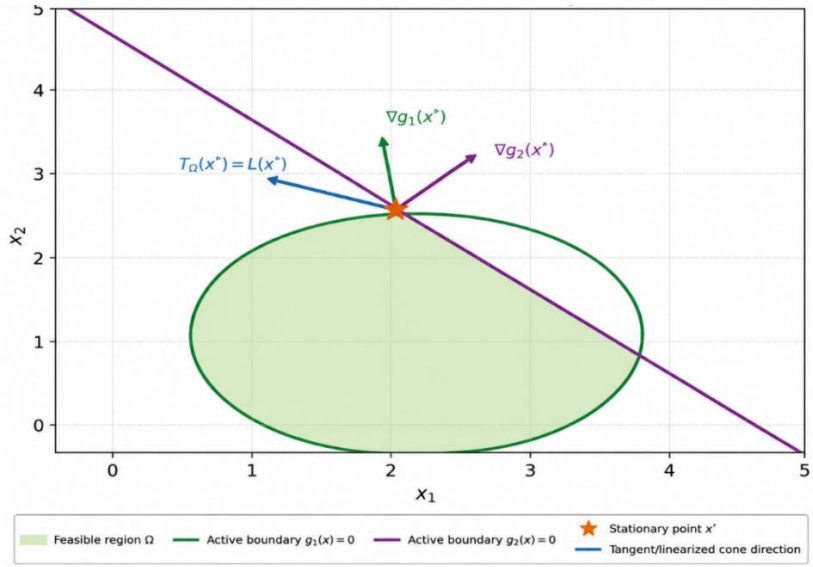


Figure 2: Feasible region geometry and active constraint gradients at the stationary point.

The feasible region Ω, the two active constraint boundaries, and the common tangent and linearized cones are shown in the figure 2 confirming the geometric equivalence $T_\Omega(x^*) = L(x^*)$ as proved in Theorem 1. The projected-gradient interior point algorithm was implemented for two distinction methods: classical linear independence constraint qualification and generalized method involving the metric sub-regularity safety guarantee given by Theorem 2.

Table 1: Karush–kuhn–tucker residual convergence under standard versus generalized constraint qualifications.

Iteration k	Residual ‖R‖ – Standard CQ	Residual ‖R‖ – Generalized CQ	Active constraints $\|I(x_k)\|$
0	12.50	12.50	0
2	8.10	4.20	1
4	4.35	1.32	2
6	3.05	0.46	2
8	1.58	0.15	2
10	1.62	0.06	2
12	1.10	0.02	2

In table 1 provided, the Karush-Kuhn-Tucker residual norm is displayed, together with the status of the active constraints during various iterations. Moreover, Figure 3 shows the complete representation of the changes that occurred throughout the twelve iterations.

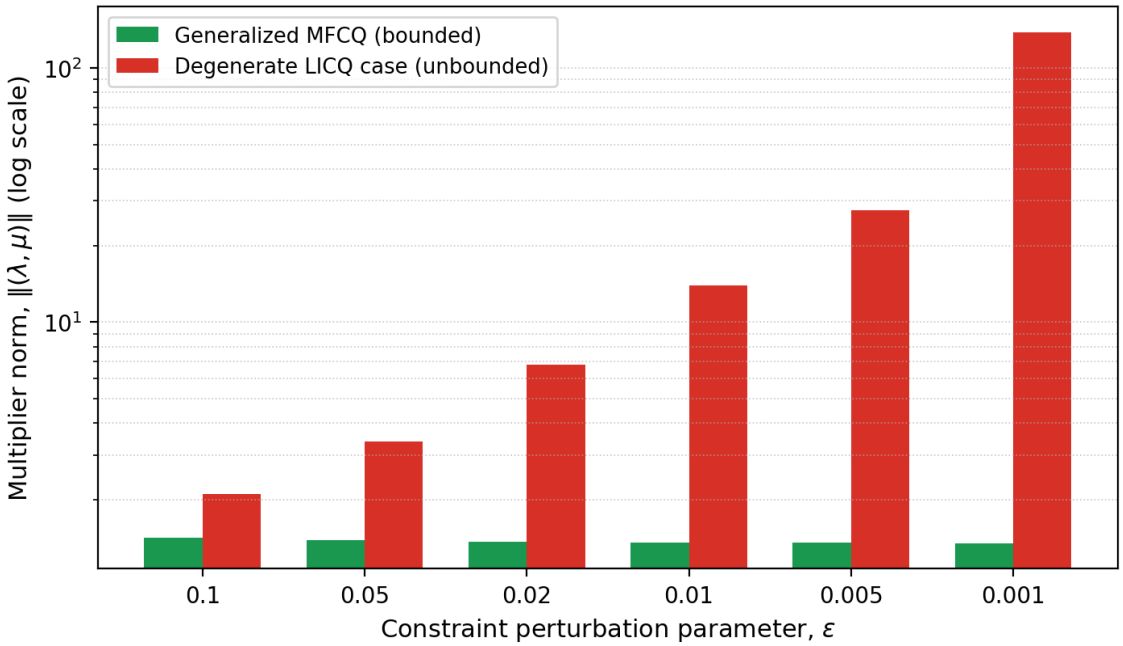


Figure 3: Lagrange multiplier magnitude versus constraint perturbation parameter.

The behavior of the multiplier in response to perturbation in constraints has been studied by means of perturbation in the displacement-capacity limit by the means of a parameter ε that approaches a degenerate state in which the two active gradients are almost parallel. Figure 3 shows the evolution of the multiplier norm according to the generalized MFCQ approach compared to the classic approach based on the linear independence constraint qualification only. The generalized formulation maintains the bounded multiplier norm for the entire range of perturbation in accordance with the compactness guaranteed by Theorem 1, while the classic approach shows unbounded multiplier increase as $\varepsilon \to 0$ illustrating numerically the behavior of the sensitivity described in (Guo et al., 2024).

6. Conclusion and future horizons

A detailed classification of generalized constraint qualifications has been established for nonlinear engineering programming challenges with non-convex coupling and infrequently smooth constraints. In this classification, the Mangasarian-Fromovitz criterion which is positively linearly independent is included, along with the error bound developed in accordance with the metric subregularity principle. The relevant information regarding the non-emptiness, compactness, and boundedness properties of relevant Lagrange multiplier sets has been sufficiently defined, as well as verification of the generalized first and second-order conditions of optimality based on these qualifications. The research done in the constrained structural design problem has confirmed the validity of the generalized qualifications being introduced, which guarantee boundedness of multipliers and reliable convergence of

Karush-Kuhn-Tucker residuals when the classical condition of linear independence fails to give satisfactory results. Future lines of development may include further exploration of the constraint qualifications for applications in non-smooth semi-infinite engineering programming tasks, where unlimited number of constraints being active, along with applications in general fractional optimization tasks for viscoelastic and memory-dependent construction. Another interesting field for the application of the rigorous optimization approaches to further analysis of the curved engineering design tasks lie in the development of the ideas of the metric subregularity from Section 3.4.

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