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Equilibrium and Quasi-Equilibrium Problems under φ-Quasimonotonicity and φ-Quasiconvexity. Existence, Stability and Applications

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We introduce a further generalization of quasimonotonicity of real-valued bifunctions and quasiconvexity of real-valued functions what we call *φ-quasimonotonicity* and *φ-quasiconvexity* respectively, φ being a bivariate function under appropriate conditions. Our generalization is on the one hand motivated by several counter-examples and examples such as the Clarke-Rockafellar directional derivative, and on the other hand includes the weak or relaxed and strong counterparts of these concepts and moreover leads to new definitions of generalized quasiconvexity and quasimonotonicity. Thereafter, this new material is exploited in the main focus of the paper, i. e., equilibrium problems. To deal with them we first consider a converse/opposed dual problem defined by a Min or Max formulation. Then, by providing conditions ensuring the emptiness of the set of solutions to the converse problem, we obtain global or local Minty solutions. Thereafter, the recourse is made to a very weak concept of sign-regularity, the so-called upper-sign property in the sense of M. Castellani and M. Giuli [*Refinements of existence results for relaxed quasimonotone equilibrium problems*, J. Global Optim. 57 (2013) 1213–1227], to ensure the passage from Minty's type solutions to standard ones. Our method discusses several kinds of classic solutions local and/or global, weak or relaxed, standard and strong ones, and moreover leads to new *strong Minty* and *eigenvalue-equilibrium* points for which we show the intimate link to φ-quasiconvexity, representing by the meantime solutions to an adequate penalized problem of the original one. Following [6, Definition 7 and Remark 5 (first point)], this new class of solutions may meet the so-called strict and star solutions for the particular case of set-valued Stampacchia variational inequalities in their quasi-convex programming context, and moreover disposes at nice stability properties under mild assumptions whenever the objective bifunction is subject to a deterministic parametric perturbation. Furthermore, the treatment we propose here contains both the compact and coercive setting under a variety of assumptions

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including also a noncoercive case. After that, we turn our attention into quasi-equilibrium problems, wherein the constraints set is not a fixed set but a set-valued map. In this quasi case, we make appeal to the Himmelberg Fixed Point Theorem which is well adapted to the coercive case instead of the Kakutani's one widely used in literature under compactness assumption as in the very recent paper by D. Aussel and J. Cotrina [*Quasimonotone quasivariational inequalities: existence results and applications*, JOTA 158 (2013) 637–652]. The obtained pattern is then applied to coercive set-valued quasi-variational inequalities and coercive quasi-minimization of second type semistrictly quasiconvex functions including the Nikaido-Isoda functions that arise in generalized Nash equilibria. Our optimization approach here is a direct one and doesn't need the passage via the normal operator to adjusted sublevels of the underlying function which not only complements but also extends and improves in many directions the study proposed in D. Aussel and J. Cotrina [see above].

Keywords: Clarke subdifferential, Clarke generalized derivative, Clarke-Rockafellar subdifferential, eigenvalues equilibrium points, eigenvalues minimizers, equilibrium problems, Minty equilibrium problems, fixed points, Lipschitz functions, parametric perturbation, penalization, φ-quasiconvexity, quasi-equilibrium problems, φ-quasimonotonicity, quasi-optimization, quantitative stability, quasiconvex programming, quasi-optimization, Stampacchia variational and quasi-variational inequalities.

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1. Introduction

Let K be a closed convex subset of a normed space X whose norm is denoted by $\|.\|$. Let $f : K \times K \to \mathbb{R}$ be a real-valued quasimonotone bifunction i.e., for all $x, y \in K$,

$$f(x,y) > 0 \implies f(y,x) \leqslant 0. \tag{1}$$

Quasimonotonicity plays an essential role in equilibrium and optimization theory, it has been extended very recently in [18] to a relaxed form, the so-called μ-relaxed quasimonotonicity for some $\mu \geqslant 0$ i.e., for all $x, y \in K$,

$$f(x,y) > 0 \implies f(y,x) \leqslant \mu\|y-x\|^2. \tag{2}$$

The authors in [18], inspired by the μ-relaxed quasimonotonicity of set-valued operators due to Bai and Hadjisavvas [11], have introduced the property (2) to refine the existence results for equilibrium problems where the μ-relaxed counterpart has been introduced also for monotone, pseudomonotone and properly quasimonotone bifunctions. This extension has been justified to be efficient in establishing the existence of equilibrium points when it is combined with a very weak regularity property namely the μ-upper sign property of f at a point $x \in K$ i.e., there exists $r > 0$ such that for all $y \in K \cap B(x,r)$,

$$f(z_t, x) \leqslant \mu\|z_t - x\|^2, \ \forall t \in (0,1) \implies f(x,y) \geqslant 0, \tag{3}$$

where $z_t = (1-t)x + ty$.

In this paper we introduce the concept of quasimonotonicity relative to another bifunction $\varphi : K \times K \to \mathbb{R}$, that is, for all $x, y \in K$,

$$f(x,y) > 0 \implies f(y,x) \leqslant \varphi(y,x). \tag{4}$$

Property (4) will be called from now on φ-*quasimonotonicity*; it is of course more general than the μ-relaxed quasimonotonicity defined in (2). Indeed, one can easily check that the class of φ-quasimonotone bifunctions is strictly larger than the μ-relaxed quasimonotone one, see Examples 3.3 below. Moreover, the introduction of the term φ is motivated by its unification feature since (4) coincides with the strong quasimonotonicity if $\varphi(x,y) = \alpha\|x-y\|^2$ with $\alpha < 0$, represents the classic quasimonotonicity in (1) if $\varphi = 0$ and collapse into the μ-relaxed quasimonotonicity if $\varphi(x,y) = \mu\|y-x\|^2$ for some $\mu \geqslant 0$. When $\varphi(x,y) = \omega\|y-x\|$ for some $\omega \in \mathbb{R}\backslash\{0\}$ we obtain another kind of quasimonotonicity named ω-quasimonotonicity-like, including the ω-relaxed quasimonotonicity-like if $\omega > 0$ and the ω-strong quasimonotonicity-like if $\omega < 0$. The latter will subsequently play a decisive role in quantitative stability of equilibrium solutions, see Theorem 6.5 in Section 6.

An important question which imposes itself is to find the right mode of generalized convexity that corresponds to φ-quasimonotonicity. It is not an easy task to find a general answer for such a question because of the presence of two variables. Nevertheless, the class of φ-relaxed properly quasimonotone bifunctions will be characterized by an extended relaxed version of the diagonal quasiconcavity in the first variable, see the subsequent Definition 5.4 and Lemma 5.6 (Section 5). Moreover, we extend the characterization of quasiconvexity of a lower semicontinuous and/or Lipschitz real-valued function $g : K \to \mathbb{R}$ in terms of the quasimonotonicity of its Clarke-Rockafellar and/or Clarke subdifferential $\partial^{CR}g$ (and hence $\partial^{C}g$) established in [8] and [2] to the φ-quasimonotone setting wherein we prove the necessity and/or sufficiency of φ-quasimonotonicity of $\partial^{CR}g$ and/or $\partial^{C}g$ to obtain the φ-quasimonotonicity of the corresponding directional derivative as a function of two variables, say $f(x,y) = g^0(x, y-x)$, covering henceforth both the relaxed, standard and strong cases, see the subsequent results in Proposition 3.12, Corollaries 3.13 and 3.14, Proposition 3.16, Corollaries 3.18 and 3.19 and Theorem 3.20 (Section 3). This characterization requires, naturally, to introduce a parallel generalization of quasiconvexity what we shall call φ-*quasiconvexity* of a function $g : K \longrightarrow \mathbb{R}$ in the sense that for any $x, y \in K$ and any $t \in [0,1]$,

$$g(tx + (1-t)y) \leqslant \max\big(g(x), g(y)\big) - t(1-t)\varphi(x,y). \tag{5}$$

When K is not necessarily convex and (5) is satisfied only for x, y in a convex neighborhood of a point $x_0 \in K$, for every $x_0 \in K$, then we obtain the φ-*approximate quasiconvex* counterpart of the well-known approximate convex functions. Visibly, (5) includes the three modes of quasiconvexity: the relaxed or weak one if $\varphi < 0$, the usual one if $\varphi = 0$ and the strong one if $\varphi > 0$. This class of quasiconvexity corresponds usually to $\varphi(x,y) = \alpha\|x-y\|^2$ where $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$ respectively. In addition, similarly to quasimonotonicity-like, when $\varphi(x,y) = \omega\|x-y\|$ we obtain the ω-*quasiconvexity-like*, see the example in Lemma 3.11 in Section 3. Notice that (5) also includes strongly $\alpha(.)$-paraconvex and $\alpha(.)$-paraconvex functions in the sense of S. Rolewicz [40], and that these concepts are extendable to the $\alpha(.)$-*para-quasiconvex* setting. Further examples of φ-quasimonotone bifunctions can also be constructed by set-valued operators with nonempty weakly compact and convex values i.e., for a map $T : K \rightrightarrows X^*$, we define a bifunction f_T by

$$f_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle, \quad x, y \in K,$$

X^* being the topological dual of X. In this case, the φ-quasimonotonicity of f_T leads to equivalent new concepts of *φ-quasimonotonicity* on T, see Definition 3.8 in Section 3.

The main motivation of the introduction of the above generalized quasimonotonicity and quasiconvexity is the application to the study of the equilibrium problem in the sense of Blum and Oettli [17], $EP(f,K)$ for short. More precisely, given a real-valued bifunction $f : K \times K \longrightarrow \mathbb{R}$, by $EP(f,K)$ we understand the problem to find $\overline{x} \in K$ such that

$$EP(f,K): \quad f(\overline{x}, y) \geq 0, \ \forall y \in K.$$

It is well known that the formulation $EP(f,K)$ is actually a convenient unifying model of several mathematical problems: optimization, saddle points, complementarity, systems of equations, fixed points, Nash equilibrium, variational inequalities and inverse optimization. This is the reason why this problem has attracted the attention of many optimizers during the last three decades and was therefore subject to intensive research, see for example [12, 14, 15, 18, 19, 27, 33] and references therein. For an algorithmic standpoint, the reader is referred to [37] and the recent survey [16]. For the stability and sensitivity analysis we refer to [1] and [13]. The equilibrium problems abstract formulation justifies once more its great interest in many applied models steaming from a variety of applied mathematics research fields such as traffic networks, engineering sciences, nonsmooth mechanics, optimal control, mathematical economics and so forth. This usefulness from the applicability point of view has led to the consideration of the important extension to the quasi-equilibrium formulation, which is also of interest to us in this paper: given a set-valued map $K : C \rightrightarrows C$, find $\overline{x} \in C$ such that $\overline{x} \in K(\overline{x})$ and

$$QEP(f,K): \quad f(\overline{x}, y) \geq 0, \ \forall y \in K(\overline{x}).$$

The problem $QEP(f,K)$ includes the special case of quasi-variational inequalities together with their applied models for example: quasi-optimization, calculus of variations, game theory and economics, transportation, mechanic and networks, see for instance multi-leader-follower games in Fukushima and Pang [28], superconductivity, thermoplasticity, electrostatic with implicit ionization threshold in Kunze and Rodrigues [35], obstacle problems in Mosco and Joly [34], quasiconvex programming and traffic network in Aussel and Cotrina [10]. However, still there are some open questions in this direction such as the Ricceri's conjecture on the quasi-variational inequality problem whose operator doesn't necessarily require any monotonicity property, see P. Cubiotti [21, 22] and P. Cubiotti and N. D. Yen [23] where the authors obtained a partial answer to this conjecture under its weak – probably the weakest – regularity assumption on the corresponding operator. This problem is of a different nature if compared with the generalized quasimonotone one and it then asks to be treated independently, since it is not easy to obtain – under generalized monotonicity conditions – a result for this problem with the same beauty of the one presented in [23].

Reverting to the existence of solutions to the problem $EP(f,K)$, one of the mostly used methods in the very recent literature makes appeal to a passage via an extended version of the so-called Minty equilibrium problem: Find $\overline{x} \in K$ such that

$$MEP(f,K): \quad f(y,\overline{x}) \leq 0 \qquad \forall y \in K.$$

Actually, one considers a local version of this problem as it is done in Bianchi and Pini [14] who considered local Minty solutions by extending the upper-sign continuity concept of set-valued maps due to N. Hadjisavvas [31] to bifunctions: for all $x \in K$, there exists a convex neighborhood $\mathcal{V}_x$ such that: for all $y \in \mathcal{V}_x$,

$$f(z_t,x) \geqslant 0, \forall t \in (0,1) \implies f(x,y) \geqslant 0, \tag{6}$$

where $z_t = (1-t)x+ty$. As observed in [14], the condition (6) allows to recapture solutions to $EP(f,K)$ from local solutions to $MEP(f,K)$, whose set is given by

$$M_L(f,K) = \{x \in K : \exists\, r > 0 \ \text{ with } \ f(y,x) \leqslant 0 \ \ \forall y \in K \cap B(x,r)\} \tag{7}$$

where $B(x,r)$ is the ball with radius $r > 0$ and centered at x. Under (6) the authors of [14] prove the inclusion

$$M(f,K) \subseteq M_L(f,K) \subseteq S(f,K), \tag{8}$$

where $M(f,K)$ denotes the set of solutions to $MEP(f,K)$ and $S(f,K)$ stands for those to $EP(f,K)$. More recently, M. Castellani and M. Giuli [18] introduced a weaker condition than (6) what they called μ-upper-sign property defined in (3) above. It is well explained in [18] that (3) improves (6) for the case of set-valued maps and moreover corresponds to the μ-relaxed quasimonotonicity defined by the above relation (2) also introduced in [18]. Under (3), the authors in [18] extended (8) to the more general inclusion

$$M^\mu(f,K) \subseteq M^\mu_L(f,K) \subseteq S(f,K), \tag{9}$$

where $M^\mu_L(f,K)$ denotes the set of μ-relaxed local Minty solutions given by

$$M^\mu_L(f,K) = \{x \in K : \exists r > 0 \text{ with } f(y,x) \leqslant \mu\|y-x\|^2 \ \ \forall y \in K \cap B(x,r)\} \tag{10}$$

and $M^\mu(f,K)$ denotes the set of μ-relaxed global Minty solutions given by

$$M^\mu(f,K) = \{x \in K : f(y,x) \leqslant \mu\|y-x\|^2 \ \ \forall y \in K\}. \tag{11}$$

In the present paper we consider generalizations of both $S(f,K)$, $M^\mu_L(f,K)$ and $M^\mu(f,K)$. More precisely, given another real-valued bifunction $\varphi : K \times K \to \mathbb{R}$, we look for a φ-*global* (resp. Minty) *equilibrium point* $\overline{x} \in K$ i.e.,

$$\varphi - EP(f,K): \quad f(\overline{x},y) \geq \varphi(\overline{x},y), \quad \forall y \in K,$$
$$\text{(respectively } \varphi - MEP(f,K): \quad f(y,\overline{x}) \leq \varphi(y,\overline{x}), \quad \forall y \in K).$$

If the latter is true only for y in a neighborhood of $\overline{x}$ we speak about φ-*Minty local solutions*. It is possible to define the φ-upper sign counterpart of the condition (3) if we replace $\mu\|x-y\|^2$ by $\varphi(x,y)$ and then obtain a more general property, than (9), which will be of the form:

$$M^{\varphi}(f,K) \subseteq M_L^{\varphi}(f,K) \subseteq S(f,K).$$

Such an eventual φ-version of (9) would ask only to overcome technical difficulties but such a generalization doesn't fit in the scope of this work because we would like to focus our attention rather on the impact of φ-quasimonotonicity and φ-quasiconvexity on equilibrium solutions. When the passage from global or local Minty to standard solutions is needed we make appeal to condition (3) for the sake of simplicity. On the other hand, apart relevant applications such as the coercive quasiconvex programming and quasi-optimization problems, the purpose here is to refine, unify and extend the recent literature on existence theory to the φ-generalized monotonicity framework. In this way, we discuss several types of solutions, all of them are assembled in Definition 4.4: weak or relaxed, standard and strong solutions. We also introduce the concept of *eigenvalues equilibrium* points (Definition 4.7), which can be regarded as penalized solutions that are close to φ-quasiconvexity: Proposition 4.9, Section 4.

We continue our analysis by observing that φ-quasiconvexity in the second argument of the objective bifunction f combined with its φ-quasimonotonicity yields the uniqueness of the solution, which is particularly useful in discretization and approximations methods as in the random variational inequalities considered by Gwinner and Raciti [30].

We want to stress here that the $\varphi - EP(f,K)$ and $\varphi - MEP(f,K)$ formulations have further advantages, for instance when one deals with Bregman Proximal Interior Point Algorithm for Equilibrium Problems they enable us the solvability of iteratively constructed regularized equilibrium subproblems. Precisely, φ takes there the form $\varphi(x,y) = c_k\langle x - x^{k-1}, y - x\rangle$, where x^{k-1} is the iterate obtained at iteration $k-1$. It should be noticed that in this algorithmic context appears the notion of undermonotonicity which is nothing else but a φ-monotonicity with $\varphi(x,y) = \theta\langle g'(x) - g'(y), x - y\rangle$, g' being the Fréchet derivative of a convex differentiable function g and $\theta > 0$. Besides the undermonotonicity, a co-coercive operator $T : X \to X^*$ (i.e., $\langle T(x) - T(y), x - y\rangle \geqslant c\|T(x) - T(y)\|^2$, $x, y \in X$, $c > 0$) enters also in the setting of φ-monotonicity. The same remark can be done for weakly monotone or hypomonotone operators that characterize the weakly convex or else the so-called lower $C^{1,\alpha}$-functions.

Our treatment considers the following assumptions:

($\mathbb{H}_0$) $f(x,x) = 0$ for all $x \in K$;

($\mathbb{H}_1$) $f(x,x) \geqslant 0$ for all $x \in K$;

($\mathbb{H}_2$) $F_\varphi(x) = \{y \in K|\ f(x,y) \leqslant \varphi(x,y)\}$ is compactly closed for all $x \in K$ i.e., $F_\varphi(x) \cap B$ is closed for every compact $B \subset K$;

($\mathbb{H}_3$) f is properly φ-quasimonotone;

($\mathbb{H}_4$) f is φ-quasimonotone and $f(x,.)$ is strictly quasiconvex for all $x \in K$;

($\mathbb{H}_5$) $f(.,y)$ is diagonally quasiconcave for every $y \in K$;

($\mathbb{H}_6$) f is φ-quasimonotone and not properly φ-quasimonotone, and $f(x,.)$ is quasiconvex for all $x \in K$;

($\mathbb{H}_7$) f is φ-pseudomonotone and $f(x,.)$ is quasiconvex for all $x \in K$.

Under these assumptions we establish solutions to $\varphi{-}MEP(f,K)$ in the following scheme:

(A) K is compact and one of the following conditions holds:
- ($\mathbb{H}_i$), $i \in \{2,3\}$, Proposition 5.10;
- ($\mathbb{H}_i$), $i \in \{0,2,5\}$ and $\varphi \geqslant 0$, Proposition 5.12;
- ($\mathbb{H}_i$), $i \in \{1,2\}$ and either ($\mathbb{H}_4$) or ($\mathbb{H}_7$), Proposition 5.13;
- f is μ-relaxed pseudomonotone and strictly convex in y, Corollary 5.15 in Section 5; **or**

(B) ($\mathbb{H}_i$), $i \in \{2,6\}$ and $\varphi \geqslant 0$, where we obtain local φ-Minty solutions, Proposition 5.23.

Thus, we obtain *approximate* global weak Minty solutions if f is convex in y and μ-strongly quasimonotone, Theorem 5.17. The following step involves the μ-upper sign property, defined earlier in (3), to conclude the solutions for the standard problem $EP(f,K)$ in Proposition 5.32, Lemma 5.30 or Corollary 5.34 with an additional sign preserving property to cover the local Minty case, and Corollary 5.36 where the conditions ensuring the equivalence between the different obtained Minty types solutions in both weak, strong, global or local formulations from one side and the standard equilibrium points from the other side are also given.

The results in case (A) above are thereafter extended in Proposition 5.37 to the coercive framework stated on the basis of the classic finite intersections property and compactness Lemma ensuring that a coercive problem $MEP(f,K)$ admits a solution when K is unbounded if $MEP(f,\tilde{K})$ admits a solution for every compact $\tilde{K} \subset K$. This enables us to formulate a second unified Theorem on the nonemptiness of standard solutions $S(f,K)$ under a coercivity condition and the μ-upper sign property by considering the particular case of $\varphi(x,y) = \mu\|x-y\|^2$ in the perspective to have in hand an easiest applicable version: Theorem 5.39 in which we equally include a noncoercive case.

Once we dispose at the existence of solutions, another aspect which is of interest to us in Section 6 of this paper is the parametric perturbation where external parameters are involved either at the level of bifunction or the constraints set, the case in which the parameters occur in both of them comes easily by the triangle inequality as observed in [1] and confirmed in [5]. We compute in a simple way a sharp error bound of the quantitative stability: Theorem 6.2 and Corollary 6.3 in concern with strong Minty type and standard solutions under perturbation of the objective bifunction, and Theorem 6.5 regarding perturbed standard solutions where the constraints set is subject to a small change, adapting and extending henceforth the results of [2] to ω-quasimonotone-like equilibrium problems under the ω-quasiconvexity-like in the second argument of the underlying bifunction. The stability obtained here is in the spirit of that investigated in N. D Yen [42] for strongly monotone finite dimensional variational inequalities by the use of geometrical arguments such as projections, extended later in [1] to abstract strongly monotone equilibrium problems in normed spaces via direct computations of the error bounds.

Thereafter, we consider extensions to the quasi-equilibrium formulation mentioned earlier, where we adapt the variational-fixed point scheme of Mosco and Joly [34] to coercive Minty quasi-equilibria under φ-quasimonotonicity by applying the Himmelberg fixed point Theorem for set-valued maps which is well adapted to the coercive case, the Kakutani's one widely used in the literature for set-valued quasi-variational inequalities, (SQVI), under compactness assumption (as in [10]) is no longer applicable if K is unbounded. Then, similarly to the above mentioned treatment of $EP(f, K)$, the solutions to $QEP(f, K)$ are concluded in Theorem 7.9 thanks to the upper-sign property. This conclusion necessitates establishing at first the auxiliary results on the solutions map needed for the Joly-Mosco scheme such as the closedness of the graph in Proposition 7.1, the upper semicontinuity in Corollary 7.5 and convexity in Proposition 7.7.

The problem (SQVI) is furthermore considered in Theorem 8.15 as a typical application of $QEP(f, K)$ wherein we provide equally star solutions required to obtain non-trivial solutions for quasi-optimization and quasiconvex programming see [2, 3, 9, 10]. Thus, we devote the last part of this work to a further application of quasi-equilibrium to a quasi-minimization problem of semistrictly quasiconvex such as those defined by a supremum of a family of Nikaido-Isoda's functions (see [10]) that arise in generalized Nash equilibrium: Theorem 9.9. In the latter we, as well, introduce and prove the existence of λ-*eigenvalue minimizers* under φ-quasiconvexity. It should be emphasized that our optimization approach here is a direct one and doesn't need the passage via Stampacchia quasi-variational inequality, (SQVI), that necessarily employ the normal operator N_a to adjusted sublevel sets of the underlying function (which complements the previous results as the one in [10] for example). However, for a complete discussion with the very nice literature on this topic we also include a treatment of the quasi-minimization of continuous quasiconvex functions expressed in terms of (SQVI) formulation whose operator is the just quoted normal one N_a: Corollary 9.13.

2. Preliminaries

Through this paper unless otherwise is specified, X is a linear normed space. For a subset $A \subset X$, when we say A is closed or compact we mean that in the sense of the strong topology of X. The topological dual of X will be denoted by X^*

with duality paring $\langle . \rangle$. We write weak-* to mean the weak star topology in X^*. B_X (resp. B_{X^*}) will stand for the closed unit ball of X (resp. X^*), and for any $x_0 \in K$ and $r > 0$ we denote by $B(x_0, r)$ the open ball centered in x_0 with radius r. For an arbitrary subset A in X, we denote by $\overline{A}$ (or $cl(A)$) the closure in X, $Conv(A)$ its convex hull. If $A \subseteq K$, by A^c the complementary of A in K that is the set $\{x \in A : x \notin K\}$. For a multifunction $F : X \longrightarrow 2^X$, the notation $Gr(F)$ stands for the graph of F that is the set of elements (x, y) in $X \times X$ satisfying $y \in F(x)$. A map $F : X \longrightarrow 2^X$ is said to be closed (resp. weakly closed) if its graph is closed in the sense of strong (resp. weak) topology, and upper semicontinuous (or upper continuous following [29, (a) Definition 2.5.1, pp. 51]) at a point $x_0 \in K$ if for every open $V \subset X$, $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that for all $x \in U : F(x) \subset V$. A map $F : X \longrightarrow 2^X$ is said to be lower semicontinuous if and only if $F(x) \subset \liminf_n F(x_n)$ for any converging sequence (x_n) with x being the limit, where $\liminf_n F(x_n)$ is the lower limit in the sense of Kuratowski defined by

$$y \in \liminf_n F(x_n) \iff \exists (u_n)_n \subset X : y = \lim_n u_n \text{ and } u_n \in F(x_n), \forall n.$$

For a further discussion on semicontinuity and closedness for set-valued maps we refer to [29]. The following fixed point Theorem for set-valued maps will also be needed.

Theorem 2.1. (Himmelberg[32] or [43, Theorem A])
Let K be a convex subset of a Locally convex Hausdorff topological space X and B a nonempty compact subset of X such that $B \cap K \neq \emptyset$. Let $F : K \rightrightarrows B$ be an upper semicontinuous set-valued map with nonempty closed convex values. Then there exists $x \in B$ with $x \in F(x)$ (i.e. F has a fixed point in B).

Theorem 2.2. *Let B a compact subset of K and $F : K \rightrightarrows X$ a set-valued map such that $F(K) \subset B$. If F is closed-valued, then it is closed if and only if it is upper semicontinuous.*

Proof. Let $x \in K$. Since $F(x)$ is closed, it is a compact (in view of the compactness of B). Hence, if F is closed at x then F is compact at x (in the sense of [29, (iv) Definition 2.5.7, pp 55]). Thus, from [29, (i) Proposition 2.5.9, pp 56], it follows that F is upper semicontinuous at x. The converse assertion comes also from [29, (i) Proposition 2.5.9, pp 56]. This ends the proof. □

Given a convex subset K of X, a function $g : K \longrightarrow \mathbb{R}$ is said to be:

- *convex*, if for all $x, y \in K$, $g(tx + (1-t)y) \leqslant tg(x) + (1-t)g(y), \ \forall t \in [0, 1]$;
- *quasiconvex*, if for all $x, y \in K$, $g(tx+(1-t)y) \leqslant \max\{g(x), g(y)\}, \ \forall t \in [0, 1]$, i.e., for any $x_1, \dots, x_n \in K$ and $\varrho_i, \dots \rho_n > 0$ such that $\sum_{i=1}^{n} \rho_i = 1$, we have

$$g(\sum_{i=1}^{n} \rho_i x_i) \leqslant \max_{1 \leqslant i \leqslant n} g(x_i); \tag{12}$$

- *strictly quasiconvex* if for all $x, y \in K$ with $x \neq y$, $g(tx + (1-t)y) < \max\{g(x), g(y)\}, \ \forall t \in]0,1[$;
- *semistrictly quasiconvex*, if for all $x, y \in K$ with $x \neq y$,

$$g(x) < g(y) \Longrightarrow g(tx + (1-t)y) < g(y), \ \forall t \in]0,1[; \tag{13}$$

- *second type semistrictly quasiconvex*, if g is quasiconvex and (13) holds;
- *concave*, if $-g$ is convex;
- *quasiconcave* if for all $x, y \in K$, $g(tx+(1-t)y) \geq \min\{g(x), g(y)\}, \ \forall t \in [0,1]$;
- *semistrictly quasiconcave* if $-g$ is semistrictly quasiconvex;
- *second type semistrictly quasiconcave*, if $-g$ is second type semistrictly quasiconvex;
- *upper-semi continuous*, at $x_0 \in K$ if $\limsup\limits_{x \longrightarrow x_0} g(x) \leqslant g(x_0)$ and lower-semicontinuous at x_0 if $-g$ is upper-semicontinuous at x_0;
- *upper-hemicontinuous*, if it is upper semicontinuous on every segment of K;
- *lower-hemicontinuous*, if $-g$ is upper hemicontinuous.

Definition 2.3. ([45]) A bifunction $f : K \times K \longrightarrow \mathbb{R}$ is said to be *diagonally quasiconvex* (resp. quasiconcave) in y, if for any finite subset $\{y_1, ..., y_n\} \subset K$ and any $y_0 \in Conv\{y_1, ..., y_n\}$ we have

$$f(y_0, y_0) \leqslant \max_{1 \leqslant i \leqslant n} f(y_0, y_i) \quad (resp.\ f(y_0, y_0) \geq \min_{1 \leqslant i \leqslant n} f(y_0, y_i)).$$

Remark 2.4. (1) The strict inequality in (13) does not ensures that f is quasiconvex, see a counterexample in [18].

(2) Semistrict quasi-convexity of second type coincides with the semistrict quasiconvexity in the sense of [3] and [10].

Remark 2.5. If g is strictly quasiconvex then the inequality in (12) is strict.

Let us now recall the definition of the Clarke-Rockafellar subdifferential of a function $g : K \longrightarrow \mathbb{R}$:

$$\partial^{CR} g(x) = \{x^* \in X^* : \langle x^*, v \rangle \leqslant g^{\uparrow}(x, v) \ \ \forall v \in K\}$$

with

$$g^{\uparrow}(x, v) = \sup_{\varepsilon > 0} \limsup_{\substack{(x', \delta) \to_g x \\ t \searrow 0}} \inf_{v' \in B(v, \varepsilon)} \frac{g(x' + tv') - \delta}{t},$$

where $(x', \delta) \longrightarrow_g x$ means that $x' \longrightarrow x$ and $\delta \longrightarrow g(x)$ with $\delta \geqslant g(x')$.

It is well known that this subdifferential coincides, whenever the function g if locally Lipschitz, with the Clarke subdifferential defined by: $\partial^C g(x) = \{x^* \in X^* / \langle x^*, v \rangle \leqslant g^0(x, v) \ \ \forall v \in X\}$, where the directional derivative g^0 is given by

$$g^0(x, v) = \lim_{\substack{x' \longrightarrow x \\ t \searrow 0}} \sup \frac{g(x' + tv) - g(x')}{t}.$$

Among the properties of ∂^C, we recall those which we need in the sequel.

Lemma 2.6. *Let $g : K \longrightarrow \mathbb{R}$ be a locally Lipschitz continuous at a given point $x \in K$ with a rank of Lipschitz L. Then, for every $x, v \in K$:*

(i) $\partial^C g(x)$ *is a nonempty compact subset of* X^* *and* $\|\xi\| \leqslant L$ *for each* $\xi \in \partial^C g(x)$;

(ii) $g^0(x, v) = \max\{\langle \xi, v\rangle : \xi \in \partial^C g(x)\}$;

(iii) *The function* $v \longmapsto g^0(x, v)$ *is finite, positively homogenous, subadditive, convex and continuous;*

(iv) $g^0(x, -v) = (-g)^0(x, v)$ *and* $g^0(x; 0) = 0$;

(v) $g^0(x, v)$ *is upper semi continuous as a function of* (x, v);

(vi) $\tilde{v} \mapsto g^0(x, \tilde{v})$ *is Lipchitz of rank* L;

(vii) *if* $g(x) = \int_0^x \beta(s)ds$ *with* $\beta \in L^\infty_{loc}(\mathbb{R})$ *then* $g^0(t, z) \leqslant \beta^+(t)z$ *if* $z > 0$ *and* $g^0(t, z) \leqslant \beta_-(t)z$ *if* $z < 0$, *where:*

- $\beta^+(t) = \lim_{\delta \longrightarrow 0} \operatorname{ess-sup}_{|s-t|\leqslant\delta} \beta(s)$, ess-sup *being the almost everywhere supremum;*
- $\beta_-(t) = \lim_{\delta \longrightarrow 0} \operatorname{ess-inf}_{|s-t|\leqslant\delta} \beta(s)$, ess-inf *being the almost everywhere infimum;*
- $L^\infty_{loc}(\mathbb{R})$ *stands for the essentially locally bounded functions from* $\mathbb{R}$ *to* $\mathbb{R}$.

3. A new class of generalized quasimonotonicity and generalized quasiconvexity

In view of the multiple motivations mentioned earlier in the introduction, we propose the following generalizations of monotonicity:

Definition 3.1. Let f, φ be two real-valued bifunctions defined on $K \times K$. f is said to be

- *φ-monotone* if for all $x, y \in K$,

$$f(x, y) + f(y, x) \leqslant \varphi(y, x); \tag{14}$$

- *φ-relaxed monotone* if $\varphi \geqslant 0$ and (14) holds for all $x, y \in K$;
- *φ-strongly monotone* if $\varphi > 0$ (i.e.,$\varphi(x, x) \geqslant 0$ and $\varphi(x, y) > 0$ for all $x \neq y$) and for all $x, y \in K$,

$$f(x, y) + f(y, x) \leqslant -\varphi(y, x); \tag{15}$$

- *φ-pseudomonotone* if for all $x, y \in K$,

$$f(x, y) \geqslant 0 \Longrightarrow f(y, x) \leqslant \varphi(y, x); \tag{16}$$

- *φ-relaxed pseudomonotone* if $\varphi \geqslant 0$ and (16) holds for all $x, y \in K$;

- *φ-strongly pseudomonotone* if $\varphi > 0$ and for all $x, y \in K$,

$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant -\varphi(y,x); \tag{17}$$

- *φ-quasimonotone* if for all $x, y \in K$,

$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant \varphi(y,x); \tag{18}$$

- *φ-relaxed quasimonotone* if f is φ-quasimonotone and $\varphi \geqslant 0$;
- *φ-strongly quasimonotone* if $\varphi > 0$ and for all $x, y \in K$,

$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant -\varphi(y,x); \tag{19}$$

- *φ-properly quasimonotone* if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in K$ and $\overline{x} \in Conv\{x_1, \ldots, x_n\}$ there exists $i \in \{1, \ldots, n\}$ such that

$$f(x_i, \overline{x}) \leqslant \varphi(x_i, \overline{x}).$$

Remark 3.2. (1) If $\varphi \geqslant 0$ then every quasimonotone bifunction is φ-quasimonotone. In particular if $\varphi = 0$, then 0-quasimonotonicity coincide with standard quasimonotonicity.

(2) If for some $\mu \geqslant 0$, $\varphi(x,y) = \mu\|x-y\|^2$, then the φ-pseudomonotonicity (resp. φ-quasimonotonicity, φ-proper quasimonotonicity) collapse into the μ-relaxed pseudomonotonicity (μ-relaxed quasimonotonicity, μ-relaxed properly quasimonotonicity) introduced very recently in [18].

(3) If for some $\alpha > 0$, $\varphi(x,y) = -\alpha\|x-y\|^2$, then the φ-quasimonotonicity is nothing else but the usual α-strong quasimonotonicity. Observe that any α-strong quasimonotone bifunction is quasimonotone and pseudomonotone.

(4) If for some real number ω, $\varphi(x,y) = \omega\|x-y\|$, then the φ-quasimonotonicity will be called *ω-quasimonotonicity-like* where one speaks about ω-strong-*quasimonotonicity-like* if $\omega < 0$ and ω-relaxed-*quasimonotonicity-like* if $\omega > 0$.

The following example shows that the class of φ-quasimonotone bifunctions is strictly larger than the μ-quasimonotone one. Precisely, we provide bifunctions which are φ-quasimonotone for some φ but never μ-quasimonotone for any $\mu \geqslant 0$.

Examples 3.3. (1) Let $K = \mathbb{R}$, $\varphi(x,y) = (x^2 + y^2)/2$, and $f(x,y) = xy$ for all $x, y \in \mathbb{R}$. It is clear that f is φ-quasimonotone (because $f(x,y) \leq \varphi(x,y)$, for all $x, y \in \mathbb{R}$). But f is not μ-quasimonotone for all $\mu \in \mathbb{R}^+$. Indeed, if there exists $\mu > 0$ such that f is μ-quasimonotone then

$$f(x,y) > 0 \implies f(y,x) \leq \mu \mid x - y \mid^2, \quad \forall x, y \in \mathbb{R}. \tag{20}$$

So, if $x = y$ in implication (20) we get $x^2 > 0$ implies that $x^2 \leq 0$ which is not true.

(2) Let $K = \mathbb{R}$. Consider the bifunctions f and φ defined by $f(x,y) = x|x-y|^2$ and $\varphi(x,y) = x|x-y|^2 + 1$, $x, y \in K$. It is clear that f is φ-quasimonotone

(indeed $f(x,y) \leq \varphi(x,y)$ for all $x, y \in \mathbb{R}$), but f is not μ-quasimonotone for all $\mu \geqslant 0$, otherwise there exists $\mu \geqslant 0$ such that f is μ-quasimonotone. Then

$$\forall x, y \in K,\ f(x,y) > 0 \implies f(y,x) \leqslant \mu|x-y|^2. \tag{21}$$

In particular, for every $y \in \mathbb{R}$ such that $y > \mu$ and any $x \in \mathbb{R}$ such that $x > 0$ and $x \neq y$, we have $f(x,y) > 0$. Then, by (21), $y|x-y|^2 = f(y,x) \leq \mu|x-y|^2$. This means that $y \leq \mu$, a contradiction with $y > \mu$.

The following further examples of relaxed φ-monotonicity are useful for instance for hemi-variational inequalities, see for example [4].

Example 3.4. Assume that $X = \mathbb{R}$. Let $\beta \in L^\infty_{loc}(\mathbb{R})$ and $j : \mathbb{R} \to \mathbb{R}$ the real-valued locally Lipschitz function defined by $j(t) = \int_0^t \beta(s)ds$. Let $\gamma > 0$ and $r \geqslant 1$. Assume that β is γ-lower essentially r-Hölder in the sense of [4, Definition 3.1] i.e., for all $t, s \in \mathbb{R}$,

$$t < s \ \Rightarrow\ \beta^+(t) \leqslant \beta_-(s) + \gamma(s-t)^r.$$

Let $\tilde{j}$ be the map defined on $\mathbb{R}\times\mathbb{R}$ by $\tilde{j}(s,t) = j^0(s, t-s)$, $s, t \in \mathbb{R}$, where $j^0(x,d)$ is the Clarke directional derivative of j at a point $x \in \mathbb{R}$ in a direction $d \in \mathbb{R}$. Then, from [4, Lemma 3.3], $\tilde{j}$ is φ-monotone with $\varphi(t,s) = \gamma|t-s|^{r+1}$, $t, s \in \mathbb{R}$.

Example 3.5. Suppose V is a Banach space, whose norm is denoted by $\|.\|$, continuously imbedded in some $L^p(\Omega)$ for some open bounded subset Ω of $\mathbb{R}^n$, and $n, p > 1$. If we denote by $\|.\|_p$ the norm of $L^p(\Omega)$, then $\forall u \in V$, $\|u\|_p \leqslant c_p\|u\|$ for some positive constant c_p. By means of the maps $\beta \in L^\infty_{loc}(\mathbb{R})$ and j defined in Example 3.4, we introduce the bifunction $J : V \times V \longrightarrow \mathbb{R}$ defined by

$$J(u,v) = \int_\Omega j^0(u(t), v(t) - u(t))dt, \quad u, v \in V.$$

Clearly, by Example 3.4, J is φ-monotone with $\varphi(u,v) = \gamma \int_\Omega \|u(x) - v(x)\|^{r+1}dx$. In particular, if $r = 1$, then, J is μ-relaxed monotone with $\mu = \gamma c_p^2(mes(\Omega))^{\frac{p-2}{p}}$, where μ is computed in [4, Remark 4.2] by elementary facts of integration theory, $mes(\Omega)$ being the measure of Ω.

Now, we come up to give extension of the μ-relaxed quasimonotonicity for set-valued maps introduced by Bain and Hadjisavvas in [11]. Such an extension will be subsequently useful in expressing the relation between the new introduced modes of generalized quasiconvexity and generalized quasimonotonicity:

Definition 3.6. Let $\varphi : X \times X \to \mathbb{R}$. A set valued map $T : K \rightrightarrows X^*$ is called

- *φ-quasimonotone*, if for all $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$ it is true that $\langle x^*, y-x \rangle > 0 \ \Rightarrow\ \langle y^*, x-y \rangle \leqslant \varphi(x,y)$;
- *φ-strongly quasimonotone*, if $\varphi \geqslant 0$ and for all $x, y \in K$, $x^* \in T(x)$, $y^* \in T(y)$, $\langle x^*, y-x \rangle > 0 \Rightarrow \langle y^*, x-y \rangle \leqslant -\varphi(x,y)$.

A subclass of φ-quasimonotonicity for operators is introduced as follows.

Definition 3.7. Let $\alpha \geqslant 0$. A set valued map $T : K \rightrightarrows X^*$ is called

- *α-quasimonotone-like* if for all $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, it is true that $\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, x - y\rangle \leqslant \alpha\|x - y\|$;
- *α-strongly quasimonotone-like* if for all $x, y \in K$, $x^* \in T(x)$ and $y^* \in T(y)$, $\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, x - y\rangle \leqslant -\alpha\|x - y\|$.

Presently, we introduce the mode of generalized quasiconvexity which will be proved to be parallel (in a sense to be precise later) to φ-quasimonotonicity.

Definition 3.8. A function $g : K \longrightarrow \mathbb{R}$ is said to be *φ-quasiconvex* if for any $x, y \in K$ and any $t \in [0, 1]$,

$$g(tx + (1 - t)y) \leqslant \max\Big(g(x), g(y)\Big) - t(1 - t)\varphi(x, y).$$

Definition 3.9. Let $\alpha \geq 0$. A function $g : K \longrightarrow \mathbb{R}$ is said to be *α-strongly quasiconvex-like* if for any $x, y \in K$ and any $t \in [0, 1]$,

$$g(tx + (1 - t)y) \leqslant \max\Big(g(x), g(y)\Big) - \alpha \min\{t, 1 - t\}\|x - y\|.$$

Remark 3.10. (1) If $\varphi \geqslant 0$, then φ-quasiconvexity is a strong quasiconvexity while its a weak (or relaxed) one if φ has nonpositive values.

(2) If for some $\alpha > 0$, $\varphi(x, y) = \alpha\|x - y\|^2$ then φ-quasiconvexity of f is nothing else but α-strong quasiconvexity i.e., for any $x, y \in K$ and any $t \in [0, 1]$,

$$g(tx + (1 - t)y) \leqslant \max\Big(g(x), g(y)\Big) - \alpha t(1 - t)\|x - y\|^2.$$

(3) Let $T : K \rightrightarrows X^*$ be a set valued operator with weakly-*compact and convex values. With the bifunction defined by $f_T(x, y) = \sup\limits_{x^* \in T(x)} \langle x^*, y - x\rangle$, if for some $\mu \geqslant 0$, $\varphi(x, y) = \mu\|x - y\|^2$, then φ-quasimonotonicity of f_T coincides with the μ-relaxed quasimonotonicity for T introduced by Bai and Hadjisavvas in [11].

(4) If a function $g : K \longrightarrow \mathbb{R}$ is α-strongly quasiconvex-like then it is φ-quasiconvex with $\varphi(x, y) = \alpha\|x-y\|$, since for all $t \in [0, 1]$, $t(1-t) \leq \min\{t, 1-t\}$.

(5) The class of φ-quasiconvex functions includes strongly $\alpha(.)$-paraconvex and $\alpha(.)$-paraconvex functions in the sense of S. Rolewicz [40]. These concepts are of course extendable to *$\alpha(.)$-para-quasiconvex* setting.

We provide in the following Lemma a general example of α-strongly quasiconvex-like functions.

Lemma 3.11. *Let $\beta \in L^\infty_{loc}(\mathbb{R})$ and j be the function defined by $j(x) = \int_0^x \beta(s)ds$ for every $x \geqslant 0$. Assume that β is bounded from above by a positive constant. Then the function j is α-strongly quasiconvex-like for some $\alpha > 0$.*

Proof. Let $\alpha > 0$ such that $\beta(s) \geqslant \alpha$ for every $s \geqslant 0$. Let $x, y \in \mathbb{R}_+$ with $y < x$. Then for $t \in]0.1[$,

$$\begin{aligned} j(tx + (1-t)y) &= \int_0^{tx+(1-t)y} \beta(s)ds \\ &= \int_0^x \beta(s)ds + \int_x^{tx+(1-t)y} \beta(s)\, ds. \end{aligned}$$

Clearly, $\int_0^x \beta(s)ds = j(x) = \max\{j(x), j(y)\}$. Let us now discuss the second term $\int_x^{tx+(1-t)y} \beta(s)ds$. Observe that

$$\int_x^{tx+(1-t)y} \beta(s)ds \leq \int_x^{tx+(1-t)y} \alpha ds = -\alpha(1-t)(x-y) \leq -\alpha \min(t, 1-t)(x-y).$$

Therefore, $j(tx + (1-t)y) \leqslant \max\{j(x), j(y)\} - \alpha \min\{t, 1-t\}|x-y|$. □

The following result, which is important for applications, generalizes [2, Proposition 4.1] and provides a general example of φ-strongly quasimonotone operators.

Proposition 3.12. *Let X be a normed space, K a convex subset of X, and $\varphi : K \times K \to \mathbb{R}$ a continuous, positive and symmetric (i.e., $\varphi(x, y) = \varphi(y, x) \geqslant 0$ for all $x, y \in K$) function. Let g be a lower semicontinuous and φ-quasiconvex function defined on K. Then the Clarke-Rockafellar subdifferential $\partial^{CR}g(x)$ of g is φ-strongly quasimonotone.*

Proof. Let $x \in dom(\partial^{CR}g)$, $y \in K$ and $x^* \in \partial^{CR}g(x)$ such that $\langle x^*, y - x\rangle > 0$. We have to show that $\langle y^*, x - y\rangle \leqslant -\varphi(y, x)$ for every $y^* \in \partial^{CR}g(y)$. Let $\eta \in \left]0, \langle x^*, y - x\rangle\right[$, then, $\langle x^*, y - x\rangle > \eta$. Evidently, for any $\varepsilon > 0$, there exists $\gamma \in]0, \frac{\varepsilon}{2}[$ such that

$$\langle x^*, v - x\rangle > \eta, \quad \forall v \in B(y, \gamma). \tag{22}$$

Fix $v \in B(y, \gamma)$ and remark that

$$g^{\uparrow}(x, v - x) \geqslant \langle x^*, v - x\rangle > \eta. \tag{23}$$

Thus, by the definition of Clarke-Rockafellar derivative, there exists $\varepsilon' : 0 < \varepsilon' < \frac{\varepsilon}{2}$ such that for any $\zeta \in]0, \varepsilon'[$ and any $\lambda \in]0, 1[$, there exists $x' \in B(x, \zeta)$, $\tau \in]0, \lambda[$ and $\delta \geqslant g(x')$ satisfying

$$\frac{g(x' + \tau(v - x')) - \delta}{\tau} > \eta. \tag{24}$$

Since $\varphi \geqslant 0$, for λ sufficiently small (then τ is also small enough), it obviously follows that

$$-(1 - \tau)\varphi(x', v) < \eta. \tag{25}$$

Therefore, by combining (24) and (25), we obtain

$$g(x' + \tau(v - x')) > \delta + \tau\eta > g(x') - \tau(1 - \tau)\varphi(x', v). \tag{26}$$

On the other hand, thanks to φ-quasiconvexity of g, we have

$$g(v + t(x' - v)) \leqslant \max\{g(x'), g(v)\} - t(1 - t)\varphi(v, x'),\ \forall t \in]0, 1[. \tag{27}$$

A fortiori $g(v) \geqslant g(x')$ otherwise,

$$g(x' + \tau(v - x')) \leqslant g(x') - \tau(1 - \tau)\varphi(x', v),$$

contradicting (26). Accordingly, (27) yields

$$g(v + t(x' - v)) \leqslant g(v) - t(1 - t)\varphi(x', v),\ \forall t \in]0, 1[. \tag{28}$$

Consequently, for any $\varepsilon > 0$, there exists $\gamma > 0$ such that for any $v \in B(y, \gamma)$, any $t \in]0, 1[$ and any $\beta \geqslant f(v)$, there exists a direction $d = x' - v \in B(y - x, \varepsilon)$ satisfying

$$\frac{g(v + t(x' - v)) - \beta}{t} \leqslant -(1 - t)\varphi(x', v). \tag{29}$$

This implies that

$$g^{\uparrow}(y, x - y) \leqslant -\varphi(x, y) = -\varphi(y, x). \tag{30}$$

Finally, the required inequality follows immediately from (30), completing the proof. □

Corollary 3.13. *Assume that X is a Banach space and let $J : X \to \mathbb{R}$ be a locally Lipschitz function. If J is α-strongly quasiconvex, for some $\alpha \geqslant 0$, then its Clarke-Rockafellar subdifferential $\partial^C J$ is α-strongly quasimonotone.*

Proof. Immediate from Proposition 3.12. □

From Proposition 3.12 we deduce the following general example of α-quasimonotonicity-like:

Corollary 3.14. *If $g : K \to \mathbb{R}$ is α-strongly quasiconvex-like on a subset K of its domain, then the Clarke-Rockafellar subdifferential $\partial^{CR} g(x)$ of g is α-strongly quasimonotone-like.*

Example 3.15. The Clarke-Rockafellar subdifferential of the function j provided in Lemma 3.11 is α-strongly quasimonotone-like for some $\alpha > 0$.

The directional derivative in the sense of Clarke of a locally Lipschitz function is interesting in applications such as in nonsmooth optimization and nonsmooth mechanic models. Here we regard this derivative as a bifunction whose second argument is the direction and present the following quasimonotonicity characterization.

Proposition 3.16. *Let $\varphi : K \times K \to \mathbb{R}$ a given bivariate function and let $j : K \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $J : K \times K \longrightarrow \mathbb{R}$ the bifunction defined by $J(u,v) = j^0(u, v-u)$. The following assertions are equivalent.*

(i) *J is φ-quasimonotone bifunction;*

(ii) *$\partial^C J$ is φ-quasimonotone operator.*

Proof. Suppose that J is φ-quasimonotone. Let $u, v \in V$ and $u^* \in \partial^C J(u)$ and suppose that $\langle u^*, v-u\rangle > 0$. From assertion (ii) of Lemma 2.6, we have

$$J(u,v) = j^0(u, v-u) \geqslant \langle u^*, v-u\rangle > 0.$$

Then $J(v,u) \leqslant \varphi(v,u)$. Therefore, for all $v^* \in \partial^C J(v)$,

$$\langle v^*, u-v\rangle \leqslant j^0(v, u-v) \leqslant \varphi(v,u)$$

implying that $\partial^C J$ is φ-quasimonotone.

For the converse, let $u, v \in V$ such that $j^0(u, v-u) > 0$. Again by Lemma 2.6, there exists $u^* \in \partial^C J(u)$ such that $j^0(u, v-u) = \langle u^*, v-u\rangle$. Take into account the φ-quasimonotonicity of $\partial^C J$ and see that

$$\begin{aligned} j^0(u, v-u) > 0 &\implies \langle u^*, v-u\rangle > 0 \\ &\implies \langle v^*, u-v\rangle \leqslant \varphi(v,u),\ \forall v^* \in \partial^C J(v) \\ &\implies \max_{v^* \in \partial J(v)} \langle v^*, u-v\rangle \leqslant \varphi(v,u),\ \forall v^* \in \partial^C J(v) \\ &\implies j^0(v, u-v) = J(v,u) \leqslant \varphi(v,u). \end{aligned}$$

This means that J is φ-quasimonotone. □

The following is a simple generalization of Proposition 3.16.

Proposition 3.17. *Let $T : X \rightrightarrows X^*$ be an operator with nonempty convex weak-*compact values and let f_T be the bifunction defined by*

$$f_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle.$$

The following assertions are equivalent:

(i) *f_T is φ-quasimonotone bifunction;*

(ii) *T is φ-quasimonotone operator.*

Proof. Similar to Proposition 3.16. □

A slight change in the proof of Proposition 3.16 leads to independence of the result in this proposition on the sign of φ, then in particular we are able to state the following consequences:

Corollary 3.18. *Let $\mu \geq 0$ and let $j : K \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $J : K \times K \longrightarrow \mathbb{R}$ the bifunction defined by $J(u,v) = j^0(u, v-u)$, $u, v \in K$. The following assertions are satisfied.*

(i) *J is μ-relaxed quasimonotone bifunction if, and only if $\partial^C J$ is μ-relaxed-quasimonotone;*

(ii) *J is μ-strongly quasimonotone bifunction if, and only if $\partial^C J$ is μ-strongly quasimonotone.*

Corollary 3.19. *Let $\mu \geq 0$ and let $j : K \longrightarrow \mathbb{R}$ be a locally Lipschitz function and $J : K \times K \longrightarrow \mathbb{R}$ the bifunction defined by $J(u,v) = j^0(u, v-u)$, $u, v \in K$. The following assertions are satisfied.*

(i) *J is μ-relaxed quasimonotone-like bifunction if, and only if $\partial^C J$ is μ-relaxed-quasimonotone-like;*

(ii) *J is μ-strongly quasimonotone-like bifunction if, and only if $\partial^C J$ is μ-strongly quasimonotone-like.*

Now we give the sufficient condition for converse of the result in Corollary 3.13.

Theorem 3.20. *Assume that X is a Banach space and K a convex subset of X. Let $J : K \subset X \to \mathbb{R}$ be a locally Lipschitz function on an open subset U containing K. Then the following assertions are true:*

(i) *J is quasiconvex if and only if its Clarke subdifferential $\partial^C J$ is quasimonotone;*

(ii) *Assume that for all $x, y \in K$, with $x \neq y$, $J^0(x, y-x)J^0(y, x-y) < 0$. Then*
   (1) *If moreover, for some $\alpha > 0$, $\tau\partial^C J$ is α-strongly quasimonotone-like for all $\tau \in]0,1]$, then J is α-strongly quasiconvex-like.*
   (2) *If moreover, for some $\alpha > 0$, $\tau\partial^C J$ is α-strongly quasimonotone for all $\tau \in]0,1]$, then J is α-strongly quasiconvex.*

To prove this Theorem, we need to recall the Lebourg's mean-value Theorem:

Theorem 3.21. ([20], Theorem 2.3.7) *Assume that X is a Banach space and K a convex subset of X. Let $J : K \subset X \to \mathbb{R}$ a locally Lipschitz function on an open subset U containing K. Then, for all $x, y \in K$, there exists $z \in (x, y)$, there exists $\xi \in \partial^C J(z)$ such that $J(y) - J(x) = \langle \xi, y - x \rangle$.*

Proof of Theorem 3.20. For (ii), we only prove (1), (2) being similar. Assume that $\partial^C J$ is α-strongly quasimonotone-like for some $\alpha > 0$ and that for all $x, y \in K$, with $x \neq y$, $J^0(x, y-x).J^0(y, x-y) < 0$. We have to show that J is α-strongly quasiconvex-like. For contradiction, suppose there exist some $x, y \in K$ and $\overline{\lambda} \in]0,1[$ such that for $\overline{x} = x + \overline{\lambda}(y - x)$ we have

$$J(\overline{x}) > \max\{J(x), J(y)\} - \alpha \min(\overline{\lambda}, 1 - \overline{\lambda})\|x - y\|.$$

Thus, we can assume that $J(x) \geqslant J(y)$ and obtain that

$$J(\overline{x}) > J(x) - \alpha \min(\overline{\lambda}, 1 - \overline{\lambda})\|x - y\| \geqslant J(y) - \alpha \min(\overline{\lambda}, 1 - \overline{\lambda})\|x - y\|. \quad (31)$$

Now, by Theorem 3.21, there exists $\hat{x} \in (y, \overline{x})$ and $\tilde{x} \in (\overline{x}, x)$ such that for some $\hat{\xi} \in \partial^C J(\hat{x})$ and $\tilde{\xi} \in \partial^C J(\tilde{x})$ we have

$$J(\overline{x}) - J(y) = \langle \hat{\xi}, \overline{x} - y \rangle \text{ and } J(\overline{x}) - J(x) = \langle \tilde{\xi}, \overline{x} - x \rangle. \tag{32}$$

By combining (31) and (32) and the definition of $\overline{x}$ we obtain

$$\langle \hat{\xi}, \overline{x} - y \rangle > -\alpha \min(\overline{\lambda}, 1 - \overline{\lambda}) \|x - y\| \tag{33}$$

and

$$\langle \tilde{\xi}, \overline{x} - x \rangle > -\alpha \min(\overline{\lambda}, 1 - \overline{\lambda}) \|x - y\|. \tag{34}$$

Therefore

$$\langle \hat{\xi}, \overline{x} - y \rangle > -\alpha \|x - y\| \tag{35}$$

and

$$\langle \tilde{\xi}, \overline{x} - x \rangle > -\alpha \|x - y\|. \tag{36}$$

Now, since $\hat{x} \in (y, \overline{x})$ and $\tilde{x} \in (\overline{x}, x)$, for some $\hat{\lambda}$ and $\tilde{\lambda}$ in $(0,1)$ with $\tilde{\lambda} < \overline{\lambda} < \hat{\lambda}$ we can write $\hat{x} = x + \hat{\lambda}(y - x)$ and $\tilde{x} = x + \tilde{\lambda}(y - x)$. Accordingly,

$$(1 - \overline{\lambda}) \langle \hat{\xi}, \tilde{x} - \hat{x} \rangle > -\alpha \|\tilde{x} - \hat{x}\| \tag{37}$$

and

$$\overline{\lambda} \langle \tilde{\xi}, \hat{x} - \tilde{x} \rangle > -\alpha \|\tilde{x} - \hat{x}\|. \tag{38}$$

Therefore, $(1 - \overline{\lambda}) J^0(\hat{x}, \tilde{x} - \hat{x}) > -\alpha \|\tilde{x} - \hat{x}\|$ and

$$\overline{\lambda} J^0(\tilde{x}, \hat{x} - \tilde{x}) > -\alpha \|\tilde{x} - \hat{x}\|. \tag{39}$$

Now, by the hypothesis, either $J^0(\hat{x}, \tilde{x} - \hat{x}) > 0$ or $J^0(\tilde{x}, \hat{x} - \tilde{x}) > 0$. Then by α-strong quasimonotonicity-like of $\tau \partial J$ $(\tau \in]0,1])$ and Proposition 3.16, either $\overline{\lambda} J^0(\tilde{x}, \hat{x} - \tilde{x}) \leq -\alpha \|\tilde{x} - \hat{x}\|$ or $(1 - \overline{\lambda}) J^0(\hat{x}, \tilde{x} - \hat{x}) \leq -\alpha \|\tilde{x} - \hat{x}\|$, contradicting (38) or (39).

Let us prove (i). By Corollary 3.13, if J is quasiconvex then $\partial^c J$ is quasimonotone. For the converse, by the same contradictory reasoning as in the case (ii) without any further condition on J^0, there exist some $x, y \in K$ and $\overline{\lambda} \in]0,1[$ such that for $\overline{x} = x + \overline{\lambda}(y - x)$, there exist $\hat{x} \in (y, \overline{x})$ and $\tilde{x} \in (\overline{x}, x)$ such that

$$J^0(\hat{x}, \tilde{x} - \hat{x}) > 0 \tag{40}$$

and

$$J^0(\tilde{x}, \hat{x} - \tilde{x}) > 0. \tag{41}$$

But $\partial^C J$ is quasimonotone, then by Proposition 3.16, the bifunction $J^0(x, y - x)$ is quasimonotone. Thus, (40) implies that $J^0(\tilde{x}, \hat{x} - \tilde{x}) \leqslant 0$, which conflicts with (41). □

Remark 3.22. (1) The assumption $J^0(x, y-x).J^0(y, x-y) < 0$ for all $x, y \in K$, $x \neq y$, implies that $J^0(., . - .)$ is quasimonotone.

(2) The assertion (i) of Theorem 3.20 presents a (technically) more simple proof of the result in [8, Theorem 4.1] for locally Lipschitz functions.

(3) If J is quasiconvex such that for all $x \in K$ and all $d \in K\backslash\{0\}$, $J^0(x, d) \neq 0$ and $-J$ is pseudoconvex then an easy verification shows that $J^0(x, y-x).J^0(y, x-y) < 0$, $\forall x, y \in K : x \neq y$. This double monotonicity condition is not too restrictive for some class of functions, see for example [2, Remark 4.4].

(4) Observe that by the same arguments, a similar result to assertions in (ii) of Theorem 3.20 can be obtained whenever $J^0(x, y-x) > 0$ for all $x, y \in K$ with $x \neq y$ instead of the condition $J^0(x, y-x)J^0(y, x-y) < 0$. Notice also that in this case the quasimonotonicity property of $\tau\partial^C J$ is required only for $\tau = 1$. Of course, one can find examples of function J such that $J^0(x, y-x) > 0$ for all $x, y \in K$ with $x \neq y$, for instance take the real-valued function $J(x) = x^3$ and K any interval bounded from below by some $m > 0$. This choice of K is done such that J be strongly convex on K.

(5) The quasimonotonicity (resp. -like) of $\tau\partial^C J$ can be seen as a kind of regularization of that on J^0. Let us give a general example of such a property. Let f be a bifunction and $\alpha > 0$. Then, for any $\tau \in]0, 1[$, the bifunction τg where g is defined by $g(x, y) = f(x, y) - \alpha\frac{1-\tau}{\tau}\|x-y\|^2$ (resp. $f(x, y) - \alpha\frac{1-\tau}{\tau}\|x-y\|$) is α-strongly quasimonotone (resp. -like) whenever f is so.

(6) Observe that for $\tau \in]0, 1[$, $\tau J^0(x, y-x) = \max\limits_{\xi \in \tau\partial^C J(x)} \langle \xi, y-x \rangle$. Then, $\tau\partial^C J(x)$ satisfies a quasimonotonicity condition if and only if the associated bifunction defined by $\hat{J}(x, y) = \tau J^0(x, y-x)$ is so.

(7) In a parallel research of the authors of this paper, strong quasimonotonicity of $\tau\partial^C J$ is shown to play a crucial role in establishing the strong quasimonotonicity of the associated normal operator to adjusted sublevels of J, see [6, Lemma 7], which allow new perspectives to quantitative stability of quasiconvex programming.

4. Solutions concepts, uniqueness and penalization

4.1. Solutions concepts

To motivate the introduction of the term φ in the equilibrium inequality, let us agree to introduce the following concept of equilibrium points:

Definition 4.1. (*φ-equilibrium*) Let f, φ be two real-valued bifunctions defined on $K \times K$. A point $\overline{x} \in K$ is said to be a *φ-global* (resp. *Minty*) *equilibrium* for f if, and only if it is true for all $y \in K$:

$$\varphi - EP(f, K): \quad f(\overline{x}, y) \geqslant \varphi(\overline{x}, y),$$

$$\Big(\text{resp. } \varphi - MEP(f, K): \quad f(y, \overline{x}) \leqslant \varphi(y, \overline{x})\Big).$$

If for some $\mu \geqslant 0$, $\varphi(x,y) = \mu\|x-y\|^2$, then the φ-global Minty equilibrium for f is said to be *μ-relaxed global Minty equilibrium* for f i.e.,

$$f(y,\overline{x}) \leqslant \mu||y-\overline{x}||^2, \;\; \forall y \in K. \tag{42}$$

If (42) is satisfied only in a neighborhood of $\overline{x}$, then $\overline{x}$ coincides with the local μ-relaxed Minty equilibrium for f introduced in [18]. In particular if $\varphi = 0$ then $\varphi - EP(f,K)$ (resp. $\varphi - MEP(f,K)$) collapses into the equilibrium problem introduced by Blum and Oettli [17] (resp. Minty or Dual equilibrium problem).

Notation: The solutions sets will be denoted as follows:

- The set of solutions to the equilibrium problem $\varphi - EP(f,K)$ is denoted by $S^\varphi(f,K)$. In particular, if $\varphi = 0$ we write $S(f,K)$ instead of $S^0(f,K)$.
- The set of solutions to Minty equilibrium problem $\varphi - MEP(f,K)$ is denoted by $M^\varphi(f,K)$. In particular, if $\varphi = 0$ we write $M(f,K)$ or $M^0(f,K)$.
- For every real bifunctions f and φ defined on $K \times K$, we denote by f_φ the bifunction defined by $f_\varphi(x,y) = f(x,y) - \varphi(x,y)$ for all x and y in K.

The local concept of solutions to $\varphi - MEP(f,K)$ is defined as follows:

Definition 4.2. A point $\overline{x}$ will be called a *local Minty φ-equilibrium* for f if, and only if there exists $r > 0$ such that

$$f(y,\overline{x}) \leqslant \varphi(y,\overline{x}), \;\; \forall y \in K \cap B(\overline{x},r).$$

Here, $B(\overline{x},r)$ is the open ball with radius $r > 0$ centered at $\overline{x}$.

The set of local Minty φ-equilibrium for f will be denoted by $M_L^\varphi(f,K)$. Of course, we always have $M^\varphi(f,K) \subset M_L^\varphi(f,K)$. The converse of this inclusion is treated as follows.

Proposition 4.3. *Let D be a closed and convex set of X and let $f : D \times D \longrightarrow \mathbb{R}$ be a bifunction. Assume that f is semistrictly quasiconcave and continuous in y and $f(x,x) = 0$ for all $x \in D$. Then the set of local Minty equilibrium points of f, say $M_L(f,D)$, coincides with the set $M(f,D)$ of global Minty equilibrium points of f i.e., $M_L(f,D) = M(f,D)$.*

Proof. Clearly, since f is null on the diagonal, a point $\overline{x}$ (in D) is a local equilibrium to $MEP(f,D)$ if, and only if $\overline{x}$ is a local maximizer of $f(\overline{x},.)$. Thus, from [24, Proposition 4], a point $\bar{x}$ is a global maximizer of $f(\overline{x},.)$ if, and only if it is a local maximizer of $f(\overline{x},.)$. □

Particular and interesting instances of Definitions 4.1 and 4.2 are as follows:

Definition 4.4. Let f be a real-valued bifunction defined on $K \times K$ and $\mu \geqslant 0$. A point $\overline{x} \in K$ is said to be

(i) a *μ-strong* equilibrium for f if, and only if

$$f(\overline{x},y) \geqslant \mu\|\overline{x}-y\|^2, \; \forall y \in K;$$

the set of μ-strong equilibrium points of f will be denoted by $S_s^\mu(f,K)$;

(ii) a *μ-weak* equilibrium or *μ-relaxed* equilibrium for f if, and only if

$$f(\overline{x}, y) \geqslant -\mu\|\overline{x} - y\|^2, \ \forall y \in K;$$

the set of μ-weak equilibrium points for f will be denoted by $S_w^\mu(f, K)$;

(iii) a *μ-global weak Minty* equilibrium or *μ-global relaxed Minty equilibrium* for f if (42) holds; the set of μ-global weak Minty equilibrium points for f will be denoted by $M_w^\mu(f, K)$;

(iv) a *μ-relaxed local Minty* equilibrium for f if and only if there exists $r > 0$ such that

$$f(y, \overline{x}) \leqslant \mu\|\overline{x} - y\|^2, \quad \forall y \in K \cap B(\overline{x}, r);$$

the set of μ-relaxed local Minty equilibrium points for f will be denoted by $M_L^\mu(f, K)$;

(v) a *μ-global strong Minty* equilibrium for f if, and only if

$$f(y, \overline{x}) \leqslant -\mu\|\overline{x} - y\|^2, \ \forall y \in K;$$

the set of μ-global strong Minty equilibrium points for f will be denoted by $M_s^\mu(f, K)$.

Lemma 4.5. *If f is μ-strongly monotone then any standard equilibrium point for f is a μ-strong Minty equilibrium point for f i.e., $S(f, K) \subset M_s^\mu(f, K)$. Moreover, if $\mu > 0$, then $S(f, K)$ reduces at most to a singleton.*

Proof. Let $\overline{x} \in S(f, K)$. Then, for all $y \in K$, $f(\overline{x}, y) \geqslant 0$. By the μ-strong monotonicity of f we have

$$\begin{aligned} f(y, \overline{x}) &\leqslant -f(\overline{x}, y) - \mu\|y - \overline{x}\|^2 \\ &\leqslant -\mu\|y - \overline{x}\|^2. \end{aligned}$$

The last point of the conclusion is immediate. □

Relaxed local (or global) Minty equilibrium are investigated in [18], strong Minty equilibrium points are also provided in Corollary 5.18 below and will be proven to have nice quantitative stability properties, see Section 6. Let us give some examples of solutions that may be obtained from strong equilibrium points.

Example 4.6. Given a set-valued map $T : X \longrightarrow 2^{X^\star}$ and a nonempty subset K of X, we consider the following Stampacchia variational inequality: Find $x \in K$ and $x^\star \in T(x)$ such that

$$SVI(T, K) : \quad \langle x^\star, y - x\rangle \geq 0, \ \forall y \in K.$$

Depending on the generalized monotonicity properties of the operator T we may consider different types of solutions, then we first fix the following concepts of solutions:

- If the (dual)-element $x^\star$ in the inequality $SVI(T,K)$ is such that $x^\star \in T(x)\backslash\{0\}$, then x is called a *star solution*, see Definition 8.1 in Section 8 where this concept is treated, see also [2, 3, 9, 10].
- If the (dual)-element $x^\star$ in the inequality $SVI(T,K)$ is such that $\langle x^\star, y - x\rangle > 0,\ \ \forall y \in K\backslash\{x\}$ then x is called a *strict solution*, see for example [2].

If T has nonempty weakly-*compact and convex values, then star or strict solutions to $SVI(T,K)$ may be obtained from strong equilibrium points of the bifunction f_T defined by $f_T(x,y) = \sup\limits_{x^*\in T(x)} \langle x^*, y - x\rangle$ (see [6, Definition 7 and Remark 5 (first point)]).

4.2. Penalization and eigenvalues solutions

Definition 4.7. Let $\varphi : K \times K \longrightarrow \mathbb{R}$ be a bivariate function and $\lambda \geq 0$. A point $\overline{x} \in K$ will be said a λ-*eigenvalue solution* to $\varphi - EP(f,K)$ if and only if

$$f(\overline{x}, x) \geqslant \lambda\varphi(\overline{x}, x), \ \ \forall x \in K. \tag{43}$$

Notation: The set of λ-eigenvalue solutions to $\varphi - EP(f,K)$ will be denoted by $S^{\lambda\varphi}(f,K)$.

Remark 4.8. If for some $\mu \geqslant 0$, $\varphi(x,y) = \mu\|x-y\|^2,\ \ x,y \in K$, then λ-eigenvalue solutions are nothing else but $\lambda\mu$-strong solutions to $EP(f,K)$.

Now we present a penalization result which by the meantime provides eigenvalues equilibrium points under the φ-quasi-convexity assumption.

Proposition 4.9. *Let K be a convex subset of X, $f : K \times K \longrightarrow \mathbb{R}$ and $\varphi : K \times K \longrightarrow \mathbb{R}$ be a bifunction such that $\varphi(x,y) \geqslant 0$ for all $x, y \in K$. Assume that the following conditions hold true:*

(i) $f(x,x) = 0,$ *for all* $x \in K$;
(ii) $f(x,.)$ *is* φ*-quasiconvex.*

Then, for all $\lambda \in [0, \frac{1}{4}]$, the problems $EP(f,K)$ and $\lambda\varphi$-$EP(f,K)$ are equivalent, i.e., $S(f,K) = S^{\lambda\varphi}(f,K)$.

Proof. Since $\varphi \geqslant 0$, the inclusion $S^{\lambda\varphi}(f,K) \subset S(f,K)$ is trivial. We have to prove the converse inclusion. Let $\overline{x}$ be a solution to $EP(f,K)$, i.e., $\overline{x} \in S(f,K)$. Then,

$$f(\overline{x}, x) \geqslant 0, \ \forall x \in K. \tag{44}$$

Let t be an element in $]0,1[$ and put $x_t = t\overline{x} + (1-t)x$. By (44) and assumption (ii) we have

$$0 \leqslant f(\overline{x}, x_t) \leqslant \max\{f(\overline{x},\overline{x}), f(\overline{x}, x)\} - t(1-t)\varphi(\overline{x}, x), \ \forall x \in K. \tag{45}$$

Hence, by assumption (i) and (45), we obtain

$$0 \leqslant f(\overline{x}, x_t) \leqslant f(\overline{x}, x) - t(1-t)\varphi(\overline{x}, x), \ \forall x \in K. \tag{46}$$

Therefore,

$$t(1-t)\varphi(\overline{x},x) \leqslant f(\overline{x},x), \ \forall x \in K, \ \forall t \in]0,1[. \tag{47}$$

Thus, given that the function $\psi(t) = t(1-t)$ is continuous with $\frac{1}{4}$ as upper bound over $[0,1]$, it follows that

$$\lambda\varphi(\overline{x},x) \leqslant f(\overline{x},x), \ \forall x \in K, \ \forall \lambda \in [0,\frac{1}{4}]. \tag{48}$$

This means that $\overline{x} \in S^{\lambda\varphi}(f,K)$ for all $\lambda \in [0,\frac{1}{4}]$, ending the proof. □

Remark 4.10. Proposition 4.9 is an extension of [6, Proposition 1] from the α-strong quasiconvexity to the general φ-quasiconvexity including the strong quasiconvexity-like mentioned earlier.

4.3. The case of uniqueness of the solution

Lemma 4.11. *Let $\varphi : K \times K \to \mathbb{R}$ such that $\varphi(x,y) > 0$ for all $x \neq y$ in K. Assume that $f(x,x) = 0$ for all $x \in K$, and that one of the following conditions is satisfied:*

(i) *f is $\lambda\varphi$-quasimonotone for some $\lambda \in]0,\frac{1}{4}[$ and φ-quasiconvex in y;*
(ii) *f is φ-quasimonotone and $\lambda\varphi$-quasiconvex in y for some $\lambda \in]0,\frac{1}{4}[$.*

Then the problem $EP(f,K)$ admits at most one solution i.e., $S(f,K)$ reduces to a singleton (at most).

Proof. Suppose for contradiction that $EP(f,K)$ admits two solutions x_1 and x_2 in K with $x_1 \neq x_2$. From Proposition 4.9, it follows that $\lambda'\varphi(x_1,x_2) \leqslant f(x_1,x_2)$, for all $\lambda' \in]0,\frac{1}{4}[$. Clearly, $f(x_1,x_2) > 0$, then by the $\lambda\varphi$-quasimonotonicity of f we deduce that

$$f(x_2,x_1) \leqslant \lambda\varphi(x_2,x_1). \tag{49}$$

Since x_2 is a solution to $EP(f,K)$ we also have

$$\lambda'\varphi(x_2,x_1) \leqslant f(x_2,x_1), \ \forall \lambda' \in]0,\frac{1}{4}[. \tag{50}$$

Hence, for any $\lambda' > \lambda$ ($\lambda' \in]0,\frac{1}{4}[$), (49) and (50) are in conflict, what completes the proof. □

Lemma 4.12. *Let $\varphi : K \times K \to \mathbb{R}$ such that $\varphi(x,y) > 0$ for all $x \neq y$ in K. Then, the equilibrium problem φ-$EP(f,K)$ has at most one solution (i.e., $S^{\varphi}(f,K)$ reduces to a singleton at most) if one of the following conditions is satisfied:*

(i) *f is quasimonotone;*
(ii) *for some $\lambda \in]0,\frac{1}{4}[$, f is $\lambda\varphi$-quasimonotone, φ-quasiconvex in y and $f(x,x) = 0$ for all $x \in K$.*

(iii) *f is φ-quasimonotone and $\frac{1}{\hat{\lambda}}\varphi$-quasiconvex in y for some $\hat{\lambda} \in]0, \lambda[$ (in particular if $\hat{\lambda}$ is close to $\frac{1}{4}$), and $f(x,x) = 0$ for all $x \in K$.*

Proof. Suppose for contradiction that φ-$EP(f,K)$ admits two solutions x_1 and x_2 in K.

First case: (i) is satisfied. On one hand we have $f(x_1,x_2) \geqslant \varphi(x_1,x_2) > 0$. Then, the quasimonotonicity of f implies that $f(x_2,x_1) \leqslant 0$. But x_2 is also a solution to φ-$EP(f,K)$, then $f(x_2,x_1) \geqslant \varphi(x_2,x_1) > 0$, a contradiction.

Second case: (ii) is satisfied. By Proposition 4.9, $\lambda'\varphi(x_1,x_2) \leqslant f(x_1,x_2)$, for all $\lambda' \in]0, \frac{1}{4}[$. Then, by the $\lambda\varphi$-quasimonotonicity of f, it follows that $f(x_2,x_1) \leqslant \lambda\varphi(x_2,x_1)$. By the meantime we have $\lambda'\varphi(x_2,x_1) \leqslant f(x_2,x_1)$. Hence, if we take $\lambda' > \lambda$ we obtain a contradiction.

Third case: (iii) is satisfied. Similarly to the previous case, $(\lambda/\hat{\lambda})\,\varphi(x_1,x_2) \leqslant f(x_1,x_2)$, for all $\lambda \in]0, \frac{1}{4}[$. Then, $(\lambda/\hat{\lambda})\,\varphi(x_2,x_1) \leqslant f(x_2,x_1) \leqslant \varphi(x_2,x_1)$. The contradiction is clearly obtained if $\lambda > \hat{\lambda}$. □

Remark 4.13. The assumption (ii) is particularly satisfied if f is α-strongly quasiconvex for some $\alpha > 0$ and μ-relaxed quasimonotone for some $\mu \in]0, \frac{1}{4}[$, and $f(x,x) = 0$ for all $x \in K$.

Remark 4.14. Lemmas 4.11 and 4.12 are respectively natural extensions of [6, Lemma 1 and Lemma 2] to the more general case of a bivariate function φ with the same properties of square of the norm.

5. Existence Theorems for $EP(f,K)$

In this section, we establish the existence of global or local φ-Minty equilibrium points under the various assumptions $(\mathbb{H}_i)$, $i \in \{1, \ldots, 7\}$ introduced in Section 1.

To link the assumption $(\mathbb{H}_3)$ to $(\mathbb{H}_5)$ we begin with the following Lemma that expresses the proper quasi-monotonicity of the equilibrium bifunction in terms of its quasi-concavity.

Lemma 5.1. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Under $(\mathbb{H}_0)$, f is properly 0-quasimonotone if, and only if f is diagonally quasiconcave in x.*

Proof. ($\Rightarrow$) Let $n \in \mathbb{N}$ and $x_1, ..., x_n \in K$ then there exist $\overline{x} \in Conv(x_1, ..., x_n)$ and $i \in \{1, ..., n\}$ such that $f(x_i, \overline{x}) \leqslant 0$. Hence, in view of $(\mathbb{H}_0)$, $f(\overline{x}, \overline{x}) \geqslant \min\limits_{1 \leqslant j \leqslant n} f(x_j, \overline{x})$, which means that f is diagonally quasiconcave in x.

($\Leftarrow$) Let $n \in \mathbb{N}$ and $x_1, ..., x_n \in K$ and $\overline{x} \in Conv(x_1, ..., x_n)$. Suppose, for contradiction, that $f(x_i, \overline{x}) > 0$ for all $i \in \{1, ..., n\}$; then $\min_{1 \leqslant i \leqslant n} f(x_i, \overline{x}) > 0$.

Equivalently (thanks to $(\mathbb{H}_0)$), $\min_{1 \leqslant i \leqslant n} f(x_i, \overline{x}) > f(\overline{x}, \overline{x})$.

This contradicts the fact that f is diagonally quasi concave in x, then there exists $i \in \{1, ..., n\}$ such that $f(x_i, \overline{x}) \leqslant 0$ and then f is clearly properly 0-quasimonotone. □

Remark 5.2. Lemma 5.1 ensures that if f satisfies $(\mathbb{H}_0)$, then $(\mathbb{H}_3) \Longleftrightarrow (\mathbb{H}_5)$.

Next, Lemma 5.1 will facilitate the comparison between local and global Minty solutions.

Corollary 5.3. *Assume that f is continuous in x and satisfies $(\mathbb{H}_0)$ and f is not properly quasimonotone. Then, the set of global Minty equilibrium points of f is strictly included in the set of its local Minty equilibrium i.e., $M(f, K) \subsetneq M_L(f, K)$.*

Proof. As noticed in Proposition 4.3, a point $\overline{x}$ (in K) is a local equilibrium to $MEP(f, K)$ if, and only if $\overline{x}$ is a local maximizer of $f(., \overline{x})$. On the other hand, since f is not properly quasimonotone, by Lemma 5.1, f is not diagonally quasiconcave in x. Then, f is not quasiconcave in x and thus f is not second type semistrictly quasiconcave. Therefore, by [24, Proposition 4], there exists $\overline{x} \in K$ which is a local and non global maximizer of $f(., \overline{x})$. Then, $\overline{x} \in M_L(f, K)$ and $\overline{x} \notin M(f, K)$, completing the proof. □

The characterization in Lemma 5.1 can be extended to the corresponding relaxed quasimontonicity and quasiconcavity counterparts. In this respect, we introduce the following, which is an extension of definition of diagonal quasiconcavity given in [45]:

Definition 5.4. Let $f : K \times K \longrightarrow \mathbb{R}$ and $\mu \geqslant 0$. f is said to be *μ-relaxed diagonally quasiconcave* in x, if for any finite subset $\{x_1, ..., x_n\} \subset K$ and any $x_0 \in Conv\{x_1, ..., x_n\}$, for some $i : 1 \leqslant i \leqslant n$, we have

$$\min_{1 \leqslant j \leqslant n} f(x_j, x_0) \leqslant \mu \|x_0 - x_i\|^2 + f(x_0, x_0).$$

Of course, quasi-concavity implies μ-relaxed diagonal quasi-concavity, the following example illustrates that the converse is not true.

Example 5.5. Take $\mu = 0$. For a given $\varphi \in X^*$ such that $\varphi \neq 0$, we consider the operator A defined by A: $K \longrightarrow X^*$, $x \longrightarrow ||x||.\varphi$. Then the bifunction f defined, for $x, y \in K$, by $f(x, y) = \langle A(y), y - x \rangle$ is diagonally quasiconcave in y but is not quasiconcave in y for some $x \in K$.

This new concept of μ-relaxed diagonal quasi-concavity will give the parallel generalized convexity to μ-relaxed proper quasimonotonicity which interprets the assumption $(\mathbb{H}_3)$ in the case when φ is defined by $\varphi(x, y) = \mu \|x - y\|^2$, $x, y \in K$.

Lemma 5.6. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction and $\mu \geq 0$. Under the assumption $(\mathbb{H}_0)$, f is μ-relaxed properly quasimonotone if, and only if f is μ-relaxed diagonally quasiconcave in x.*

Proof. ($\Rightarrow$) Let $n \in \mathbb{N}$ and $x_1, ..., x_n \in K$ then there exists $\overline{x} \in Conv(x_1, ..., x_n)$ and there exists $i \in \{1, ..., n\}$ such that $f(x_i, \overline{x}) \leqslant \mu\|x_i - \overline{x}\|^2$. Hence,

$$\min_{1 \leqslant j \leqslant n} f(x_j, \overline{x}) \leqslant f(x_i, \overline{x}) \leqslant \mu\|x_i - \overline{x}\|^2.$$

Thus,

$$\min_{1 \leqslant j \leqslant n} f(x_j, \overline{x}) \leqslant f(x_i, \overline{x}) \leqslant \mu\|x_i - \overline{x}\|^2$$

which, in view of $(\mathbb{H}_0)$, implies that

$$\min_{1 \leqslant j \leqslant n} f(x_j, \overline{x}) \leqslant f(\overline{x}, \overline{x}) + \mu\|x_i - \overline{x}\|^2.$$

This means that f is μ-relaxed diagonally quasiconcave in x.

($\Leftarrow$) Let $n \in \mathbb{N}$ and $x_1, ..., x_n \in K$ and $\overline{x} \in Conv(x_1, ..., x_n)$. Suppose, for contradiction, that $f(x_i, \overline{x}) > \mu\|x_i - \overline{x}\|^2$ for all $i \in \{1, ..., n\}$, then

$$\min_{1 \leqslant i \leqslant n} f(x_i, \overline{x}) > \mu \max_{1 \leqslant j \leqslant n} \|x_j - \overline{x}\|^2.$$

Thus, thanks to $(\mathbb{H}_0)$, it follows that

$$\min_{1 \leqslant j \leqslant n} f(x_j, \overline{x}) > f(\overline{x}, \overline{x}) + \mu \max_{1 \leqslant j \leqslant n} \|x_j - \overline{x}\|^2.$$

This contradicts the μ-relaxed diagonal quasiconcavity of f in x. Then there exists $i \in \{1, ..., n\}$ such that $f(x_i, \overline{x}) \leqslant 0$ and hence f is μ-relaxed properly quasimonotone. □

Examples 5.7. Every bifunction null on the diagonal and diagonally quasiconcave in x is μ-relaxed diagonally quasiconcave in x. Now, we give another example: Let g be a Lipschitz function with a rank of Lipschitz equal to some $L > 0$. Then, the bifunction defined by $f(x, y) = g^0(x, (y - x)\|x - y\|)$ is L-relaxed diagonally quasiconcave in x thanks to the properties (iii) and (vi) of Lemma 2.6. Actually we have the following more general property:

$$f(x, y) \leqslant L\|x - y\|^2, \quad \forall x, y \in K. \tag{51}$$

The following result provides a general class of μ-relaxed diagonal quasiconcave bivariate functions with respect to the first variable.

Proposition 5.8. *Let $f : K \times K \to \mathbb{R}$ and $\mu \geqslant 0$. If f is quasiconvex in y and μ-relaxed pseudomonotone then f is μ-relaxed diagonally quasiconcave in x.*

Proof. By [18, Lemma 2], f is μ-relaxed properly quasimonotone. Then, Lemma 5.6 implies that f is μ-relaxed diagonally quasiconcave in x. □

Now, we start our treatment of existence by proving an elementary result which defines the opposite problem to φ-(MEP). So, under two topological assumptions only one of them has a solution. More precisely we claim that:

Lemma 5.9. *Let $f, \varphi : K \times K \longrightarrow \mathbb{R}$ be two bifunctions. If K is compact and $(\mathbb{H}_2)$ is satisfied, then* ***either*** *$\varphi - MEP(f, K)$ admits at least one solution* ***or*** *there exist $y_1, \dots y_m$ in K with $m \in \mathbb{N}$ and $\overline{x} \in Conv\{y_1, \dots y_m\}$ such that*

$$\min_{1 \leqslant i \leqslant m} f_\varphi(y_i, \overline{x}) > 0.$$

Proof. Classical by means of the well known KKM Lemma. □

As a consequence of Lemma 5.9 we obtain the following:

Proposition 5.10. *Let $f, \varphi : K \times K \longrightarrow \mathbb{R}$ be two bifunctions. If K is compact and $(\mathbb{H}_i)$ is verified for $i \in \{2, 3\}$, then $\varphi - MEP(f, K)$ admits at least one solution.*

A particular case of Lemma 5.9 is as follows:

Lemma 5.11. *Let $f, \varphi : K \times K \longrightarrow \mathbb{R}$ be two bifunctions. If K is compact, $\varphi(x, y) \geqslant 0$ for all $x, y \in K$ and $(\mathbb{H}_2)$ is satisfied, then*

either *$\varphi - MEP(f, K)$ admits at least one solution*

or *there exist $y_1, \dots y_m$ in K with $m \in \mathbb{N}$ and $\overline{x} \in Conv\{y_1, \dots y_m\}$ such that $\min_{1 \leqslant i \leqslant m} f(y_i, \overline{x}) > 0$.*

Proof. Follows immediately from Lemma 5.9. □

Proposition 5.12. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. If K is compact, $\varphi(x, y) \geqslant 0$ for all $x, y \in K$ and $(\mathbb{H}_i)$ is satisfied for $i \in \{0, 2, 5\}$, then $\varphi - MEP(f, K)$ admits at least one solution.*

Proof. By Remark 5.2, under the assumption $(\mathbb{H}_0)$, we have $(\mathbb{H}_3) \Longleftrightarrow (\mathbb{H}_5)$. Then, the required result is a consequence of Proposition 5.10. □

The same conclusion of Proposition 5.12 is true if we replace $(\mathbb{H}_5)$ by $(\mathbb{H}_4)$ or $(\mathbb{H}_7)$. More precisely, we state the following:

Proposition 5.13. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Assume that K is compact and $(\mathbb{H}_i)$ is satisfied for $i \in \{1, 2\}$ and either $(\mathbb{H}_4)$ or $(\mathbb{H}_7)$ is satisfied. Then, $\varphi - MEP(f, K)$ admits at least one solution.*

Proof. If, for contradiction, $\varphi - (MEP)$ has no solutions then by Lemma 5.9 there exist $y_1, \dots, y_m$ and $\overline{x} \in Conv\{y_1, \dots, y_m\}$ such that

$$f(y_i, \overline{x}) > \varphi(y_i, \overline{x}), \quad \forall i : 1 \leqslant i \leqslant m. \tag{52}$$

First case: $(\mathbb{H}_4)$. In this case, the (φ-quasimonotonicity) of f and (52) ensure that

$$f(\overline{x}, y_i) \leqslant 0, \quad \forall i : 1 \leqslant i \leqslant m. \tag{53}$$

Equivalently,

$$\max_{1\leqslant i\leqslant m} f(\overline{x}, y_i) \leqslant 0. \tag{54}$$

On the other hand, $(\mathbb{H}_1)$ and the assumption that $f(x, .)$ is strictly quasiconvex imply that

$$0 \leqslant f(\overline{x}, \overline{x}) < \max_{1\leqslant i\leqslant m} f(\overline{x}, y_i). \tag{55}$$

Then we see that (54) and (55) are in conflict.

Second case: $(\mathbb{H}_7)$. Thanks to the equivalent formulation of φ-pseudomonotonicity of f, the inequality in (52) implies that

$$f(\overline{x}, y_i) < 0, \quad \forall i : 1 \leqslant i \leqslant m. \tag{56}$$

Thus

$$\max_{1\leqslant i\leqslant m} f(\overline{x}, y_i) < 0. \tag{57}$$

Now involve $(\mathbb{H}_1)$ and the quasiconvexity of f and see that

$$0 \leqslant f(\overline{x}, \overline{x}) \leqslant \max_{1\leqslant i\leqslant m} f(\overline{x}, y_i). \tag{58}$$

Clearly, (58) and (57) are in contradiction, finishing the proof. □

Remark 5.14. When φ has positive values, the solutions to $\varphi - MEP(f, K)$ are indeed relaxed Minty solutions or else first type Minty solutions. Instead, if φ has negative values then the corresponding Minty solutions are of a strong type or second type, they may be obtained under stronger assumptions but they will later prove to enjoy nice stability properties, see Theorem 6.5 in the next Section. Next, we deal with both the first and second type Minty solutions in the particular case where φ is defined by the square of the norm.

Corollary 5.15. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction such that f is strictly convex in y and μ-relaxed pseudomonotone for some $\mu > 0$. Assume moreover that K is compact, $(\mathbb{H}_1)$ is satisfied and $(\mathbb{H}_2)$ is fulfilled with $\varphi(x, y) = \mu\|x - y\|^2$, $x, y \in K$. Then f admits at least one μ-global weak Minty equilibrium.*

Proof. Since f is strictly convex in y and μ-relaxed pseudomonotone for some $\mu > 0$, f satisfies $(\mathbb{H}_4)$ with $\varphi(x, y) = \mu\|x - y\|^2$, $x, y \in K$. Then, the required result is a consequence of Proposition 5.13. □

Remark 5.16. Of course, Corollary 5.15 remains true if f is μ-relaxed quasimonotone instead of μ-relaxed pseudomonotone but the statement of this Corollary will be exactly needed to derive $\tilde{\mu}$-weak Minty solutions in the next result, for $\tilde{\mu} < \mu$, leading to approximate relaxed Minty solutions.

Theorem 5.17. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction such that f is convex in y and μ-strongly quasimonotone for some $\mu > 0$. Assume moreover that K is compact, $(\mathbb{H}_1)$ is satisfied and $(\mathbb{H}_2)$ is verified with $\varphi(x, y) = (\mu - \frac{\delta}{2})\|x - y\|^2$, $x, y \in K$ and δ any number in $\in]0, 2\mu[$. Then, for all $\delta < 2\mu$, f admits at least one $(\mu - \frac{\delta}{2})$-global weak Minty equilibrium.*

Proof. Let $\delta \in]0, 2\mu[$ and g be the bifunction defined by $g(x, y) = f(x, y) + \frac{\delta}{2}\|x - y\|^2$. We first claim that g is μ-relaxed pseudomonotone. Indeed, let $x, y \in K$ such that $g(x, y) \geqslant 0$. Then,

$$f(x, y) \geqslant -\frac{\delta}{2}\|x - y\|^2 > -\mu\|x - y\|^2.$$

Hence, the μ-strong quasimonotonicity of f implies that $f(y, x) \leqslant 0$, and thus $g(y, x) \leqslant \frac{\delta}{2}\|x - y\|^2 \leqslant \mu\|x - y\|^2$, proving the claim. In addition, given that $\frac{\delta}{2}\|x - y\|^2$ is δ-strongly convex and that f is convex (in y), g is δ-strongly convex in y. The bifunction g also trivially satisfies $(\mathbb{H}_1)$. Moreover, by assumption, g verifies $(\mathbb{H}_2)$ with $\varphi(x, y) = \mu\|x - y\|^2$. Therefore, by Corollary 5.15, g admits a μ-global weak Minty equilibrium which is in turn a $(\mu - \frac{\delta}{2})$-global weak Minty equilibrium for f. □

Proposition 5.13 allows also to obtain strong Minty solutions by weakening the closedness assumption but under a stronger quasimonotonicity hypothesis.

Corollary 5.18. *Let $\mu > 0$. Assume that f is convex in y, for all $x \in K$, the set $\{y \in K : f(x, y) \leqslant -\mu\|x - y\|^2\}$ is closed and the bifunction g defined by $g(x, y) = f(x, y) + \mu\|x - y\|^2$, $x, y \in K$, is quasimonotone. Then, f admits at least one μ-global strong Minty equilibrium.*

Proof. Clearly, g is 2μ-strongly convex in y then it is strictly convex in y. In addition, g verifies trivially $(\mathbb{H}_1)$ and also $(\mathbb{H}_2)$ with $\varphi = 0$ by assumption. Thus, by Proposition 5.13, g admits a global Minty equilibrium which is nothing else but a global μ-strong Minty equilibrium for f. □

Remark 5.19. Let g be the bifunction used in Corollary 5.18 defined by $g(x, y) = f(x, y) + \mu\|x - y\|^2$, $x, y \in K$. As shown in the proof of Theorem 5.17, μ-strong quasimonotonicity of f implies the relaxed pseudomonotonicity of the bifunction g but it is not enough to ensures its quasimonotonicity. The latter property requires an additional condition as shows Proposition 5.20 below.

Proposition 5.20. *Assume that the following conditions are satisfied:*

(i) *f is μ-strongly quasimonotone for some $\mu > 0$;*
(ii) *for all $x, y \in K$ such that $x \neq y$, $f(x, y).f(y, x) < 0$.*

Then the bifunction g defined by $g(x, y) = f(x, y) + \mu\|x - y\|^2$, $x, y \in K$, is quasimonotone.

Proof. Let $x, y \in K$ such that $g(x, y) > 0$. Then,

$$f(x, y) > -\mu\|x - y\|^2. \tag{59}$$

Hence (by the μ-strongly quasimonotonicity of f), $f(y,x) \leqslant 0$. Consequently, if $x \neq y$ then by assumption (ii), $f(x,y) > 0$. Moreover, thanks to (59), we also have $f(x,y) > 0$ if $x = y$. Therefore, by applying again the μ-strongly quasimonotonicity of f, it results that $f(y,x) \leqslant -\mu\|x-y\|^2$, which means that $g(y,x) \leqslant 0$. □

Remark 5.21. Observe that assumption (ii) implies that f is quasimonotone.

Example 5.22. (1) An example of bifunction satisfying conditions (i) and (ii) of Proposition 5.20 is provided earlier in Remark 3.22 with respect to Clark's generalized derivative.

(2) Let $K = \{0\} \times [0,1] \subset \mathbb{R}^2$ and $T : K \to 2^{\mathbb{R}^2}$ be defined by

$$T(0,y) = \left\{ \left(1, \frac{1+y}{2}\right) \right\}.$$

Then, the bifunction defined by $f_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x \rangle$, $x, y \in K$ is strongly quasimonotone and $-f$ is strictly pseudomonotone, and hence verifies conditions (i) and (ii) of Proposition 5.20.

In the following result we deal with the case of $(\mathbb{H}_6)$ where a local solution is established without compactness or coercivity assumptions.

Proposition 5.23. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. If $\varphi(x,y) \geqslant 0$ for all $x, y \in K$ and $(\mathbb{H}_i)$ for $i \in \{2,6\}$ is satisfied, then $M_L^{\varphi}(f,K)$ is nonempty.*

To prove this result we make the following observation:

Lemma 5.24. *Let $f : K \times K \longrightarrow \mathbb{R}$ be a non properly-φ-quasimonotone bifunction. Then there exist $\{y_1, \ldots, y_m\}$ in K and $\overline{x} \in Conv\{y_1, \ldots, y_m\}$ such that*

$$\min_{1 \leqslant i \leqslant m} f_{\varphi}(y_i, \overline{x}) > 0. \tag{60}$$

Proof. Straightforward from the definition of the non-properly φ-quasimonotonicity of f. □

Proof of Proposition 5.23. By Lemma 5.24 there exist $\{y_1, \ldots, y_m\}$ in K and $\overline{x} \in Conv\{y_1, \ldots, y_m\}$ such that

$$f_{\varphi}(y_i, \overline{x}) > 0, \quad \forall i : 1 \leqslant i \leqslant m. \tag{61}$$

The inequalities in (61) mean that $\overline{x} \in F_{\varphi}(y_i)^c$ for all $i : 1 \leqslant i \leqslant m$. But $F_{\varphi}(y_i)^c$ is open in K for all $i : 1 \leqslant i \leqslant m$ (thanks to assumption $(\mathbb{H}_2)$). Then there exists a neighborhood $\mathcal{V}_{\overline{x}}$ of $\overline{x}$ such that

$$f_{\varphi}(y_i, y) > 0, \ \ \forall y \in K \cap \mathcal{V}_{\overline{x}}, \ \ \forall i : 1 \leqslant i \leqslant m. \tag{62}$$

Now, the first part of assumption ($\mathbb{H}_6$) (φ-quasimonotonicity) combined with (62) leads to

$$f(y, y_i) \leqslant 0, \quad \forall y \in K \cap \mathcal{V}_{\overline{x}}, \quad \forall i : 1 \leqslant i \leqslant m. \tag{63}$$

Then, taking into account the second part of ($\mathbb{H}_6$) and the fact that φ is a positive bifunction, the inequality (63) in turn implies that

$$f(y, \overline{x}) \leqslant \max_{0 \leq i \leq m} f(y, y_i) \leq 0 \leq \varphi(y, \overline{x}), \qquad \forall y \in K \cap \mathcal{V}_{\overline{x}}. \tag{64}$$

The proof is therefore complete. □

Remark 5.25. (1) In Proposition 5.23, the compactness of K is dropped.

(2) Proposition 5.23 is indeed an extension of [18, Theorem 3] from μ-relaxed quasimonotonicity to φ-quasimonotonicity.

We are now in a position to present a unified result of existence of Minty φ-equilibria.

Theorem 5.26. *Let K be a convex subset of X and $f, \varphi : K \times K \longrightarrow \mathbb{R}$ two bifunctions such that ($\mathbb{H}_2$) is satisfied. Then:*

(1) *$M^{\varphi}(f, K)$ is nonempty if K is compact and one of the following conditions is satisfied:*
- (i) ($\mathbb{H}_3$)*;*
- (ii) ($\mathbb{H}_i$) *for* $i \in \{1, 4\}$;
- (iii) ($\mathbb{H}_i$) *for* $i \in \{0, 5\}$ *and* $\varphi(x, y) \geqslant 0$ *for all* $x, y \in K$;
- (iv) ($\mathbb{H}_i$) *for* $i \in \{1, 7\}$.

(2) *$M^{\varphi}_L(f, K)$ is nonempty if ($\mathbb{H}_6$) holds true and $\varphi(x, y) \geqslant 0$ for all $x, y \in K$.*

Proof. For the assertion in (1), in the forth cases (i), (ii), (iii) and (iv), the required conclusion comes from Propositions 5.10, 5.12 and 5.13 respectively i.e., the set of global φ-Minty solutions $M^{\varphi}(f, K)$ is nonempty. For the assertion in (2), Proposition 5.23 ensures that $M^{\varphi}_L(f, K)$ is nonempty. □

Remark 5.27. (1) The result in assertion (1) of Theorem 5.26 remains true if K is assumed to be weakly compact and instead of ($\mathbb{H}_2$), the set $F_{\varphi}(x) = \{y \in K | \ f(x, y) \leqslant \varphi(x, y)\}$ is weakly compactly closed for all $x \in K$ i.e., $F_{\varphi}(x) \cap B$ is weakly closed for every weakly compact $B \subset K$.

(2) According to the previous point of this remark, the case (i) of assertion (1) in Theorem 5.26 improves the previous results in the literature as in [18, Corollary 2] by weakening the closedness assumption and by considering the more general φ-quasimonotonicity. Moreover, our case provides various possibilities offering global φ-Minty solutions that generalize μ-local Minty considered there.

To ensure the passage from Minty solutions to standard ones, we need to recall the concept of local μ-upper sign property introduced in [18, Definition 5 pp 1223] stated by the introduction in (3).

Definition 5.28. Assume that X is a normed space and K a convex subset of X. Let $\mu \geqslant 0$ and f be a bifunction defined on $K \times K$. f will be said to have the local *μ-upper sign property* at a point $x \in K$ if there exists a convex neighborhood $\mathcal{V}_x$ such that: for all $y \in \mathcal{V}_x$,

$$f(z_t, x) \leqslant \mu \|z_t - x\|^2, \forall t \in (0,1) \Longrightarrow f(x,y) \geqslant 0, \tag{65}$$

where $z_t = (1-t)x + ty$.

Remark 5.29. (1) Generally, the (local) μ-upper sign property in x, is not a kind of weak continuity, but if for every $x \in K$ there exists $r > 0$ such that for every $y \in K \cap B(x,r)$ the following implication is satisfied:

$$(1-t)f(z_t, x) + tf(z_t, y) \geq 0 \qquad \forall t \in]0,1[\tag{66}$$

then the upper hemicontinuity of f in x implies that the μ-upper sign property of f at x for every $\mu \geq 0$. See [18, Lemma 4, pp 1223] for more details.

(2) Whenever (65) is satisfied for every $y \in K$, f will be said to have the *global μ-upper sign property* in x, which is an extension of the upper sign continuity of operators introduced in [31], see Definition 8.4 below in Section 8.

Lemma 5.30. *Let $\mu \geqslant 0$. The following assertions are true:*

(i) *If f has the global μ-upper sign property in x, then any μ-relaxed global Minty solution is a standard solution to $EP(f,K)$ i.e., $M_w^\mu(f,K) \subset S(f,K)$.*

(ii) *If f is μ-relaxed pseudomonotone, then $S(f,K) \subset M_w^\mu(f,K)$.*

(iii) *If $\mu > 0$ and f is μ-strongly monotone and has the global μ-upper sign property in x, then $S(f,K), M_w^\mu(f,K)$ and $M_s^\mu(f,K)$ coincide and reduce at most to a singleton $\{x\}$.*

Proof. (i) and (ii) are immediate. For (iii), from Lemma 4.5, we obtain $S(f,K) \subset M_s^\mu(f,K)$. Of course we always have $M_s^\mu(f,K) \subset M_w^0(f,K) \subset M_w^\mu(f,K)$. Thus,

$$S(f,K) = M_s^\mu(f,K) \subset M_w^0(f,K) \subset M_w^\mu(f,K).$$

Finally, in view of the μ-strong monotonicity of f, $S(f,K)$ is reduced at most to a singleton. This ends the proof. □

Remark 5.31. If f is quasimonotone and satisfies $(\mathbb{H}_0)$, then for all $\mu > 0$, all $\lambda > 0$ and all positive bivariate function φ defined on $K \times K$, both λ-eigenvalue solutions to $\varphi - EP(f,K)$ and μ-strong global solutions to $EP(f,K)$ are global Minty equilibrium points for f i.e., $S^{\lambda\varphi}(f,K) \subset M_w^0(f,K) \subset M_w^\mu(f,K)$ and $S_s^\mu(f,K) \subset M_w^0(f,K) \subset M_w^\mu(f,K)$.

If f has the local μ-upper sign property in x, then μ-relaxed local Minty equilibrium points for f are also standard equilibrium ones for f under the following additional condition: for all $x, y \in K$,

$$f(x,y) < 0 \Longrightarrow f(x, (1-t)x + ty) < 0, \ \ \forall t \in]0,1[. \tag{67}$$

More precisely, we state:

Proposition 5.32. *Let $\mu > 0$. Assume that f has the local μ-upper sign property in x and (67) is satisfied. Then we have*

(1) $M_L^\mu(f, K) \subset S(f, K)$.

(2) *If, in addition, $(\mathbb{H}_0)$ is satisfied and f is μ-relaxed quasimonotone and strictly quasiconvex in y then*

$$M_w^\mu(f, K) = M_L^\mu(f, K) = S(f, K). \tag{68}$$

Proof. The first point is already established in [18, Theorem 2, pp 1223]. We only have to prove the second one. Let $\overline{x} \in S(f, K)$. Then,

$$f(\overline{x}, x) \geqslant 0, \ \ \forall x \in K. \tag{69}$$

Let t be an element in $]0, 1[$ and put $x_t = t\overline{x} + (1 - t)x$. By (69) and the strict quasiconvexity of f we obtain

$$0 \leqslant f(\overline{x}, x_t) < \max\{f(\overline{x}, \overline{x}), f(\overline{x}, x)\}, \ \ \forall x \in K. \tag{70}$$

Hence, from assumption $(\mathbb{H}_0)$ and (70), it follows that

$$0 \leqslant f(\overline{x}, x_t) < f(\overline{x}, x), \ \ \forall x \in K. \tag{71}$$

Therefore,

$$0 < f(\overline{x}, x), \ \ \forall x \in K. \tag{72}$$

Accordingly, the μ-relaxed quasimonotonicity of f together with (72) lead to

$$f(x, \overline{x}) \leqslant \mu\|x - \overline{x}\|^2, \ \ \forall x \in K. \tag{73}$$

This means that $\overline{x} \in M_w^\mu(f, K)$ and then $S(f, K) \subset M_w^\mu(f, K)$. But every time we have $M_w^\mu(f, K) \subset M_L^\mu(f, K)$, which in turn implies, thanks to (1), that

$$S(f, K) \subset M_w^\mu(f, K) \subset M_L^\mu(f, K) \subset S(f, K). \tag{74}$$

Finally,

$$S(f, K) = M_w^\mu(f, K) = M_L^\mu(f, K). \qquad \square \tag{75}$$

Remark 5.33. Let us emphasize that if f is quasiconvex (in particular if f is c onvex) in y and satisfies $(\mathbb{H}_0)$ (null on the diagonal) then f satisfies (67).

Corollary 5.34. *Let $\mu \geqslant 0$. Assume that $(\mathbb{H}_0)$ and $(\mathbb{H}_i)$ for $i \in \{2, 4\}$ are verified with $\varphi(x, y) = \mu\|x - y\|^2$, f has the local μ-upper sign property in x and (67) is satisfied. Then $\emptyset \neq M_w^\mu(f, K) = M_L^\mu(f, K) = S(f, K)$.*

Proof. Since $(\mathbb{H}_0)$ implies $(\mathbb{H}_1)$, the required conclusion is a quick combination of Propositions 5.13 and 5.32. $\square$

Remark 5.35. In view of Corollary 5.3, if $\mu = 0$, Proposition 5.32 is no longer true if f is continuous in x and non-properly quasimonotone since Corollary 5.3 ensures in this case that $M(f,K) \subsetneq M_L(f,K)$. We bring a more precision in Corollary 5.36 below.

Corollary 5.36. *Assume that* $(\mathbb{H}_0)$ *and* $(\mathbb{H}_i)$ *for* $i \in \{2,4\}$ *are verified with* $\varphi = 0$, f *has the local upper sign property in* x *and (67) is satisfied and moreover either* f *is non-continuous in* x *or* f *is properly quasimonotone. Then* $\emptyset \neq M(f,K) = M_L^0(f,K) = S(f,K)$.

Proof. Consequence of Corollary 5.34. □

Presently, we deal with extensions to the coercive case. So, in the following K will be a convex and (possibly) unbounded subset of X. In this respect, we shall introduce the following coercivity conditions (see for example [19] and [14]):

(C_1) There exists a compact and convex subset B of K such that for all $x \in K \backslash B$, $\exists y \in B,\ f(y,x) > \varphi(y,x)$.

(C_2) There exists a compact and convex subset B of K such that for all $x \in K \backslash B$, $\exists y \in B,\ f(x,y) < \varphi(x,y)$.

Notation: For any $D \subset K$, the restriction of f on $D \times D$, will be denoted again by f.

The extension of φ-Minty global to the coercive framework is stated in the following

Proposition 5.37. *Let* K *be a closed (unbounded) subset of* X. *Under the assumption* $(\mathbb{H}_2)$, *if for every compact and convex subset* $D \subset K$, $\varphi - MEP(f,D)$ *admits a solution and* (C_1) *is satisfied then* $MEP(f,K)$ *admits also a solution.*

Remark 5.38. The result of Proposition 5.37 remains true if (C_1) is satisfied with a weakly compact B and instead of $(\mathbb{H}_2)$, the set $F_\varphi(x) = \{y \in K|\ \ f(x,y) \leqslant \varphi(x,y)\}$ is weakly compactly closed for all $x \in K$ i.e., $F_\varphi(x) \cap B$ is weakly closed for every weakly compact $B \subset K$.

Next, we state a unified Theorem for the nonemptiness of $S(f,K)$.

Theorem 5.39. *Let* $\mu \geq 0$ *and* φ *the bivariate function defined by* $\varphi(x,y) = \mu\|x-y\|^2$. *Assume that* K *is a convex subset of* X *and* $f : K \times K \longrightarrow \mathbb{R}$ *satisfies* $(\mathbb{H}_2)$. *Then the following assertions are satisfied:*

(1) $M_w^\mu(f,K)$ *is nonempty and* $M_w^\mu(f,K) \subset B$ *if* (C_1) *and one of the following conditions are satisfied:*
  (i) $(\mathbb{H}_3)$;
  (ii) $(\mathbb{H}_i)$ *for* $i \in \{1,4\}$;
  (iii) $(\mathbb{H}_i)$ *for* $i \in \{0,5\}$.
  (iv) $(\mathbb{H}_i)$ *for* $i \in \{1,7\}$;

  If, in addition, f *has the global* μ*-upper sign property in* x, *then* $\emptyset \neq M_w^\mu(f,K) \subset S(f,K)$.

(2) *If* $(\mathbb{H}_6)$ *holds,* f *has the local* μ*-upper sign property but non-continuous in* x *and the condition (67) is verified, then* $\emptyset \neq M_L^\mu(f,K) \subset S(f,K)$*. If in addition,* f *is strictly quasiconvex and satisfies* $(\mathbb{H}_0)$ *then* $M^\mu(f,K) = S(f,K) = M_L^\mu(f,K)$*. If moreover* f *satisfies* (C_1) *or* (C_2) *then* $S(f,K) \subset B$*.*

Proof. For (2), let D be a convex and compact subset of K. Evidently, for any $i \in \{0,\ldots,5\}$, the restriction of f on $D \times D$ verifies $(\mathbb{H}_i)$ if f is so. Then, by Theorem 5.26, $M_w^\mu(f,D)$ is nonempty for the cases (i), (ii) (or (iv)) and (iii). Thus, the assertion (i) of Proposition 5.37 ensures that $M^\mu(f,K)$ is nonempty. The required conclusion is then achieved by the assertion (i) of Lemma 5.30. The inclusion $S(f,K) \subset B$ is immediate from the condition (C_1).

Concerning (2), by Theorem 5.26 we have $M_L^\mu(f,K) \neq \emptyset$. The additional required conclusion is achieved by Proposition 5.32. Finally, the inclusion $S(f,K) \subset B$ is quickly obtained from (C_1) or (C_2). □

Corollary 5.40. *Assume that the conditions of Theorem* 5.39 *are satisfied and moreover* f *is* φ*-strongly quasiconvex. Then for all* $\lambda \in]0,\frac{1}{4}[$, $S(f,K) = S^{\lambda\varphi}(f,K)$ *i.e., standard equilibrium points of* f *coincide with* λ*-eigenvalue equilibrium ones.*

Proof. Direct from Proposition 4.9. □

Remark 5.41. (1) Thanks to Remarks 5.27 and 5.38, we obtain the same conclusion in the (unified) Theorem above if the subset B in (C_1) is weakly compact and instead of $(\mathbb{H}_2)$ the subset $F_\varphi(x)$ is weakly compactly closed for all $x \in K$.

(2) The case of $(\mathbb{H}_6)$:

(a) Neither compactness on K nor coercivity on f are necessary. Notice that in this case, the requirement that f must not be properly relaxed quasimonotone is fundamental; see the counterexample after Corollary 1 in [18].

(b) In Theorem 5.39 (2), f is supposed to be non-continuous in x because f is non-properly quasimonotone, otherwise the equality of the sets of solutions in the assertion (2) of the conclusion in this Theorem is no longer true in view of Corollary 5.36.

(c) In Theorem 5.39 (1), given that $M_w^\mu(f,K) \subset S(f,K)$, the condition (C_1) can be replaced by (C_2) (since in this case (C_2) implies that $S(f,K) \subset B$).

(d) Covers the result in [18, Corollary 1, pp 1225].

6. Parametric stability

In this paragraph, inspired by the technic introduced in [1] for quantitative stability of strongly monotone equilibrium problems, we first show that strong Minty solutions provided in Corollary 5.18 have 'nice' stability properties. For any nonempty subset A of X and any $x \in X$, $d(x,A) = \inf\{\|x-y\|,\ y \in A\}$ will stand for the distance from x to A, and if B is another subset of X, $e(A,B)$

denotes the excess of A on B given by $e(A, B) = \sup\{d(a, B) : \ a \in A\}$. Finally, the Hausdorff distance between two bounded subsets A and B of X is given by

$$\begin{aligned} haus(A, B) &= \max\{e(A, B), e(B, A)\} \\ &= \inf\{\nu \geq 0 \ : A \subset B + \nu B_X, B \subset A + \nu B_X\}. \end{aligned}$$

Let Λ be a linear normed space (space of a parameter p). For every $p \in \Lambda$, consider a bifunction $f_p : K \times K \to \mathbb{R}$. We denote by $S(p)$ the set of solutions to the (perturbed) problem $EP(f_p, K)$ and by $M_s^\mu(p)$ the set of μ-strong Minty equilibrium points (see v) in Definition 4.4) of f_p under the constraints set K. Recall that when $\mu = 0$, we simply write $M^0(p)$ to stand for $M(f_p, K)$.

The following assumption will be needed: $(\mathcal{A}_1)$ there exist $\theta > 0$, $1 \geqslant \gamma > 0$ and $2 > \delta > 0$ and a neighborhood N_3 of $\bar{p}$ such that: for all $x, y \in K$ and all $p, p' \in N_3$ we have

$$|f_p(x, y) - f_{p'}(x, y)| \leqslant \theta \left\|p - p'\right\|^\gamma \left\|y - x\right\|^\delta .$$

Example 6.1. Consider a family of operators $T(., p) : X \to X^*$ and their associated bifunctions $f_p(x, y) = \langle T(x, p), y - x\rangle$, $p \in \Lambda$. If, uniformly in x, the map $p \mapsto T(x, p)$ is locally Lipschitz and/or γ-Hölder, with rank of Lipschitz and/or Hölder equal to some $l > 0$. Then f_p satisfies $(\mathcal{A}_1)$ with $\theta = l$ and $\delta = 1$. For more discussion on this assumption see [1].

Let us now state our first main result on quantitative stability results for perturbed μ-strong Minty solutions around a reference value $\overline{p}$ of the parameter p for which we write $f_{\overline{p}} := \overline{f}$.

Theorem 6.2. *Let $\mu \geq 0$. Assume that $(\mathcal{A}_1)$ holds, $M^0(p)$ and $M_s^\mu(p)$ are nonempty for all $p \in N_3$ and moreover the following conditions are satisfied:*

(i) $\overline{f}$ *has the global upper sign property in x;*
(ii) $\theta = \mu^2$.

Then the corresponding strong μ-Minty solutions satisfy the following Hölder estimate:

$$haus(M_s^\mu(p), M^0(\overline{p})) \leqslant \kappa \|p - \overline{p}\|^{\tilde{\gamma}}, \ \ \forall p \in N_3. \tag{76}$$

In particular,

$$haus(M_s^\mu(p), M_s^u(\overline{p})) \leqslant \kappa \|p - \overline{p}\|^{\tilde{\gamma}}, \ \ \forall p \in N_3. \tag{77}$$

Here, $\kappa = \mu^{\tilde{\gamma}}$ with $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Proof. Let $x \in M_s^\mu(p)$ and $\overline{x} \in M^0(\overline{p})$. Since $f_{\overline{p}}$ has the upper sign property in x, from Lemma 5.30, we obtain that $\overline{x} \in M^\mu(\overline{p}) \subset S(\overline{p})$. Thus

$$f_{\overline{p}}(\overline{x}, x) \geqslant 0. \tag{78}$$

Now, since $x \in M_s^\mu(p)$, by definition of M_s^μ we have

$$f_p(y,x) \leqslant -\mu\|x-y\|^2, \ \forall y \in K. \tag{79}$$

In particular, with $y = \overline{x}$ in (79) we obtain

$$f_p(\overline{x},x) \leqslant -\mu\|x-\overline{x}\|^2 \quad \text{or else} \quad -f_p(\overline{x},x) \geqslant \mu\|x-\overline{x}\|^2. \tag{80}$$

Therefore, the sum of (78) and (80) lead to

$$f_{\overline{p}}(\overline{x},x) - f_p(\overline{x},x) \geqslant \mu\|x-\overline{x}\|^2, \tag{81}$$

which, thanks to the assumption $(\mathcal{A}_1)$ and (ii), yields

$$\left\|x-\overline{x}\right\| \leqslant \mu^{\frac{1}{2-\delta}} \left\|p-\overline{p}\right\|^{\frac{\gamma}{2-\delta}}. \tag{82}$$

Since x and $\overline{x}$ are arbitrarily taken in $M_s^\mu(p)$ and $M^0(\overline{p})$, respectively, we get

$$\inf_{u\in M_s^\mu(p)} \|u-\overline{x}\| = d(\overline{x}, M_s^\mu(p)) \leqslant \|x-\overline{x}\| \leqslant \mu^{\frac{1}{2-\delta}} \|p-\overline{p}\Big\|^{\frac{\gamma}{2-\delta}}.$$

Now, pass to supremum over $\overline{x}$ in the latter and see that

$$e(M^0(\overline{p}), M_s^\mu(p)) = \sup_{\overline{x}\in M^0(\overline{p})} d(\overline{x}, M_s^\mu(p)) \leqslant \mu^{\frac{1}{2-\delta}} \|p-\overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{83}$$

Similarly, we also have

$$e(M_s^\mu(p), M^0(\overline{p})) = \sup_{x\in M_s^\mu(p)} d(x, M^0(\overline{p}) \leqslant \mu^{\frac{1}{2-\delta}} \left\|p-\overline{p}\right\|^{\frac{\gamma}{2-\delta}}. \tag{84}$$

From (83) and (84) we immediately obtain

$$\begin{aligned} haus(M_s^\mu(p), M^0(\overline{p})) &= \max\Big(e(M^0(\overline{p}), M_s^\mu(p)), e(M_s^\mu(p), M^0(\overline{p})\Big) \\ &\leqslant \mu^{\frac{1}{2-\delta}} \left\|p-\overline{p}\right\|^{\frac{\gamma}{2-\delta}}. \end{aligned} \tag{85}$$

Finally, (77) is obtained by the same arguments as in (85) since $M_s^\mu(\overline{p}) \subset M^0(\overline{p})$. This completes the proof. □

We now derive from Theorem 6.2 the following variant of step I in [1, Theorem 2.2.1].

Corollary 6.3. *Assume that the assumptions of Theorem* 6.2 *are satisfied and moreover the family $(f_p)_p$ is uniformly (in p) μ-strongly monotone. Then, for all $p \in N_3$, the solutions sets $M_s^\mu(p)$, $M_w^\mu(p)$ and $S(f_p, K)$ coincide and reduce to a singleton $\{x_p\}$ and the following estimate is satisfied*

$$\|x_p - x_{\overline{p}}\| \leqslant \kappa\|p-\overline{p}\|^{\tilde{\gamma}}, \ \forall p \in N_3.$$

Here, $\kappa = \mu^{\tilde{\gamma}}$ with $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Proof. By (iii) of Lemma 5.30, $M_s^\mu(p) = M_w^\mu(p) = S(f_p, K)$ and these sets reduce to a singleton $\{x_p\}$. Then, the required conclusion descends immediately from Theorem 6.2. □

We end this section by a second result in concern with perturbation on the constraints set for the class of strongly quasimonotone-like bifunctions. So, given another linear normed space M whose norm is again denoted by $\|.\|$, assume that the constraint K depends on a parameter $\lambda \in M$, i.e., $K : M \rightrightarrows X$, $\lambda \longmapsto K_\lambda := K(\lambda)$, while f is a fixed bivariate function with a relaxation of the domain to the whole space X i.e., $f : X \times X \to \mathbb{R}$. We denote the set of solutions to $EP(f, K_\lambda)$ by S_λ for any $\lambda \in M$, and consider the following assumptions:

($\mathcal{A}_2$) $K(\lambda) = K_\lambda$ is Lipschitz or Hölder continuous at $\bar{\lambda}$, that is, for a neighborhood $\overline{M}$ of $\bar{\lambda}$ and some constants $L > 0$ and $1 \geqslant \xi > 0$, we have

$$K_\lambda \subset K_{\lambda'} + L \|\lambda - \lambda'\|^\xi B_X, \quad \text{for all} \quad \lambda, \lambda' \in M;$$

($\mathcal{A}_3$) for some $\beta > 0$, there exists $R \geq 0$ such that for all $x, y, y' \in X$, one has

$$|f(x,y) - f(x,y')| \leqslant R \|y - y'\|^\beta .$$

Examples 6.4. (1) An example of a map $K(\lambda)$ satisfying the condition ($\mathcal{A}_2$) is (even with a more general property) stated in [5, Proposition 1] in the context of elastic traffic network models.

(2) Take a similar example to the one given above for ($\mathcal{A}_1$) where $f(x,y) = \langle T(x), y - x\rangle$. If T is bounded i.e., for some $c > 0$, $\sup\limits_{x \in X} \|T(x)\| \leq c$, then f verifies ($\mathcal{A}_3$) with $\beta = 1$ and $R = c$.

Theorem 6.5. *Assume that the following conditions are satisfied:*

(i) ($\mathcal{A}_2$) *holds and for a convex subset $D \subset X$, for all $\lambda \in \overline{M}$, $K_\lambda \subset D$;*

(ii) *for some $\omega > 0$, f is ω-strongly quasimonotone-like on $D \times D$;*

(iii) *f is α-strongly quasiconvex-like for some $\alpha \leqslant \omega$;*

(iv) ($\mathcal{A}_3$) *is verified with $\beta = 1$.*

Then, for all $\lambda \in \overline{M}$, S_λ is reduced at most to a singleton $\{x_\lambda\}$, and for all $\lambda, \lambda' \in \overline{M}$, we have

$$\|x_\lambda - x_{\lambda'}\| \leqslant \varrho \Big\| \lambda - \lambda' \Big\|^\xi . \tag{86}$$

Here, $\varrho = L \max\{\frac{4R}{\omega\alpha} + 1, \frac{R}{\omega}\}$.

Proof. Let $\lambda, \lambda' \in \overline{M}$. Put $\varphi(x,y) = \omega\|x - y\|$, for $x, y \in K$. Since f is ω-strongly quasimonotone-like and $\alpha \leqslant \omega$, it follows that f is $\lambda\alpha$-quasimonotone-like for all $\lambda \in]0, \frac{1}{4}[$, then the uniqueness of the solution to $EP(f, K_\lambda)$ is ensured by Lemma 4.11. Set $S_\lambda = \{x\}$ and $S_{\lambda'} = \{x'\}$ such that $x \neq x'$.

Suppose $f(x,x')>0$, then by the assumptions (ii) and $(\mathcal{A}_2)$ there exists $u'\in K_{\lambda'}$ such that

$$f(x',x)\leqslant -\omega\|x-x'\| \quad \text{and} \quad \|x-u'\|\leqslant L\,\|\lambda-\lambda'\|^{\xi}. \tag{87}$$

Then, since $x'\in S_{\lambda'}$, it follows that

$$\omega\,\|x-x'\|\leqslant f(x',u')-f(x',x). \tag{88}$$

Now involve assumption $(\mathcal{A}_3)$ in (88) we get

$$\omega\|x-x'\|\leqslant LR\|\lambda-\lambda'\|^{\xi}. \tag{89}$$

Thus,

$$\|x-x'\|\leqslant \frac{LR}{\omega}\Big\|\lambda-\lambda'\Big\|^{\xi}\leqslant \varrho\|\lambda-\lambda'\Big\|^{\xi}. \tag{90}$$

Otherwise suppose $f(x,x')\leqslant 0$. By assumption $(\mathcal{A}_2)$, there exists $u\in K_{\lambda}$ such that

$$\|x'-u\|\leqslant L\,\|\lambda-\lambda'\|^{\xi}. \tag{91}$$

Hence, assumption (iii) and Proposition 4.9 yield

$$f(x,u)-f(x,x')\geqslant \frac{\omega\alpha}{4}\|x-u\|. \tag{92}$$

Next, by assumption $(\mathcal{A}_3)$ and (92), it results that

$$R\|x'-u\|\geqslant \frac{\omega\alpha}{4}\|x-u\| \text{ or else } \|x-u\|\leqslant \frac{4R}{\omega\alpha}\|x'-u\|. \tag{93}$$

Then, thanks to (91) and (93), we conclude that

$$\begin{aligned}\|x-x'\|&\leqslant \|x-u\|+\|x'-u\|\leqslant \Big(1+\frac{4R}{\omega\alpha}\Big)\Big\|x'-u\Big\|\\ &\leqslant L\Big(1+\frac{4R}{\omega\alpha}\Big)\Big\|\lambda-\lambda'\Big\|^{\xi}\leqslant \varrho\Big\|\lambda-\lambda'\Big\|^{\xi}.\end{aligned}$$

This ends the proof. □

7. Quasi equilibrium problems

In this section and also in the sequel of the paper till otherwise is specified, X still is a normed space equipped with the strong topology. To define the problem, let C be a closed convex in X, $f: C\times C\longrightarrow \mathbb{R}$ be a real-valued function and $K: C\longrightarrow 2^C$ be a closed set-valued map with convex and nonempty values.

The strong convergence in X will be denoted by $\to$ and the weak one by $\rightharpoonup$. The quasi-equilibrium problem we consider consists on finding $\overline{x} \in C$ such that $\overline{x} \in K(\overline{x})$ and

$$QEP(f, K) : \quad f(\overline{x}, y) \geq 0, \forall y \in K(\overline{x}).$$

When K has a constant value D with D a closed convex subset of X i.e., $K(x) = D$ for all $x \in C$, then $QEP(f, K)$ collapses into the equilibrium problem $EP(f, D)$. Introduce now the variational selection $S : C \rightrightarrows C$ defined for each $x \in C$ by

$$S(x) := \{y \in K(x) \mid f(y, z) \geq 0, \ \forall z \in K(x)\}, \tag{94}$$

and see that the solutions to $QEP(f, K)$ coincide with the fixed points of S. However, in our case this variational selection-fixed point scheme will be used rather for Minty quasi-equilibrium: find $\overline{x} \in C$ such that $\overline{x} \in K(\overline{x})$ and

$$MQEP(f, K) : \quad f(y, \overline{x}) \leq 0, \ \forall y \in K(\overline{x}).$$

Then in the second step we recapture the solutions to $QEP(f, K)$ from those of $MQEP(f, K)$ under the upper-sign property condition.

Let us begin with the closeness of the graph of the solutions map of the generated family of Minty equilibrium problems.

Proposition 7.1. *Assume that the following conditions hold true:*

(h_1) *The map K is closed and lower semicontinuous.*

(h_2) *For all $x, y \in C$, for all $x_n \rightharpoonup x$ and $y_n \to y$, there exists a subsequence (n_k) such that*

$$\liminf_k f(x_{n_k}, y_{n_k}) \leq 0 \implies f(x, y) \leq 0.$$

Then the Minty solution map $M(f, K(.))$ is closed with respect to strong topology of $X \times X$.

Remark 7.2. In particular, in Proposition 7.1, for every $x \in C$, $M(f, K(x))$ is closed. This closedeness can also be directly obtained if ($\mathbb{H}_2$) is satisfied, which is the case here since (h_2) $\implies$ ($\mathbb{H}_2$).

Proof of Proposition 7.1. Let us prove that the graph of the set valued map $M(f, K(.))$ is closed. To do that, let (x_n) and (u_n) be two sequences in C such that $u_n \in M(f, K(x_n)$ and (x_n, u_n) strongly converges to a point (x, u). Of course, $u_n \in K(x_n)$, then by the assumption (h_1) it follows that $u \in K(x)$. Now, let $y \in K(x)$. Again by (h_1), there exists $y_n \in K(x_n)$ such that $y_n \to y$. But $u_n \in M(f, K(x_n))$, thus $f(y_n, u_n) \leq 0$ which leads to $\liminf_n f(y_n, u_n) \leq 0$. Then $f(y, u) \leq 0$, thanks to (h_2), which means that $u \in M(f, K(x))$. This completes the proof. $\square$

Remark 7.3. If X is a Banach (resp. reflexive Banach) space, by assuming in Proposition 7.1 that K is weakly closed and weakly lower semicontinuous (resp. for all $x_n \rightharpoonup x$, $K(x_n)$ Mosco-converges to $K(x)$) instead of assumption (h_1), with slight adjustments we obtain that the map $M(f, K(.))$ is weakly closed.

We have to introduce by now the coercivity adapted to the quasi-variational case:

(C_3) There exists a strongly compact and convex subset B of C such that for all $x \in C$, $K(x) \cap B \neq \emptyset$, for all $z \in K(x) \setminus B$, there exists $y \in K(x) \cap B$, $f(y, z) > 0$.

Remark 7.4. Variants of (C_3) are considered for similar problems in [41, Theorem 3] and also in [26, Theorems 8 and 9].

From Proposition 7.1 we derive the following:

Corollary 7.5. *Assume that* (C_3) *and the assumptions of Proposition* 7.1 *hold true. Then the (global) Minty solution map* $M(f, K(.))$ *is strongly upper semicontinuous.*

Proof. Since f satisfies (C_3), for every $x \in C$, the restriction of f on $K(x) \times K(x)$, also denoted by f, verifies the condition (C_1) with $\varphi = 0$ and $B_x := K(x) \cap B$ plays the role of B of (C_1). Then, for every $x \in C$, whenever the set of solutions to $EP(f, K(x))$ is nonempty it is included in the compact B (defined in (C_3)). Thus, the solution map $M(f, K(.))$ ranges in B. Therefore, the required conclusion is direct from Proposition 7.1 and Theorem 2.2. □

Remark 7.6. Assume that the set B in the condition (C_3) is weakly compact. If X is a Banach (resp. reflexive Banach) space, by assuming that K is weakly closed and weakly lower semicontinuous (resp. for all $x_n \rightharpoonup x$, $K(x_n)$ Mosco-converges to $K(x)$) instead of assumption (h_1), we obtain that the map $M(f, K(.))$ is weakly upper semicontinuous.

Proposition 7.7. *Let D be a closed and convex set of X and let $f : D \times D \longrightarrow \mathbb{R}$ be a bifunction. Then, the set $M(f, D)$ of global Minty solutions to $EP(f, D)$ is convex whenever f is quasiconvex in y (or semistrictly quasiconvex).*

Proof. The convexity of $M(f, D)$ comes immediately from quasiconvexity or semistrict quasiconvexity of f in y. □

Remark 7.8. If f is semistrictly quasiconcave and continuous in x and $f(x, x) = 0$ for all $x \in D$ then the solution set $M_L(f, D)$ of local Minty solutions is also convex (thanks to Proposition 4.3).

The following is a result on the existence of quasi-equilibrium points with a slight modification in the assumption ($\mathbb{H}_6$):

($\mathbb{H}'_6$) f is quasimonotone, not properly quasimonotone and non-continuous in x, and $f(x, .)$ is strictly quasiconvex for all $x \in K$.

Theorem 7.9. *Let C be a convex subset of X, $K : C \longrightarrow 2^C$ be a closed set-valued map with convex and nonempty values and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction. Assume that:*

(i) (C_3) *and* (h_i) *for* $i \in \{1, 2\}$ *hold true;*
(ii) *one of the following conditions is satisfied:*

(a) $(\mathbb{H}_3)$ *and* f *is quasiconvex in* y*;*
(b) $(\mathbb{H}_i)$ *for* $i \in \{1, 5\}$ *and* f *is quasiconvex in* y*;*
(c) $(\mathbb{H}_i)$ *for* $i \in \{1, 4\}$*;*
(d) $(\mathbb{H}_i)$ *for* $i \in \{1, 7\}$;
(e) $(\mathbb{H}_0)$ *and* $(\mathbb{H}'_6)$.

Then $MQEP(f, K)$ *admits at least one solution. Moreover, the set of solutions to* $MQEP(f, K)$ *is included in* B*. If in addition* f *has the upper sign property in* x *then* $QEP(f, K)$ *has also a solution.*

Proof. We have already observed in Remark 7.2 that (h_2) implies $(\mathbb{H}_2)$. By Theorem 5.39, Proposition 7.7 and (1) of Remark 7.2, for all $x \in C$, the set $M(f, K(x))$ is nonempty convex and closed. Moreover, in view of (C_3), for all $x \in C$, the set $M(f, K(x)) \subset B$ (B being the compact subset provided by the assumption (C_3)). On the other hand, Corollary 7.5 implies that the map $M(f, K(.))$ is upper semicontinuous. Hence, Theorem 2.1 (Himmelberg Theorem) ensures that $M(f, K(.))$ admits a fixed point $\overline{x}$ in $B \subset C$. Of course, $\overline{x}$ is a solution to $MQEP(f, K)$. Then, using the upper sign property of f in x, $\overline{x}$ is equally a solution to $EP(f, K(\overline{x}))$, which means that $\overline{x}$ is a solution to $QEP(f, K)$, ending the proof. □

Remark 7.10. The technique of variational selection-fixed point scheme is inspired by Joly and Mosco [34] on quasi-variational inequalities. It has been subsequently exploited in the literature as in [10] for quasi-variational inequalities under compactness. Here, we consider the case of coercive quasi-equilibrium problem and use Himmelberg fixed points instead of Kakutani's ones.

8. Set-valued variational and quasi-variational inequalities

The classical example of equilibrium problem is the variational inequality problem, say (VI), which we recall as follows. Given a set-valued map $T : K \rightrightarrows X^*$, we seek at $x \in K$ such that there exists $x^* \in T(x)$ with

$$VI(T, K) : \quad \langle x^*, y - x \rangle \geq 0, \quad \forall y \in K.$$

The associated Minty problem to $VI(T, K)$ is as follows: find $x \in K$ such that:

$$MVI(T, K) : \quad \langle y^*, x - y \rangle \leq 0 \quad \forall y \in K, \forall y^* \in T(y).$$

The formulation of set-valued quasi-variational inequalities is as follows: Find $x \in K(x)$ such that there exists $x^* \in T(x)$ and

$$SQVI(T, K) : \quad \langle x^*, y - x \rangle \geq 0, \quad \forall y \in K(x).$$

Here, $T : C \rightrightarrows X^*$ and $K : C \rightrightarrows C$ are two set-valued maps.

The corresponding Minty formulation to $SQVI(T, K)$ is as follows: Find $x \in K(x)$ such that

$$MQVI(T, K) : \quad \langle y^*, x - y \rangle \leq 0 \quad \forall y \in K(x), \ \forall y^* \in T(y).$$

Notation: The set of solutions to $SQVI(T, K)$ (resp. $MQVI(T, K)$) will be denoted by $SQ(T, K)$ (resp. $MQ(T, K)$).

Definition 8.1. ([3]) If the point x in $VI(T, K)$ is such that $x^* \in T(x)\backslash\{0\}$ then x is called a *star solution* to $VI(T, K)$.

Here $T : K \rightrightarrows X^*$ is a set valued operator which will always be supposed to have nonempty weakly-*compact and convex values.

Now denote by f_T the associated bifunction to T, defined by

$$f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle,$$

and consider the following

Lemma 8.2. *Assume that T is weakly-*compact and convex-valued set-valued map. Then, the following assertions hold true.*

(a) *The Minty problems $MVI(T, K)$ and $MEP(f_T, K)$ are equivalent i.e., any solution to $MVI(T, K)$ is a solution to $MEP(f_T, K)$ and vice versa.*

(b) *The variational problems $VI(T, K)$ and $EP(f_T, K)$ are equivalent.*

Proof. (a) is immediate while (b) is a consequence of Sion's Theorem since T is weakly-*compact and convex-valued set-valued map (see also [18, Section 1.1 pp. 126]). □

Remark 8.3. Suppose that $\varphi = 0$. Then the bifunction f_T:

(1) satisfies automatically the assumptions $(\mathbb{H}_0)$ and $(\mathbb{H}_1)$;

(2) satisfies also the assumption $(\mathbb{H}_2)$ since $f_T(x, .)$ is lower semicontinuous as a pointwise supremum of the family of the lower semicontinuous functions $y \longmapsto \langle x^*, y - x \rangle$, $x^* \in T(x)$ (see for instance [39, 1.26 Proposition]). Moreover, in view of the convexity of $f_T(x, .)$, $(\mathbb{H}_2)$ is also satisfied by f_T with respect to the weak topology;

(3) is quasiconvex (it is even convex) in y;

(4) fulfills the condition (67) in view of Remark 5.33 and the previous point;

(5) is properly quasimonotone if and only if T is properly-quasimonotone (i.e., for all $n \in \mathbb{N}$, $x_1, x_2, ..., x_n \in K$ and $\overline{x} \in Conv(x_1, x_2, ..., x_n)$ there exists i such that $\langle x_i^*, \overline{x} - x_i \rangle \leq 0$, $\forall x_i^* \in T(x_i)$);

(6) is quasimonotone if and only if T is so (i.e., for all $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$ it results that: $\langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, x - y \rangle \leq 0$).

The following definitions are needed for our application in this section.

Definition 8.4. ([31]) An operator $T \colon K \rightrightarrows X^*$ is said to have the *(global) upper sign continuity* at a point $x \in K$ if for every $y \in K$, with $z_t = (1 - t)x + ty$ for $t \in]0, 1[$, the following implication holds true:

$$\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x \rangle \geq 0, \ \forall t \in]0, 1[\quad \Longrightarrow \quad \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0. \tag{95}$$

Definition 8.5. ([18]) An operator $T : K \rightrightarrows X^*$ is said to have the *local upper sign continuity* at a point $x \in K$ if there exists $r > 0$ such that for every $y \in K \cap B(x, r)$ the implication (95) holds true.

Definition 8.6. A set-valued map $T : X \rightrightarrows X^*$ is said to be *locally sub-upper sign-continuous* at a point $x \in X$ if there exists a convex neighborhood $\mathcal{V}$ of x and an upper sign-continuous submap $\Phi_x : \mathcal{V} \rightrightarrows X^*$, with nonempty convex weakly-*compact values such that $\Phi_x(v) \subset T(v)\backslash\{0\}$, for every $v \in \mathcal{V}$.

Remark 8.7. (1) Local sub-upper sign-continuity is called in [3, 10] local upper sign-continuity, here we make the difference with Definition 8.5.

(2) The local upper sign continuity is known in the literature to play an important role in the study of normal operator in the context of quasiconvex programming, see [3, Definition 4.1] and [9] as well as the references therein.

The interest of local sub-upper sign-continuity is that it enables us to obtain star solutions also in the quasi-variational case in the sense of the following:

Definition 8.8. A point $x \in C$ is called a *star solution* to $SQVI(T, K)$ if $x \in K(x)$ and there exists $x^* \in T(x)\backslash\{0\}$ such that

$$SQVI^*(T, K) : \quad \langle x^*, y - x\rangle \geq 0, \qquad \forall y \in K(x).$$

The set of star solutions to $SQVI(T, K)$ will be denoted by $SQ^*(T, K)$.

Star solutions can be regarded as nontrivial solutions to $SQVI(T, K)$), they enjoy under mild conditions very nice stability properties, see [2, 3], and moreover they characterize optimality conditions in quasiconvex programming.

Lemma 8.9. *Assume that T is locally sub-upper sign-continuous, then*

$$MQ(T, K) \subseteq SQ^*(T, K).$$

Proof. Let $\overline{x} \in C$ be a solution to $MQVI(T, K)$. Then, $\overline{x}$ is a fixed point for K and, by the meantime, $\overline{x}$ is a solution to $MVI(T, K(\overline{x}))$. On the other hand, since T is locally sub-upper sign-continuous it follows from [9, (i) Lemma 3. 1] that $\overline{x}$ is a star solution to $VI(T, K(x))$, and hence $\overline{x}$ is a solution to $SQVI^*(T, K)$. □

Lemma 8.10. *Let $T : K \rightrightarrows X^*$ be a set valued map. Then*

(i) *T has the global upper sign continuity at x if and only if the function $f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle$ has the global upper sign property at x.*

(ii) *T has the local upper sign continuity at x if and only if the function $f_T(x, y)$ has the local upper sign property at x.*

Proof. Assertion (i) is immediate. For (ii), see [18, Example 3]. Indeed, as observed there, the upper sign continuity for a set-valued operator T can be equivalently reformulated as follows: there exists $\mathcal{V}_x$ a convex neighborhood of x such that for every $y \in K \cap \mathcal{V}_x$ the implication (95) is true. Now, let f_T be the equilibrium bifunction associated to T and $y \in K \cap \mathcal{V}_x$. Then if we take $z_t = (1 - t)x + ty$ with $t \in]0, 1[$ we see that

$$f_T(z_t, x) = \sup_{z_t^* \in T(z_t)} \langle z_t^*, x - z_t\rangle = -t \inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x\rangle$$

and the inequality $f_T(z_t, x) \leq 0$ coincides with $\inf_{z_t^* \in T(z_t)} \langle z_t^*, y - x\rangle \geq 0$. Hence, the upper sign property of f_T is equivalent to the upper sign continuity of T. □

To achieve our aim we need a suitable coercivity condition on the operator T:

(C_4) There exists a strongly compact and convex subset B of K such that $\inf_{y \in B} \inf_{y^* \in T(y)} \left\langle y^*, y - x\right\rangle < 0$ for all $x \in K \setminus B$.

Lemma 8.11. *Let K be a convex and closed subset of X and $T : K \rightrightarrows X^*$ be a set valued map with nonempty and weakly-*compact and convex values. Then T satisfies (C_4) if, and only if f_T satisfies (C_1).*

Proof. It suffice to observe that for all $x \in K \setminus B$ we have

$$\begin{aligned} \inf_{y \in B} \inf_{y^* \in T(y)} < y^*, y - x > < 0 \quad &\Longleftrightarrow \quad \exists y \in B, \quad \inf_{y^* \in T(y)} < y^*, y - x > < 0 \\ &\Longleftrightarrow \quad \exists y \in B, \quad \sup_{y^* \in T(y)} < y^*, x - y > > 0 \\ &\Longleftrightarrow \quad \exists y \in B, \quad f_T(y, x) > 0. \end{aligned}$$ □

We derive now from Theorem 5.39 the following result on the variational case:

Corollary 8.12. *Let K be a convex subset of X and $T : K \rightrightarrows X^*$ be a set valued map with nonempty and weakly-* compact and convex values. Then $VI(T, K)$ has at least one solution in one of the two following cases:*

(1) *T is quasimonotone and not properly quasimonotone and has the local upper sign continuity;*
(2) *T satisfies (C_4) and T is properly quasimonotone and has the global upper sign continuity.*

If in addition T is locally sub-upper sign-continuous then $VI(T, K)$ admits a star solution.

Proof. The bifunction $f_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle$ satisfies the condition (67) (see Remark 8.3 (4)). The hypothesis ($\mathbb{H}_2$) is also satisfied in view of 8.3 (2). In addition, from Lemma 8.10, T is (locally or globally) upper sign continuous at x if and only if f_T has the local or global upper sign property in x. Then, two cases are possible:

- *First case*: If T is quasimonotone and not properly quasimonotone, then f_T satisfies the condition ($\mathbb{H}_6$).
- *Second case*: If T is properly quasimonotone, then f_T satisfies the condition ($\mathbb{H}_3$).

In both cases, $S(f_T, K)$ is not empty in view of Theorem 5.39, Lemma 8.11 and Remark 8.3, which means that $V(T, K)$ has at least one solution. The last point of the required conclusion comes from Lemma 8.9. □

On the basis of Theorem 7.9, we next establish the existence of solutions to $SQVI(T, K)$ for which we organize the proof by the following Lemmas:

Lemma 8.13. *Assume that T is weakly-*compact and convex-valued set-valued map. Then, the following assertions hold true.*

(a) *The Minty problems $MQVI(T, K)$ and $MQEP(f_T, K)$ are equivalent.*

(b) *The problems $SQVI(T, K)$ and $QEP(f_T, K)$ are equivalent.*

Proof. Let $\overline{x} \in C$ be a fixed point of K. Then by Lemma 8.2, $\overline{x}$ is a solution to $MQVI(T, K)$ (resp. $SQVI(T, K)$) if, and only if $\overline{x}$ is a solution to $MQEP(f_T, K)$ (resp. $QEP(f_T, K)$). □

The following further coercivity condition will equally be needed for our discussion:

(C_5) There exists a strongly compact and convex subset B of C such that for all $x \in C$, $K(x) \cap B \neq \emptyset$, for all $z \in K(x) \setminus B$,

$$\inf_{y \in B \cap K(x)} \inf_{y^* \in T(y)} \langle y^*, y - z \rangle < 0.$$

Lemma 8.14. *Let C be a convex and closed subset of X, let $T : C \rightrightarrows X^*$ be a set valued map with nonempty, weakly-*compact and convex values and let $K : C \rightrightarrows C$ be a set valued map with nonempty and convex values. Then T satisfies the condition* (C_5) *if, and only if f_T satisfies* (C_3).

Proof. Immediate. □

Theorem 8.15. *Let C be a convex and closed subset of X, $T : C \rightrightarrows X^*$ be a set valued map with nonempty and weakly-*compact and convex values and $K : C \rightrightarrows C$ be a set valued map with nonempty and convex values. Suppose that the following assumptions are satisfied:*

(i) (h_1)*, i.e, K is closed and lower semicontinuous;*

(ii) (c_1): *For all $x, y \in C$, for all $x_n \rightharpoonup x$, for all $y_n \longrightarrow y$,*

$$\liminf_{n \to +\infty} \sup_{x_n^* \in T(x_n)} \langle x_n^*, y_n - x_n \rangle \leq 0 \implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0;$$

(iii) *T is properly quasimonotone and the condition* (C_5) *is verified.*

Then $MQ(T, K)$ is nonempty. Moreover, $SQ(T, K)$ is nonempty whenever T is upper sign continuous while $SQ^*(T, K)$ is nonempty if T is locally sub-upper sign-continuous.

Proof. We consider in Theorem 7.9 the bifunction $f = f_T$. Clearly, f_T is convex in y, satisfies (h_2) thanks to (c_1) and ($\mathbb{H}_3$) (indeed, f_T is properly quasimonotone thanks to the proper quasimonotonicity of T, see Remark 8.3). Moreover, in view of (C_5) and Lemma 8.14, f_T verifies (C_3). Therefore, all the conditions of Theorem 7.9 are satisfied, then $MQEP(f_T, K)$ has at least one solution which is, thanks to Lemma 8.13, a solution to $MQVI(T, K)$. If in addition, T is upper sign continuous at x, then the bifunction f_T has the upper sign property at x. Then, again Theorem 7.9 ensures that $QEP(f_T, K)$ has a solution which is also a solution to $SQVI(T, K)$ (by Lemma 8.13). Finally, Lemma 8.9 implies that $SQ^*(T, K)$ is nonempty if T is locally sub-upper sign-continuous, ending the proof. □

Corollary 8.16. *The conclusion of Theorem* 8.15 *remains true if* T *is pseudomonotone and* (C_5) *is replaced by one of the following:*

$(\hat{C}_5)$ *There exists a strongly compact and convex subset* B *of* C *such that for all* $x \in C$, $K(x) \cap B \neq \emptyset$, *for all* $z \in K(x) \setminus B$,

$$\sup_{y \in B \cap K(x)} \inf_{z^* \in T(z)} \langle z^*, z - y \rangle > 0.$$

$(\tilde{C}_5)$ *There exist nonempty (strongly) compact and convex subsets* B *and* K_1 *of* C, *with* $K_1 \subset B$, *such that* K_1 *is finite-dimensional and has nonempty intersection with* $K(x)$ *for all* $x \in C$, *for all* $z \in K(x) \setminus B$,

$$\sup_{y \in K_1 \cap K(x)} \inf_{z^* \in T(z)} \langle z^*, z - y \rangle > 0.$$

Proof. With the pseudomonotonicity and the upper sign continuity of T, the problems $SQVI(T, K)$ and $MQVI(T, K)$ are equivalent. Then the proof of Theorem 8.15 is valid for standard solutions instead of Minty ones by replacing (C_5) by $(\hat{C}_5)$. □

Remark 8.17. (1) The assumption $(\hat{C}_5)$ is an adaptation of (C_2) (on $EP(f, K)$) to the problem $QEP(f, K)$.

(2) In particular, $(\tilde{C}_5)$ implies the following condition: $(\check{C}_5)$ there exist nonempty compact and convex subsets B and K_1 of C such that K_1 is finite-dimensional and $K_1 \subset B$, for all $x \in K(x) \backslash B$,

$$\sup_{y \in K_1 \cap K(x)} \inf_{x^* \in T(x)} \langle x^*, x - y \rangle > 0.$$

(3) If $C = X$ and in addition, for any $x \in C$, $K(x)$ has a nonempty interior in the affine hull of X then $(\check{C}_5)$ meets the coercivity condition considered in [21, 23] in the treatment of Ricceri's conjecture on generalized quasi-variational inequalities [38]. As commented in the introductory section, this problem seems to have very few chances to be solved with similar ideas to those used in the setting of generalized quasimonotonicity and quasiconvexity, wherein the (expected) results consider different hypotheses. The interested reader may compare for example Corollary 8.16 with [23, Theorem 2.1].

9. Quasi-convex programs and quasi-optimization

In this paragraph, we keep X as before (linear normed space) and consider extension to the coercive case of the quasi-optimization problem treated in [10, Section

4]. We present two types of solutions: standard (star) and strong ones. Let us introduce the problem: given a real-valued function $g : X \longrightarrow \mathbb{R}$ and a set-valued map $K : C \rightrightarrows C$, where C is a closed convex subset of X, the quasi-optimization problem subject to our treatment is as follows:

$$QOpt(g, K): \quad \text{find } \overline{x} \in K(\overline{x}) \text{ such that } \min_{x \in K(\overline{x})} g(x) = g(\overline{x}).$$

An important instance arising in generalized Nash equilibrium is when g is defined by a supremum of Nikaido-Isoda functions, i.e., $g(x) = \sup_{y \in K(x)} \Psi(x, y)$; see for example [10] and references therein for more details.

If for a subset $D \subset X$, $K(x) = D$ for all $x \in C$, then we obtain the standard quasiconvex programming problem over the constraints set D :

$$Opt(g, D): \quad \text{find } \overline{x} \in D \text{ such that } \min_{x \in D} g(x) = g(\overline{x}).$$

With the bifunction $f_g : C \times C \to \mathbb{R}$, defined for $x, y \in C$ by $f_g(x, y) = g(y) - g(x)$, this problem can equivalently be formulated as follows:

$$QEP(f_g, K): \ \exists\, \overline{x} \in C \text{ such that } \overline{x} \in K(\overline{x}) \text{ and } f_g(\overline{x}, y) \geqslant 0, \ \forall y \in K(\overline{x}). \quad (96)$$

To solve the problem $QOpt(g, K)$ in the framework of quasi-equilibrium problems we need to fix the following observations:

Lemma 9.1. *Assume that g is quasiconvex. Then the associated bifunction f_g is properly quasimonotone.*

Proof. It is clear that the bifunction f_g is quasiconcave in y and satisfies $(\mathbb{H}_0)$. Then, from Lemma 5.1, f_g is properly quasimonotone. □

Remark 9.2. If g is semistrictly quasiconvex then the conclusion of Lemma 9.1 comes also from [15, Proposition 1.1] since f_g is monotone and then pseudomonotone.

Lemma 9.3. *Assume that g is second type semistrictly quasiconvex and lower hemicontinuous. Then the associated bifunction f_g has the upper sign property in x.*

Proof. Remark that the bifunction f_g is upper hemicontinuous in x. Moreover, in view of the quasiconvexity and semistrict quasiconvexity of g, f_g satisfies also the sign preserving property in the sense of [18, (5) Remark 1]. Then, the conclusion is a consequence of [18, Lemma 3]. □

Introduce now the coercivity conditions corresponding to the considered quasi-minimization problem:

(C_6) There exists a strongly compact and convex subset B of C, for all $x \in C \setminus B$, $K(x) \cap B \neq \emptyset$, for all $z \in K(x) \setminus B$, $\exists y \in K(x) \cap B$, $g(y) > g(z)$.

($\hat{C}_6$) There exists a strongly compact and convex subset B of C, for all $x \in C \setminus B$, $K(x) \cap B \neq \emptyset$, for all $z \in K(x) \setminus B$, $\exists y \in K(x) \cap B$, $g(y) < g(z)$.

Lemma 9.4. *The bifunction f_g satisfies* (C_3) *if and only if g satisfies* (C_6).

Proof. Immediate. □

Remark 9.5. ($\hat{C}_6$) will be an alternative in Theorem 9.9 below.

Lemma 9.6. *If g is continuous with respect to strong topology then f_g satisfies the condition* (h_2).

Proof. Obviously, the continuity of g implies the lower semicontinuity of f_g in (x, y). Then, condition (h_2) is automatically satisfied if g is continuous. □

At this stage, all the ingredients are prepared to state the existence of solutions to the problem $QOpt(g, K)$ for which we are also able to provide strong minimizers or else eigenvalue minimizers in the following sense:

Definition 9.7. Let $g : C \longrightarrow \mathbb{R}$ a real-valued function and $\lambda \geq 0$. A point $\overline{x} \in C$ will be said a λ-*eigenvalue minimizer* of f over a constraints set $D \subset C$ if

$$g(x) \geqslant g(\overline{x}) + \lambda\|x - \overline{x}\|^2, \ \ \forall x \in D. \tag{97}$$

Remark 9.8. (1) Definition 9.7 provides an example of λ-eigenvalue equilibrium points in the sense of Definition 4.7 above (see Section 4, Paragraph 4.2).

(2) When D is a neighborhood of $\overline{x}$, then the eigenvalue minimization meets the so-called strict minimization of order 2 also known in the literature as 2-order sharp minimization with a certain modulus following [44]. We believe that "eigenvalue" terminology is more significant since this definition comes from our (general new) introduced concept of eigenvalue-equilibrium that include many other eigenvalue solutions such as the eigenvalues problems for hemivariational inequalities arising in nonsmooth mechanic and engineering sciences [4].

Theorem 9.9. *Assume that:*

(m_0) *the set-valued map K verifies h_1);*
(m_1) *g is continuous and second type semistrictly quasiconvex;*
(m_2) *g satisfies either (C_6) or ($\hat{C}_6$).*

Then, the quasi-optimization problem $QOpt(g, K)$ admits at least a solution. If, in addition g is α-strongly quasiconvex for some $\alpha > 0$, then $QOpt(g, K)$ admits λ-eigenvalue (i.e., $\lambda\alpha$-strong) minimizer for all $\lambda \in [0, \frac{1}{4}]$.

Proof. Assume that (C_6) holds. By Lemmas 9.1, 9.3, 9.4 and 9.6, the hypotheses (i) (i.e., (C_3) and (h_i) for $i \in \{1, 2\}$) and the case (a) of (ii) in Theorem 7.9 are satisfied. Then, by this Theorem, $QEP(f_g, K)$ admits a solution which is in turn a minimizer of $QOpt(g, K)$ over $K(\overline{x})$. The second assertion of the conclusion comes directly from Proposition 4.9. Now assume that ($\hat{C}_6$) is satisfied. Since f_g is continuous and pseudomonotone (even monotone), similar arguments to those of Corollary 8.16 allow to obtain the required result. This ends the proof. □

Remark 9.10. Since (m_2) is particularly satisfied if C is weakly compact, with the unique additional assumption of semistrict quasi-convexity (second type), Corollary 9.12 improves [10, Proposition 4.2] by:

(1) Working on the general setting of normed spaces.
(2) Removing the assumption $C \subset X \backslash Argmin_X g$.
(3) Extending the result to the coercive setting.
(4) Using a direct approach different (more simple) from that of the normal operator N_a to adjusted sublevel sets.
(5) Dropping the regularity assumption on the normal operator N_a. Note that in our case, the continuity of g is enough to get the property h_2) which replaces this regularity.
(6) Obtaining a better conclusion that is λ-eigenvalue minimizers.

To discuss our treatment on quasiconvex-programming problems with the contribution of [10], we fix (only for the next result) X as a Banach space (to ensure the nonsmooth analysis background). We should also recall the set-valued variational inequality formulation of these optimization problems that necessitates the introduction of the normal operator associated to g : $N_f^a : X \rightrightarrows X^*$ defined as follows: for $x \in X$,

$$N_g^a(x) = \{v \in X^* : \langle v, y - x\rangle \leqslant 0, \, \forall y \in S_g^a(x)\},$$

where $S_g^a(x)$ is the adjusted sublevel set of g at x given by:

$$S_g^a(x) = \begin{cases} S_g(x) & \text{if } x \in argmin_X g \\ S_{g(x)} \bigcap cl\Big(B(S^<_{g(x)}, \rho_x)\Big) & \text{if } x \notin argmin_X g. \end{cases}$$

Here $S_{g(x)} = \{y \in X \mid g(y) \leqslant g(x)\}$, and $S^<_{g(x)} = \{y \in X \mid g(y) < g(x)\}$; $B(S^<_{g(x)}, \rho_x) = S^<_{g(x)} + \varrho_x B(0,1)$ with $\rho_x = dis(x, S^<_{g(x)})$, where $B(0,1)$ is the open unit ball of X.

Lemma 9.11. *If g is continuous and second type semistrictly quasiconvex then a point $\overline{x}$ is a solution to the quasi-optimization problem ($QOpt(g, K)$) if, and only if it is a solution to the set-valued variational inequality $SVI(N_g^a \backslash \{0\}, K(\overline{x}))$ i.e., $\overline{x}$ is a star solution to $SQVI(N_g^a, K)$.*

Proof. See [10, (iv) Proposition 4.1]. □

The conclusion of the considered quasi-optimization problem is the same via variational formulation with the same improvements apart the need of Banach structure. Precisely, as a consequence of Theorem 9.9 and Lemma 9.11, we derive the following:

Corollary 9.12. *Assume that the conditions* (m_i), *for* $i \in \{0, 1, 2\}$, *of Theorem* 9.9 *are satisfied. Then,*

(a) $SVI(N_g^a\backslash\{0\}, K) \Leftrightarrow QOpt(g, K) \Leftrightarrow QEP(f_g, K)$.
(b) $QOpt(g, K)$ *has a solution.*

If, in addition g is strongly quasiconvex then $QOpt(g, K)$ *admits a* λ*-eigenvalue (i.e.,* $\lambda\alpha$*-strong) minimizer for all* $\lambda \in [0, \frac{1}{4}]$.

Notice that similar arguments of Theorem 7.9 with the weak convergence allow to extend [10, Proposition 4.2] to the coercive case for the class of continuous quasiconvex functions (without being semistrictly quasiconvex). Precisely we state the following:

Corollary 9.13. *Let* $g : X \to \mathbb{R}$ *be a continuous quasiconvex function,* C *be a weakly closed and convex subset of* X *and* $K : C \rightrightarrows C$ *a weakly lower semicontinuous set-valued map with weakly closed and convex nonempty values such that* $K(x) \subset X\backslash Argmin_X g$ *for any* $x \in C$*. Assume moreover that* g *verifies the coercivity condition* (C_6) *with a weakly compact* $B \subset C$*, and that the normal operator* N_a *satisfies the regularity assumption* (C_1) *with respect to the weak convergence. Then,* $QOpt(g, K)$ *admits a solution.*

Proof. By [10, Proposition 4.2]), for $x \in C$, $MVI(N_g^a\backslash\{0\}, K(x))$ admits a solution. Moreover, in view of (C_6), the solution set to $MVI(N_g^a\backslash\{0\}, K(x))$ is included in the weakly compact B. On the other hand, with a slight change in Proposition 7.1 and Corollary 7.5, the weak-convergence with respect to (y_n) in (C_1) ensures the weak upper semicontinuity of the solution map $MVI(N_g^a, K(.))$. Then, thanks to Remark 7.6, similar arguments to those of Theorem 7.9 ensures that the set-valued Minty quasi-variational inequality $MVI(N_g^a, K)$ admits a solution. In addition, the operator N_a is locally sub-upper sign continuous (see again the proof of [10, Proposition 4.2]), then by the use of Lemma 8.9 the inequality $SVI(N_g^a\backslash\{0\}, K)$ admits a solution. Therefore, $QOpt(g, K)$ has a solution thanks to Corollary 9.12 . □

10. Conclusion

In this paper we have introduced new modes of quasiconvexity and quasimonotonicity, motivated them by several examples and counter-examples. In particular, our characterization of strong quasiconvexity (Theorem 3.20) seems to be new in the literature.

The new material underlined above have been applied to equilibrium problems regarding the existence of solutions by discussing the classic or introducing new solution concepts. Quantitative stability results have also been established of the introduced strong Minty solutions under the perturbation on the objective function. A similar quantitative stability result is thereafter presented for the case of perturbation on the feasibility set under strong quasiconvexity-like and strong quasimonotonicity-like on the objective function. The latter is improved in [6, Theorem 7] by replacing strong quasiconvexity-like by standard strong quasiconvexity and strong quasimonotonicity by (simple) quasimonotonicity.

Extensions to quasi-equilibrium are also considered and applications to set-valued variational and quasi-variational inequalities, and to quasi-convex programming. Quasi-optimization problems are also established by a new and direct approach discussing the very recent literature.

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