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Existence of Entire Solutions for Quasilinear Equations in the Heisenberg Group

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The paper deals with the existence of entire solutions for a quasilinear equation $(\mathcal{E}_\lambda)$ in $\mathbb{H}^n$, depending on a real parameter λ, which involves a general elliptic operator $\mathbf{A}$ in divergence form and two main nonlinearities. The competing nonlinear terms combine each other. Under some conditions, we prove the existence of a critical value $\lambda_* > 0$ with the property that $(\mathcal{E}_\lambda)$ admits nontrivial nonnegative entire solutions if and only if $\lambda \geq \lambda_*$. Furthermore, under the further assumption that the potential $\mathscr{A}$ of $\mathbf{A}$ is uniform convex, we give the existence of a second independent nontrivial nonnegative entire solution of $(\mathcal{E}_\lambda)$, when $\lambda > \lambda_*$.

Keywords: Heisenberg group, entire solutions, critical exponents.

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1. Introduction

Differential equations arising from variational principles have a strong geometric content and Geometric Analysis in the Heisenberg group, and more in general in sub-Riemannian manifolds, represents one of the currently most active and exciting areas of mathematics. Indeed, the Heisenberg group $\mathbb{H}^n$, $n = 1, 2, 3, \ldots$, plays a crucial role for instance in representation theory, harmonic analysis, complex variables, partial differential equations and quantum mechanics.

Recently, great interest has been directed to the study of nonlinear elliptic problems involving critical nonlinearities, in the context of stratified groups. We briefly refer to [3, 13, 14] and to the references therein. However, up-to-date results have been also produced by many authors in the Euclidean elliptic setting. We mention [17] and related references and comments given there, since this paper is an extension of [17] to the Heisenberg setting.

In this paper we study the *one parameter elliptic equation* in $\mathbb{H}^n$

$$(\mathcal{E}_\lambda) \qquad -\operatorname{div}_H \mathbf{A}(q, D_H u) + a(q)\,|u|^{p-2}\,u = \lambda w(q)\,|u|^{m-2}\,u - h(q)\,|u|^{\mathfrak{m}-2}\,u,$$

where $\lambda \in \mathbb{R}$ and $\mathbf{A}\colon \mathbb{H}^n \times \mathbb{H}_H \to \mathbb{H}_H$ admits a potential $\mathscr{A}$, with respect to its

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second variable ξ, while $\mathbb{H}_H$ is the span generated by $\{X_j, Y_j\}_{j=1}^n$, and

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

are vector fields for any $j = 1, \ldots, n$. The horizontal vector field $\mathbf{A}$ satisfies the following assumption, required throughout the paper.

($\mathcal{A}_1$) The *potential* $\mathscr{A} = \mathscr{A}(q, \xi)$ is a continuous function in $\mathbb{H}^n \times \mathbb{H}_H$, with continuous derivative with respect to ξ, $\mathbf{A} = \partial_\xi \mathscr{A}(q, \xi)$, and verifies:
- (i) $\mathscr{A}(q, 0_H) = 0$ and $\mathscr{A}(q, \xi) = \mathscr{A}(q, -\xi)$ for all $(q, \xi) \in \mathbb{H}^n \times \mathbb{H}_H$;
- (ii) $\mathscr{A}(q, \cdot)$ is strictly convex in $\mathbb{H}_H$ for all $q \in \mathbb{H}^n$;
- (iii) There exist positive constants c_1, $c_2 > 0$ and an exponent p such that

$$c_1 |\xi|_H^p \le (\mathbf{A}(q, \xi), \xi)_H, \qquad |\mathbf{A}(q, \xi)|_H \le c_2 |\xi|_H^{p-1} \tag{1}$$

for all $(q, \xi) \in \mathbb{H}^n \times \mathbb{H}_H$, where $1 < p < Q$, and $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, while $(\cdot, \cdot)_H$ is the natural inner product in $\mathbb{H}_H$ and $|\cdot|_H$ the related norm introduced properly in Section 2.

Furthermore, we denote by r the *Heisenberg norm*, defined for all $q = (z, t) \in \mathbb{H}^n$ by

$$r(q) = (|z|^4 + t^2)^{1/4}, \quad z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \; t \in \mathbb{R},$$

and by $|\cdot|$ the *Euclidean norm* in $\mathbb{R}^{2n}$. In the entire paper we assume for $(\mathcal{E}_\lambda)$

($\mathcal{H}_1$)
- (i) $a \in L^\infty_{\mathrm{loc}}(\mathbb{H}^n)$, and $a(q) \ge a_1[1 + r(q)]^{-p}$ for all $q \in \mathbb{H}^n$ and some constant $a_1 \in (0, 1]$.
- (ii) The exponents m and $\mathfrak{m}$ are such that $1 < m < \mathfrak{m} < \infty$.
- (iii) $0 < h \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and $w \not\equiv 0$

Equation $(\mathcal{E}_\lambda)$ is a kind of quasilinear elliptic problem involving a possibly super Sobolev critical nonlinearity, with combined effects. In the Euclidean setting several papers are devoted on problems of this kind, we just quote, for example, [1, 16, 17] and the comments and references therein. Motivated by these papers, we extend the general results of [17] from the Euclidean to the Heisenberg setting. However, $(\mathcal{E}_\lambda)$ is more delicate to handle, since the general framework produces new interesting complications.

Finally, we introduce further mild structure conditions, which are assumed *only* when specifically required.

($\mathcal{A}_2$) $\mathscr{A}$ is uniformly convex, i.e. for any $\varepsilon \in (0, 1)$ there exists a number $\delta = \delta(\varepsilon) \in (0, 1)$ such that either $|\xi - \eta| \le \varepsilon \max\{|\xi|, |\eta|\}$, or

$$\mathscr{A}(q, (\xi + \eta)/2) \le \frac{1}{2}(1 - \delta)[\mathscr{A}(q, \xi) + \mathscr{A}(q, \eta)]$$

for any $q \in \mathbb{H}^n$ and all ξ, $\eta \in \mathbb{H}_H$.

$(\mathcal{H}_2)$ The coefficients h and w are related by the condition that either

$$\int_{\mathbb{H}^n} \frac{w(q)^{\mathfrak{m}/[\mathfrak{m}-m]}}{h(q)^{m/[\mathfrak{m}-m]}}\, dq = \mathscr{H} \in \mathbb{R}^+, \text{ or} \tag{2}$$

$$\begin{cases} w(w/h)^{(m-1)/(\mathfrak{m}-m)} \in L^{p^{*\prime}}(\mathbb{H}^n), \quad p^* = \dfrac{nQ}{n-Q}, \\ \left\| w(w/h)^{(m-1)/(\mathfrak{m}-m)} \right\|_{p^{*\prime}} = \mathscr{W} \in \mathbb{R}^+ \end{cases} \tag{3}$$

holds, where $p^{*\prime}$ is the Hölder conjugate of p^*.

$(\mathcal{H}_2^+)$ $w \in L^{p^*/(p^*-m)}(\mathbb{H}^n)$ and $p < m < p^*$.

The standard convexity condition of *Simon* type, assumed for instance in [1], implies that $\mathscr{A}$ satisfies condition $(\mathcal{A}_2)$, as shown in the Remark of Section 3 of [17]. Concerning the famous condition (2) we refer to the historical comments given in [17].

A typical operator covered in the paper is $\mathbf{A}(q,\xi) = \alpha(q)\, |\xi|_H^{p-2}\, \xi$, where $\alpha \in C(\mathbb{H}^n)$ and $0 < \alpha_1 \le \alpha \le \alpha_2$ in $\mathbb{H}^n$ for suitable numbers $\alpha_1,\ \alpha_2 \in \mathbb{R}^+$, and $\mathbf{A}(q,\xi) = \alpha(q)\, |\xi|_H^{p-2}\, \xi$ satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, provided that $1 < p < Q$. Therefore, in the subcase $\alpha \equiv 1$, along any $\varphi \in C_c^\infty(\mathbb{H}^n)$

$$\mathrm{div}_H \mathbf{A}(q, D_H\varphi) = \mathrm{div}_H(|D_H\varphi|^{p-2}\, D_H\varphi) = \Delta_{\mathbb{H}^n,p}\varphi$$

reduces to the so called *horizontal p-Laplacian on the Heisenberg group.*

Throughout the paper we require assumptions $(\mathcal{A}_1)$ and $(\mathcal{H}_1)$, without further mentioning. We present the main results in the following statement.

Theorem 1.1. *Let $(\mathcal{H}_2)$ hold.*

(i) *There exists $\lambda_* \ge 0$ such that $(\mathcal{E}_\lambda)$ has at least a nontrivial nonnegative entire solution for all $\lambda > \lambda_*$ and has no nontrivial nonnegative entire solutions for $\lambda < \lambda_*$;*

(ii) *If $(\mathcal{H}_2^+)$ is also satisfied, then there exists $\lambda_* > 0$ such that $(\mathcal{E}_\lambda)$ admits at least a nontrivial nonnegative entire solution if and only if $\lambda \ge \lambda_*$;*

(iii) *Let $(\mathcal{A}_2)$ be also satisfied. If $(\mathcal{E}_\lambda)$ has a nontrivial nonnegative entire solution for some $\lambda > 0$, then $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions.*

Further multiplicity results for $(\mathcal{E}_\lambda)$ in the subcase $m < p$ are given in the forthcoming paper [15].

It is worthwhile to see that (2) implies (3), when w satisfies $(\mathcal{H}_2^+)_1$, but only if $1 < m < p^*$ and $m < \mathfrak{m}$. Indeed, the Hölder inequality, with $(p^*-1)m/(p^*-m)$ and $m'/p^{*\prime}$, where $p^{*\prime}$ and m' are the Hölder conjugates of p^* and m, implies

$$\int_{\mathbb{H}^n}\left[w(q)\left(\frac{w(q)}{h(q)}\right)^{(m-1)/(\mathfrak{m}-m)}\right]^{p^{*\prime}} dq = \int_{\mathbb{H}^n} w^{p^{*\prime}/m}\left[\left(\frac{w^{\mathfrak{m}}}{h^m}\right)^{1/(\mathfrak{m}-m)}\right]^{p^{*\prime}/m'} dq$$
$$\leq \|w\|_{p^*/(p^*-m)}^{p^{*\prime}/m}\mathscr{H}^{p^{*\prime}/m'}.$$

Therefore (3) holds by virtue of (2) and $(\mathcal{H}_2^+)_1$, but asking only the request that $1 < m < p^*$ and $m < \mathfrak{m}$.

The paper is divided into four sections. Section 2 contains some relevant definitions and notations related to the Heisenberg group functional setting. Section 3 gives some useful properties of functionals and operators. Section 4 presents the proof of Theorem 1.1.

2. Preliminaries

We briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [7, 10, 11, 12].

Let $\mathbb{H}^n$ be the Heisenberg group of dimension $2n+1$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non-Abelian group law

$$q \circ q' = \left(z + z', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i')\right)$$

for all q, $q' \in \mathbb{H}^n$, with

$$q = (z,t) = (x_1, \dots, x_n, y_1, \dots, y_n, t), \quad q' = (z', t') = (x_1', \dots, x_n', y_1', \dots, y_n', t').$$

In $\mathbb{H}^n$ the natural origin is denoted by $O = (0,0)$. Define

$$r(q) = r(z,t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } q = (z,t) \in \mathbb{H}^n,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{2n}$. The *Korányi norm* is homogeneous of degree 1, with respect to the dilations $\delta_R\colon (z,t) \mapsto (Rz, R^2t)$, $R > 0$. Indeed, for all $q = (z,t) \in \mathbb{H}^n$

$$r(\delta_R(q)) = r(Rz, R^2t) = (|Rz|^4 + R^4t^2)^{1/4} = R\, r(q).$$

Hence, the *Korányi distance* is

$$d_K(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q, q') \in \mathbb{H}^n \times \mathbb{H}^n,$$

and the *Korányi open ball* of radius R centered at q_0 is

$$B_R(q_0) = \{q \in \mathbb{H}^n : d_K(q, q_0) < R\}.$$

For simplicity B_R denotes the ball of radius R centered at $q_0 = O$.

The Jacobian determinant of δ_R is R^{2n+2}. The natural number $Q = 2n + 2$, which is the so called *homogeneous dimension of* $\mathbb{H}^n$, plays a role analogous to the topological dimension in the Euclidean context.

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n}\times\mathbb{R}$. It is invariant under left translations and Q-homogeneous with respect to dilations. Hence, as noted in [11], the topological dimension $2n+1$ of $\mathbb{H}^n$ is strictly less than its Hausdorff dimension Q. We denote by $|U|$ the measure of any measurable set $U\subset\mathbb{H}^n$. Then

$$|\delta_R(U)| = R^Q|U|, \qquad d(\delta_R q) = R^Q dq.$$

In particular, if $U = B_R$, then $|B_R| = R^Q w_Q$, where w_Q is the measure of the unit sphere of $\mathbb{H}^n$.

The vector fields for $j = 1,\dots,n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t},$$

constitute a basis for the real Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero. The span of $\{X_j, Y_j\}_{j=1}^n$ is briefly denoted by $\mathbb{H}_H$ and a vector field in $\mathbb{H}_H$ is called *horizontal*.

Let $u\in C^1(\mathbb{H}^n)$ be fixed. The *horizontal gradient* $D_H u$ is

$$D_H u = \sum_{j=1}^n \left[(X_j u)X_j + (Y_j u)Y_j\right],$$

that is, it is an element of $\mathbb{H}_H$. Furthermore, if $f\in C^1(\mathbb{R})$, then $D_H(f\circ u) = f'(u)D_H u$. The natural inner product in $\mathbb{H}_H$

$$\big(W, Z\big)_H = \sum_{j=1}^n \left(w^j z^j + \widetilde{w}^j\widetilde{z}^j\right)$$

for $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ and $Z = \{z^j X_j + \widetilde{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm

$$|D_H u|_H = \sqrt{\big(D_H u, D_H u\big)_H}$$

for the horizontal vector field $D_H u$. Moreover, if also $v\in C^1(\mathbb{H}^n)$ then the Cauchy-Schwarz inequality

$$\left|\big(D_H u, D_H v\big)_H\right| \le |D_H u|_H |D_H v|_H$$

continues to be valid.

For horizontal vector fields $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$ we define the *horizontal divergence* by

$$\mathrm{div}_H W = \sum_{j=1}^n [X_j(w^j) + Y_j(\widetilde{w}^j)].$$

If furthermore $g\in C^1(\mathbb{H}^n)$, then the *Leibnitz formula* holds, namely

$$\mathrm{div}_H(gW) = g\,\mathrm{div}_H W + \big(D_H g, W\big)_H.$$

The *Kohn-Spencer Laplacian* Δ_H, or equivalently the *horizontal Laplacian in* $\mathbb{H}^n$, of a function u of class $C^2(\mathbb{H}^n)$ is defined by

$$\Delta_H u = \sum_{j=1}^n (X_j^2 + Y_j^2) u = \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},$$

and Δ_H is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* in [9]. In particular, $\Delta_H u = \mathrm{div}_H D_H u$ for each $u \in C^2(\mathbb{H}^n)$.

A well known generalization of the Kohn-Spencer Laplacian is the *horizontal p-Laplacian* on the Heisenberg group given by

$$\Delta_{H,p} u = \mathrm{div}_H (|D_H u|_H^{p-2} D_H u),$$

which is well defined for all function $u \in C^2(\mathbb{H}^n)$ and $p \in (1, \infty)$. But in this paper $1 < p < Q$.

Since we are interested in weighted Lebesgue spaces, denoted by ω a generic weight on $\mathbb{H}^n$ of class $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we put for any σ, with $1 < \sigma < \infty$,

$$L^\sigma(\mathbb{H}^n, \omega) = \{u \colon \mathbb{H}^n \to \mathbb{R} \text{ measurable} : \omega^{1/\sigma} |u| \in L^\sigma(\mathbb{H}^n)\},$$

endowed with the norm $\|u\|_{\sigma,\omega} = \|\omega^{1/\sigma} u\|_\sigma$. In particular, the next result summarizes the main properties of the weighted spaces $L^p(\mathbb{H}^n, a)$, $L^m(\mathbb{H}^n, w)$ and $L^{\mathfrak{m}}(\mathbb{H}^n, h)$ we are interested in, see [10] for a complete discussion. By Proposition A.6 in [1], which still holds in the context of the Heisenberg group, we have

Lemma 2.1. *Let the weights a, w, h be of class $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, and let p, m, $\mathfrak{m}$ be finite Lebesgue exponents strictly greater than* 1*. Then $L^p(\mathbb{H}^n, a)$, $L^m(\mathbb{H}^n, w)$ and $L^{\mathfrak{m}}(\mathbb{H}^n, h)$ are separable uniformly convex Banach spaces.*

We recall that throughout the paper we assume that $(\mathcal{A}_1)$ and $(\mathcal{H}_1)$ are satisfied, without further mentioning. Now conditions $(\mathcal{A}_1)$(i) and (ii) imply that

$$\mathscr{A}(q, \xi) \le (\mathbf{A}(q, \xi), \xi)_H \quad \text{for all } (q, \xi) \in \mathbb{H}^n \times \mathbb{H}_H. \tag{4}$$

Furthermore, $(\mathcal{A}_1)$(ii) *is weaker than the request that $\mathscr{A}$ is uniformly convex, that is that $(\mathcal{A}_2)$ holds*, i.e. for any $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$\mathscr{A}\left(q, \frac{\xi + \eta}{2}\right) \le (1 - \delta) \frac{\mathscr{A}(q, \xi) + \mathscr{A}(q, \eta)}{2}$$

for all $q \in \mathbb{H}^n$ and ξ, $\eta \in \mathbb{H}_H$, with $|\xi - \eta|_H \ge \varepsilon \max\{|\xi|_H, |\eta|_H\}$.

By $(\mathcal{A}_1)$(i) and (iii), we have

$$\mathscr{A}(q, \xi) = \int_0^1 \frac{d}{dt} \mathscr{A}(q, t\xi) dt = \int_0^1 \frac{1}{t} (\mathbf{A}(q, t\xi), t\xi)_H dt \ge \frac{c_1}{p} |\xi|_H^p,$$

which, together with (1) and (4), implies

$$c\,|\xi|_H^p \le \mathscr{A}(q,\xi) \le (\mathbf{A}(q,\xi),\xi)_H \le c_2\,|\xi|_H^p \tag{5}$$

for all $(q,\xi) \in \mathbb{H}^n \times \mathbb{H}_H$.

We are now able to introduce the main spaces $E = (E, \|\cdot\|_E)$ and $X = (X, \|\cdot\|)$ naturally associated to $(\mathcal{E}_\lambda)$. Let E denote the completion of $C_c^\infty(\mathbb{H}^n)$, with respect to the norm $\|u\|_E = \left(\|D_H u\|_p^p + \|u\|_{p,a}^p\right)^{1/p}$, with

$$\|D_H u\|_p = \left(\int_{\mathbb{H}^n} |D_H u(q)|_H^p dq\right)^{1/p}, \quad \|u\|_{p,a} = \left(\int_{\mathbb{H}^n} a(q)|u(q)|^p dq\right)^{1/p},$$

and the main solution space

$$X = \left\{u \in E \,:\, \int_{\mathbb{H}^n} h(q)|u(q)|^{\mathfrak{m}} dq < \infty\right\},$$

endowed with the norm $\|u\| = \|u\|_E + \|u\|_{\mathfrak{m},h}$.

Clearly E and X are well defined, since $C_c^\infty(\mathbb{H}^n) \subset X \subset E$.

Lemma 2.2. *The space E is a separable uniformly convex Banach space and X is a separable reflexive Banach space.*

Proof. Clearly E is a Banach space. The linear operator $T\colon E \to L^p(\mathbb{H}^n, a) \times L^p(\mathbb{H}^n, \mathbb{R}^{2n})$, defined for all $u \in E$ by $T(u) = (u, D_H u)$, is isometric. Hence E is a closed subspace of $L^p(\mathbb{H}^n, a) \times L^p(\mathbb{H}^n, \mathbb{R}^{2n})$. Thus E is a separable uniformly convex Banach space since $1 < p < \infty$.

For the latter part, put $Y = E \times L^{\mathfrak{m}}(\mathbb{H}^n, h)$ and endow the space Y with the norm $\|u\|_Y = \|u\|_E + \|u\|_{\mathfrak{m},h}$. It is easy to see that Y is separable and reflexive, since both E and $L^{\mathfrak{m}}(\mathbb{H}^n, h)$ are separable and reflexive.

Consider now the operator $T\colon X \to Y$ defined by $T(u) = (u,u)$. Clearly T is well defined and linear. Moreover T is an isometry, being X endowed exactly with the norm $\|\cdot\|_Y$. Therefore, $T(X)$ is a closed subspace of the separable reflexive space Y, and so $T(X)$ is separable and reflexive. Consequently, also $(X, \|\cdot\|_Y) = (X, \|\cdot\|)$ is separable and reflexive, being isomorphic to a separable reflexive Banach space. $\square$

Let us recall that throughout the paper $1 < p < Q$ and that B_R denotes the ball in $\mathbb{H}^n$ of center O and radius $R > 0$. Let $S^{1,p}(\mathbb{H}^n)$ be the Folland-Stein space, that is the completion of $C_c^\infty(\mathbb{H}^n)$, with respect to the norm $\|D_H\varphi\|_p$, $\varphi \in C_c^\infty(\mathbb{H}^n)$. The homogeneous dimension Q plays an important role also for the sub-elliptic version of the classical Sobolev embedding theorem.

Lemma 2.3. (i) *The embeddings $X \hookrightarrow E \hookrightarrow S^{1,p}(\mathbb{H}^n) \hookrightarrow L^{p^*}(\mathbb{H}^n)$ are continuous, with $\|D_H u\|_p \le \|u\|_E$ for all $u \in E$, $\|u\|_E \le \|u\|$ for all $u \in X$ and*

$$\|u\|_{p^*} \le C_{p^*}\|D_H u\|_p \quad \textit{for all } u \in S^{1,p}(\mathbb{H}^n). \tag{6}$$

(ii) *For any $R > 0$ and for any $q_0 \in \mathbb{H}^n$ the embeddings $E \hookrightarrow\hookrightarrow L^\sigma(B_R(q_0))$ and $X \hookrightarrow\hookrightarrow L^\sigma(B_R(q_0))$ are compact for all σ, with $1 \le \sigma < p^*$.*

Proof. (i) By [4, 5, 10, 18, 19], the *Sobolev inequality* asserts that

$$\|\varphi\|_{p^*} \le C_{p^*}\|D_H\varphi\|_p \tag{7}$$

for all $\varphi \in C_c^\infty(\mathbb{H}^n)$, where $C_{p^*} = C_{Q,p}$ is a positive constant depending only on Q and p. Consequently, (7) gives at once the validity of (6).

(ii) If Ω is a bounded PS domain in $\mathbb{H}^n$, the embedding

$$HW^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\sigma(\Omega) \tag{8}$$

is compact, when $1 \le \sigma < p^*$ by [8, 10, 20]. By [6, 10, 20] the property (8) holds in Carnot-Carathéodory balls, which are special bounded PS domains of $\mathbb{H}^n$. Since the Carnot-Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$ by [2, 12], then (8) can be applied when Ω is any Korányi ball $B_R(q_0)$, $q_0 \in \mathbb{H}^n$, and $R > 0$.

Thus, since by $(\mathcal{H}_1)$(i) the weight a has the property that $a \ge c_{R,q_0} > 0$ in $B_R(q_0)$ and a is bounded in $B_R(q_0)$, the embedding (8) and (i) yield at once that $E \hookrightarrow\hookrightarrow L^\sigma(B_R(q_0))$ and $X \hookrightarrow\hookrightarrow L^\sigma(B_R(q_0))$ are compact for all σ, with $1 \le \sigma < p^*$ and for any $R > 0$ and $q_0 \in \mathbb{H}^n$. □

Lemma 2.4. *Every weakly convergent sequence either in E or in X admits a subsequence converging a.e. in $\mathbb{H}^n$ to the same limit.*

Proof. Similar to the proof of Theorem A.10 of [1], thanks to Lemma 2.3(ii). We omit it here. □

Lemma 2.5. *If $(\mathcal{H}_2)$ is satisfied, then the embedding $X \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact. Moreover, if $(\mathcal{H}_2^+)$ is satisfied, then $E \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact.*

Proof. We divide the proof into three steps.

Step 1: First assume that (2) is satisfied. Fix $u \ne 0$ in X. By the Hölder inequality and (2) we have

$$\begin{aligned}
1 &= \int_{\mathbb{H}^n} w(q)\left|\frac{u}{\|u\|_{m,w}}\right|^m dq = \int_{\mathbb{H}^n} \frac{w(q)}{h(q)^{m/\mathfrak{m}}} h(q)^{m/\mathfrak{m}}\left|\frac{u}{\|u\|_{m,w}}\right|^m dq \\
&\le \left(\int_{\mathbb{H}^n} \frac{w(q)^{\mathfrak{m}/[\mathfrak{m}-m]}}{h(q)^{m/[\mathfrak{m}-m]}} dq\right)^{1-m/\mathfrak{m}} \left(\int_{\mathbb{H}^n} h(q)\frac{|u|^{\mathfrak{m}}}{\|u\|_{m,w}^{\mathfrak{m}}} dq\right)^{m/\mathfrak{m}} \\
&= \mathscr{H}^{1-m/\mathfrak{m}}\|u\|_{m,w}^{-m}\|u\|_{\mathfrak{m},h}^m,
\end{aligned}$$

since $1 < m < \mathfrak{m} < \infty$ by $(\mathscr{H}_1)$(ii). Consequently,

$$\|u\|_{m,w} \le C_{\mathscr{H}}\|u\|_{\mathfrak{m},h}, \quad C_{\mathscr{H}} = \mathscr{H}^{1/m-1/\mathfrak{m}}. \tag{9}$$

Hence $X \hookrightarrow L^m(\mathbb{H}^n, w)$ is continuous.

To prove that it is compact, fix a sequence $(u_k)_k$ and u in X, with $u_k \rightharpoonup u$ in X. Up to a subsequence, still denoted by $(u_k)_k$, then $u_k \to u$ a.e. in $\mathbb{H}^n$ by Lemma 2.4. We claim that $\big(w|u_k - u|^m\big)_k$ is uniformly integrable in $\mathbb{H}^n$.

Since $(u_k)_k$ is bounded in X, then $\|u_k\| + \|u\| \le M$ for all $k = 1, 2, \cdots$ and some a constant $M > 0$. Of course, $(u_k)_k$ is also bounded in $L^{\mathfrak{m}}(\mathbb{H}^n, h)$.

Fix $\varepsilon > 0$. Since $w/h^{m/\mathfrak{m}} \in L^{\frac{\mathfrak{m}}{\mathfrak{m}-m}}(\mathbb{H}^n)$ by (2), there exists $\delta = \delta(\varepsilon) > 0$ such that, for any measurable subset $U \subset \mathbb{H}^n$, with $|U| < \delta$, we have

$$\int_U \frac{w(q)^{\mathfrak{m}/(\mathfrak{m}-m)}}{h(q)^{m/(\mathfrak{m}-m)}} dq = \int_U \left|\frac{w(q)}{h(q)^{\frac{m}{\mathfrak{m}}}}\right|^{\frac{\mathfrak{m}}{\mathfrak{m}-m}} dq < \left(\frac{\varepsilon}{M^m}\right)^{\mathfrak{m}/(\mathfrak{m}-m)},$$

and similarly there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\mathbb{H}^n \setminus B_R} \frac{w(q)^{\mathfrak{m}/(\mathfrak{m}-m)}}{h(q)^{m/(\mathfrak{m}-m)}} dq < \left(\frac{\varepsilon}{M^m}\right)^{\mathfrak{m}/(\mathfrak{m}-m)}. \tag{10}$$

Let U be any fixed measurable subset of $\mathbb{H}^n$ with $|U| < \delta$. Similar to the proof of (9),

$$\begin{aligned}\int_U w(q)|u_k - u|^m dq &\le \left(\int_U \frac{w(q)^{\mathfrak{m}/(\mathfrak{m}-m)}}{h(q)^{m/(\mathfrak{m}-m)}} dq\right)^{1-m/\mathfrak{m}} \left(\int_U h(q)|u_k - u|^{\mathfrak{m}} dq\right)^{m/\mathfrak{m}} \\ &< \frac{\varepsilon}{M^m} M^m = \varepsilon\end{aligned}$$

for all $k = 1, 2, \dots$. Using (10), we also obtain for all $k = 1, 2, \dots$

$$\begin{aligned}\int_{\mathbb{H}^n \setminus B_R} w(q)\,|u_k - u|^m\, dq \le \left(\int_{\mathbb{H}^n \setminus B_R} \frac{w(q)^{\mathfrak{m}/(\mathfrak{m}-m)}}{h(q)^{m/(\mathfrak{m}-m)}} dq\right)^{1-m/\mathfrak{m}} \times \\ \times \left(\int_{\mathbb{H}^n \setminus B_R} h(q)|u_k - u|^{\mathfrak{m}} dq\right)^{m/\mathfrak{m}} < \varepsilon.\end{aligned}$$

In conclusion, the claim is proved. Finally, $w\,|u_k - u|^m \to 0$ a.e. in $\mathbb{H}^n$ and so the Vitali convergence theorem assures that $u_k \to u$ in $L^q(\mathbb{H}^n, w)$ as $k \to \infty$. Therefore, $X \hookrightarrow\hookrightarrow L^q(\mathbb{H}^n, w)$ is compact.

Step 2: Assume that (3) is satisfied. Let us recall the following elementary inequality: *for every $k_1 \ge 0$, $k_2 > 0$ and α, β, with $0 < \alpha < \beta$, there exists*

$$\begin{aligned}&C_{\alpha\beta} = \frac{\beta - \alpha}{\beta} \left(\frac{\alpha}{\beta}\right)^{\alpha/(\beta-\alpha)} \in (0,1) \quad \textit{such that for all } t \in \mathbb{R} \\ &k_1\,|t|^{\alpha} - k_2\,|t|^{\beta} \le C_{\alpha\beta} k_1 \left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \le k_1 \left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)}.\end{aligned} \tag{11}$$

Hence, taking $k_1=w$, $k_2=h/2$, $\alpha=m-1$, $\beta=\mathfrak{m}-1$ in (11), we get for all $u\in X$

$$\begin{aligned}\|u\|_{m,w}^m &= \int_{\mathbb{H}^n}\Big(w(q)\,|u|^{m-1}-\frac{1}{2}h(q)\,|u|^{\mathfrak{m}-1}\Big)\,|u|\,dq+\frac{1}{2}\int_{\mathbb{H}^n}h(q)\,|u|^{\mathfrak{m}}dq\\ &\le C_1\int_{\mathbb{H}^n}w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u|\,dq+\frac{1}{2}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\\ &\le C_1\left\|w(w/h)^{(m-1)/(\mathfrak{m}-m)}\right\|_{\frac{p^*}{p^*-1}}\|u\|_{p^*}+\frac{1}{2}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\\ &\le C_1C_{p^*}\mathscr{W}\|D_Hu\|_p+\frac{1}{2}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\end{aligned}\tag{12}$$

by (6), where $C_1=2^{(m-1)/(\mathfrak{m}-m)}$. Therefore, for any $v\in X$, $v\ne 0$, putting $u=v/\|v\|$ and $C_{\mathscr{W}}=C_1C_{p^*}\mathscr{W}$, we have

$$\begin{aligned}\int_{\mathbb{H}^n}w(q)\left|\frac{v}{\|v\|}\right|^m dq &\le C_{\mathscr{W}}\left\|D_H\frac{v}{\|v\|}\right\|_p+\frac{1}{2}\int_{\mathbb{H}^n}h(q)\left|\frac{v}{\|v\|}\right|^{\mathfrak{m}}dq\\ &\le C_{\mathscr{W}}\left\|D_H\frac{v}{\|v\|_E}\right\|_p+\frac{1}{2}\int_{\mathbb{H}^n}h(q)\left|\frac{v}{\|v\|_{\mathfrak{m},h}}\right|^{\mathfrak{m}}dq\le C_{\mathscr{W}}+1.\end{aligned}$$

Hence,

$$\begin{aligned}1&=\int_{\mathbb{H}^n}w(q)\left|\frac{v(q)}{\|v\|_{m,w}}\right|^m dq=\left(\frac{\|v\|}{\|v\|_{m,w}}\right)^m\int_{\mathbb{H}^n}w(q)\left|\frac{v(q)}{\|v\|}\right|^m dq\\ &\le\left(\frac{\|v\|}{\|v\|_{m,w}}\right)^m\big(C_{\mathscr{W}}+1\big).\end{aligned}$$

Thus $\|v\|_{m,w}\le\big(C_{\mathscr{W}}+1\big)^{1/m}\|v\|$ and so X is continuously embedded in $L^m(\mathbb{H}^n,w)$.

It remains to prove that the embedding is actually compact. To this aim let $u_k\rightharpoonup u$ in X. Again, up to a subsequence, still denoted by $(u_k)_k$, we have $u_k\to u$ a.e. in $\mathbb{H}^n$ by Lemma 2.4. We claim that for all $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that

$$\int_{\mathbb{H}^n}w(q)\,|u_k-u|^m\,dq<\varepsilon\qquad\text{for all } k\ge N.\tag{13}$$

Since $u_k\rightharpoonup u$ in X, then $\int_{\mathbb{H}^n}h(q)\,|u_k|^{\mathfrak{m}}\,dq\le M$ for all $k=1,2,\cdots$ and a suitable positive constant M. Clearly also $\int_{\mathbb{H}^n}h(q)\,|u|^{\mathfrak{m}}\,dq\le M$ by Fatou's lemma.

Fix $\varepsilon>0$. By (11), as in the proof of (12), taking $k_1=w$, $k_2=\varepsilon h/2^{\mathfrak{m}+1}M$, $\alpha=m-1$, $\beta=\mathfrak{m}-1$, we get for all $k=1,2,\cdots$

$$\begin{aligned}\int_{\mathbb{H}^n}w(q)\,|u_k-u|^m dq &\le\int_{\mathbb{H}^n}C_\varepsilon w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k-u|\,dq\\ &\qquad+\frac{\varepsilon}{2^{\mathfrak{m}+1}M}\int_{\mathbb{H}^n}h(q)\,|u_k-u|^{\mathfrak{m}}dq\\ &\le\int_{\mathbb{H}^n}C_\varepsilon w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k-u|\,dq+\frac{\varepsilon}{2},\end{aligned}\tag{14}$$

where $C_\varepsilon = \left(2^{\mathfrak{m}+1}M/\varepsilon\right)^{(m-1)/(\mathfrak{m}-m)}$. We assert that $\left(w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k - u|\,\right)_k$ is uniformly integrable in $\mathbb{H}^n$. Indeed, for any measurable subset $U \subset \mathbb{H}^n$,

$$\int_U w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k - u|\,dq \le \left(\int_U [w(w/h)^{(m-1)/(\mathfrak{m}-m)}]^{p^{*\prime}}\right)^{1-1/p^*} \|u_k - u\|_{p^*}.$$

Similar to the proof of the first step we get from the last inequality at once the assertion. Hence the Vitali convergence theorem yields

$$\lim_{k\to\infty}\int_{\mathbb{H}^n} C_\varepsilon w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k - u|\,dq = \int_{\mathbb{H}^n}\lim_{k\to\infty} C_\varepsilon w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k - u|\,dq = 0.$$

Thus there exists $N = N(\varepsilon)$ such that

$$\int_{\mathbb{H}^n} C_\varepsilon w(w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u_k - u|\,dq < \frac{\varepsilon}{2}$$

for all $k \ge N$. In conclusion the claim (13) is valid by (14). Therefore, the embedding $X \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact.

Step 3: Assume $(\mathcal{H}_2^+)$. Similarly, $E \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact, if $(\mathcal{H}_2^+)$ is satisfied. It is simply enough to replace in the main argument $w(w/h)^{(m-1)/(\mathfrak{m}-m)}$ by w. Indeed, for all $u \in E$ the Hölder inequality, Lemma 2.3 and (6) give

$$\begin{aligned}\int_{\mathbb{H}^n} w(q)|u|^m dq &\le \|w\|_{p^*/(p^*-m)}\|u\|_{p^*}^m \le C_{p^*}^m \|w\|_{p^*/(p^*-m)}\|D_H u\|_p^m \\ &\le C_{p^*}^m \|w\|_{p^*/(p^*-m)}\|u\|_E^m.\end{aligned}$$

Thus, $\|u\|_{m,w} \le C_{p^*}\|w\|_{p^*/(p^*-m)}^{1/m}\|u\|_E$, that is the embedding $E \hookrightarrow L^m(\mathbb{H}^n, w)$ is continuous. Let us prove that it is actually compact. Fix $(u_k)_k$, $u \in E$ and assume that $u_k \rightharpoonup u$ in E. Then, up to a subsequence, still denoted by $(u_k)_k$, we get that $u_k \to u$ a.e. in $\mathbb{H}^n$ as $k \to \infty$ by Lemma 2.4. Moreover, there exists a constant $M > 0$ such that $\|u_k\|_E + \|u\|_E \le M$. We claim that $u_k \to u$ in $L^m(\mathbb{H}^n, w)$. Fix $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon) > 0$ and $r = r(\varepsilon) > 0$ such that for any measurable subset $U \subset \mathbb{H}^n$, with $|U| < \delta$,

$$\int_U w(q)^{p^*/(p^*-m)}dq < \left[\frac{\varepsilon}{(MC_{p^*})^m}\right]^{p^*/(p^*-m)},$$

$$\int_{\mathbb{H}^n\setminus B_R} w(q)^{p^*/(p^*-m)}dq < \left[\frac{\varepsilon}{(MC_{p^*})^m}\right]^{p^*/(p^*-m)} \quad \text{for all } R \ge r,$$

since $w \in L^{p^*/(p^*-m)}(\mathbb{H}^n)$ by $(\mathcal{H}_2^+)$. Consequently, for any measurable subset $U \subset \mathbb{H}^n$, with $|U| < \delta$,

$$\int_U w(q)|u_k(q) - u(q)|^m dq \le \left(\int_U w(q)^{p^*/(p^*-m)} dq\right)^{(p^*-m)/p^*} \|u_k - u\|_{p^*}^m < \varepsilon,$$
$$\int_{\mathbb{H}^n \setminus B_R} w(q)|u_k(q) - u(q)|^m dq \le \left(\int_{\mathbb{H}^n \setminus B_R} w(q)^{p^*/(p^*-m)} dq\right)^{(p^*-m)/p^*} \|u_k - u\|_{p^*}^m < \varepsilon$$

for all $R \ge r$. Therefore, the fact that $w|u_k - u| \to 0$ a.e. in $\mathbb{H}^n$ and the Vitali convergence theorem yield

$$\lim_{k\to\infty} \int_{\mathbb{H}^n} w(q)|u_k(q) - u(q)|^m dq = 0,$$

that is $u_k \to u$ in $L^m(\mathbb{H}^n, w)$, as claimed. This completes the proof. $\square$

3. Preliminary results for existence

Let us first note that for all $u \in E$

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H u)_H + \|u\|_{p,a}^p \ge \kappa\big(\|D_H u\|_p^p + \|u\|_{p,a}^p\big), \tag{15}$$

where $\kappa = \min\{c_1, 1\} > 0$ and c_1 is given in (1). Using (5), for all $u \in E$, we get

$$\int_{\mathbb{H}^n} \mathscr{A}(q, D_H u) dq + \frac{1}{p}\|u\|_{p,a}^p \ge \frac{\kappa}{p}\big(\|D_H u\|_p^p + \|u\|_{p,a}^p\big). \tag{16}$$

Lemma 3.1. *Assume that $(\mathcal{H}_2)$ holds. If $u \in X\backslash\{0\}$ and $\lambda \in \mathbb{R}$ satisfy*

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H u)_H dq + \|u\|_{p,a}^p + \|u\|_{\mathfrak{m},h}^{\mathfrak{m}} = \lambda \|u\|_{m,w}^m, \tag{17}$$

then
$$0 < \kappa\big(\|D_H u\|_p^p + \|u\|_{p,a}^p\big) + \|u\|_{\mathfrak{m},h}^{\mathfrak{m}} \le \lambda \|u\|_{m,w}^m \tag{18}$$

and $\lambda > 0$. Moreover,

(i) *if (2) holds, then $\|u\|_{m,w} \le \kappa_1 \lambda^{1/(\mathfrak{m}-m)}$;*

(ii) *if (3) is valid, then $\|u\|_{m,w} \le \kappa_2 \lambda^{[p(m-1)+\mathfrak{m}-m]/(\mathfrak{m}-m)(p-1)m}$;*

(iii) *if $(\mathcal{H}_2^+)$ is satisfied, then $\kappa_3 \lambda^{1/(p-m)} \le \|u\|_{m,w}$,*

and the positive constants κ_1, κ_2 and κ_3 are independent of u.

Proof. Let $u \in X\backslash\{0\}$ and $\lambda \in \mathbb{R}$ satisfy (17). By (15), we have at once (18). Hence $\lambda > 0$, since $u \ne 0$.

(i) Assume that (2) is satisfied.

By (17) and (18), the Young inequality, with $\sigma = \mathfrak{m}/m > 1$, $\sigma' = \mathfrak{m}/(\mathfrak{m} - m)$, $\varepsilon = \mathfrak{m}/2m > 0$ and $C = C_\varepsilon = (2m/\mathfrak{m})^{m/(\mathfrak{m}-m)}(\mathfrak{m} - m)/m > 0$ yields

$$\begin{aligned}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} &\le \int_{\mathbb{H}^n} \lambda w(q)\,|u|^m\,dq = \int_{\mathbb{H}^n} h(q)^{m/\mathfrak{m}}|u|^m \frac{\lambda w(q)}{h(q)^{m/\mathfrak{m}}}dq \\ &\le \int_{\mathbb{H}^n}\left[\frac{1}{2}h(q)|u|^{\mathfrak{m}} + C\left(\frac{\lambda w(q)}{h(q)^{m/\mathfrak{m}}}\right)^{\mathfrak{m}/(\mathfrak{m}-m)}\right]dq \\ &\le \frac{1}{2}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} + C\mathscr{H}\lambda^{\mathfrak{m}/(\mathfrak{m}-m)}.\end{aligned}$$

By (9) and the above inequality, we have

$$\|u\|_{m,w} \le C_{\mathscr{H}}\|u\|_{\mathfrak{m},h} \le \kappa_1\lambda^{1/(\mathfrak{m}-m)},$$

with $\kappa_1 = 2C_{\mathscr{H}}C\mathscr{H}$, as stated.

(ii) Assume that (3) is satisfied.

By (1), (11) and (17), putting $\tilde{C}_1 = \max\{1, 1/c_1\}$, we have

$$\begin{aligned}\|u\|_E^p &\le \tilde{C}_1\int_{\mathbb{H}^n}\left[(\mathbf{A}(q, D_H u), D_H u)_H + a(q)\,|u|^p + \frac{1}{2}h(q)\,|u|^{\mathfrak{m}}\right]dq \\ &= \tilde{C}_1\int_{\mathbb{H}^n}\left[\lambda w(q)\,|u|^{m-1} - \frac{1}{2}h(q)\,|u|^{\mathfrak{m}-1}\right]\cdot|u|\,dq \\ &\le \tilde{C}_1 C_1\int_{\mathbb{H}^n}\lambda w(\lambda w/h)^{(m-1)/(\mathfrak{m}-m)}\,|u|\,dq \\ &\le \tilde{C}_1 C_1\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)}\left\|w(w/h)^{(m-1)/(\mathfrak{m}-m)}\right\|_{p^{*\prime}}\|u\|_{p^*} \\ &\le C_2\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)}\|D_H u\|_p \le C_2\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)}\|u\|_E,\end{aligned} \tag{19}$$

where $C_2 = \tilde{C}_1 C_1 C_{p^*}\mathscr{W}$, $C_1 = 2^{(m-1)/(\mathfrak{m}-m)}$, as in (12). Thus

$$\|u\|_E \le C\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)(p-1)}, \tag{20}$$

with $C = C_2^{1/(p-1)}$. By (15) and (17), since $\lambda > 0$, we finally get, thanks to (19) and (20),

$$\begin{aligned}\|u\|_{m,w} &\le \left(\frac{2}{\lambda}\int_{\mathbb{H}^n}\left[(\mathbf{A}(q, D_H u), D_H u)_H + a(q)\,|u|^p + \frac{1}{2}h(q)\,|u|^{\mathfrak{m}}\right]dq\right)^{1/m} \\ &\le \left(2C_{\mathscr{W}}\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)-1}C\lambda^{(\mathfrak{m}-1)/(\mathfrak{m}-m)(p-1)}\right]^{1/m} \\ &= \kappa_2\lambda^{[p(m-1)+\mathfrak{m}-m]/(\mathfrak{m}-m)(p-1)m},\end{aligned}$$

where $C_{\mathscr{W}} = C_1 C_{p^*}\mathscr{W}$ as above, and $\kappa_2 = (2C_{\mathscr{W}}C)^{1/m}$.

(iii) Assume that $(\mathcal{H}_2^+)$ holds. Then Lemma 2.5, and (18) imply

$$\|u\|_{m,w} \le C_m \|u\|_E \le C_m \left(\lambda/\kappa\right)^{1/p} \|u\|_{m,w}^{m/p},$$

where $C_m > 0$ is the constant of the embedding $E \hookrightarrow L^m(\mathbb{H}^n, w)$. In conclusion, since $u \neq 0$ and $p < m$, we get $\|u\|_{m,w} \ge \kappa_3 \lambda^{1/(p-m)}$, where $\kappa_3 = (C_m^p \kappa)^{1/(p-m)}$. This completes the proof. $\square$

We say that $u \in X$ is a *(weak) entire solution of* $(\mathcal{E}_\lambda)$ if, for all $v \in X$,

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H v)_H dq + \int_{\mathbb{H}^n} a(q)\, |u|^{p-2} uv dq \\ = \lambda \int_{\mathbb{H}^n} w(q)\, |u|^{m-2} uv dq - \int_{\mathbb{H}^n} h(q)\, |u|^{\mathfrak{m}-2} uv dq\,.$$

Hence the entire solutions of $(\mathcal{E}_\lambda)$ correspond to the critical points of the energy functional $\Phi_\lambda \colon X \to \mathbb{R}$, defined by

$$\Phi_\lambda(u) = \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u) dq + \frac{1}{p}\|u\|_{p,a}^p - \frac{\lambda}{m}\|u\|_{m,w}^m + \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\,.$$

If $(\mathcal{E}_\lambda)$ *admits a nontrivial entire solution* $u \in X$*, then* $\lambda > 0$*. From now on we consider only the case* $\lambda > 0$*.*

Lemma 3.2. *Assume that* $(\mathcal{H}_2)$ *holds.*

(i) *The functional* Φ_λ *is coercive in* X *and any sequence* $(u_k)_k$ *in* X*, with* $(\Phi_\lambda(u_k))_k$ *bounded, admits a weakly convergent subsequence in* X*.*

(ii) *Fixed* $\lambda > 0$*, all the critical points of* Φ_λ *are uniformly bounded in* X*.*

Proof. (i) We divide the proof into two parts.

Part (a): Let (2) be satisfied. For all $q \in \mathbb{H}^n$ we apply the elementary inequality (11), now with $k_1 = \lambda w(q)/m,\ k_2 = h(q)/p'\mathfrak{m},\ \alpha = m$ and $\beta = \mathfrak{m}$. Then

$$\lambda \frac{w(q)}{m}\, |u(q)|^m - \frac{h(q)}{p'\mathfrak{m}}\, |u(q)|^{\mathfrak{m}} \le \lambda \frac{w(q)}{m} \left[\frac{\lambda w(q)/m}{h(q)/p'\mathfrak{m}}\right]^{m/(\mathfrak{m}-m)} \\ = p' \frac{\mathfrak{m}}{m} \lambda^{\mathfrak{m}/(\mathfrak{m}-m)} \left[\frac{w(q)^{\mathfrak{m}}}{h(q)^m}\right]^{1/(\mathfrak{m}-m)}.$$

Integrating the above inequality over $\mathbb{H}^n$, we get by (2)

$$\frac{\lambda}{m}\|u\|_{m,w}^m - \frac{1}{p'\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} \le C_\lambda,$$

where $C_\lambda = \mathscr{H} p' \mathfrak{m} \lambda^{\mathfrak{m}/(\mathfrak{m}-m)}/m$.

Therefore, by (16), putting $p\tilde{c} = \min\{\kappa, 1/\mathfrak{m}\}$, for all $u \in X$ we have

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p - \left[\frac{\lambda}{m}\|u\|_{m,w}^m - \frac{1}{p'\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\right] \\
&\qquad\qquad - \frac{1}{p'\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} + \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} \\
&\geq \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p - C_\lambda + \frac{1}{p\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} \\
&\geq \tilde{c}\left(\|u\|_E^p + \|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\right) - C_\lambda \geq \tilde{c}(\|u\|_E^{1+\epsilon} - 1) + \tilde{c}\big(\|u\|_{\mathfrak{m},h}^{1+\epsilon} - 1\big) - C_\lambda \\
&\geq \tilde{c}\|u\|^{1+\epsilon} - C_\lambda - 2\tilde{c},
\end{aligned}$$

where $\epsilon = \min\{p-1, \mathfrak{m}-1\} > 0$. In conclusion, Φ_λ is coercive in X.

The last part of the claim follows at once by the coercivity of Φ_λ and the reflexivity of the Banach space X.

Part (b): Let (3) hold. Again by (11) and (16), putting now $\tilde{c} = \min\{\kappa/p, 1/2\mathfrak{m}\}$, arguing as in (12) and using the same notations, we get for all $u \in X$

$$\begin{aligned}
\Phi_\lambda(u) &= \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p - \left[\frac{\lambda}{m}\|u\|_{m,w}^m - \frac{1}{2\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\right] + \frac{1}{2\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} \\
&\geq \frac{\kappa}{p}\|u\|_E^p + \frac{1}{2\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} - \int_{\mathbb{H}^n}\left[\lambda\frac{w(q)}{m}|u|^{m-1} - \frac{h(q)}{2\mathfrak{m}}|u|^{\mathfrak{m}-1}\right] \cdot |u|\, dq \\
&\geq \tilde{c}\big(\|u\|_E^p + \|u\|_{\mathfrak{m},h}^{\mathfrak{m}}\big) - C_1\int_{\mathbb{H}^n} \lambda w(\lambda w/h)^{(m-1)/(\mathfrak{m}-m)}|u|\, dq \\
&\geq \tilde{c}(\|u\|_E^{1+\epsilon} - 1) + \tilde{c}\big(\|u\|_{\mathfrak{m},h}^{1+\epsilon} - 1\big) - C_{\mathscr{W}}\|D_H u\|_p \\
&\geq \tilde{c}\|u\|^{1+\epsilon} - C_{\mathscr{W}}\|u\| - 2\tilde{c},
\end{aligned}$$

with again $\epsilon = \min\{p-1, \mathfrak{m}-1\} > 0$ and $C_{\mathscr{W}} = C_1 C_{p^*}\mathscr{W}$. Thus Φ_λ is coercive in X also in this latter case.

(ii) Fix $\lambda > 0$ and let $\mathcal{S}_\lambda = \{u \in X : u \text{ is a critical point of } \Phi_\lambda\}$. Clearly every $u \in \mathcal{S}_\lambda$ is a solution of $(\mathcal{E}_\lambda)$ and so satisfies (17). Hence either (i) or (ii) of Lemma 3.1 is valid. In conclusion, $\mathcal{S}_\lambda$ is bounded in $L^m(\mathbb{H}^n, w)$, and so in E and also in $L^{\mathfrak{m}}(\mathbb{H}^n, h)$ by (18). In particular, $\mathcal{S}_\lambda$ is bounded in X, as required. □

Define the functionals $\Phi_{\mathscr{A}}$, Φ_a, Φ_w, $\Phi_h\colon X \to \mathbb{R}$ by

$$\Phi_{\mathscr{A}}(u) = \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq, \qquad \Phi_a(u) = \frac{1}{p}\|u\|_{p,a}^p,$$

$$\Phi_w(u) = \frac{1}{m}\|u\|_{m,w}^m, \qquad \Phi_h(u) = \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}.$$

Lemma 3.3. *The functional $\Phi_{\mathscr{A}}$ is convex, of class $C^1(X)$ and sequentially weakly lower semi-continuous in X.*

Proof. Fix $(u_k)_k$ and u in X, with $u_k \to u$ in X. Then $\|D_H u_k - D_H u\|_p \to 0$ and, up to a subsequence, $D_H u_k \to D_H u$ a.e. in $\mathbb{H}_H$, and $\big(|D_H u_k - D_H u|_H^p\big)_k$ is uniformly integrable by the Vitali convergent theorem. By (5)

$$0 \le \mathscr{A}(q, D_H u_k) \le c_2\big(|D_H u|_H^p + |D_H u_k - D_H u|_H^p\big).$$

Hence it is easy to prove that also the sequence $\big(\mathscr{A}(q, D_H u_k)\big)_k$ is uniformly integrable and $\mathscr{A}(q, D_H u_k) \to \mathscr{A}(q, D_H u)$ a.e. in $\mathbb{H}_H$ for all q by $(\mathcal{A}_1)$. Thus by the Vitali convergent theorem $\lim\limits_{k\to\infty} \Phi_{\mathscr{A}}(u_k) = \Phi_{\mathscr{A}}(u)$, that is $\Phi_{\mathscr{A}}$ is continuous.

Similarly, we can see that as $k \to \infty$

$$\begin{aligned}\|\Phi'_{\mathscr{A}}(u_k) - \Phi'_{\mathscr{A}}(u)\| &= \sup_{\varphi\in X, \|\varphi\|=1} \left| \int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_k) - \mathbf{A}(q, D_H u), D_H\varphi)_H dq \right| \\ &\le \|\mathbf{A}(q, D_H u_k) - \mathbf{A}(q, D_H u)\|_{p'} \to 0.\end{aligned}$$

Therefore, $\Phi'_{\mathscr{A}}$ is also continuous in X. Thus $\Phi_{\mathscr{A}}$ is of class $C^1(X)$.

Obviously, $\Phi_{\mathscr{A}}$ is convex in X by $(\mathcal{A}_1)$(ii). Finally, $\Phi_{\mathscr{A}}$ is sequentially weakly lower semi-continuous in X by Theorem A.2 of [17]. $\square$

Lemma 3.4. *The functionals Φ_a and Φ_h are strictly convex, of class $C^1(X)$ and sequentially weakly lower semi-continuous in X. Moreover, if $(u_k)_k$, u are in X and $u_k \rightharpoonup u$ in X, then $\Phi'_a(u_k) \overset{*}{\rightharpoonup} \Phi'_a(u)$ in X', where X' is the dual space of X, and $\Phi'_h(u_k) \overset{*}{\rightharpoonup} \Phi'_h(u)$ in X'.*

Proof. It is enough to prove the lemma for Φ_a, since the same argument applies to Φ_h with obvious changes.

Arguing as in the proof of Lemma 3.3, we show that also Φ_a is strictly convex, of class $C^1(X)$ and sequentially weakly lower semi-continuous in X. Fix $v \in X$. For any measurable subset $U \subset \mathbb{H}^n$, we have

$$\begin{aligned}&\int_U a(q) \left||u_k(q)|^{p-2} u_k(q) - |u(q)|^{p-2} u(q)\right| \cdot |v(q)|\, dq \\ &\qquad \le \int_U a(q)^{1/p'} (|u_k(q)|^{p-1} + |u(q)|^{p-1}) a(q)^{1/p} |v(q)|\, dq \\ &\qquad \le 2^{1/p} \left(\int_U a(q)\big[\,|u_k(q)|^p + |u(q)|^p\big] dq\right)^{1/p'} \left(\int_U a(q)|v(q)|^p dq\right)^{1/p}.\end{aligned}$$

Similar to the proof of Lemma 2.5, we can see that

$$\big(a(q)\left||u_k|^{p-2} u_k - |u|^{p-2} u\right| \cdot |v|\big)_k$$

is uniformly integrable in $\mathbb{H}^n$. Hence, Lemma 2.4 and an application of Vitali's convergence theorem give

$$\lim_{k\to\infty}\int_{\mathbb{H}^n} a(q)\left||u_k(q)|^{p-2}\,u_k(q)-|u(q)|^{p-2}\,u(q)\right|\cdot|v(q)|\,dq=0.$$

Thus, $\Phi_a'(u_k)\overset{*}{\rightharpoonup}\Phi_a'(u)$ in X'. □

Lemma 3.5. *If $(\mathcal{H}_2)$ holds, then Φ_w is convex, of class $C^1(X)$ and sequentially weakly continuous in X, i.e. $u_k\rightharpoonup u$ in X implies $\Phi_w(u_k)\to\Phi_w(u)$ as $k\to\infty$.*

Proof. Obviously, Φ_w is convex, $C^1(X)$ continuity can be proved as in the proof of Lemma 3.4. Hence, we have only to show that Φ_w is sequentially weak continuous in X. To this aim, let $u_k\rightharpoonup u$ in X. Then $u_k\to u$ in $L^m(\mathbb{H}^n,w)$ by Lemma 2.5, so that $\Phi_w(u_k)\to\Phi_w(u)$ as $k\to\infty$, as required. □

Lemma 3.6. *Assume $(\mathcal{H}_2)$ holds. Then Φ_λ is of class $C^1(X)$ and sequentially weakly lower semi-continuous in X. Hence, if $u_k\rightharpoonup u$ in X, then*

$$\Phi_\lambda(u)\le\liminf_{k\to\infty}\Phi_\lambda(u_k).$$

Proof. Lemmas 3.3 – 3.5 yield that Φ_λ is of class $C^1(X)$. Hence it remains to show that Φ_λ is the sequentially weakly lower semi-continuous in X. To this aim fix $(u_k)_k$, u in X, with $u_k\rightharpoonup u$ as $k\to\infty$.

Since $(\mathcal{H}_2)$ is satisfied, Φ_w is weakly continuous in X by Lemma 3.5. Thus, by Lemmas 3.3 – 3.5 and the fact that $\lambda>0$,

$$\begin{aligned}\liminf_{k\to\infty}\Phi_\lambda(u_k)&\ge\liminf_{k\to\infty}\big[\Phi_{\mathscr{A}}(u_k)+\Phi_a(u_k)+\Phi_h(u_k)\big]-\lambda\limsup_{k\to\infty}\Phi_w(u_k)\\&\ge\Phi_{\mathscr{A}}(u)+\Phi_a(u)+\Phi_h(u)-\lambda\Phi_w(u)=\Phi_\lambda(u).\end{aligned}$$

Therefore Φ_λ is sequentially weakly lower semi-continuous in X. □

Lemma 3.7. *Suppose that $\mathscr{A}$ satisfies also $(\mathcal{A}_2)$, i.e. $\mathscr{A}$ is a uniformly convex function.*

(i) *Then $\Phi_{\mathscr{A}}\colon X\to\mathbb{R}$ is a uniformly convex functional, i.e. for any $\varepsilon\in(0,1)$ there exists $\delta=\delta(\varepsilon)\in(0,1)$ such that for all $u,v\in X$ either*

$$\Phi_{\mathscr{A}}\left(\frac{u-v}{2}\right)\le\varepsilon\,\frac{\Phi_{\mathscr{A}}(u)+\Phi_{\mathscr{A}}(v)}{2}\quad\text{or}\quad\Phi_{\mathscr{A}}\left(\frac{u+v}{2}\right)\le(1-\delta)\,\frac{\Phi_{\mathscr{A}}(u)+\Phi_{\mathscr{A}}(v)}{2}.$$

(ii) *If $u_k\rightharpoonup u$ in X and if $\limsup_{k\to\infty}\langle\Phi_{\mathscr{A}}'(u_k)-\Phi_{\mathscr{A}}'(u),u_k-u\rangle\le0$, then we have $\Phi_{\mathscr{A}}(u_k-u)\to0$ as $k\to\infty$.*

Proof. (i) The statement follows directly from $(\mathcal{A}_2)$.

(ii) Fix $(u_k)_k$, u in X, with $\limsup_{k\to\infty} \langle \Phi'_{\mathscr{A}}(u_k) - \Phi'_{\mathscr{A}}(u), u_k - u\rangle \le 0$ and $u_k \rightharpoonup u$ as $k \to \infty$. Hence, $\lim_{k\to\infty} \langle \Phi'_{\mathscr{A}}(u_k) - \Phi'_{\mathscr{A}}(u), u_k - u\rangle = 0$ by convexity, so that

$$\lim_{k\to\infty} \langle \Phi'_{\mathscr{A}}(u_k), u_k - u\rangle = 0$$

by Lemma 3.3. Since $(\|u_k\|)_k$ is bounded, then $(u_k)_k \subset B_R \subset X$ for some $R > 0$ large enough. Thus also $\big(\Phi_{\mathscr{A}}(u_k)\big)_k$ is bounded by (5) and there exists $c \in \mathbb{R}$ and

$$\Phi_{\mathscr{A}}(u) \le \liminf_{k\to\infty} \Phi_{\mathscr{A}}(u_k) = c$$

by Lemma 3.3. On the other hand, since $\Phi_{\mathscr{A}}$ is convex in X, for all k

$$\Phi_{\mathscr{A}}(u) \ge \Phi_{\mathscr{A}}(u_k) + \langle \Phi'_{\mathscr{A}}(u_k), u - u_k\rangle\,,$$

that is $\Phi_{\mathscr{A}}(u) \ge c$. In conclusion $\Phi_{\mathscr{A}}(u) = c$.

Clearly also $(u_k + u)/2 \rightharpoonup u$ as $k \to \infty$, and again by the weak lower semicontinuity of $\Phi_{\mathscr{A}}$ in X

$$c = \Phi_{\mathscr{A}}(u) \le \liminf_{k\to\infty} \Phi_{\mathscr{A}}\left(\frac{u_k + u}{2}\right). \tag{21}$$

Assume by contradiction that the assertion is false. Then there exists $\epsilon \in (0,1)$ and a subsequence $(u_{k_j})_j$ of $(u_k)_k$ such that

$$\Phi_{\mathscr{A}}(u - u_{k_j}) \ge \epsilon\,\frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(u_{k_j})}{2}$$

for all j. Let $\delta = \delta(\epsilon) \in (0,1)$ be the corresponding number of the uniform convexity of $\Phi_{\mathscr{A}}$ on the ball $B_R \subset X$. Then as $j \to \infty$

$$\Phi_{\mathscr{A}}\left(\frac{u_{k_j} + u}{2}\right) \le (1-\delta)\,\frac{\Phi_{\mathscr{A}}(u) + \Phi_{\mathscr{A}}(u_{k_j})}{2} \to (1-\delta)c < c,$$

which contradicts (21). Thus $\Phi_{\mathscr{A}}(u_k - u) \to 0$ as $k \to \infty$ in X. $\square$

Let $\mathscr{I}\colon X \to \mathbb{R}$ be defined by

$$\mathscr{I}(u) = \frac{1}{p}\int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p + \frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}} = \Phi_{\mathscr{A}}(u) + \Phi_a(u) + \Phi_h(u)$$

for all $u \in X$ and let L be its derivative operator, i.e. $L = \mathscr{I}'\colon X \to X'$, where X' is the dual space of X. Moreover, L can be represented, for all u, $v \in X$, as

$$\langle L(u), v\rangle = \int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H v)_H dq + \int_{\mathbb{H}^n} a(q)\,|u|^{p-2}\,uvdq + \int_{\mathbb{H}^n} h(q)\,|u|^{\mathfrak{m}-2}\,uvdq\,.$$

Lemma 3.8. (i) *$L\colon X\to X'$ is a continuous, bounded, strictly monotone operator. Moreover, if $\mathscr{A}$ satisfies also $(\mathcal{A}_2)$, that is $\mathscr{A}$ is uniform convex, then*

(ii) *L is a map of type (S_+), i.e.*
if $u_k \rightharpoonup u$ in X and $\limsup_{k\to\infty}\langle L(u_k)-L(u), u_k-u\rangle \le 0$, then $u_k\to u$ in X;

(iii) *$L\colon X\to X'$ is a homeomorphism.*

Proof. (i) From Lemmas 3.3 – 3.5 it follows that L is a continuous and bounded operator.

Fix u, $v\in X$, with $u\neq v$, and put $g(t)=\mathscr{I}(t(u-v)+v)$, $t\in\mathbb{R}$. It is easy to check that $t\mapsto \Phi_{\mathscr{A}}(t(u-v)+v)$ is strictly convex in $\mathbb{R}$, since $\mathscr{A}(q,\cdot)$ is strictly convex in $\mathbb{H}_H$ for all $q\in\mathbb{H}^n$ by $(\mathcal{A}_1)$(ii). Moreover, Φ_a and Φ_h are strictly convex in X by Lemma 3.4. Therefore g is strictly convex in $\mathbb{R}$, and its derivative is strictly increasing in $\mathbb{R}$. In particular,

$$0<g'(1)-g'(0)=\langle L(u)-L(v),u-v\rangle.$$

Thus L is strictly monotone in X.

(ii) Property (S_+) for L is a direct consequence of Lemma 3.7.

(iii) The strict monotonicity of L implies that L is an injection operator. Clearly

$$\lim_{\|u\|\to\infty}\frac{\langle Lu,u\rangle}{\|u\|}=\lim_{\|u\|\to\infty}\frac{\int_{\mathbb{H}^n}(\mathbf{A}(q,D_Hu),D_Hu)_H dq+\|u\|_{p,a}^p+\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}}{\|u\|}=\infty,$$

so that L is coercive in X. Hence L is a surjection in view of the Minty-Browder theorem, see Theorem 26A of [21]. Thus L has an inverse operator $L^{-1}\colon X'\to X$ and the continuity of L^{-1} is sufficient to ensure that L is a homeomorphism.

To this aim, fix $(f_k)_k$, $f\in X'$, with $f_k\to f$ in X'. Put $u_k=L^{-1}(f_k)$ for all k and $u=L^{-1}(f)$. Then $L(u_k)=f_k$, $L(u)=f$ and $(u_k)_k$ is bounded in X, since L is coercive in X. Without loss of generality, we assume that $u_k\rightharpoonup u_0$ in X for some $u_0\in X$, being X reflexive. Consequently, as $k\to\infty$

$$\begin{aligned}\langle L(u_k)-L(u_0),u_k-u_0\rangle&=\langle L(u_k)-L(u),u_k-u_0\rangle+\langle L(u)-L(u_0),u_k-u_0\rangle\\&=\langle f_k-f,u_k-u_0\rangle+\langle L(u)-L(u_0),u_k-u_0\rangle=o(1),\end{aligned}$$

since $f_k\to f$ in X' and $u_k\rightharpoonup u_0$ in X. Therefore, $u_k\to u_0$ as $k\to\infty$ by the (S_+) property of L. But $f_k\to f$ in X' and L is continuous in X, so that $L(u_0)=\lim_{k\to\infty}L(u_k)=\lim_{k\to\infty}f_k=f$. Since L is bijective, we conclude that $u_0=u$. Hence L^{-1} is continuous and this completes the proof. □

Lemma 3.9. *If $(\mathcal{A}_2)$ and $(\mathcal{H}_2)$ hold, then Φ_λ satisfies the (PS) condition, namely, $(u_k)_k\subset X$, with $\Phi_\lambda(u_k)\to c$ and $\Phi'_\lambda(u_k)\to 0$ in X', admits a convergent subsequence in X.*

Proof. Fix $(u_k)_k \subset X$ such that $\Phi_\lambda(u_k) \to c$ and $\Phi'_\lambda(u_k) \to 0$ in X'. Then $(u_k)_k$ is bounded in X, being Φ_λ coercive in X by Lemma 3.2 and $(\mathcal{H}_2)$. Thus $(u_k)_k$ has a weakly convergent subsequence, still denoted by $(u_k)_k$, since X is reflexive by Lemma 2.2. Hence $u_k \rightharpoonup u$ as $k \to \infty$ for some u in X.

By Lemma 2.5 and $(\mathcal{H}_2)$ the embedding $X \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact, so that $\Phi'_w(u_k) \to \Phi'_w(u)$ in X'. Consequently, being L^{-1} continuous from X' to X by Lemma 3.8 and $(\mathcal{A}_2)$, it follows that $u_k \to L^{-1} \circ \Phi'_w(u)$ in X and we are done. □

For all $\lambda > 0$ and $(q, u) \in \mathbb{H}^n \times \mathbb{R}$ put

$$g_\lambda(q, u) = -a(q)\,|u|^{p-2}\,u + \lambda w(q)\,|u|^{m-2}\,u - h(q)\,|u|^{\mathfrak{m}-2}\,u.$$

Lemma 3.10. *Assume that $(\mathcal{H}_2)$ holds. Then $g_\lambda(\cdot, u) \in L^1_{\text{loc}}(\mathbb{H}^n)$ along any $u \in X$.*

Proof. Fix $\lambda > 0$, $u \in X$ and $R > 0$. Clearly

$$\int_{B_R} a(q)\,|u|^{p-1}\,dq \le \int_{B_R} a(q)(|u|^p + 1)dq \le \|u\|^p_{p,a} + |B_R| \operatorname*{ess\,sup}_{q \in B_R} a(q);$$
$$\int_{B_R} h(q)\,|u|^{\mathfrak{m}-1}\,dq \le \int_{B_R} h(q)(|u|^{\mathfrak{m}} + 1)dq \le \|u\|^{\mathfrak{m}}_{\mathfrak{m},h} + \int_{B_R} h(q)dq.$$

By Lemma 2.5 and $(\mathcal{H}_2)$

$$\begin{aligned}\lambda \int_{B_R} w(q)\,|u|^{m-1}\,dq &\le \lambda \int_{B_R} w(q)(|u|^m + 1)dq \le \lambda \left(\|u\|^m_{m,w} + \int_{B_R} w(q)dq \right) \\ &\le \lambda C \left(\|u\|^m + \int_{B_R} w(q)dq \right),\end{aligned}$$

where $C = \max\{1, C_m\}$. Hence, summarizing the above inequalities, we obtain

$$\int_{B_R} |g_\lambda(q, u)|\,dq < \infty.$$

Thus $g_\lambda(\cdot, u) \in L^1_{\text{loc}}(\mathbb{H}^n)$ along $u \in X$. □

4. Proof of Theorem 1.1

Throughout the section we assume also condition $(\mathcal{H}_2)$, without further mentioning. Let us introduce the *crucial* value

$$\overline{\lambda} = \inf_{\substack{u \in X \\ \Phi_w(u)=1}} \left\{ \int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|^p_{p,a} + \frac{1}{\mathfrak{m}}\|u\|^{\mathfrak{m}}_{\mathfrak{m},h} \right\} = \inf_{\substack{u \in X \\ \Phi_w(u)=1}} \mathscr{I}(u).$$

We claim that $\overline{\lambda} > 0$. Indeed, for any $u \in X$, with $\Phi_w(u) = 1$, then $\|u\|_{m,w} = m^{1/m} > 1$. By Lemma 2.5 and $(\mathcal{H}_2)$, there exists a constant $C_m > 0$ such that

$\|u\|_{m,w} \leq C_m\|u\|$ for any $u \in X$. Thus $\|u\| \geq 1/C_m$ for any $u \in X$, with $\Phi_w(u) = 1$. Since $\|u\| = \|u\|_E + \|u\|_{\mathfrak{m},h}$, it follows that either

$$\|u\|_E \geq \frac{1}{2C_m}, \quad \text{or} \quad \|u\|_{\mathfrak{m},h} \geq \frac{1}{2C_m}.$$

The first case and (16) imply

$$\int_{\mathbb{H}^n} \mathscr{A}(q, D_H u)dq + \frac{1}{p}\|u\|_{p,a}^p \geq \frac{\kappa}{p}\|u\|_E^p \geq \frac{\kappa}{p}\left(\frac{1}{2C_m}\right)^p.$$

Put $C = \min\{\kappa/p(2C_m)^p, 1/\mathfrak{m}(2C_m)^{\mathfrak{m}}\}$. Then, in both cases, $\mathscr{I}(u) \geq C$ for all $u \in X$, with $\Phi_w(u) = 1$. Thus $\overline{\lambda} \geq C > 0$ and the claim is proved.

Lemma 4.1. *For all $\lambda > \overline{\lambda}$ there exists a global nontrivial nonnegative minimizer $e \in X$ of Φ_λ with negative energy, that is $\Phi_\lambda(e) < 0$.*

Proof. By Lemma 3.2 and $(\mathcal{H}_2)$ the functional Φ_λ is coercive in X and Lemma 3.6 gives that Φ_λ is sequentially weakly lower semi-continuous in X. Hence for all $\lambda > 0$ there exists a global minimizer $e \in X$ of Φ_λ, that is

$$\Phi_\lambda(e) = \inf_{v \in X} \Phi_\lambda(v).$$

Clearly e is a solution of $(\mathcal{E}_\lambda)$. The definition of $\overline{\lambda}$ yields that $\inf_{v\in X}\Phi_\lambda(v) < 0$ for all $\lambda > \overline{\lambda}$. Thus $e \neq 0$. In conclusion, for any $\lambda > \overline{\lambda}$ equation $(\mathcal{E}_\lambda)$ has a nontrivial solution $e \in X$ such that $\Phi_\lambda(e) < 0$. Finally, we may assume $e \geq 0$ a.e. in $\mathbb{H}^n$, since $|e| \in X$ and $\Phi_\lambda(e)= \Phi_\lambda(|e|)$ by $(\mathcal{A}_1)$(i). □

Put $\mathcal{E} = \{\lambda \in \mathbb{R} : (\mathcal{E}_\lambda) \text{ admits a nontrivial nonnegative entire solution}\}$.

Lemma 4.1 assures that $\mathcal{E}$ is nonempty. Set

$$\lambda_* = \sup\{\lambda : (\mathcal{E}_\mu) \text{ admits only the trivial solution for all } \mu < \lambda\},$$
$$\lambda_{**} = \inf\{\lambda : \lambda \in \mathcal{E}\}.$$

Clearly, $\lambda_* \geq 0$ and $\lambda_{**} \geq 0$ by Lemma 3.1.

Theorem 4.2. *For all $\lambda > \lambda_{**}$ equation $(\mathcal{E}_\lambda)$ admits a nontrivial nonnegative entire solution $u_\lambda \in X$. Moreover, $\lambda_*=\lambda_{**}$.*

Proof. Fix $\lambda > \lambda_{**}$. Then there exist $\mu \in \mathcal{E}$, with $0 \leq \lambda_{**} < \mu < \lambda$, and a nontrivial nonnegative entire solution $u_\mu \in X$ of $(\mathcal{E}_\mu)$. We next show that $(\mathcal{E}_\lambda)$ admits a nontrivial nonnegative entire solution $u_\lambda \in X$. For all $q \in \mathbb{H}^n$ put

$$f_{u_\mu}(q,t) = \begin{cases} w(q)\,|t|^{m-2}\,t, & \text{if } t \geq u_\mu(q), \\ w(q)\,|u_\mu(q)|^{m-2}\,u_\mu(q), & \text{if } t < u_\mu(q), \end{cases}$$

and

$$F_{u_\mu}(q,t) = \int_0^t f_{u_\mu}(q,\tau)d\tau.$$

Let us consider

$$(\mathcal{E}_\lambda)^+ \qquad -\operatorname{div}\mathbf{A}(q,D_H u)+a(q)\,|u|^{p-2}\,u+h(q)\,|u|^{\mathfrak{m}-2}\,u \;=\lambda f_{u_\mu}(q,u).$$

The solutions of $(\mathcal{E}_\lambda)^+$ are the critical points of $\Phi_{\lambda,u_\mu}\colon X\to\mathbb{R}$ defined by

$$\Phi_{\lambda,u_\mu}(u)=\int_{\mathbb{H}^n}\mathscr{A}(q,D_H u)dq+\frac{1}{p}\|u\|_{p,a}^p+\frac{1}{\mathfrak{m}}\|u\|_{\mathfrak{m},h}^{\mathfrak{m}}-\lambda\int_{\mathbb{H}^n}F_{u_\mu}(q,u(q))dq.$$

As in the proof of Lemmas 3.2 and 3.6, it is not hard to see that Φ_{λ,u_μ} is coercive and sequentially weakly lower semi-continuous in X. Furthermore Φ_{λ,u_μ} achieves its infimum in X at $u_\lambda\in X$. Therefore u_λ is a nontrivial nonnegative entire solution of $(\mathcal{E}_\lambda)$ provided that $u_\lambda\ge u_\mu$.

To see this, first we show that u_μ is a sub-solution for $(\mathcal{E}_\lambda)^+$. Indeed, for all $v\in X$, with $v\ge 0$, we have

$$\begin{aligned}
&\int_{\mathbb{H}^n}(\mathbf{A}(q,D_H u_\mu),D_H v)_H dq+\int_{\mathbb{H}^n}a(q)\,|u_\mu|^{p-2}u_\mu v dq+\int_{\mathbb{H}^n}h(q)\,|u_\mu|^{\mathfrak{m}-2}\,u_\mu v dq\\
&=\mu\int_{\mathbb{H}^n}w(q)\,|u_\mu|^{m-2}\,u_\mu v dq\le\lambda\int_{\mathbb{H}^n}w(q)\,|u_\mu|^{m-2}\,u_\mu v dq=\lambda\int_{\mathbb{H}^n}f_{u_\mu}(q,u_\mu)v dq.
\end{aligned}$$

As test function v take $(u_\mu-u_\lambda)^+=\max\{u_\mu-u_\lambda,0\}\in X$. Then

$$\begin{aligned}
&\langle L(u_\lambda)-L(u_\mu),(u_\mu-u_\lambda)^+\rangle\\
&\quad=\int_{\mathbb{H}^n}(\mathbf{A}(q,D_H u_\lambda)-\mathbf{A}(q,D_H u_\mu),D_H(u_\mu-u_\lambda)^+)dq\\
&\qquad+\int_{\mathbb{H}^n}a(q)(|u_\lambda|^{p-2}\,u_\lambda-|u_\mu|^{p-2}\,u_\mu)(u_\mu-u_\lambda)^+dq\\
&\qquad+\int_{\mathbb{H}^n}h(q)(|u_\lambda|^{\mathfrak{m}-2}\,u_\lambda-|u_\mu|^{\mathfrak{m}-2}\,u_\mu)(u_\mu-u_\lambda)^+dq\\
&\quad\ge\lambda\int_{\mathbb{H}^n}(f_{u_\mu}(q,u_\lambda)-f_{u_\mu}(q,u_\mu))(u_\mu-u_\lambda)^+dq=0.
\end{aligned}$$

Hence $(u_\mu-u_\lambda)^+\equiv 0$ in $\mathbb{H}^n$, since L is a strictly monotone operator by Lemma 3.8(i). Thus $u_\mu\le u_\lambda$, as claimed.

The first part of the statement shows that $\lambda^{**}\ge\lambda^*$. Suppose by contradiction that $\lambda^{**}>\lambda^*$. Then $(\mathcal{E}_\lambda)$ cannot admit a nontrivial solution $u\in X$ if $\lambda<\lambda^{**}$, since this would contradict the minimality of λ^{**}. Hence, for all $\lambda\in[\lambda^*,\lambda^{**})$ the unique solution of $(\mathcal{E}_\lambda)$ is $u\equiv 0$. But this is again impossible since it would contradict the maximality of λ^*. Hence $\lambda^{**}=\lambda^*$. □

Theorem 4.3. *Let also $(\mathcal{H}_2^+)$ hold. Then $(\mathcal{E}_{\lambda_*})$ admits a nontrivial nonnegative entire solution $u\in X$ and so $\lambda_*>0$.*

Proof. The argument is similar to the proof of Lemma 3.4 of [14], see also Lemma 4.3 of [3], given both in the Heisenberg setting. For the reader's convenience we present it also in this simpler but different context. Let $(\lambda_k)_k$ be a strictly decreasing sequence converging to λ_* and let $u_k \in X$ be a nontrivial nonnegative entire solution of $(\mathcal{E}_{\lambda_k})$. Then for all $v \in X$

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_k), D_H v)_H dq = \int_{\mathbb{H}^n} g_k(q) v dq, \tag{22}$$

where $g_k(q) = -a(q)u_k^{p-1} + \lambda_k w(q) u_k^{m-1} - h(q)]u_k^{\mathfrak{m}-1}$ for all k.

By (17) and the monotonicity of $(\lambda_k)_k$, we obtain

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_k), D_H u_k)_H dq + \|u_k\|_{p,a}^p + \|u_k\|_{\mathfrak{m},h}^{\mathfrak{m}} = \lambda_k \|u_k\|_{m,w}^m \leq C,$$

where $C = \kappa_1^m \lambda_1^{\mathfrak{m}/(\mathfrak{m}-m)}$ by Lemma 3.1(i) if (2) holds, while

$$C = \kappa_2^m \lambda_1^{p(\mathfrak{m}-1)/(\mathfrak{m}-m)(p-1)},$$

thanks to Lemma 3.1(ii) if (3) is satisfied. Therefore, the sequences $(\|u_k\|_E)_k$ and $(\|u_k\|_{\mathfrak{m},h})_k$ are bounded, and in turn also $(\|u_k\|)_k$ is bounded. Hence, $(g_k)_k$ is bounded in $L^1_{\text{loc}}(\mathbb{H}^n)$, since also $(\lambda_k)_k$ is bounded. Moreover, by $(\mathcal{A}_1)$(iii), Lemmas 2.2 and 2.5, it is possible to extract a subsequence, still relabeled $(u_k)_k$, satisfying

$$\begin{aligned} &u_k \rightharpoonup u \text{ in } X, && u_k \to u \text{ in } L^m(\mathbb{H}^n, w), \\ &u_k \rightharpoonup u \text{ in } L^{\mathfrak{m}}(\mathbb{H}^n, h), && u_k \to u \text{ a.e. in } \mathbb{H}^n, \\ &D_H u_k \rightharpoonup D_H u \text{ in } L^p(\mathbb{H}^n, \mathbb{R}^{2n}), && \mathbf{A}(q, D_H u_k) \rightharpoonup \Theta \text{ in } L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n}), \end{aligned} \tag{23}$$

for some $u \in X$ and $\Theta \in L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n})$. We claim that $\Theta = \mathbf{A}(\cdot, D_H u)$ and that u, which is clearly nonnegative in $\mathbb{H}^n$ by (23), is the solution we are looking for.

Step 1. Fix $R > 0$. Let $\varphi_R \in C_c^\infty(\mathbb{H}^n)$ be such that $0 \leq \varphi_R \leq 1$ in $\mathbb{H}^n$ and $\varphi_R \equiv 1$ in B_R. Given $\epsilon > 0$ define for each $t \in \mathbb{R}$

$$\eta_\epsilon(t) = \begin{cases} t, & \text{if } |t| < \epsilon, \\ \epsilon \dfrac{t}{|t|}, & \text{if } |t| \geq \epsilon. \end{cases}$$

Put $v_k = \varphi_R \eta_\epsilon \circ (u_k - u)$, so that $v_k \in X$ for all k. Taking $v = v_k$ in (22), we get

$$\begin{aligned} &\int_{\mathbb{H}^n} \varphi_R (\mathbf{A}(q, D_H u_k) - \mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H dq \\ &\quad = - \int_{\mathbb{H}^n} (\eta_\epsilon \circ (u_k - u))(\mathbf{A}(q, D_H u_k), D_H \varphi_R)_H dq \\ &\qquad - \int_{\mathbb{H}^n} \varphi_R (\mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H dq + \int_{\mathbb{H}^n} g_k(q) v_k dq. \end{aligned} \tag{24}$$

Moreover,

$$\int_{\mathbb{H}^n} (\eta_\epsilon \circ (u_k - u))(\mathbf{A}(q, D_H u_k), D_H \varphi_R)_H dq \to 0 \quad \text{as } k \to \infty,$$

since $\eta_\epsilon \circ (u_k - u) D_H \varphi_R \to 0$ in $L^p(\operatorname{supp} \varphi_R, \mathbb{R}^{2n})$ and $\mathbf{A}(q, D_H u_k) \rightharpoonup \Theta$ in $L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n})$ by (23). Furthermore, $\varphi_R D_H(\eta_\epsilon \circ (u_k - u)) \rightharpoonup 0$ in $L^p(\mathbb{H}^n, \mathbb{R}^{2n})$ by Lemma 2.3 and (23), since $u_k \rightharpoonup u$ in X. Consequently,

$$\int_{\mathbb{H}^n} \varphi_R(\mathbf{A}(q, D_H u), D_H(\eta \circ (u_k - u)))_H dq \to 0 \quad \text{as } k \to \infty,$$

being $\mathbf{A}(\cdot, D_H u) \in L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n})$.

In conclusion, the first two terms in the right hand side of (24) go to zero as $k \to \infty$. Now, recalling that $0 \le \varphi_R \le 1$ in $\mathbb{H}^n$, we have

$$\int_{\mathbb{H}^n} g_k(q) v_k dq \le \int_{\operatorname{supp} \varphi_R} |g_k(q)| \cdot |\eta_\epsilon \circ (u_k - u)| \, dq \le \epsilon \int_{\operatorname{supp} \varphi_R} |g_k(q)| \, dq \le \epsilon \, C_R,$$

since $(g_k)_k$ is bounded in $L^1_{\text{loc}}(\mathbb{H}^n)$ by Lemma 3.10. Now $(\mathcal{A}_1)$(ii) and the definitions of φ_R and η_ϵ yield

$$\varphi_R(\mathbf{A}(q, D_H u_k) - \mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H \ge 0 \qquad \text{a.e. in } \mathbb{H}^n,$$

and in turn

$$\int_{B_R} (\varphi_R \mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H dq \\ \le \int_{\mathbb{H}^n} \varphi_R(\mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H dq.$$

Combining all these facts with (24), we find that

$$\limsup_{k \to \infty} \int_{B_R} \varphi_R(\mathbf{A}(q, D_H u), D_H(\eta_\epsilon \circ (u_k - u)))_H dq \le \epsilon \, C_R. \tag{25}$$

Define the nonnegative functions e_k for all k by

$$e_k(q) = (\mathbf{A}(q, D_H u_k) - \mathbf{A}(q, D_H u), D_H(u_k - u))_H.$$

We claim that $(e_k)_k$ is bounded in $L^1(\mathbb{H}^n)$. Indeed,

$$0 \le \int_{\mathbb{H}^n} e_k dq \le \|\mathbf{A}(\cdot, D_H u_k) - \mathbf{A}(\cdot, D_H u)\|_{p'} \|D_H u_k - D_H u\|_p \le C_0, \tag{26}$$

where C_0 is an appropriate constant, independent of k, since $(D_H u_k)_k$ in the space $L^p(\mathbb{H}^n, \mathbb{R}^{2n})$ and $\big(\mathbf{A}(\cdot, D_H u_k)\big)_k$ in $L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n})$ are both bounded by (23).

Fix $\theta \in (0,1)$. Split the ball B_R into

$$S_k^\epsilon(R) = \{q \in B_R : |u_k(q) - u(q)| \le \epsilon\}, \quad G_k^\epsilon(R) = B_R \backslash S_k^\epsilon(R).$$

By the Hölder inequality,

$$\begin{aligned}\int_{B_R} e_k^\theta dq &\le \left(\int_{S_k^\epsilon(R)} e_k dq\right)^\theta |S_k^\epsilon(R)|^{1-\theta} + \left(\int_{G_k^\epsilon(R)} e_k dq\right)^\theta |G_k^\epsilon(R)|^{1-\theta} \\ &\le (\epsilon C_R)^\theta |S_k^\epsilon(R)|^{1-\theta} + C_0^\theta |G_k^\epsilon(R)|^{1-\theta},\end{aligned}$$

by (25), since $\varphi_R \equiv 1$ and $D_H(\eta_\epsilon \circ (u_k - u)) = D_H(u_k - u)$ in $S_k^\epsilon(R)$, and by (26). Since $u_k \to u$ in measure in B_R, then $|G_k^\epsilon(R)|$ tends to zero as $k \to \infty$. Therefore

$$0 \le \limsup_{k\to\infty} \int_{B_R} e_k^\theta dq \le (\epsilon\, C_R)^\theta |B_R|^{1-\theta}.$$

Letting ϵ tend to 0^+ we find that $e_k^\theta \to 0$ in $L^1(B_R)$ and so, since $R > 0$ is arbitrary, we deduce that, up to a subsequence,

$$e_k \to 0 \quad \text{a.e. in } \mathbb{H}^n.$$

Thus, Lemma 2.1 of [1] yields that also

$$D_H u_k \to D_H u \quad \text{a.e. in } \mathbb{H}^n,$$

and so by $(\mathcal{A}_1)$: $\quad \mathbf{A}(q, D_H u_k) \to \mathbf{A}(q, D_H u) \quad$ a.e. in $\mathbb{H}^n$.
Hence, Proposition A.7. of [1] implies that $\Theta = \mathbf{A}(\cdot, D_H u)$ and consequently

$$\int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_k), D_H v)_H dq \to \int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u), D_H v) dq \tag{27}$$

for all $v \in X$ as $k \to \infty$, since $\mathbf{A}(\cdot, D_H u_k) \to \mathbf{A}(\cdot, D_H u)$ in $L^{p'}(\mathbb{H}^n, \mathbb{R}^{2n})$ by (23).

Step 2. Since $u_k \rightharpoonup u$ in X, Lemma 3.4 yields in particular that for all $v \in X$

$$\int_{\mathbb{H}^n} a(q) |u_k|^{p-2} u_k v dq \to \int_{\mathbb{H}^n} a(q) |u|^{p-2} u v dq \tag{28}$$

as $k \to \infty$. Moreover, Lemmas 3.4 and 3.5 imply that for all $v \in X$

$$\begin{aligned}\int_{\mathbb{H}^n} w(q) |u_k|^{m-2} u_k v dq &\to \int_{\mathbb{H}^n} w(q) |u|^{m-2} u v dq, \\ \int_{\mathbb{H}^n} h(q) |u_k|^{\mathfrak{m}-2} u_k v dq &\to \int_{\mathbb{H}^n} h(q) |u|^{\mathfrak{m}-2} u v dq,\end{aligned} \tag{29}$$

as $k \to \infty$. In conclusion, passing to the limit as $k \to \infty$ in (22), we get by (27) – (29)

$$\int_{\mathbb{H}^n} \mathbf{A}(q, D_H u) D_H v dq + \int_{\mathbb{H}^n} a(q) |u|^{p-2} uv dq = \lambda_* \int_{\mathbb{H}^n} w(q) |u|^{m-2} uv dq - \int_{\mathbb{H}^n} h(q) |u|^{\mathfrak{m}-2} uv dq$$

for all $v \in X$, that is u is a nonnegative entire solution of $(\mathcal{E}_{\lambda_*})$.

Step 3. We claim that $u \neq 0$. Indeed, since $u_k \rightharpoonup u$ in X by (23), Lemma 2.5 yields in particular that $\|u\|_{m,w} = \lim_{k\to\infty} \|u_k\|_{m,w}$. Moreover, Lemma 3.1(iii) applied to each $u_k \neq 0$ implies that $\|u_k\|_{m,w} \geq \kappa_3 \lambda_k^{1/(p-m)} \geq \kappa_3 \lambda_1^{1/(p-m)}$, since $\lambda_k \searrow \lambda_*$ and $p < m$ by $(\mathcal{H}_2^+)$. Consequently, $\|u\|_{m,w} \geq \kappa_3 \lambda_1^{1/(p-m)} > 0$. Hence u is nontrivial and nonnegative by (23). Lemma 3.1 yields now that $\lambda_* > 0$. $\square$

Proof of Theorem 1.1. (i) Theorem 4.2 says that there exists $\lambda_* \geq 0$ such that $(\mathcal{E}_\lambda)$ has at least a nontrivial nonnegative entire solution for $\lambda > \lambda_*$ and $(\mathcal{E}_\lambda)$ has no nonnegative entire solution for $\lambda < \lambda_*$ by definition of λ_*.

(ii) Lemmas 3.1 and 4.1, Theorems 4.2 and 4.3 show the existence of $\lambda_* > 0$ such that $(\mathcal{E}_\lambda)$ admits at least a nontrivial nonnegative entire solution if and only if $\lambda \geq \lambda_*$.

(iii) Assume that $\mathscr{A}$ is uniformly convex, and $(\mathcal{E}_\lambda)$ has a nontrivial nonnegative entire solution u_1. We claim that $(\mathcal{E}_\lambda)$ admits at least two nontrivial nonnegative entire solutions.

To this aim let us consider

$$-\mathrm{div}\, \mathbf{A}(q, D_H u) + a(q) |u|^{p-2} u + h(q) |u|^{\mathfrak{m}-2} u = \lambda w(q) |u|^{m-2} u^+, \tag{30}$$

where $u^+ = \max\{u, 0\}$. The embedding $X \hookrightarrow\hookrightarrow L^m(\mathbb{H}^n, w)$ is compact by Lemma 2.5 and $(\mathcal{H}_2)$. Thus $\Phi'_w \colon X \to (L^m(\mathbb{H}^n, w))'$ is compact, i.e. if $u_k \rightharpoonup u$ in X then $\Phi'_w(u_k) \to \Phi'_w(u)$ in $(L^m(\mathbb{H}^n, w))'$.

Since $\mathscr{A}$ is assumed to be uniformly convex, $L \colon X \to X'$ is a homeomorphism by Lemma 3.8(iii). Hence u is a solution of (30) if and only if u is a solution of the operator equation $u = L^{-1} \circ \Phi'_w(u^+)$. Let $\delta \in [0, 1]$ and consider

$$u = L^{-1} \circ \Phi'_{\delta w}(u^+). \tag{31}$$

Define $G \colon [0, 1] \times X \to X$ by $G(\delta, u) = L^{-1} \circ \Phi'_{\delta w}(u^+)$ for all $(\delta, u) \in [0, 1] \times X$. Thus G is continuous and compact. Lemma 3.2(ii) yields that all the solutions of (31) are uniformly bounded in X, so that there exists $R > 0$ sufficiently large such that (31) has no solution on $\partial B_R \subset X$. Therefore,

$$\deg_{LS}(I - G(1, \cdot), B_R, 0) = \deg_{LS}(I - G(0, \cdot), B_R, 0) = 1.$$

Since (30) has the trivial solution zero and the nontrivial nonnegative entire solution u_1, then (30) has another nontrivial entire solution $u_2 \in X$.

We claim that u_2 is nonnegative. Suppose the contrary. Put $u_2^- = \min\{u_2, 0\}$. Then $u_2^- \in X$ and take u_2^- as a test function. Therefore by (30), $(\mathcal{A}_1)$(i) and (15)

$$
\begin{aligned}
0 &= \lambda \int_{\mathbb{H}^n} w(q)\,|u_2(q)|^{m-2}\, u_2^+(q) u_2^-(q) dq \\
&= \int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_2), D_H u_2^-(q))_H dq + \int_{\mathbb{H}^n} a(q)\,|u_2|^{p-2} u_2 u_2^-(q) dq \\
&\qquad\qquad\qquad + \int_{\mathbb{H}^n} h(q)\,|u_2|^{\mathfrak{m}-2} u_2 u_2^-(q) dq \\
&= \int_{\mathbb{H}^n} (\mathbf{A}(q, D_H u_2^-), D_H u_2^-)_H dq + \int_{\mathbb{H}^n} a(q) |u_2^-|^p dq + \int_{\mathbb{H}^n} h(q) |u_2^-|^{\mathfrak{m}} dq \\
&\geq \kappa \left(\int_{\mathbb{H}^n} \left| D_H u_2^- \right|_H^p dq + \int_{\mathbb{H}^n} a(q) \left| u_2^- \right|^p dq \right) + \int_{\mathbb{H}^n} h(q) |u_2^-|^{\mathfrak{m}} dq \\
&\geq \kappa \| u_2^- \|_E^p + \| u_2^- \|_{\mathfrak{m},h}^{\mathfrak{m}} \geq 0.
\end{aligned}
$$

In conclusion, $\|u_2^-\| = 0$, that is $u_2^- = 0$, as required. Thus u_2 is nonnegative. Finally, u_2 is a nontrivial nonnegative solution of $(\mathcal{E}_\lambda)$. □

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