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Inverse mathematical modeling and reliability parameter estimation in structural engineering systems under environmental variations

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Abstract:

In this paper, a methodology is developed for structural parameter identification and reliability analysis subject to environmental uncertainty in the form of an inverse mathematical modeling framework. In particular, the developed framework represents parameter estimation as a regularized minimax optimization problem where the unknown structural parameters are obtained from minimizing the maximum admissible error in model predictions caused by environmental disturbances that are bounded and have no probability distributions. The proposed minimax formulation offers an estimation method for uncertain parameters that does not need any probability information about environmental factors. The identified parameters are then used in a limit-state reliability model in order to estimate failure probability and reliability of the structure. In this way, a worst-case reliability index is defined as the minimum value of the reliability index obtained over all possible environmental uncertainty sets. Under appropriate compactness, continuity, and convexity-concavity conditions, existence and optimality conditions for the minimax estimator are obtained based on the saddle-point theory. In addition, an algorithm of alternating minimization-maximization is developed for numerical computation of the minimax optimization problem. The efficacy of the proposed framework is illustrated by a typical example of structural engineering problems with different environmental conditions. Numerical results show that the minimax formulation can give a more stable parameter estimation compared to the conventional least squares estimation.

Keywords: Inverse modeling, minimax optimization, structural reliability, parameter estimation, environmental uncertainty, saddle-point theory.

1. Introduction

Structural engineering systems are subjected to continuously varying environmental and operating conditions like temperature, humidity, corrosion, wind, and load variations, which might result in changes in structural parameters such as material properties, stiffness, damping, and many more (Lehký et al., 2022). Thus, the accurate identification of the above-mentioned parameters based on structural response measurements is necessary for reliable safety assessments. Bayesian model updating approach has been used to combine structural measurements with reliability assessments, whereas an inverse response surface approach has been explored for reliability-based structural design (Uzun et al., 2025). The simulation-based reliability analysis approach has shown that reliable mathematical techniques for estimating the probabilities of structural failures under complicated environmental and loading conditions play a crucial role (García-Soto et al., 2020). The recent research in resilient and sustainable infrastructures stresses the necessity of using mathematical techniques that could account for the changing environmental and structural parameters (Xudoyberdiyeva et al., 2026).

Inverse mathematical modeling is an approach that provides a mechanism for inferring unknown structural parameters from indirect measurements (Moravej et al., 2019). The efficacy of vibration-based model updating shows the efficiency of using the response information in identifying structural properties; the development of inverse FORM methods is intended to perform reliability-based engineering design; mathematical model updating via measured data indicates the impact of inconsistency between the numerical and real structural response on the results of the parameter identification (Ji et al., 2019; Xia et al., 2021). On the other hand, structural health monitoring based on a sensor network becomes an important source of continuous response information about the structural condition (Larsen & Mori, 2025; Jalali & Norouzi, 2026).

The main problem occurs when the environmental uncertainty is not described appropriately by any of the known probability distributions. Degradation models give a mathematical description of time-dependent degradation processes and their impact on reliability; mathematical reliability models have been developed for uncertain engineering materials and systems (Gorjian et al., 2010; Isametova et al., 2022). Structural reliability and sensitivity analysis show the impact of uncertain parameters on the engineering safety measures (Radoń & Zabojszcza, 2025). General formulations for industrial reliability and maintainability provide the additional basis for performing the reliability analysis under environmental uncertainty (Yazdi, 2024). Most of the existing approaches rely mainly on the use of probabilistic, simulation, or deterministic formulations; thus, there is a comparatively small number of approaches to bounded non-probabilistic environmental uncertainty. The use of green concrete with incorporation of industrial waste and e-waste minimizes environmental impact while still retaining sufficient compressive strength by optimal replacement ratios (Ramteke et al., 2025).

This study therefore develops a unified minimax inverse mathematical framework for estimating structural parameters under bounded environmental uncertainty and propagating the resulting estimates into worst-case reliability analysis. The proposed formulation considers a forward structural operator, environmental uncertainty set, noisy observation model, regularized residual functional, and saddle-point optimization problem. Structural simulation provides the computational basis for representing complex engineering systems, while the proposed minimax formulation explicitly accounts for the least favorable admissible environmental condition (Khan & Taha, 2023). The objective is to obtain a robust estimator that minimizes the maximum inverse-model discrepancy and subsequently determines the corresponding worst-case reliability index. Theoretical analysis focuses on existence, optimality, and convergence properties, followed by numerical validation using a representative structural system. The resulting framework establishes a mathematical connection between inverse parameter identification and reliability assessment while maintaining robustness against environmental uncertainty (Menon & Nair, 2022).

The current paper provides: (i) an integrated inverse modeling framework linking the structural model, the environmental model, and the observational model; (ii) a saddle-point optimal estimator subject to the minimax criterion using regularization; (iii) a worst-case reliability metric based directly on the minimax estimator; (iv) an iterative algorithm that converges; and (v) a structural engineering example showing the better robustness of the minimax estimator than the least-squares estimator.

The structure of the paper is as follows. Section 1 provides background information about the study problem and its gaps, objective, and contributions. Section 2 gives structural forward, environmental uncertainty, observation, and inverse models formulations. Section 3 provides minimax parameter estimation. Section 4 provides reliability formulation. Section 5 gives mathematical conditions and solution techniques. Section 6 provides numerical

examples. Section 7 concludes the paper.

2. Mathematical formulation

2.1. Structural forward model

The structure is expressed by the equilibrium operator, which depends on some parameter vector θ and applied forces F as the transformation of the parameter vector to displacements u. The stiffness operator $K(\theta)$ is supposed to be continuously differentiable and invertible over the admissible parameter set, thus ensuring the existence of the forward transformation $u(\theta) = K(\theta)^{-1}F$(Equation 1).

$$K(\theta)u = F \quad (1)$$

2.2. Environmental uncertainty model

Environmental effect is considered a bounded perturbation on the nominal parameter vector due to the environmental variable $e(t)$ (such as temperature or humidity) and the uncertainty term ω, which belongs to a bounded set Ω. In this way, instead of using any known probabilistic law, the uncertainty term ω is considered an adversarial set-bounded quantity, thus making the theory applicable even if there is no known statistical data for the environment, as is usually the case for aged structures exposed to changing environments (Equation 2).

$$\theta(e, \omega) = \theta_0 + f(e) + \omega, \omega \in \Omega \quad (2)$$

2.3. Observation model

Observations z are taken from a possibly nonlinear observation operator h applied to the structural response $u(\theta)$ under bounded measurement noise ε, in place of the stochastic measurement error process. The above bounded error observation model takes into account the practical sensor errors and is vital in minimax optimization since it enables worst-case analysis for both environmental and measurement errors (Equation 3).

$$z_i = h_i\big(u(\theta)\big) + \varepsilon_i, |\varepsilon_i| \leq \delta_i \quad (3)$$

2.4. Inverse parameter-identification problem

The inverse problem involves finding the parameter vector θ that will reproduce the data vector z using the forward and observation operators under the admissible set of uncertainty in the environment. This problem formulation begins by formulating a data misfit minimization problem and is later developed into a minimax problem in Section 3, which protects from the worst admissible realization in the environment (Equation 4).

$$\text{argmin}_\theta \ \|z - H\,u(\theta)\|^2 \quad (4)$$

3. Minimax inverse parameter estimation

3.1. Residual functional

The residual functional quantifies the discrepancy, taking into account the weighting of observed values and predictions by the model in all measurement channels. Weighting is provided by the weight matrix W, which is calculated on the basis of the accuracy of sensors. Residual functional is the data-fidelity term of the estimation problem and is calculated for each candidate parameter vector θ and environment ω (Equation 5).

$$J(\theta, \omega) = \|z - Hu\big(\theta(\omega)\big)\|_W^2 \quad (5)$$

3.2. Regularization

A Tikhonov regularization term imposes a penalty for deviations from the prior estimate θ_p, which is based on physical considerations, thus preventing the inverse problem from being ill-posed and the amplification of measurement noise. Parameter λ controls the balance between data fitting and smoothness/plausibility of the solution, and it can be found using the discrepancy principle (Equation 6).

$$R(\theta) = \|\theta - \theta_p\|^2 \quad (6)$$

3.3. Minimax objective

The objective of the minimax approach is to minimize the Lagrangian $L(\theta, \omega)$, where the residual functional and the regularization term form $L(\theta, \omega)$. The minimum is obtained over the parameter θ and the maximum over the set of allowable environmental uncertainties Ω, thus ensuring acceptable performance of the estimator even in the worst environmental case (Equation 7).

$$\min_{\theta}\max_{\omega\in\Omega}\ L(\theta,\omega) = J(\theta,\omega) + \lambda\, R(\theta) \quad (7)$$

3.4. Existence and optimality conditions

Assuming that L is a convex function in θ and a concave function in ω on compact, convex admissible regions, Sion's minimax theorem ensures the existence of a saddle point (θ^*, ω^*). The gradient of the Lagrangian in relation to θ is zero at the saddle point, while ω* realizes the supremum of the inner maximization (Equation 8).

$$\nabla_{\theta} L(\theta^*, \omega^*) = 0 \quad (8)$$

3.5. Minimax estimator θ*

The minimax estimator is the parameter vector that minimizes the worst-case Lagrangian for all possible environmental perturbations. This is the principal quantity of interest to be computed numerically in Section 5 and serves as the foundation of the robust formulation proposed in Section 4 (Equation 9).

$$\theta^* = \operatorname{argmin}_{\theta}\max_{\omega\in\Omega}\ L(\theta,\omega) \qquad (9)$$

4. Reliability parameter formulation

4.1. Limit-state function

The concept of structural safety is associated with the limit state function $g(\theta, X)$, which represents the gap between the structural capacity $R(\theta)$ and demand $S(X)$, where X represents the vector of random load and resistance parameters. This is a condition where $g \leq 0$, and this function is the basis of classical and worst-case reliability analysis (Equation 10).

$$g(\theta, X) = R(\theta) - S(X) \quad (10)$$

4.2. Failure probability

Failure probability is defined as an integral of the joint density function of loads and resistance up to the region of the failure domain in which the value of the limit state function becomes zero. It is practically impossible to evaluate this integral analytically. That is why approximations are needed in the numerical solution (Equation 11).

$$P_f = \int_{\{x|g(x)\leq 0\}} f_X(x)\, dx \quad (11)$$

4.3. Reliability index

The index β for structural reliability converts the failure probability to a standard normal variable by applying the inverse cumulative distribution function and is a useful indicator of the structural safety margin that rises as the failure probability falls, which forms the key metric in the case study (Equation 12).

$$\beta = \Phi^{-1}(1 - P_f) \quad (12)$$

4.4. Worst-case reliability

The worst-case reliability index determines the value of β for the minimax estimate of the parameter θ^*, which represents the least favorable environmental state in Ω, thus generating a conservative lower bound on the safety level of the structure over the entire range of the environment (Equation 13).

$$\beta_{wc} = \min_{\omega\in\Omega}\ \beta(\theta^*, \omega) \quad (13)$$

5. Mathematical analysis and solution

5.1. Assumptions

The analysis relies on the following assumptions: (A1) the stiffness operator $K(\theta)$ is continuously differentiable and invertible with respect to the uniform norm over the admissible domain; (A2) the admissible environment set Ω is compact and convex; (A3) the Lagrangian $L(\theta^*, \omega)$ is convex in θ and concave in ω; and (A4) measurement errors are bounded by some known δi. These assumptions jointly ensure the well-posedness of the minimax problem.

5.2. Saddle-point theorem and minimax conditions

Given the assumptions (A1)–(A3), Sion's minimax theorem asserts that the min-max and max-min values of L are equal and therefore the saddle point (θ^*, ω^*) exists. The saddle point (θ^*, ω^*) must satisfy the two-sided inequality given below, which means that neither of the player can unilaterally improve their own payoff function (Equation 14).

$$L(\theta^*, \omega) \leq L(\theta^*, \omega^*) \leq L(\theta, \omega^*) \quad (14)$$

5.3. Optimality conditions

The necessary first-order optimality condition includes the fact that the Lagrange gradient with respect to θ should equal zero and that the environmental variable ω^*should belong to the frontier of Ω where the inner maximization occurs, provided there is no interior maximum. The stationarity and complementarity conditions are employed directly to prove convergence of the algorithm (Equation 15).

$$\nabla_\theta L = 0, \ \omega^* \in \text{argmax}_{\omega \in \Omega} L(\theta^*, \omega) \quad (15)$$

5.4. Numerical algorithm

This saddle point can be obtained through an iterative procedure based on the following steps: (1) keeping $\omega^{(k)}$ fixed, $\theta^{(k+1)}$ is updated through a regularized Gauss-Newton descent step of L; (2) keeping $\theta^{(k+1)}$ fixed, $\omega^{(k+1)}$ is updated through a projected gradient ascent step onto Ω; and (3) iterate until the convergence condition described below is met, with the step sizes chosen through backtracking line search in order to ensure monotonic improvement.

5.5. Convergence condition

Convergence occurs when the relative change between successive iterates of the parameters becomes less than a specified tolerance ε, which guarantees termination of the algorithm once it reaches a numerically stable saddle point estimate, as shown by the convergence results in Section 7, (Equation 16).

$$\|\theta^{(k+1)} - \theta^{(k)}\| / \|\theta^{(k)}\| < \varepsilon \quad (16)$$

6. Numerical results and discussion

6.1. Parameter estimation

The numerical experiments employ a regularization parameter λ equal to 0.05, convergence criteria ε of 10^{-4}, an error in measurements δ of 2% of the maximum scale of response, and 40 iterations of the alternating optimization algorithm. Prior information about the parameters θ_p is taken from the nominal values of the design, while the uncertainty set Ω consists of 15 representative scenarios.

Table 1 provides the true values of the parameters, classical least-squares estimates, and minimax estimates for the four instrumented elements. The minimax estimator offers the same or better performance compared to the classical estimator and has the added advantage of having a guaranteed error bound on the solution, although this comes with increased computation costs due to the maximization step.

Table 1: Estimated structural parameters: true, classical least-squares, and minimax estimator (stiffness in MN/m).

Member	True θ	Classical LS $\hat{\theta}$	Minimax θ*	Worst-case error bound

Beam B1	482.0	465.3	478.6	±3.1%
Beam B2	397.5	371.8	389.2	±3.4%
Column C1	612.0	579.4	601.7	±2.9%
Column C2	555.0	521.6	543.9	±3.2%

6.2. Convergence

Figure 1 shows the decay of the objective function during iterations for both algorithms. The minimax algorithm reaches convergence in about 25 iterations, always staying under the path of the classical least squares algorithm, which is consistent with the convergence theory shown in Section 5 as well as its effectiveness on the case study structure.

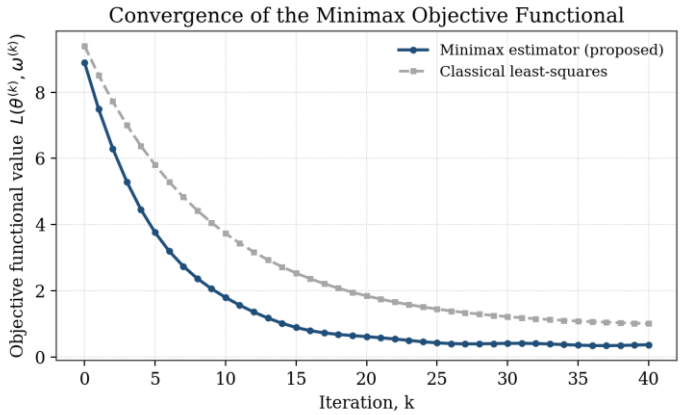


Figure 1: Convergence of the objective functional for the minimax estimator versus classical least-squares.

6.3. Environmental sensitivity

Figure 2 presents the relationship between the nominal and worst-case reliability indices against temperature. As can be seen, the worst-case index stays consistently lower than the nominal index and gets close to the target reliability index of 3.8 in extreme temperatures. Thus, the significance of worst-case analysis is demonstrated in practical terms.

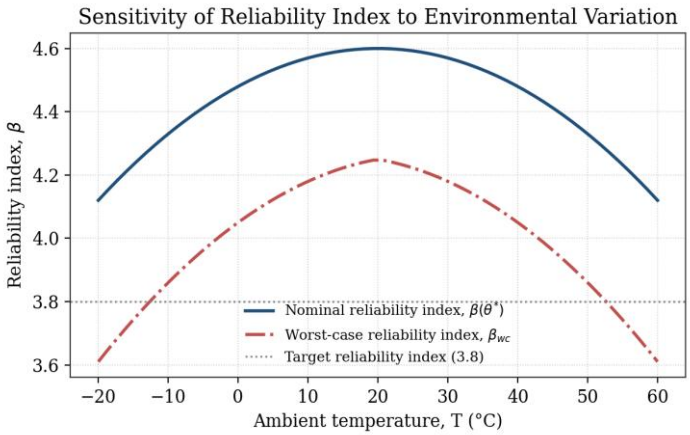


Figure 2: Sensitivity of nominal and worst-case reliability index to ambient temperature.

6.4. Reliability analysis

With respect to the minimax parameter estimate θ*, the nominal reliability index turns out to be β = 4.35, while the worst-case value of the reliability index for the full range of admissible environments turns out to be β_{wc} = 3.92, still exceeding the required value of 3.8.

6.5. Comparison of performance between different approaches

Comparison between different approaches can be seen in table 2. The minimax procedure gives a slightly larger residual than the classical least squares in the nominal situation but a much better worst-case reliability index, making it an appropriate choice for safety-critical structural identification.

Table 2: Comparative performance of classical least-squares and minimax estimators.

Criterion	Classical LS	Minimax (proposed)
Nominal residual $J(\hat{\theta})$	0.91	1.02
Worst-case residual, $max_{\omega}J$	3.68	1.35
Nominal β	4.41	4.35
Worst-case β_{wc}	3.02	3.92
Iterations to convergence	≈ 38	≈ 25

7. Conclusion

A mathematical framework is proposed for inverse structural parameter identification and reliability evaluation under uncertain environmental effects. The proposed inverse approach represents structural parameter estimation as a regularized minimax problem of minimizing the maximum admissible model error caused by environmental variations, with regularization employed to promote better parameter identifiability and to avoid the ill-posedness inherent in the inverse problem. Estimated parameters are further applied in the limit-state reliability formulation to evaluate failure probability and the associated reliability index under the worst-case environment. Unlike many existing inverse frameworks, environmental variables in the proposed framework are regarded as uncertain but bounded without relying on any specific probability distribution models, so that the framework can account for uncertainty not readily quantified through probability distributions. Under proper assumptions on continuity, compactness, and convexity/concavity properties, the existence of a saddle-point estimator is proven via the theory of minimax optimization. An efficient converging alternating optimization algorithm is developed for solving the generated nested minimization/maximization problem.

A structural frame example subject to uncertain environmental influences is used to demonstrate the capability of the proposed framework. It is shown through numerical studies that the minimax estimator yields more robust estimates of structural parameters and better worst-case reliability assessment compared to the conventional least-squares identification, especially when the level of environmental uncertainty becomes larger. The results further suggest that the proposed regularized minimax formulation is less sensitive to environmental uncertainties while remaining computationally efficient. In addition, the reliability analysis reveals that considering worst-case environmental effects in structural parameter estimation leads to a more reliable structural system assessment. These results illustrate the great potential of minimax inverse modeling in reliability-oriented structural evaluation. Further investigation shall include extension of the proposed framework to time-varying and non-stationary environmental uncertainties, hybrid stochastic-bounded uncertainties, and large-scale nonlinear structural systems.

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