

Multiple Nonradial Solutions for a Nonlinear Elliptic Radial Problem: an Improved Result

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Received: October 22, 2018
Accepted: November 10, 2018

We obtain an improved version of a recent result concerning the existence of nonnegative nonradial solutions $u \in D^{1,2}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^{-\alpha} dx)$ to the equation

$$-\Delta u + \frac{A}{|x|^\alpha} u = f(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \quad A, \alpha > 0,$$

where f is a continuous nonlinearity satisfying $f(0) = 0$.

Keywords: Semilinear elliptic PDE, singular vanishing potential, symmetry breaking.

2010 Mathematics Subject Classification: Primary 35J60; secondary 35Q55, 35J20.

1. Introduction and main result

In this paper we consider the following semilinear elliptic problem:

$$(\mathcal{P}) \quad \begin{cases} -\Delta u + \frac{A}{|x|^\alpha} u = f(u) & \text{in } \mathbb{R}^N, \quad N \geq 3 \\ u \in H_\alpha^1 \setminus \{0\}, \quad u \geq 0 \end{cases}$$

where $A, \alpha > 0$ are real constants, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies $0 < f(s) \leq (\text{const.}) s^{p-1}$ for some $p > 2$ and all $s > 0$ (hence $f(0) = 0$), and H_α^1 is the natural energy space related to the equation, i.e.,

$$H_\alpha^1 := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{u^2}{|x|^\alpha} dx < \infty \right\}.$$

Here and in the rest of the paper, $D^{1,2}(\mathbb{R}^N)$ is the usual Sobolev space, which identifies with the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the L^2 norm of the gradient.

We are interested in *weak solutions* to (\mathcal{P}) , i.e., functions $u \in H_\alpha^1 \setminus \{0\}$ such that $u \geq 0$ almost everywhere in \mathbb{R}^N and

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} uv dx = \int_{\mathbb{R}^N} f(u) v dx \quad \text{for all } v \in H_\alpha^1. \quad (1)$$

As is well known, problems like (\mathcal{P}) arise in many branches of mathematical physics, such as population dynamics, nonlinear optics, plasma physics, condensed matter physics and cosmology (see e.g. [9, 12, 22]). In this context, (\mathcal{P}) is a prototype for problems exhibiting radial potentials which are singular at the origin and/or vanishing at infinity (sometimes called the *zero mass case*; see e.g. [6, 16]).

Although it can be considered as a quite recent investigation, the study of problem (\mathcal{P}) has already some history and, currently, the problem of existence and nonexistence of *radial* solutions has been essentially solved, through various subsequent contributions, in the *pure-power case* $f(u) = |u|^{p-2}u$, where the results obtained rest upon compatibility conditions between α and p . The first results in this direction are probably due to Terracini [21], who proved that (\mathcal{P}) has no solution if

$$\left\{ \begin{array}{l} \alpha = 2 \\ p \neq 2^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha \neq 2 \\ p = 2^* \end{array} \right. , \quad 2^* := \frac{2N}{N-2},$$

and explicitly found all the radial solutions of (\mathcal{P}) for $(\alpha, p) = (2, 2^*)$. As usual, 2^* denotes the critical exponent for the Sobolev embedding in dimension $N \geq 3$. The problem was then considered again in [11], where the authors proved that (\mathcal{P}) has at least a radial solution if

$$\left\{ \begin{array}{l} 0 < \alpha < 2 \\ 2^* + \frac{\alpha-2}{N-2} < p < 2^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha > 2 \\ 2^* < p < 2^* + \frac{\alpha-2}{N-2} \end{array} \right. ,$$

while it has no solution if

$$\left\{ \begin{array}{l} 0 < \alpha < 2 \\ p > 2^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha > 2 \\ 2 < p < 2^* \end{array} \right. .$$

The existence and nonexistence results of [11] were subsequently extended in [7], by showing that (\mathcal{P}) has no solution also if

$$\left\{ \begin{array}{l} 0 < \alpha < 2 \\ 2 < p \leq 2_\alpha \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} 2 < \alpha < N \\ p \geq 2_\alpha \end{array} \right. , \quad 2_\alpha := \frac{2N}{N-\alpha},$$

and obtaining at least a radial solution for every pair (α, p) such that

$$\left\{ \begin{array}{l} 0 < \alpha < 2 \\ 2^* + 2\frac{\alpha-2}{N-2} < p < 2^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha > 2 \\ 2^* < p < 2^* + 2\frac{\alpha-2}{N-2} \end{array} \right. .$$

A further extension of this existence condition was found in [19, 20], where the authors proved that (\mathcal{P}) has at least a radial solution for all the pairs (α, p) satisfying

$$\left\{ \begin{array}{l} 0 < \alpha < 2 \\ 2_\alpha^* < p < 2^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} 2 < \alpha < 2N-2 \\ 2^* < p < 2_\alpha^* \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha \geq 2N-2 \\ p > 2^* \end{array} \right. , \quad (2)$$

where

$$2_\alpha^* := 2\frac{2N-2+\alpha}{2N-2-\alpha}.$$

Finally, the problem of radial solutions in the left open cases was solved in [3] and [10], where, respectively, it was proved that the problem has no radial solutions for both

$$\begin{cases} 0 < \alpha < 2 \\ 2_\alpha < p \leq 2_\alpha^* \end{cases} \quad \text{and} \quad \begin{cases} 2 < \alpha < 2N - 2 \\ 2_\alpha^* \leq p < 2_\alpha \end{cases} .$$

All these results are portrayed in the picture of the αp -plane given in Fig.1.1, where nonexistence regions are shaded in gray (nonexistence of radial solutions) and light gray (nonexistence of solutions at all, which includes both the lines $p = 2^*$ and $p = 2_\alpha$ except for the pair $(\alpha, p) = (2, 2^*)$), whereas white color (of course above the line $p = 2$) means existence of radial solutions.

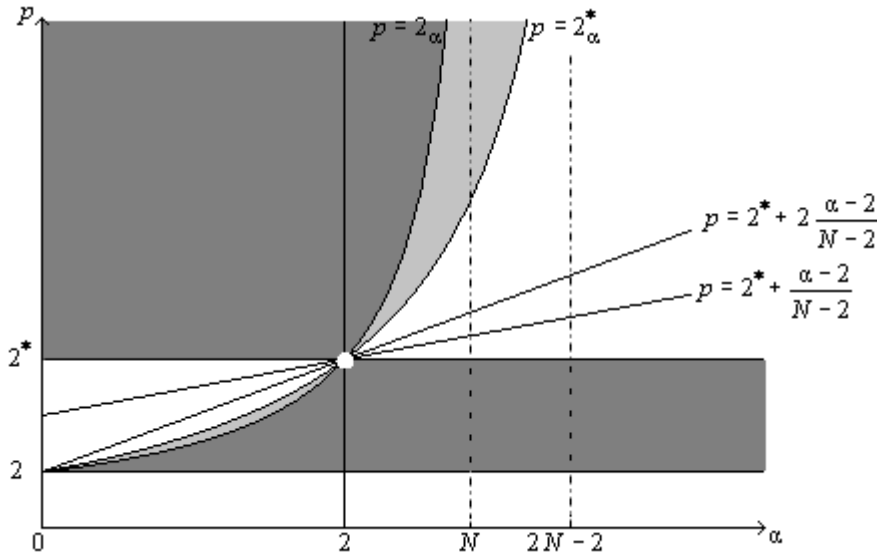


Figure 1.1: Regions of nonexistence of solutions (light gray), and existence (white with $p > 2$) and nonexistence (dark gray) of radial solutions.

All the above results on the existence of radial solutions are obtained by variational techniques which can be extended in a standard way (see [7, 20]) to general continuous nonlinearities $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the so-called *Ambrosetti-Rabinowitz condition* (i.e., assumption (\mathbf{h}_2) below) and $0 < f(s) \leq (\text{const.}) s^{p-1}$ for all $s > 0$ and for some $p > 2$ such that the pair (α, p) satisfies (2).

Concerning nonradial solutions, Terracini proved in [21] that problem (\mathcal{P}) with $N \geq 4$, $\alpha = 2$ and $f(u) = |u|^{2^*-2}u$ has at least one nonradial solution for every A large enough. This just concerns the point $(\alpha, p) = (2, 2^*)$ in Fig.1.1, which brings Catrina to say, in the introduction of his paper [10]: “*Two questions still remain: whether one can find non-radial solutions in the case when radial solutions do not exist, or in the case when radial solutions exist*”.

A first result in this direction has been obtained in [18], for nonlinearities satisfying the following assumption:

- (\mathbf{h}_0) there exist $p_1, p_2 > 2$ and $M > 0$ such that $0 < f(s) \leq M \min\{s^{p_1-1}, s^{p_2-1}\}$ for all $s > 0$.

Assumption (\mathbf{h}_0) is the so-called *double-power growth condition* and seems to be typical in nonlinear problems with potentials vanishing at infinity (see e.g. [1, 2, 4, 5, 6, 14, 16, 18] and the references therein). It obviously implies the single-power growth condition $0 < f(s) \leq Ms^{p-1}$ for all $p \in [p_1, p_2]$ and $s > 0$, but it is actually more stringent than that whenever it is assumed with $p_1 \neq p_2$.

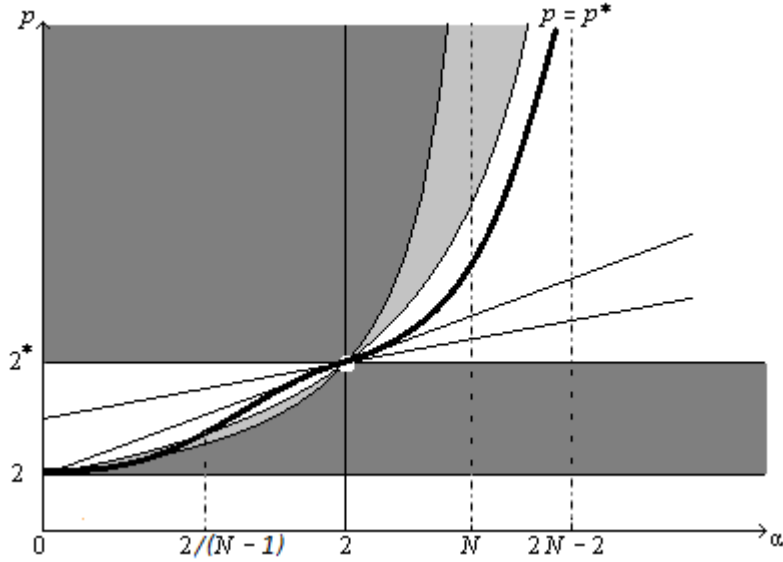


Figure 1.2: The curve $p = p_\alpha^*$ in the same αp -plane of Fig.1.1.

The result of [18] is the following. For every $\alpha \in (0, 2N - 2) \setminus \{2\}$ and $p_1, p_2 > 2$, define

$$p_\alpha^* := \begin{cases} 2 \frac{\alpha^2(N-1) - 2\alpha(N-1) + 4N}{\alpha^2(N-1) - 2\alpha(N+1) + 4N} & \text{if } 0 < \alpha < 2 \\ 2 \frac{2N+2-\alpha}{2N-2-\alpha} & \text{if } 2 < \alpha < 2N - 2 \end{cases}$$

and $\nu := \nu_{N,\alpha,p_1,p_2} :=$

$$\begin{cases} \left\lceil \left[2 \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{2-\alpha} \frac{2^*-p_1}{p_1-2} \right\} - 2N \left(\frac{1}{\alpha} - \frac{1}{2} \right) \right] - 1 \right\rceil & \text{if } 0 < \alpha < 2 \\ \left\lceil \left[2 \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{\alpha-2} \frac{p_2-2^*}{p_2-2} \right\} \right] - 1 \right\rceil & \text{if } 2 < \alpha < 2N - 2 \end{cases} \quad (3)$$

where $\lceil \cdot \rceil$ denotes the ceiling function (i.e., $\lceil x \rceil := \min \{n \in \mathbb{Z} : n \geq x\}$). Fig.1.2 shows the curve $p = p_\alpha^*$ in the same αp -plane of Fig.1.1. Note that $p_\alpha^* = 2_\alpha^*$ if $\alpha = 2/(N - 1)$.

Theorem 1.1. *Let $N \geq 4$ and $\alpha \in (2/(N - 1), 2N - 2)$, $\alpha \neq 2$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying assumption (\mathbf{h}_0) with $2 < p_1 < p_\alpha^*$ and $p_2 > 2^*$ if $\alpha \in (2/(N - 1), 2)$, or $2 < p_1 < 2^*$ and $p_2 > p_\alpha^*$ if $\alpha \in (2, 2N - 2)$. Assume furthermore that:*

- (\mathbf{h}_1) *the function $f(s)/s$ is strictly increasing on $(0, +\infty)$*
- (\mathbf{h}_2) *$\exists \mu > 2$ such that the function $F(s)/s^\mu$ is decreasing on $(0, +\infty)$*
- (\mathbf{h}_3) *$\exists \eta > 2$ such that the function $F(s)/s^\eta$ is increasing on $(0, +\infty)$*

where $F(s) := \int_0^s f(t) dt$. Then there exists $A_* > 0$ such that for every $A > A_*$ problem (\mathcal{P}) has both a radial solution and ν different nonradial solutions.

Under the assumptions of Theorem 1.1, ν turns out to be strictly positive (see [18, Lemma 5.2]), so that at least one nonradial solution actually exists. Unfortunately, the theorem does not encompass the case of pure-power nonlinearities, since it always requires $p_1 < p_2$ in assumption (\mathbf{h}_0) . Observe that assumption (\mathbf{h}_3) is the well known *Ambrosetti-Rabinowitz condition*, as the increasingness of $F(s)/s^\eta$ on $(0, +\infty)$ amounts to $\eta F(s) \leq f(s)s$ for all $s > 0$.

The aim of this paper is to improve Theorem 1.1, by proving the following result.

Theorem 1.2. *Let $N \geq 4$ and $\alpha \in (2/(N - 1), 2N - 2)$, $\alpha \neq 2$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (\mathbf{h}_0) with p_1, p_2 as in Theorem 1.1. Assume furthermore that:*

- (\mathbf{h}'_1) *the function $f(s)/s$ is increasing on $(0, +\infty)$*
- (\mathbf{h}'_2) *$\exists \mu > 2$ and $\exists s_* > 0$ such that the function $F(s)/s^\mu$ is decreasing on $(0, s_*)$.*

Then the same conclusion of Theorem 1.1 holds true.

As for Theorem 1.1, Theorem 1.2 does not concern pure-power nonlinearities (owing to assumption (\mathbf{h}_0) with $p_1 < p_2$), and we have $\nu \geq 1$ and $\lim_{N \rightarrow \infty} \nu = +\infty$ (for α, p_1, p_2 fixed).

Clearly Theorem 1.2 includes Theorem 1.1, and in particular it removes the Ambrosetti-Rabinowitz condition from the assumptions. This inclusion is actually strict, for instance because Theorem 1.2 applies to the nonlinearities

$$f(s) = \min\{|s|, |s|^{p_2-1}\} \quad \text{and} \quad f(s) = \frac{|s|^{p_2-1}}{1 + |s|^{p_2-2}}$$

for any p_2 (and p_1) as in the theorem, while Theorem 1.1 does not.

Theorem 1.2 will be proved in Section 4. Our proof is variational, since the weak solutions to problem (\mathcal{P}) are (at least formally) the critical points of the Euler functional associated to the equation of (\mathcal{P}) , i.e.,

$$I(u) := \frac{1}{2} \|u\|_A^2 - \int_{\mathbb{R}^N} F(u) dx, \quad \text{where}$$

$$\|u\|_A^2 := (u, u)_A \quad \text{and} \quad (u, v)_A := \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + \frac{A}{|x|^\alpha} uv \right) dx$$

define the norm and the scalar product of the Hilbert space H_α^1 . More precisely, our argument is the following. We modify the function f by setting $f(s) = 0$ for all $s < 0$, still denoting by f the modified function. This modification is not restrictive in proving Theorem 1.2 and will always be assumed in the rest of the paper. Then by (\mathbf{h}_0) there exists $M' > 0$ such that, for all $s \in \mathbb{R}$,

$$|f(s)| \leq M \min\{|s|^{p_1-1}, |s|^{p_2-1}\} \quad \text{and} \quad |F(s)| \leq M' \min\{|s|^{p_1}, |s|^{p_2}\}$$

which yields in particular, for all $p \in [p_1, p_2]$ and $s \in \mathbb{R}$,

$$|f(s)| \leq M |s|^{p-1} \quad \text{and} \quad |F(s)| \leq M' |s|^p \tag{4}$$

By the continuous embeddings $H_\alpha^1 \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, condition (4) with $p = 2^*$ implies that I is of class C^1 on H_α^1 and has Fréchet derivative $I'(u)$ at any $u \in H_\alpha^1$ given by

$$I'(u)v = (u, v)_A - \int_{\mathbb{R}^N} f(u)v \, dx \quad \text{for all } v \in H_\alpha^1.$$

This yields that critical points of $I: H_\alpha^1 \rightarrow \mathbb{R}$ satisfy (1). A standard argument shows that such critical points are nonnegative (cf. the proof of Theorem 1.2 below) and therefore nonzero critical points of I are weak solutions to problem (\mathcal{P}) .

Then our proof proceeds essentially as follows. Given any integer K such that $1 \leq K \leq N - 1$, we write $x \in \mathbb{R}^N$ as $x = (y, z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}$ and in the space H_α^1 we define the following closed subspaces of symmetric functions:

$$H_r := \{u \in H : u(x) = u(|x|)\} \quad \text{and} \quad H_K := \{u \in H : u(x) = u(y, z) = u(|y|, |z|)\}.$$

Considering the restrictions $I|_{H_r}$ and $I|_{H_K}$ with $2 \leq K \leq N - 2$, we prove that each of them has a nonzero mountain-pass critical point (Section 4), which is a weak solution to (\mathcal{P}) , since H_r and H_K are natural constraints for I thanks to the classical Palais' Principle of Symmetric Criticality [17]. So the proof of Theorem 1.2 is accomplished by estimating the mountain-pass critical levels we find for $I|_{H_K}$ and the nonzero critical levels of $I|_{H_r}$ (Sections 2 and 3 respectively), in order to show that their sets are disjoint. This clearly implies that the mountain-pass critical points u_K of $I|_{H_K}$ are not radial, which also ensures that they are multiple, namely $u_{K_1} \neq u_{K_2}$ for $K_1 \neq K_2$, since $H_{K_1} \cap H_{K_2} = H_r$ (see [18, Lemma 2.1]).

The above argument is essentially the same of [18]. The main differences here are due to the removal of the Ambrosetti-Rabinowitz condition (\mathbf{h}_3) , which gives rise to the problem of the boundedness of the Palais-Smale sequences of the functionals $I|_{H_r}$ and $I|_{H_K}$. This will be overcome in Section 4 by applying an abstract result from [13] about the existence of bounded Palais-Smale sequences for C^1 functionals on Banach spaces. The weakening of assumptions (\mathbf{h}_1) and (\mathbf{h}_2) will be tackled, instead, by slightly modifying the arguments of [18] in estimating the radial critical levels of I (Section 2) and the cylindrical mountain-pass levels $I(u_K)$ (Section 3).

2. Estimate of radial critical levels

Let $N \geq 3$ and $\alpha, A > 0$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy (\mathbf{h}_0) and (\mathbf{h}'_1) .

This section is devoted to deriving an estimate from below for the nonzero critical levels of $I|_{H_r}$, namely for the value

$$m_A := \inf \{I(u) : u \in H_r \setminus \{0\}, I'(u) = 0\}.$$

Lemma 2.1. *Let $u \in H_r \setminus \{0\}$ is a critical point for I . Then $I(u) = \max_{t \geq 0} I(tu)$.*

Proof. As already observed, u is nonnegative. For $t \geq 0$ define

$$g(t) := I(tu) = \frac{1}{2}t^2 \|u\|_A^2 - \int_{\mathbb{R}^N} F(tu) dx.$$

Then

$$g'(t) = I'(tu)u = t \|u\|_A^2 - \int_{\mathbb{R}^N} f(tu)u dx = t \|u\|_A^2 - \int_{\{u>0\}} f(tu)u dx$$

for all $t \geq 0$ and $g'(1) = I'(u)u = 0$. If $t \leq 1$, then $0 \leq tu \leq u$ and assumption (h'_1) gives

$$\begin{aligned} g'(t) &= t \left(\|u\|_A^2 - \int_{\{u>0\}} \frac{f(tu)}{tu} u^2 dx \right) \\ &\geq t \left(\|u\|_A^2 - \int_{\{u>0\}} \frac{f(u)}{u} u^2 dx \right) = I'(u)u = 0. \end{aligned}$$

Similarly we get $g'(t) \geq 0$ for $t \geq 1$, so that we conclude $g(1) = \max_{t \geq 0} g(t)$, which is the claim. \square

Proposition 2.2. *Assume $0 < \alpha < 2N - 2$, $\alpha \neq 2$, and let $p = \max\{2^*_\alpha, p_1\}$ or $p = \min\{2^*_\alpha, p_2\}$ if $0 < \alpha < 2$ or $2 < \alpha < 2N - 2$, respectively. Then there exists a constant $C_0 > 0$, independent from A , such that*

$$m_A \geq C_0 A^{\frac{N-2}{\alpha-2} \frac{p-2^*}{p-2}}.$$

Proof. The argument is similar to the one of [18, Proposition 3.2], except for the conclusion, so we omit most computations. Let $u \in H_r \setminus \{0\}$. Here C will denote any positive constant independent from A and u . By the radial lemma [18, Lemma 3.1], we have

$$|u(x)| \leq \frac{C}{A^{1/4}} \frac{\|u\|_A}{|x|^{\frac{2N-2-\alpha}{4}}} \quad \text{almost everywhere in } \mathbb{R}^N$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx &= \int_{\mathbb{R}^N} |u|^{2^*_\alpha-2} u^2 dx \leq C \frac{\|u\|_A^{2^*_\alpha-2}}{A^{(2^*_\alpha-2)/4}} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{\frac{2N-2-\alpha}{4}(2^*_\alpha-2)}} dx \\ &\leq \frac{C}{A^{\frac{2N-2}{2N-2-\alpha}}} \|u\|_A^{2^*_\alpha}. \end{aligned}$$

Then, both for $p = \max\{2^*_\alpha, p_1\} < 2^*$ and $p = \min\{2^*_\alpha, p_2\} > 2^*$, we can use Sobolev inequality and argue by interpolation. We get that there exists $\lambda \in [0, 1)$ such that

$$\int_{\mathbb{R}^N} |u|^p dx \leq \left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^\lambda \left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx \right)^{1-\lambda} \leq C \frac{\|u\|_A^p}{A^{\frac{(2N-2)(1-\lambda)}{2N-2-\alpha}}}.$$

Recalling condition (4), this implies

$$\left| \int_{\mathbb{R}^N} F(u) dx \right| \leq M_2 \int_{\mathbb{R}^N} |u|^p dx \leq C \frac{\|u\|_A^p}{A^{\frac{(2N-2)(1-\lambda)}{2N-2-\alpha}}}$$

and therefore $I(u) \geq \|u\|_A^2/2 - CA^{-\frac{(2N-2)(1-\lambda)}{2N-2-\alpha}} \|u\|_A^p$. Then for every $t \geq 0$ we have

$$I(tu) \geq \frac{1}{2}t^2 \|u\|_A^2 - CA^{-\frac{(2N-2)(1-\lambda)}{2N-2-\alpha}} t^p \|u\|_A^p$$

where the function of the variable t on the r.h.s. has a maximum which can be easily computed and it is given by $CA^{\frac{N-2}{\alpha-2} \frac{p-2^*}{p-2}}$. Hence, if u is a critical point of I , Lemma 2.1 gives

$$I(u) = \max_{t \geq 0} I(tu) \geq CA^{\frac{N-2}{\alpha-2} \frac{p-2^*}{p-2}}.$$

This yields the conclusion, since $u \in H_r \setminus \{0\}$ is an arbitrary critical point and C does not depend on u . □

Remark 2.3. If p is as in Proposition 2.2, it is easy to check that

$$\frac{N-2}{\alpha-2} \frac{p-2^*}{p-2} = \begin{cases} \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{2-\alpha} \frac{2^*-p_1}{p_1-2} \right\} & \text{if } 0 < \alpha < 2 \\ \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{\alpha-2} \frac{p_2-2^*}{p_2-2} \right\} & \text{if } 2 < \alpha < 2N-2. \end{cases}$$

3. Estimate of cylindrical mountain-pass levels

Let $N \geq 3$, $2 \leq K \leq N-2$ and $\alpha > 0$, $\alpha \neq 2$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (\mathbf{h}_0) and (\mathbf{h}'_2) .

In this section we show that the functional $I|_{H_K}$ has a mountain-pass geometry and provide an estimate from above for the corresponding mountain-pass level.

The mountain-pass geometry of $I|_{H_K}$ near the origin is straightforward, since condition (4) with $p = 2^*$ and the continuous embedding $H_K \hookrightarrow L^{2^*}(\mathbb{R}^N)$ imply that $\exists C > 0$ such that $I(u) \geq \|u\|_A^2/2 - C \|u\|_A^{2^*}$ for all $u \in H_K$. Hence $\exists R > 0$ such that

$$\inf_{u \in H_K, \|u\|_A \leq R} I(u) = 0 \quad \text{and} \quad \inf_{u \in H_K, \|u\|_A = R} I(u) > 0. \tag{5}$$

Now we define a suitable $\bar{u}_K \in H_K$ such that $I(\bar{u}_K) < 0$. Then one has $\|\bar{u}_K\|_A > R$ by (5), so that $I|_{H_K}$ has the mountain-pass geometry and we can define the mountain-pass level

$$c_{A,K} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0$$

where $\Gamma := \{\gamma \in C([0,1]; H_K) : \gamma(0) = 0, \gamma(1) = \bar{u}_K\}$. (6)

The definition of \bar{u}_K is inspired by some arguments of [8]. Denote by $\phi: D \rightarrow \mathbb{R}^2 \setminus \{0\}$ the change to polar coordinates in $\mathbb{R}^2 \setminus \{0\}$, namely $\phi(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta)$ for all $(\rho, \theta) \in D := (0, +\infty) \times [0, 2\pi)$. Define $E := (1/4, 3/4) \times (\pi/6, \pi/3)$ and take any $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\psi \in C_c^\infty(E)$, $\psi \neq 0$ and $0 \leq \psi < s_*$, where s_* is given in assumption (\mathbf{h}'_2) . For every $A > 1$, define

$$E_A := \left\{ (\rho, \theta) \in \mathbb{R}^2 : \left(\frac{1}{4}\right)^{1/\sqrt{A}} < \rho < \left(\frac{3}{4}\right)^{1/\sqrt{A}}, \frac{\pi}{6\sqrt{A}} < \theta < \frac{\pi}{3\sqrt{A}} \right\}$$

and a function $\psi_A \in C_c^\infty(E_A)$ by setting

$$\psi_A(\rho, \theta) := \psi(\rho^{\sqrt{A}}, \theta^{\sqrt{A}}).$$

Finally define

$$v_A(y, z) := \psi_A(\phi^{-1}(|y|, |z|)) \quad \text{for } x = (y, z) \in (\mathbb{R}^K \times \mathbb{R}^{N-K}) \setminus \{0\}, \quad v_A(0) := 0.$$

Then $v_A \in C_c^\infty(\Omega_A) \cap H_K$, where $\Omega_A := \{(y, z) \in \mathbb{R}^K \times \mathbb{R}^{N-K} : (|y|, |z|) \in \phi(E_A)\}$. Notice that $v_A \neq 0$ and $0 \leq v_A < s_*$.

By using spherical coordinates in \mathbb{R}^K and \mathbb{R}^{N-K} , and then making the change of variables $(r, \varphi) = (\rho^{\sqrt{A}}, \theta^{\sqrt{A}})$, one computes

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{v_A^2}{|x|^\alpha} dx &= \sigma_K \sigma_{N-K} \int_{E_A} \frac{\psi(\rho^{\sqrt{A}}, \theta^{\sqrt{A}})^2}{\rho^{\alpha-N+1}} H(\theta) d\rho d\theta \\ &= \frac{\sigma_K \sigma_{N-K}}{A} \int_E \frac{\psi(r, \varphi)^2}{r^{(\alpha-N)/A^{1/2}+1}} H(A^{-1/2}\varphi) dr d\varphi \end{aligned} \quad (7)$$

where $H(\theta) := (\cos \theta)^{K-1} (\sin \theta)^{N-K-1}$ (see [18] for more detailed computation). Here and in the following σ_d denotes the $(d-1)$ -dimensional measure of the unit sphere of \mathbb{R}^d . Similarly one obtains

$$\int_{\mathbb{R}^N} F(v_A) dx = \frac{\sigma_K \sigma_{N-K}}{A} \int_E F(\psi(r, \varphi)) r^{N/A^{1/2}-1} H(A^{-1/2}\varphi) dr d\varphi \quad (8)$$

and
$$\int_{\mathbb{R}^N} |\nabla v_A|^2 dx =$$

$$\sigma_K \sigma_{N-K} \int_E \left(\psi_r(r, \varphi)^2 + \frac{1}{r^2} \psi_\varphi(r, \varphi)^2 \right) r^{(N-2)/A^{1/2}+1} H(A^{-1/2}\varphi) dr d\varphi \quad (9)$$

where we denote $\psi_r = \frac{\partial \psi}{\partial r}$ and $\psi_\varphi = \frac{\partial \psi}{\partial \varphi}$ for brevity. Then we have

$$\begin{aligned} &\frac{\|v_A\|_A^2}{\int_{\mathbb{R}^N} F(v_A) dx} = \\ &= A \frac{\int_E \left((\psi_r^2 + \frac{1}{r^2} \psi_\varphi^2) r^{(N-2)/A^{1/2}+1} + \psi^2 r^{(N-\alpha)/A^{1/2}-1} \right) H(A^{-1/2}\varphi) dr d\varphi}{\int_E F(\psi) r^{N/A^{1/2}-1} H(A^{-1/2}\varphi) dr d\varphi}. \end{aligned}$$

Since in the integration set E one has $A^{-1/2}\pi/6 < A^{-1/2}\varphi < A^{-1/2}\pi/3$, for every $A > 1$ large enough we have that $A^{-1/2}\varphi/2 < \sin(A^{-1/2}\varphi) < A^{-1/2}\varphi$ and that $1/2 < (\cos A^{-1/2}\varphi) < 1$, and thus there exist two constants $\bar{C}_1, \bar{C}_2 > 0$ such that

$$\bar{C}_1 A^{-\frac{N-K-1}{2}} < H(A^{-1/2}\varphi) < \bar{C}_2 A^{-\frac{N-K-1}{2}}.$$

Similarly, since $1/4 < r < 3/4$ in E , the terms $r^{(N-2)A^{-1/2}+1}$, $r^{(N-\alpha)A^{-1/2}-1}$ and $r^{NA^{-1/2}-1}$ are bounded and bounded away from zero by positive constants independent of $A > 1$, say \bar{C}_3 and \bar{C}_4 respectively, so that we conclude

$$\frac{\|v_A\|_A^2}{\int_{\mathbb{R}^N} F(v_A) dx} \geq A \frac{\bar{C}_1 \bar{C}_4 \int_E (\psi_r^2 + \frac{1}{r^2} \psi_\varphi^2 + \psi^2) dr d\varphi}{\bar{C}_2 \bar{C}_3 \int_E F(\psi) dr d\varphi} \rightarrow +\infty \quad \text{as } A \rightarrow +\infty.$$

This allows us to fix, hereafter, a threshold $A_K > 0$ such that

$$\frac{\|v_A\|_A^2}{\int_{\mathbb{R}^N} F(v_A) dx} > 1 \quad \text{for every } A > A_K. \tag{10}$$

Proposition 3.1. *Assume $0 < \alpha < 2$ and let $A > A_K$. Define $\bar{u}_K \in H_K$ by setting*

$$\bar{u}_K(x) := v_A\left(\frac{x}{\lambda}\right) \quad \text{with} \quad \lambda = \frac{\|v_A\|_A^{2/\alpha}}{\left(\int_{\mathbb{R}^N} F(v_A) dx\right)^{1/\alpha}}.$$

Then $I(\bar{u}_K) < 0$ and the corresponding mountain-pass level (6) satisfies

$$c_{A,K} \leq C_1 A^{\frac{K-1}{2} + N(\frac{1}{\alpha} - \frac{1}{2})}$$

where the constant $C_1 > 0$ does not depend on A .

Proof. As $\lambda > 1$, an obvious change of variables yields

$$\begin{aligned} I(\bar{u}_K) &= \frac{\lambda^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_A|^2 dx + \frac{\lambda^{N-\alpha}}{2} \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} v_A^2 dx - \lambda^N \int_{\mathbb{R}^N} F(v_A) dx \\ &\leq \frac{\lambda^{N-\alpha}}{2} \left(\int_{\mathbb{R}^N} |\nabla v_A|^2 dx + \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} v_A^2 dx \right) - \lambda^N \int_{\mathbb{R}^N} F(v_A) dx \\ &= \frac{\lambda^N}{2} \left(\lambda^{-\alpha} \|v_A\|_A^2 - 2 \int_{\mathbb{R}^N} F(v_A) dx \right) = -\frac{\lambda^N}{2} \int_{\mathbb{R}^N} F(v_A) dx < 0. \end{aligned}$$

In order to estimate $c_{A,K}$, observe that hypothesis (\mathbf{h}'_2) implies $F(ts) \geq t^\mu F(s)$ for all $0 \leq s < s_*$ and $t \in [0, 1]$. Then consider the straight path $\gamma(t) := t\bar{u}_K$, $t \in [0, 1]$. Since $0 \leq \bar{u}_K < s_*$ (recall that $0 \leq v_A < s_*$), we have

$$I(\gamma(t)) = \frac{1}{2} t^2 \|\bar{u}_K\|_A^2 - \int_{\mathbb{R}^N} F(t\bar{u}_K) dx \leq \frac{1}{2} t^2 \|\bar{u}_K\|_A^2 - t^\mu \int_{\mathbb{R}^N} F(\bar{u}_K) dx. \tag{11}$$

Note that $F(\bar{u}_K) \neq 0$ since $\bar{u}_K \neq 0$. The function of the variable t on the r.h.s of (11) has the maximum

$$m \|\bar{u}_K\|_A^2 \left(\frac{\|\bar{u}_K\|_A^2}{\int_{\mathbb{R}^N} F(\bar{u}_K) dx} \right)^{2/(\mu-2)}$$

where $m := (1/\mu)^{2/(\mu-2)} (1/2 - 1/\mu)$ for brevity. Therefore, recalling that $\lambda > 1$, we obtain

$$\begin{aligned} c_{A,K} &\leq \max_{t \in [0,1]} I(\gamma(t)) \leq m \frac{\|\bar{u}_K\|_A^{2\mu/(\mu-2)}}{\left(\int_{\mathbb{R}^N} F(\bar{u}_K) dx\right)^{2/(\mu-2)}} \\ &\leq m \frac{\lambda^{\mu(N-\alpha)/(\mu-2)} \left(\int_{\mathbb{R}^N} |\nabla v_A|^2 dx + \int_{\mathbb{R}^N} A|x|^{-\alpha} v_A^2 dx\right)^{\mu/(\mu-2)}}{\lambda^{2N/(\mu-2)} \left(\int_{\mathbb{R}^N} F(v_A) dx\right)^{2/(\mu-2)}} \\ &= m \lambda^{\frac{\mu(N-\alpha)-2N}{\mu-2}} \frac{\|v_A\|_A^{2\mu/(\mu-2)}}{\left(\int_{\mathbb{R}^N} F(v_A) dx\right)^{2/(\mu-2)}}. \end{aligned}$$

Inserting the definition of λ and using computations (7)–(9), we get

$$\begin{aligned} c_{A,K} &\leq m \frac{\|v_A\|_A^{\frac{2N}{\alpha}}}{\left(\int_{\mathbb{R}^N} F(v_A) dx\right)^{\frac{N-\alpha}{\alpha}}} = m \sigma_K \sigma_{N-K} \frac{(I_1)^{\frac{N}{\alpha}}}{A^{-\frac{N-\alpha}{\alpha}} (I_2)^{\frac{N-\alpha}{\alpha}}}, \quad \text{where} \\ I_1 &= \int_E \left(\left(\psi_r^2 + \frac{1}{r^2} \psi_\varphi^2 \right) r^{(N-2)A^{-1/2}+1} + \psi^2 r^{(N-\alpha)A^{-1/2}-1} \right) H(A^{-1/2}\varphi) dr d\varphi, \\ \text{and} \quad I_2 &= \int_E F(\psi) r^{NA^{-1/2}-1} H(A^{-1/2}\varphi) dr d\varphi. \end{aligned}$$

Finally, taking as above the four constants $\bar{C}_1, \dots, \bar{C}_4 > 0$ independent of A , we conclude

$$\begin{aligned} c_{A,K} &\leq m \sigma_K \sigma_{N-K} \frac{\left(\bar{C}_2 \bar{C}_3 \int_E \left((\psi_r^2 + \frac{1}{r^2} \psi_\varphi^2) + \psi^2 r \right) A^{-\frac{N-K-1}{2}} dr d\varphi\right)^{\frac{N}{\alpha}}}{A^{-\frac{N-\alpha}{\alpha}} \left(\bar{C}_1 \bar{C}_4 \int_E F(\psi) A^{-\frac{N-K-1}{2}} dr d\varphi\right)^{\frac{N-\alpha}{\alpha}}} \\ &= C A^{\frac{K-1}{2} + N(\frac{1}{\alpha} - \frac{1}{2})} \frac{\left(\int_E \left((\psi_r^2 + \frac{1}{r^2} \psi_\varphi^2) + \psi^2 r \right) dr d\varphi\right)^{\frac{N}{\alpha}}}{\left(\int_E F(\psi) dr d\varphi\right)^{\frac{N-\alpha}{\alpha}}} \end{aligned}$$

with obvious definition of the constant C . As the last ratio does not depend on A , the conclusion ensues. \square

Proposition 3.2. Assume $\alpha > 2$ and let $A > A_K$. Define $\bar{u} \in H_K$ by setting

$$\bar{u}_K(x) := w_A \left(\frac{x}{\lambda} \right) \quad \text{with} \quad \lambda = \frac{\|v_A\|_A}{\left(\int_{\mathbb{R}^N} F(v_A) dx\right)^{1/2}}.$$

Then $I(\bar{u}_K) < 0$ and the corresponding mountain-pass level (6) satisfies

$$c_{A,K} \leq C_2 A^{\frac{K-1}{2}}$$

where the constant $C_2 > 0$ does not depend on A .

Proof. The proof is very similar to the one of Proposition 3.1 above (see also [18, Proposition 4.3]), so we leave it to the reader for brevity. \square

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, so assume all the hypotheses of the theorem.

Let K be any integer such that $2 \leq K \leq N - 2$. Assume $A > A_K$ (where A_K is defined by (10)) and consider the mountain-pass level $c_{A,K}$ defined by (6), with $\bar{u}_K \in H_K$ given by Proposition 3.1 or 3.2 if $\alpha \in (2/(N - 1), 2)$ or $\alpha \in (2, 2N - 2)$, respectively.

We first show that $c_{A,K}$ is a critical level for the energy functional I . To do this, we will make use of the sum space

$$L^{p_1} + L^{p_2} := \{u_1 + u_2 : u_1 \in L^{p_1}(\mathbb{R}^N), u_2 \in L^{p_2}(\mathbb{R}^N)\}.$$

We recall that it is a Banach space with respect to the norm

$$\|u\|_{L^{p_1} + L^{p_2}} := \inf_{u_1 + u_2 = u} \max \left\{ \|u_1\|_{L^{p_1}(\mathbb{R}^N)}, \|u_2\|_{L^{p_2}(\mathbb{R}^N)} \right\}$$

(see [6, Corollary 2.11]) and that the continuous embedding $L^p(\mathbb{R}^N) \hookrightarrow L^{p_1} + L^{p_2}$ holds for all $p \in [p_1, p_2]$ (see [6, Proposition 2.17]), in particular for $p = 2^*$. Moreover, for every $u \in L^{p_1} + L^{p_2}$ and every $\varphi \in L^{p'_1}(\mathbb{R}^N) \cap L^{p'_2}(\mathbb{R}^N)$ one has

$$\int_{\mathbb{R}^N} |u\varphi| dx \leq \|u\|_{L^{p_1} + L^{p_2}} \left(\|\varphi\|_{L^{p'_1}(\mathbb{R}^N)} + \|\varphi\|_{L^{p'_2}(\mathbb{R}^N)} \right) \quad (12)$$

where $p'_i = p_i/(p_i - 1)$ is the Hölder conjugate exponent of p_i (see [6, Lemma 2.9]).

We will also need the following abstract result from [13], concerning the existence of bounded Palais-Smale sequences for C^1 functionals on Banach spaces.

Lemma 4.1. *Let $J: X \rightarrow \mathbb{R}$ be a C^1 functional on a Banach space $(X, \|\cdot\|)$, having the form*

$$J(u) = \frac{1}{q} \|u\|^q - B(u)$$

for some $q > 0$ and some continuous functional $B: X \rightarrow \mathbb{R}$. Assume that there exists a sequence of continuous mappings $\psi_n: X \rightarrow X$ such that for all n there exist $\alpha_n > \beta_n > 0$ satisfying

$$\|u\|^q \geq \alpha_n \|\psi_n(u)\|^q \quad \text{and} \quad B(u) \leq \beta_n B(\psi_n(u)) \quad \text{for all } u \in X \quad (13)$$

and
$$\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 1, \quad \liminf_{n \rightarrow \infty} \frac{|1 - \beta_n|}{\alpha_n - \beta_n} < \infty.$$

If there exist $r > 0$ and $\bar{u} \in X$ with $\|\bar{u}\| > r$ such that

$$\inf_{\|u\|=r} J(u) > J(0) \geq J(\bar{u}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\psi_n(0)\| = \lim_{n \rightarrow \infty} \|\psi_n(\bar{u}) - \bar{u}\| = 0,$$

then J has a bounded Palais-Smale sequence $\{w_n\} \subset X$ at level

$$c_{J,\bar{u}} = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J(u), \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

Proof. This is a particular case of [13, Theorem 1.1]. □

Lemma 4.2. $c_{A,K}$ is a critical level for the functional $I|_{H_K}$.

Proof. We want to apply Lemma 4.1 to the functional $I|_{H_K}$ with $\bar{u} = \bar{u}_K$ (and of course $q = 2$ and $B(u) = \int_{\mathbb{R}^N} F(u) dx$). Take any real sequence $\{t_n\}$ such that $t_n \rightarrow 1$ and $t_n > 1$, and for every $u \in H_K$ define $\psi_n(u) \in H_K$ by setting $\psi_n(u)(x) := u(x/t_n)$. We have

$$\|\psi_n(u)\|_A^2 = t_n^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + t_n^{N-\alpha} \int_{\mathbb{R}^N} \frac{A}{|x|^\alpha} u^2 dx \leq t_n^{\max\{N-2, N-\alpha\}} \|u\|_A^2 \quad (14)$$

and
$$\int_{\mathbb{R}^N} F(\psi_n(u)) dx = t_n^N \int_{\mathbb{R}^N} F(u) dx, \quad (15)$$

so that (13) holds with $\alpha_n = t_n^{-\max\{N-2, N-\alpha\}}$ and $\beta_n = t_n^{-N}$. Note that (14) also ensures that the linear mapping $\psi_n : H_K \rightarrow H_K$ is continuous. Since $t_n > 1$ and $\max\{N-2, N-\alpha\} < N$, we have $\alpha_n > \beta_n$. Moreover, $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 1$ and

$$\lim_{n \rightarrow \infty} \frac{|1 - \beta_n|}{\alpha_n - \beta_n} = \lim_{n \rightarrow \infty} \frac{1 - t_n^{-N}}{t_n^{-\max\{N-2, N-\alpha\}} - t_n^{-N}} = \frac{N}{N - \max\{N-2, N-\alpha\}} < \infty.$$

Recalling (5) and the fact that $I(\bar{u}_K) < 0$, it remains to check that

$$\lim_{n \rightarrow \infty} \|\psi_n(\bar{u}_K) - \bar{u}_K\|_A = 0.$$

By definition of \bar{u}_K , there exist $l_1, l_2 > 0$ such that $\bar{u}_K \in C_c^\infty(\Omega_{l_1, l_2})$ where $\Omega_{l_1, l_2} := \{x \in \mathbb{R}^N : l_1 < |x| < l_2\}$. Then both \bar{u}_K and $\psi_n(\bar{u}_K)$ belong to the space $C_c^\infty(\Omega_{l_1/2, 2l_2})$ for n sufficiently large, and thus we get

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(\psi_n(\bar{u}_K) - \bar{u}_K)^2}{|x|^\alpha} dx &= \int_{\Omega_{l_1/2, 2l_2}} \frac{(\psi_n(\bar{u}_K) - \bar{u}_K)^2}{|x|^\alpha} dx \\ &\leq \frac{2^\alpha}{l_1^\alpha} \int_{\Omega_{l_1/2, 2l_2}} (\psi_n(\bar{u}_K) - \bar{u}_K)^2 dx \rightarrow 0 \end{aligned}$$

and

$$\int_{\mathbb{R}^N} |\nabla \psi_n(\bar{u}_K) - \nabla \bar{u}_K|^2 dx = \int_{\Omega_{t_1/2, 2t_2}} |\nabla \psi_n(\bar{u}_K) - \nabla \bar{u}_K|^2 dx \rightarrow 0$$

as $n \rightarrow \infty$, since $\psi_n(\bar{u}_K) = \bar{u}_K(t_n^{-1} \cdot) \rightarrow \bar{u}_K$ e $\nabla(\psi_n(\bar{u}_K)) = t_n^{-1} \nabla \bar{u}_K(t_n^{-1} \cdot) \rightarrow \nabla \bar{u}_K$ in $L^\infty(\mathbb{R}^N)$. So we can apply Lemma 4.1 and deduce that there exists a sequence $\{u_n\} \subset H_K$ such that $\{\|u_n\|_A\}$ is bounded, $I(u_n) \rightarrow c_{A,K}$ and $I'(u_n) \rightarrow 0$ in the dual space of H_K . Now we exploit the fact that the space H_K is compactly embedded into $L^{p_1} + L^{p_2}$, since $p_1 < 2^* < p_2$ and so is the subspace of $D^{1,2}(\mathbb{R}^N)$ made up of the mappings with the same symmetries of H_K (see [1, Theorem A.1]). Hence there exists $u \in H_K$ such that (up to a subsequence) $u_n \rightharpoonup u$ in H_K and $u_n \rightarrow u$ in $L^{p_1} + L^{p_2}$. This implies that $\{f(u_n)\}$ is bounded in both $L^{p'_1}(\mathbb{R}^N)$ and $L^{p'_2}(\mathbb{R}^N)$, since assumption (\mathbf{h}_0) ensures that the Nemytskiĭ operator $v \mapsto f(v)$ is continuous from $L^{p_1} + L^{p_2}$ into $L^{p'_1}(\mathbb{R}^N) \cap L^{p'_2}(\mathbb{R}^N)$ (see [6, Corollary 3.7]). Then by (12) we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u) dx \right| &\leq \|u_n - u\|_{L^{p_1} + L^{p_2}} \left(\|f(u_n)\|_{L^{p'_1}(\mathbb{R}^N)} + \|f(u_n)\|_{L^{p'_2}(\mathbb{R}^N)} \right) \\ &\leq (\text{const.}) \|u_n - u\|_{L^{p_1} + L^{p_2}} \rightarrow 0 \end{aligned}$$

and therefore

$$\begin{aligned} \|u_n - u\|_A^2 &= (u_n, u_n - u)_A - (u, u_n - u)_A \\ &= I'(u_n)(u_n - u) + \int_{\mathbb{R}^N} f(u_n)(u_n - u) dx - (u, u_n - u)_A \rightarrow 0, \end{aligned}$$

where $(u, u_n - u)_A \rightarrow 0$ since $u_n \rightharpoonup u$ in H_K , and $I'(u_n)(u_n - u) \rightarrow 0$ because $I'(u_n) \rightarrow 0$ in the dual space of H_K and $\{u_n - u\}$ is bounded in H_K . This implies that $u_n \rightarrow u$ in H_K and thus concludes the proof. \square

Now, using again Lemma 4.1 and the compactness results of [20], we show that I also has a radial critical point.

Lemma 4.3. *The functional $I|_{H_r}$ has at least a nonzero critical point.*

Proof. The arguments are absolutely similar to the ones used in proving Lemma 4.2, so we will be very sketchy. By condition (4) with $p = 2^*$ and the continuous embedding $H_r \hookrightarrow L^{2^*}(\mathbb{R}^N)$, we readily have that $\exists R > 0$ such that $\inf_{\|u\|_A=R} I|_{H_r}(u) > 0$ and $I|_{H_r}(u) \geq 0$ for $\|u\|_A \leq R$. For every $u \in H_r$ and $t > 1$ define $\psi_t(u)(x) := u(x/t)$. Take $u_0 \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap H_r$ such that $u_0 \geq 0$ and $u_0 \neq 0$. We have

$$I(\psi_t(u_0)) \leq \frac{1}{2} t^{\max\{N-2, N-\alpha\}} \|u_0\|_A^2 - t^N \int_{\mathbb{R}^N} F(u) dx$$

and therefore $I(\bar{u}) < 0$ for $\bar{u} := \psi_t(u_0)$ with t large enough. Then, letting $\{t_n\}$ be any real sequence such that $t_n \rightarrow 1$ and $t_n > 1$, as in the proof of Lemma

4.2 one checks that Lemma 4.1 applies with $\psi_n = \psi_{t_n}$, $\alpha_n = t_n^{-\max\{N-2, N-\alpha\}}$ and $\beta_n = t_n^{-N}$. So $I_{|H_r}$ has a Palais-Smale sequence $\{u_n\}$ at a nonzero level. Now observe that assumption (\mathbf{h}_0) with p_1, p_2 as in the theorem ensures that one can find $p \in [p_1, p_2]$ such that $|f(u)| \leq (\text{const.}) u^{p-1}$ (recall (4)) and (2) holds, in such a way that the embedding $H_r \hookrightarrow L^p(\mathbb{R}^N)$ is compact by the compactness results of [20]. Hence it is a standard exercise to conclude that, up to a subsequence, u_n converges to a nonzero critical point of $I_{|H_r}$. \square

Proof of Theorem 1.1. The proof is essentially the same of Theorem 1.1 of [18]; we repeat it here just for the sake of completeness. On the one hand, the restriction $I_{|H_r}$ has a critical point $u_r \neq 0$ by Lemma 4.3. On the other hand, one checks that for $\alpha \in (\frac{2}{N-1}, 2N-2) \setminus \{2\}$ the integer ν defined in (3) is at least 1 (see [18, Lemma 5.2]), so that there are ν integers K (precisely $K = 2, \dots, \nu + 1$) such that

$$\frac{K-1}{2} + N \left(\frac{1}{\alpha} - \frac{1}{2} \right) < \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{2-\alpha} \frac{2^* - p_1}{p_1 - 2} \right\} \quad \text{if } \frac{2}{N-1} < \alpha < 2$$

and

$$\frac{K-1}{2} < \min \left\{ \frac{N-1}{\alpha}, \frac{N-2}{\alpha-2} \frac{p_2 - 2^*}{p_2 - 2} \right\} \quad \text{if } 2 < \alpha < 2N-2.$$

Let K be any of such integers. By Lemma 4.2, there exists $u_K \in H_K \setminus \{0\}$ such that $I(u_K) = c_{A,K} > 0$ and $I'_{|H_K}(u_K) = 0$. Both u_r and u_K are also critical points for the functional $I: H_\alpha^1 \rightarrow \mathbb{R}$, by the Palais' Principle of Symmetric Criticality [17]. Moreover, it is easy to check that they are nonnegative: test $I'(u_K)$ with the negative part $u_K^- \in H_\alpha^1$ of u_K and use the fact that $f(s) = 0$ for $s < 0$ to get $I'(u_K)u_K^- = -\|u_K^-\|_A^2 = 0$; the same for u_r . Therefore u_r and u_K are weak solutions to problem (\mathcal{P}) . By Remark 2.3 and Propositions 2.2, 3.1 and 3.2, there exists $\tilde{A}_K > A_K$ such that $c_{A,K} < m_A$ for every $A > \tilde{A}_K$. Now, setting $A_* := \max \{ \tilde{A}_K : 2 \leq K \leq \nu + 1 \}$, we have $c_{A,K} < m_A$ for every $A > A_*$ and $K = 2, \dots, \nu + 1$, so that all the ν weak solutions u_K with $K = 2, \dots, \nu + 1$ are nonradial, since otherwise Lemma 2.1 would yield the contradiction $c_{A,K} = I(u_K) \geq m_A$. This also implies $u_{K_1} \neq u_{K_2}$ for $K_1 \neq K_2$, since $H_{K_1} \cap H_{K_2} = H_r$ (see [18, Lemma 2.1]). \square

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