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A Quick Survey on Recent Results for the NLS Equation on Metric Graphs

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We survey some recent results about the Nonlinear Schrödinger Equation on a metric graph, obtained in collaboration with D. Noja and S. Rolando. We also provide some general information about some analytic tools that are useful to set a convenient environment.

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Dedicated to Francesca

1. Introduction

While the study of the (formal) Nonlinear Schrödinger equation

$$\mathrm{i}\,\frac{d\psi}{dt} = -\Delta\psi - f(|\psi|)\psi$$

is nowadays a classical research topic on domains of $\mathbb{R}^N$ ($N \geq 1$), its counterpart on more complicated structures like *metric graphs* is still a topic in strong development. We present some basic tools for the analysis of differential equations on a metric graph, and we summarize some existence result we have obtained in the paper [7].

Let us recall some basic facts about metric graphs, so as to be ready for working with differential equations.

Definition 1.1. A *graph* $\mathcal{G}$ consists of a finite or countably infinite set of *vertices* $V = \{v_i\}$ and a set $E = \{e_j\}$ of *edges* connecting the vertices. Each edge e can be identified with a pair (v_i, v_k) of vertices. We denote by E_v the set of all edges of Γ that are incident to the vertex v. We assume that the *degree* of the vertex v, defined as $d_v = \#E_v$, is finite and positive. In particular, this excludes "isolated" vertices.

This is a set-theoretic definition. To work with differential equations on a graph, we need to add (at least) a distance, i.e. a metric structure.

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Definition 1.2. A graph $\mathcal{G}$ is a *metric graph* if each edge e is assigned a positive length $l_e \in (0, \infty]$.

In a metric graph, each edge e is naturally identified with the segment $[0, l_e]$. We highlight that we do not think of a metric graph $\mathcal{G}$ as a subset of some Euclidean space. Furthermore, a distance is easily introduced on $\mathcal{G}$.

Definition 1.3. If a family $\{e_j \mid j = 1, \dots, M\}$ forms a path, its *length* is defined as $\sum_{j=1}^{M} l_{e_j}$. For two vertices v and w of $\mathcal{G}$, the distance $d(v, w)$ is defined as the minimal length of a path that connects v and w. This distance is extended to any two points of the graph in a natural way.

To avoid unnatural situations, it is customary to introduce a couple of conditions

(A) The "infinite" ends of infinite edges are assumed to have degree one. Thus, the graph can be thought of as a graph with finite length edges with additional infinite "leads" or "ends" going to infinity attached to some vertices. We can just assume that each infinite edge is a ray with a single vertex.

(B) For any positive number r and any vertex v there is only a finite set of vertices w at a distance less than r from v. In particular, the distance between any two distinct vertices is positive, and there are no finite length paths of infinitely many edges.

From its very definition, it is easy to define the Lebesgue measure on a metric graph. Functions on a graph are defined along the edges (while in discrete models they are defined only at the vertices of the graph).

Definition 1.4. The space $L^2(\mathcal{G})$ consists of functions f that are measurable and square integrable on each edge e of $\mathcal{G}$ such that

$$\|f\|_{L^2}^2 = \sum_{e \in \mathcal{G}} \|f\|_{L^2(0,l_e)}^2 < \infty.$$

A similar definition can be given to define $L^p(\mathcal{G})$, for any $1 < p < \infty$.

Definition 1.5. The Sobolev space $H^1(\mathcal{G})$ consists of *continuous* functions f that belong to $H^1(0, l_e)$ for every edge e of $\mathcal{G}$ such that

$$\sum_{e \in \mathcal{G}} \|f\|_{H^1(0,l_e)}^2 < \infty.$$

In particular, the continuity of f means that on all edges adjacent to a vertex v, the function f assumes the same value at v.

1.1. Operators

Once a metric structure has been introduced on a graph, we need to define suitable operators to study differential equations. It is natural to define differential (or even pseudodifferential) operators on a metric graph piecewise, in the sense that a differential operator is defined on each edge. For example, letting x denote the coordinate x_e along the edge e of the graph $\mathcal{G}$, we can define the *Schrödinger operator*

$$f \mapsto -\frac{d^2}{dx^2} f + V(x) f(x)$$

or the *magnetic Schrödinger operator*

$$f \mapsto \left(-\mathrm{i}\frac{d}{dx} - A(x)\right)^2 f + V(x) f(x).$$

These and similar definitions, however, require *boundary conditions* in order to define *self-adjoint* operators on the whole graph. Let us discuss shortly this issue.

We will consider for simplicity a *finite* metric graph $\mathcal{G}$, so that the number $|\mathcal{G}|$ of edges is a finite number. Of course, the length of an edge may be infinite. A complete description of all the vertex conditions (i.e. the boundary conditions that involve a single vertex at a time) that make the operator

$$f \mapsto -\frac{d^2}{dx^2} f$$

self-adjoint via the standard von Neumann theory of extensions of symmetric operators, see [5, 6], probably the simplest condition goes under the name of Kirchhoff:

(Kir) f is continuous on $\mathcal{G}$ and at each vertex v there results

$$\sum_{e \in E_v} \frac{df}{dx_e}(v) = 0\,,$$

where the sum is taken over all edges e containing the vertex v. This sum is finite due to our assumption. Here *the derivatives are taken in the directions away from the vertex* (we will call these "outgoing directions"), the agreement we will adhere to in all cases when these conditions are involved.

This condition reduces to a Neumann boundary condition at "loose ends". Furthermore, it is *natural*, in the sense that the domain of the quadratic form of the corresponding operator does not require any conditions on a function besides being in $H^1(\mathcal{G})$.

Several vertex boundary conditions can be defined, like

(δ) f is continuous on $\mathcal{G}$ and at each vertex v there results

$$\sum_{e \in E_v} \frac{df}{dx_e}(v) = \alpha_v f(v),$$

where α_v is a fixed number. One can recognize these conditions as an analogue of conditions one obtains from a Schrödinger operator on the line with a δ potential, which explains the name.

Similarly,

(δ') The value of the derivative df/dx_e is the same for all edges e, and at each vertex v there results

$$\sum_{e \in E_v} \frac{df}{dx_e}(v) = \alpha_v f(v),$$

and even the *vertex Dirichlet conditions*: those where at each vertex it is required that the boundary values of the function on each edge are equal to zero.

2. Ground states on a metric graph

Let us consider a connected metric graph $\mathcal{G}$, and define the NLS energy functional

$$E(u, \mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}$$

with the *mass constraint* $\|u\|^2_{L^2(\mathcal{G})} = \mu.$

Here $\mu > 0$ is a given number, and $2 < p < 6$. In the "trivial" case $\mathcal{G} = \mathbb{R}$, the variational problem

$$\min \left\{ E(u, \mathbb{R}) \mid u \in H^1(\mathbb{R}),\ \|u\|^2_{L^2(\mathbb{R})} = \mu \right\}$$

is well understood. The solutions are usually called *solitons*, and it turns out that they are unique up to translations and phase multiplication. Similar results exist when $\mathcal{G} = [0, \infty)$ and, partially, when $\mathcal{G} = [a, b]$, a compact interval.

The case of a non-compact metric graph is more challenging. The following result was proved in [1].

Theorem 2.1. *If $\mathcal{G}$ contains at least one half-line, then*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) \leq \min_{\phi \in H^1_\mu(\mathbb{R})} E(\phi, \mathbb{R}) \quad \textit{and} \quad \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) \geq \min_{\phi \in H^1_\mu(\mathbb{R}^+)} E(\phi, \mathbb{R}^+).$$

Here $$H^1_\mu(\mathcal{G}) = \left\{ u \in H^1(\mathcal{G}) \mid \|u\|^2_{L^2(\mathcal{G})} = \mu \right\}.$$

In order to decide whether $\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G})$ is attained, Adami et al. introduced the following geometric condition:

(H) After removal of any edge $e \in E$, every connected component of the graph $(V, E \setminus e)$ contains at least one vertex at infinity $v \in V_\infty$.

The rôle of condition (H) is explained in the following results.

Theorem 2.2. (Adami, Serra, Tilli) *If $\mathcal{G}$ satisfies condition* (H)*, then*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}) = \min_{\phi \in H^1_\mu(\mathbb{R})} E(\phi, \mathbb{R}).$$

However, the infimum
$$\inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G})$$
is attained only in three cases, see [1, Theorem 2.5]. Very recently, the same authors proved in [2] that on any non-compact metric graph $\mathcal{G}$, when μ is large enough there exist at least as many bound (but not necessarily ground) states of mass μ as the number of bounded edges of $\mathcal{G}$.

We refer to the paper [3] for a more complete review of nonlinear dynamics on metric structures.

3. Standing waves on the double-bridge graph

While the construction of standing wave solutions to the NLS equation on a general metric graph strongly depends on the *shape* of the graph, a more detailed analysis can be performed on simple but significant structures.

We consider a metric graph $\mathcal{G}$ made up of two half-lines joined by two bounded edges, see Figure 3.1.

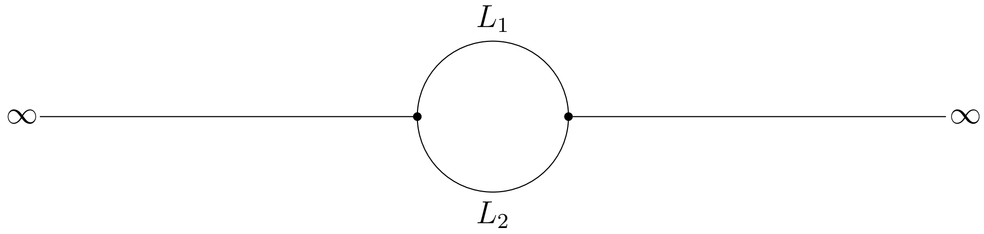

Figure 3.1: The double-bridge graph.

Equivalently, $\mathcal{G}$ can be imagined as a ring to which two half-lines are attached in two distinct vertices. The half-lines will be identified with two copies of $[0,+\infty)$, while the two bounded edges will be identified with two bounded intervals of lenght $L_1>0$ and $L_2\geq L_1$, namely $[0,L_1]$ and $[L_1,L]$, with $L=L_1+L_2$.

A Schrödinger operator $H_\mathcal{G}$ can be defined on $\mathcal{G}$ as
$$H_\mathcal{G}\psi(x_1,\dots,x_4)=(-\psi_1''(x_1),\dots,-\psi_4''(x_4)),\quad x_j\in I_j,$$
where $I_1=[0,L_1]$, $I_2=[L_1,L]$, $I_3=I_4=[0,+\infty)$. The domain $D(H_\mathcal{G})$ of $H_\mathcal{G}$ consists of functions $\psi=(\psi_1,\dots,\psi_4)$ on $\mathcal{G}$ such that $\psi_j\in H^2(I_j)$ for $j=1,2,3,4$ together with Kirchhoff boundary conditions:
$$\begin{gathered}\psi_1(0)=\psi_2(L)=\psi_3(0),\quad \psi_1(L)=\psi_2(L_1)=\psi_4(0)\\ \psi_1'(0)-\psi_2'(L)+\psi_3'(0)=\psi_1'(L_1)-\psi_2'(L_1)-\psi_4'(0)=0.\end{gathered}$$

We perturb the linear dynamics generated by $H_\mathcal{G}$ with a focusing cubic term:
$$\mathrm{i}\frac{d\psi}{dt}=H_\mathcal{G}\psi-|\psi|^2\psi, \tag{1}$$
where of course
$$|\psi|^2\psi=(|\psi_1|^2\psi_1,\dots,|\psi_4|^2\psi_4).$$

The results obtained by Adami et al. show in particular that no ground state exists on the double-bridge graph. We therefore look for standing waves of (1) in the form $\psi(t,x) = e^{-\mathrm{i}t\omega} u(x,\omega)$, where $\omega \in \mathbb{R}$ and u is a purely space function defined on $\mathcal{G}$. The sign of the frequency ω turns out to play an important rôle. For convenience, we state explicitely the system of equations to be solved:

$$\begin{cases} -u_j'' - u_j^3 = \omega u_j, & u_j \in H^2(I_j) \\ u_1(0) = u_2(L) = u_3(0), & u_1(L_1) = u_2(L_1) = u_4(0) \\ u_1'(0) - u_2'(L) + u_3'(0) = 0, & u_1'(L_1) - u_2'(L_1) - u_4'(0) = 0. \end{cases} \tag{2}$$

When $\omega > 0$, we remark that non-vanishing L^2 solutions of the stationary focusing NLS on the half-line do not exist. As a consequence, any solution of our problem must be supported on the circle. We thus see that additional Dirichlet boundary conditions arise at the vertices of $\mathcal{G}$ and make (2) an overdetermined system. A solution u is necessarily periodic on $[0,L]$, with Dirichlet conditions at 0, L_1 and L. It is natural to work with special functions that solve NLS equations on an interval, and "play" with parameters in order to match all the boundary conditions.

We recall that periodic solutions of the NLS equation on an interval are the Jacobi snoidal, cnoidal and dnoidal special functions. Since we work with a focusing equation, we rule out snoidal solutions. Furthermore, dnoidal functions have constant sign, and they are ruled out as well. Now, the cnoidal function $v(y) = \mathrm{cn}(y;k)$ with parameter k solves the equation

$$-v'' - 2k^2 v^3 = (1-2k^2)v. \tag{3}$$

Up to translations, v is the unique periodic solution of (3) oscillating around zero. Its period is given by the formula

$$T(k) = 4K(k) = 4\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where K is the so-called complete elliptic integral of first kind. There results $\mathrm{cn}(0;k) = 1$. The first zero of $\mathrm{cn}(y;k)$ is $y = T/4$. If we define

$$u(x) = \sqrt{2}\, kp\, v(px), \quad p = \sqrt{\frac{\omega}{1-2k^2}}, \quad k \in (0, 1.2), \quad \omega > 0,$$

then
$$u(x) = \sqrt{\frac{2\omega k^2}{1-2k^2}}\, \mathrm{cn}\left(\sqrt{\frac{\omega}{1-2k^2}}x; k\right)$$

solves for every $k \in (0, 1/\sqrt{2})$ the equation $-u'' - u^3 = \omega u$.

The period of u is thus given by $T_\omega(k) = 4\sqrt{\dfrac{1-2k^2}{\omega}}K(k)$.

If we now impose periodicity on $[0,L]$ and Dirichlet boundary conditions at 0 and at L, we have a sequence of parameterk $\{k_n\}_n$ and functions $\{u^{\pm}_{n,\omega}\}_n$ defined by

$$u^{\pm}_{n,\omega}(x) = \sqrt{\frac{2\omega k_n^2}{1-2k_n^2}}\,\mathrm{cn}\left(\sqrt{\frac{\omega}{1-2k_n^2}}\left(x \pm \frac{T_\omega(k_n)}{4}\right); k_n\right), \quad nT_\omega(k_n) = L.$$

The additional Dirichlet condition at L_1 can be satisfied if, and only if, there exists a natural number $m < n$ such that $mT_\omega(k_n) = L_1$, or equivalently

$$\frac{L_1}{L} = \frac{m}{2n}.$$

If we now add this condition to our assumptions, an infinite strict subset of the above families of cnoidal functions $u^{\pm}_{n,\omega}$ satisfies the problem (2) for every $\omega > 0$, namely those with $n \in \mathbb{N}q_0$, where p_0, q_0 are the unique coprime numbers such that $L_1/L = p_0/(2q_0)$.

Similarly, solutions to (2) when $\omega = 0$ exists only if $L_1/L \in \mathbb{Q}$. We refer to [4] for details.

The situation is drastically different in the case $\omega < 0$. First of all, the previous cnoidal solutions can be defined by the formula

$$u^{\pm}_{n,\omega}(x) = \sqrt{\frac{2|\omega| k_n^2}{1-2k_n^2}}\,\mathrm{cn}\left(\sqrt{\frac{|\omega|}{1-2k_n^2}}\left(x \pm \frac{T_\omega(k_n)}{4}\right); k_n\right), \quad \frac{1}{\sqrt{2}} < k_n < 1.$$

As before, we have an infinite number of global bifurcation branches, compactly supported on the graph. These solutions exists only when L_1/L is a rational number, and correspond to $u_3 = u_4 = 0$.

However, more solutions can be constructed, since for $\omega < 0$ non-vanishing solutions on the two half-lines are admissible. For instance, we might shift a cnoidal solutions on the ring and then attach to this shifted cnoidal function a half-soliton on each tail with the correct height, in order to restore continuity at the vertices. The Kirchhoff condition is trvially satisfied, since half-solitons have zero derivative at the vertices. Thus, a second bifurcation branch with non-trivial components on the half-lines arises at $\omega = 0$.

Such a construction works easily for $L_1/L \in \mathbb{Q}$, but the problem is non-trivial when L_1/L is an irrational number.

We have so far obtained partial results under restrictive assumptions. More precisely, we look for standing waves on the double-bridge graph such that

(P_1) the components u_3 and u_4 are non-trivial;

(P_2) the components u_1 and u_2 are the restriction to I_1 and I_2 respectively of some function $u \in H^2_{\mathrm{per}}([0,L])$.

Here $H^2_{\mathrm{per}}([0,L]) = \{u \in H^2([0,L]) \mid u(0) = u(L),\ u'(0) = u'(L)\}$.

Condition (P_1) forces $\omega < 0$ and

$$u_j(x) = \pm\sqrt{2|\omega|}\,\operatorname{sech}\left(\sqrt{|\omega|}(x+a_j)\right) \tag{4}$$

for some $a_j \in \mathbb{R}$, $j = 3, 4$. Condition (P_2) yields

$$u_1'(0) - u_2'(L) = u_1'(L_1) - u_2'(L_1) = 0,$$

so that $a_3 = a_4 = 0$ in (4) in view of the Kirchhoff boundary condition. The system (2) reduces therefore to

$$\begin{cases} -u'' - u^3 = \omega u, & u \in H^2_{\mathrm{per}}([0,L]), \\ \omega < 0 \\ u(0) = \pm u(L_1) = \sqrt{2|\omega|}. \end{cases} \tag{5}$$

The ± sign distinguishes the case of u_3 and u_4 of the same sign from the case in which u_3 and u_4 have different signs.

To state the main results recently proved in [7] we need some notation. We define the function

$$S\colon (1/\sqrt{2}, 1) \longrightarrow \mathbb{R}, \qquad k \mapsto 4\sqrt{2k^2-1}K(k).$$

It is an easy exercise to check that the function S is strictly increasing, continuous, and $S\left((1/\sqrt{2},1)\right) = (0,+\infty)$. We will use $[x]$ to denote the *floor* function at x, namely the greatest integer smaller than or equal to x.

Theorem 3.1. *Assume that $L_1/L \in \mathbb{Q}$. The solutions (ω, u) to problem* (5) *form a countable family. More precisely, there exist two sequences $\{\omega_n^+\}_n$ and $\{\omega_n^-\}_n$ such that the solutions of* (5) *are given by $\{(\omega_n^\pm, u_n^\pm) \mid n \in \mathbb{N}\}$ with*

$$u_n^\pm(x) = \sqrt{\frac{2|\omega_n^\pm|k_n^2}{2k_n^2-1}}\,\operatorname{cn}\left(\sqrt{\frac{|\omega_n^\pm|}{2k_n^2-1}}\left(x - s_n^\pm\right); k_n\right),$$

$$k_n = S^{-1}\left(L\frac{\sqrt{|\omega_n^\pm|}}{n}\right), \quad r_n = \frac{L_1}{L}n - \left[\frac{L_1}{L}n + \frac{1}{2}\right],$$

$$s_n^+ = \frac{L}{2n}r_n, \quad s_n^- = \frac{L}{2n}\left(|r_n| - \frac{1}{2}\right)\operatorname{sgn} r_n$$

Remark 3.2. An explicit construction of the two sequences $\{\omega_n^+\}_n$ and $\{\omega_n^-\}_n$ is also possible. We refer to [7] for more details.

Theorem 3.3. *Assume that $L_1/L = p/q \in \mathbb{Q}$, with p and q coprime integers. The set of solutions to* (5) *with the + sign is $\{(\omega, \tilde{u}_{n,\omega}^\pm) \mid \omega < 0,\ n \in \mathbb{N}q\} \cup \{(\omega_n^+, u_n^+) \mid n \in \mathbb{N},\ n \notin \mathbb{N}q,\ np/q + 1/2 \notin \mathbb{N}\}$, where ω_n^+, u_n^+ are the same as in Theorem* 3.1, *and*

$$\tilde{u}^{\pm}_{n,\omega} = \sqrt{\frac{2|\omega| k^2_{n,\omega}}{2k^2_{n,\omega} - 1}}\, \mathrm{cn}\left(\sqrt{\frac{|\omega|}{2k^2_{n,\omega} - 1}}\,(x \pm \gamma_{n,\omega}; k_{n,\omega})\right)$$

$$k_{n,\omega} = S^{-1}\left(L\frac{\sqrt{|\omega|}}{n}\right)$$

$$\gamma_{n,\omega} = \sqrt{\frac{2k^2_{n,\omega} - 1}{|\omega|}} \int_{\left[1,\sqrt{\frac{2k^2_{n,\omega}-1}{k^2_{n,\omega}}}\,\right]} \frac{dt}{\sqrt{(1-t^2)(1-k^2_{n,\omega}(1-t^2))}}.$$

The set of solutions to (5) *with the* $-$ *sign is* $\{(\omega^-_n, u^-_n) \mid n \in \mathbb{N},\ n \notin \mathbb{N}q\}$ *if* q *is odd, and* $\{(\omega, \tilde{u}^{\pm}_{n,\omega}) \mid \omega < 0,\ n \in (2\mathbb{N}-1)q/2\} \cup \{(\omega^-_n, u^-_n) \mid n \in \mathbb{N},\ n \notin \mathbb{N}q/2\}$ *if* q *is even, where* ω^-_n *and* ω^+_n *are the same as in Theorem* 3.1.

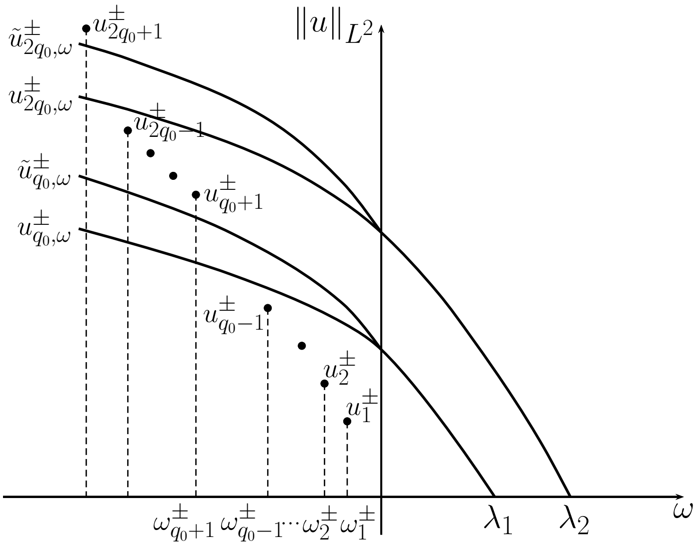

Figure 3.2: Bifurcation diagram for $L_1/L = p/q = p_0/(2q_0)$ with p, q and p_0, q_0 coprime. The functions $\tilde{u}^{\pm}_{hq_0,\omega}$ solve (5) "+" for all h if q is odd, and solve (5) "+" or (5) "-" according as h is even or odd if q is odd. All of them have period $L/(hq_o)$, the same of $u^{\pm}_{hq_0,\omega}$ and the eigenfunctions related to λ_h. The functions $u^{\pm}_n$, $n \notin \mathbb{N}q_0$, solve (5) respectively, and have period L/n. The frequencies $\omega^{\pm}_1, ..., \omega^{\pm}_q$ need not to be ordered as in the figure.

These results give a rather precise picture of the set of solutions to problem (5). Surprisingly enough, these solutions are a countable set if the geometry of the double-bridge graph L_1/L is irrational. We can also explain the content of Theorem 3.3 in terms of a bifurcation diagram. As before, it is convenient to denote by p_0, q_0 the unique coprime naturals such that $L_1/L = p_0/(2q_0)$, in such a way that $q_0 = q$ if q is odd and $q_0 = q/2$ if q is even; note also that $n \notin \mathbb{N}q$, $np/q + 1/2 \notin \mathbb{N}$ just means $n \notin \mathbb{N}q_0$. From every bifurcation branch $\{(\omega, u^{\pm}_{hq_0,\omega}) : \omega < 0\}$ originating from the eigenvalues λ_h, $h \in \mathbb{N}$, a secondary bifurcation branch $\{(\omega, \tilde{u}^{\pm}_{hq_0,\omega}) : \omega < 0\}$ bifurcates at $\omega = 0$ (a pitchfork bifurcation).

Such a branch of solutions solves (5) with "+" for all h if q is odd, and (5) with "+" or (5) with "-" according as h is even or odd if q is odd. Away from these secondary branches, we find isolated solutions to problems (5) coming from the countable families $\{(\omega_n^\pm, u_n^\pm) : n \in \mathbb{N}\}$, namely the ones with $n \notin \mathbb{N}q_0$.

Observe that we have solutions that oscillate any number of times on $[0, L]$. This situation is portrayed in Figure 3.2.

3.1. Frequencies

The analysis of the sequence $\{\omega_n\}_n$ of frequencies can be a rather challenging problem. Up to now, only some partial – but interesting – results have been obtained in [7].

Theorem 3.4. *Assume that* $L_1/L \in \mathbb{R} \setminus \mathbb{Q}$*. The frequencies* $\{\omega_n^\pm\}_n$ *of Theorem 3.1 form two sequences that are unbounded from below. Moreover, theses sequences have (at least) a cluster point falling into the intervals*

$$I^+ = \left[-\frac{1}{L^2} K \left(\frac{1}{\sqrt{2}} \right)^4, 0 \right] \quad \textit{and} \quad I^- = \left[-\frac{16}{5L^2} K \left(\frac{1}{\sqrt{2}} \right)^4, 0 \right].$$

The proof of this theorem offers a nice crossover with Elementary Number Theory. Indeed, after some simple computations, it is possible to reduce the proof of Theorem 3.4 to a fine analysis of the sequence of real numbers defined by the formula

$$\xi_n = \left| 2 \left\{ \alpha n + \frac{1}{2} \right\} - 1 \right|,$$

where $\alpha = L_1/L$ is irrational and $\{z\}$ denotes the fractional part of the number z, namely $\{z\} = z - [z]$. In particular, $0 < \xi_n < 1$ for every index n, since $\alpha \notin \mathbb{Q}$. Using Dirichlet's Theorem for Diophantine Approximation, we can show that the sequences $\{\omega_n^\pm\}_n$ are unbounded from below.

The construction of a *finite* cluster point for these sequences is non-trivial, and makes use of some fundamental result for *uniformly distributed* sequences in the interval $(0, 1)$. Hence, there is a strong interplay between the qualitative behavior of the frequencies of solutions to (5) and the analytic properties of a suitable sequence of numbers in $(0, 1)$. This is not a true surprise, but it would deserve a deeper investigation. A challenging question, that is intimately related to our results, is the following:

Open problem. *Let* $\alpha \in (0, 1)$ *be a given irrational number. Consider the sequence*

$$\tilde{\xi}_n = n \left(\{n\alpha\} - \frac{1}{2} \right). \tag{6}$$

For what choices of $0 < \alpha < 1$ *does the sequence* $\{\tilde{\xi}_n\}_n$ *have a cluster point larger than* $1/4$*?*

Actually, the existence of such a point would automatically provide a *second* finite cluster point for the frequencies $\omega_n^{\pm}$.

Another intriguing result that we have proved in [7] is somehow an "inverse" problem. If we drop the assumption that the *geometry* of the double-bridge graph, i.e. the ratio L_1/L, is fixed, can we pick any (negative) frequency ω_0 and construct a geometry such that $\omega_n^{\pm} \to \omega_0$? The next result shows that this is indeed possible.

Theorem 3.5. *For any $\omega_0 \leq 0$, there exists a number $\alpha = L_1/L \in (0,1)$ such that the sequence of frequencies $\{\omega_n^{\pm}\}_n$ of Theorem 3.1 possesses a subsequence that converges to ω_0.*

The proof is based on the following remark: it suffices to prove that, for any $\ell \geq 0$, there exists $\alpha \in (0,1)$ such that the sequence $\{\tilde{\xi_n}\}_n$ converges up to a subsequence to ℓ. Recall that $\tilde{\xi_n}$ was defined in (6). The number α can be constructed by looking at its binary expansion. Once again, the proof make use of some basic tool of Number Theory. We refer to [7, Section 4] for the details.

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