

# Existence Results and Strong Maximum Principle for a Resonant Sublinear Elliptic Problem

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Let  $\Omega$  be a bounded smooth connected open set in  $\mathbb{R}^N$  and let  $\lambda_1$  be the first eigenvalue of the Laplacian on  $\Omega$ . We study the resonant elliptic problem

$$\begin{cases} -\Delta u = \lambda_1 u + u^{s-1} - \mu u^{r-1}, & \text{in } \Omega \\ u \geq 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where  $s \in ]1, 2[$ ,  $r \in ]1, s[$ , and  $\mu \in ]0, +\infty[$ . An existence result of nonzero solutions is established via minimax and perturbation methods. Furthermore, for  $\mu$  large enough, we prove a Strong Maximum Principle for the solutions of this problem. In particular, we extend to higher dimension an analogous recent result obtained in the one-dimensional case via the time-mapping method.

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## 1. Introduction

Here and in the following,  $\Omega$  is a bounded open connected set in  $\mathbb{R}^N$  of class  $C^2$ . Let  $f: [0, +\infty[ \rightarrow \mathbb{R}$  be a continuous function and let consider the elliptic problem

$$(P) \quad \begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

By a *weak solution* to problem (P) above we mean any nonnegative function  $u \in C^0(\bar{\Omega}) \cap W_0^{1,2}(\Omega)$  which satisfies the equation

$$\int_{\Omega} (\nabla u \nabla v - f(u)v) dx = 0 \quad (1)$$

for each  $v \in W_0^{1,2}(\Omega)$ . A solution to problem (P) will be always understood in the sense specified above.

A standard technique to detect positive solutions to problem (P) consists in verifying that the following two properties hold true:

- (i) *there exists at least a nonzero solution to problem (P);*
- (ii) *every solution  $v$  to problem (P) such that  $v(x_0) = 0$ , for some  $x_0 \in \Omega$ , is identically zero in  $\Omega$ .*

Property (ii) is the well known Strong Maximum Principle for the solutions of problem (P). It can be reformulated in the following equivalent form:

- (iii) *every nonnegative and nonzero solution  $v \in C^0(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$  to problem (P) is positive in  $\Omega$ .*

Theorem 1 of [9] (see also [7]) provides a sufficient condition on the nonlinearity  $f$  for the validity of the Strong Maximum Principle for the nonnegative solutions of the equation  $-\Delta u = f(u)$  (with no boundary condition). For our convenience, we report here Theorem 1 of [9] in the following form:

**Theorem 1.1.** *Let  $f: [0, +\infty[ \rightarrow \mathbb{R}$  be a continuous function. A sufficient condition for the validity of the Strong Maximum Principle for the nonnegative continuous functions  $u: \overline{\Omega} \rightarrow \mathbb{R}$  satisfying  $\Delta u \in L_{\text{loc}}^1(\Omega)$  and the equation  $-\Delta u = f(u)$  a.e. in  $\Omega$ , is the following one:*

- (1) *There exist an interval  $[0, a]$ ,  $a > 0$ , and a continuous nonincreasing function  $g: [0, a] \rightarrow \mathbb{R}$ , with  $g(0) = 0$ , such that:*
  - (a)  *$f(t) \geq g(t)$ , for each  $t \in [0, a]$ , and*
  - (b) *either  $g(t) = 0$ , for each  $t \in [0, a]$ ,*

$$\text{or } g(t) < 0, \text{ for each } t \in ]0, a], \text{ and } \int_0^a \left( -\int_0^t g(\tau) d\tau \right)^{-\frac{1}{2}} dt = +\infty.$$

In the same paper, the author shows that condition (1) is also necessary for the validity of the Strong Maximum Principle. Indeed, Theorem 3 of [9] states that:

**Theorem 1.2.** *If  $a > 0$  and if  $f: [0, a] \rightarrow \mathbb{R}$  is a nonincreasing function with  $f(0) = 0$  and  $f(t) < 0$  for  $t \in ]0, a]$ , such that*

$$\int_0^a \left( -\int_0^t f(\tau) d\tau \right)^{-\frac{1}{2}} dt < +\infty, \quad (2)$$

*then there exists a nonzero nonnegative function  $u \in C^1(\overline{\Omega})$ , with  $\Delta u \in L^\infty(\Omega)$ , such that  $-\Delta u = f(u)$  a.e. in  $\Omega$  and  $u$  vanishes in a open nonempty subset of  $\Omega$ .*

For a special class of nonlinearities  $f$  which are nonincreasing in an interval of the type  $[0, a]$ ,  $a > 0$ , and satisfying  $f(0) = 0$ ,  $f(t) < 0$ , for  $t \in ]0, a]$ , and condition (2), it is also possible to exhibit nonzero solutions to problem (P) which vanish

in some nonempty subset of  $\Omega$ . We mention, for instance, Theorem 1 of [6] which gives, for nonlinearities of the type

$$f(t) = \lambda t^{s-1} - t^{r-1}, \quad t \in ]0, +\infty[, \tag{3}$$

where  $s \in ]1, 2[$  and  $r \in ]1, s[$  satisfy  $\frac{rs}{(2-s)(2-r)} < N$ , and  $\lambda \in ]0, \infty[$  is a sufficiently large parameter, the existence of a continuum of nonnegative and nonzero solutions to problem  $(P)$  which have compact support in  $\Omega$ . Note that the range of every nonzero weak solution  $u$  to problem  $(P)$ , with  $f$  given by (3), strictly contains the set  $[0, \lambda^{-\frac{1}{s-r}}] = f^{-1}(]-\infty, 0])$ . To see this, assume on the contrary that

$$u(\bar{\Omega}) = [0, \max_{\bar{\Omega}} u] \subseteq f^{-1}(]-\infty, 0]).$$

Then, testing (1) with  $v = u$ , we should have

$$0 < \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} f(u)u dx < 0.$$

More generally we have, of course, the same situation if we consider any continuous function  $f: [0, +\infty[ \rightarrow \mathbb{R}$  such that

$$f^{-1}(]-\infty, 0]) \supseteq ]0, a], \quad \text{for some } a > 0, \tag{4}$$

in the sense that, if  $u$  is any nonzero solution to problem  $(P)$ , then necessarily  $\max_{\bar{\Omega}} u \in f^{-1}(]0, +\infty[) \neq 0$  and  $\max_{\bar{\Omega}} u > a$ . In particular, if  $f$  satisfies (4) and nonzero solutions to  $(P)$  exist, then  $f$  is not nonincreasing in the interval  $]0, \max_{\bar{\Omega}} u[$ . In light of Theorem 3 of [9], it is quite natural to investigate whether the Strong Maximum Principle for the solutions of problem  $(P)$ , with  $f$  satisfying (4), holds at least in some special case.

Actually, we will prove that this is true when  $f$  is of the type

$$f(t) = \lambda_1 t + \lambda t^{s-1} - t^{r-1}, \quad t \in [0, +\infty[, \tag{5}$$

for  $\lambda > 0$  sufficiently small. Here,

$$\lambda_1 := \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx} > 0 \tag{6}$$

is the first eigenvalue of the Laplacian on  $\Omega$ . We will also prove that, for all  $\lambda > 0$ , problem  $(P)$ , with  $f$  given by (5), always admits at least a nonzero solution. This fact highlights a sort of opposite behavior with respect to the non-resonant case, i.e. when the resonant term  $\lambda_1 t$  is missing in (5). Indeed, in this case, Theorem 3.13 of [5] (see also [3]) states that:

**Theorem 1.3.** *Let  $s \in ]1, 2[$  and  $r \in ]1, s[$ . There exists  $\Lambda > 0$  such that problem (P), with  $f(t) = \lambda t^{s-1} - t^{r-1}$ , admits:*

(a) *at least one solution  $u \in \mathcal{P}$ , for all  $\lambda \in ]\Lambda, +\infty[$ .*

$$\text{Here, } \mathcal{P} := \{u \in C_0^1(\bar{\Omega}) : u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\}, \tag{7}$$

*is the interior of the positive cone of  $C_0^1(\bar{\Omega}) := \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$  ( $\nu$  being the unit outer normal to  $\partial\Omega$ );*

(b) *no positive solution for all  $\lambda \in ]0, \Lambda[$ .*

Note that, using the rescaling  $u = \lambda^{\frac{1}{2-s}}v$  and setting  $\mu = \lambda^{-\frac{2-r}{2-s}}$ , the problem

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda_1 u + \lambda u^{s-1} - u^{r-1}, & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

turns into the problem

$$(P_\mu) \quad \begin{cases} -\Delta v = \lambda_1 v + v^{s-1} - \mu v^{r-1}, & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \\ v|_{\partial\Omega} = 0, \end{cases}$$

Since problems  $(P_\lambda)$  and  $(P_\mu)$  are equivalent from the point of view of existence as well as of positivity of the solutions, we choose to study problem  $(P_\mu)$  in order to make a more clear comparison with the results obtained in [4]. Some open questions issued by this comparison will be proposed.

In the rest of the paper, we will use the following notations:

- $\|\cdot\| := \left(\int_\Omega |\nabla(\cdot)|^2 dx\right)^{\frac{1}{2}}$  denotes the norm of the Sobolev space  $W_0^{1,2}(\Omega)$ ;
- for each  $p \geq 1$ ,  $\|\cdot\|_p := \left(\int_\Omega |\cdot|^p dx\right)^{\frac{1}{p}}$  denotes the norm of the space  $L^p(\Omega)$ ;
- $\|\cdot\|_\infty := \text{ess sup}_{x \in \Omega} |u(x)|$  denotes the norm of the space  $L^\infty(\Omega)$ ;
- more generally, given any other functional space  $X$ ,  $\|\cdot\|_X$  denotes its standard norm;
- for each  $p \geq 1$ , with  $p \leq \frac{2N}{N-2}$ , if  $N \geq 3$ ,

$$c_p := \sup_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\|u\|_p}{\|u\|}$$

denotes the best Sobolev constant for the embedding  $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ ;

- given a function  $u: \Omega \rightarrow \mathbb{R}$ , we denote by  $u_+$  the function defined by

$$u_+(x) := \max\{u(x), 0\}, \text{ for each } x \in \Omega;$$

- finally, we denote by  $\phi_1 \in C^1(\bar{\Omega})$  the normalized (with respect to  $\|\cdot\|_\infty$ ) positive eigenfunction associated to the first eigenvalue of the Laplacian operator on  $\Omega$ .

## 2. Main Results

Let  $r, s \in ]1, 2[$ , with  $r < s$ . As announced above, we are concerned in studying problem  $(P_\mu)$  on varying of the positive parameter  $\mu$ . Recall that the solutions of problem  $(P_\mu)$  are exactly the critical points of the associated energy functional

$$I_\mu(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{2}\|u_+\|_2^2 + \frac{\mu}{r}\|u_+\|_r^r - \frac{1}{s}\|u_+\|_s^s, \quad u \in W_0^{1,2}(\Omega).$$

Standard results ensure that  $I_\mu$  is  $C^1$  and sequentially weakly lower semicontinuous in  $W_0^{1,2}(\Omega)$ .

Our main first result concerns the existence of nonzero solutions to problem  $(P_\mu)$ .

**Theorem 2.1.** *For each  $\mu > 0$ , problem  $(P_\mu)$  admits at least a nonzero solution.*

**Proof.** Fix  $\mu \in ]0, +\infty[$  and let  $n \in \mathbb{N}$ , with  $n > \frac{1}{\lambda_1}$ . Consider the functional

$$\begin{aligned} I_n(u) &\stackrel{\text{def}}{=} I_\mu(u) + \frac{1}{2n}\|u_+\|_2^2 = \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\lambda_1 - \frac{1}{n}\right)\|u_+\|_2^2 + \frac{\mu}{r}\|u_+\|_r^r - \frac{1}{s}\|u_+\|_s^s, \quad u \in W_0^{1,2}(\Omega). \end{aligned}$$

Clearly, the functional  $I_n$  has the same regularity of  $I_\mu$ , that is  $I_n$  is continuously differentiable and sequentially weakly lower semicontinuous in  $W_0^{1,2}(\Omega)$ . Moreover, one has the estimate

$$I_n(u) \geq \frac{1}{n\lambda_1}\|u\|^2 - \frac{c^s}{s}\|u\|^s, \text{ for each } u \in W_0^{1,2}(\Omega),$$

which implies  $\lim_{\|u\| \rightarrow +\infty} I_n(u) = +\infty$ .

In particular,  $I_n$  satisfies the Palais-Smale condition (see Example 38.25 of [10]). Now, we show that, for each  $n \in \mathbb{N}$  large enough,  $I_n$  has the mountain pass geometry. To this end, fix  $q \in ]2, 2^*[$  and put

$$M \stackrel{\text{def}}{=} \sup_{t>0} \frac{\frac{\lambda_1}{2}t^2 - \frac{\mu}{r}t^r + \frac{1}{s}t^s}{\frac{1}{q}t^q}.$$

It is easy to check that  $M$  is a positive number. Moreover, if we put

$$R \stackrel{\text{def}}{=} (Mc^q)^{\frac{1}{2-q}},$$

the function  $f(t) = \frac{1}{2}t^2 - \frac{Mc_q^q}{q}t^q, \quad t \in [0, +\infty[$ ,

is increasing in  $]0, R[$  and decreasing in  $]R, +\infty[$ , with

$$f(R) = \max_{t \geq 0} f(t) = \left(\frac{1}{2} - \frac{1}{q}\right) R^2 > 0.$$

Hence, for each  $n \in \mathbb{N}$  and for each  $u \in W_0^{1,2}(\Omega)$ , with  $\|u\| = R$ , one has

$$\begin{aligned} I_n(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\Omega} \left( \frac{\lambda_1}{2}u_+(x)^2 - \frac{\mu}{r}u_+(x)^r + \frac{1}{s}u_+(x)^s \right) dx = \\ &= \frac{1}{2}\|u\|^2 - \int_{u>0} \frac{\frac{\lambda_1}{2}u(x)^2 - \frac{\mu}{r}u(x)^r + \frac{1}{s}u(x)^s}{\frac{1}{q}u(x)^q} \frac{1}{q}u(x)^q dx \geq \\ &\geq \frac{1}{2}\|u\|^2 - \frac{M}{q} \int_{u>0} u(x)^q dx \geq \frac{1}{2}\|u\|^2 - \frac{M}{q}\|u\|_q^q \geq \\ &\geq \frac{1}{2}\|u\|^2 - \frac{Mc_q^q}{q}\|u\|^q = f(\|u\|) = f(R) = \left(\frac{1}{2} - \frac{1}{q}\right) R^2 > 0. \end{aligned} \tag{8}$$

Next, we show that there is a function  $v \in W_0^{1,2}(\Omega)$ , with  $\|v\| > R$ , and an integer  $\bar{n} \in \mathbb{N}$  such that

$$I_n(v) < 0, \quad \text{for each } n \in \mathbb{N}, \quad \text{with } n \geq \bar{n}. \tag{9}$$

Indeed, fix a positive number  $t$  with

$$t > \max \left\{ \frac{R}{\|\phi_1\|}, \left( \mu \frac{s}{r} \frac{\|\phi_1\|_r^r}{\|\phi_1\|_s^s} \right)^{\frac{1}{s-r}} \right\}.$$

Then, by an easy computation we see that

$$\frac{\mu t^r}{r} \|\phi_1\|_r^r - \frac{t^s}{s} \|\phi_1\|_s^s < 0.$$

Consequently, we can fix  $\bar{n} \in \mathbb{N}$  in such way that for each  $n \in \mathbb{N}$  with  $n \geq \bar{n}$

$$I_n(t\phi_1) = \frac{t^2}{2n} \|\phi_1\|^2 + \frac{\mu t^r}{r} \|\phi_1\|_r^r - \frac{t^s}{s} \|\phi_1\|_s^s < 0.$$

Hence, inequality (9) follows by choosing  $v = t\phi_1$ . This proves that, for each  $n \in \mathbb{N}$ , with  $n \geq \bar{n}$ ,  $I_n$  has the mountain pass geometry. Since  $I_n$  satisfies the Palais-Smale condition, by applying the classical Ambrosetti-Rabinowitz Mountain Pass Theorem ([2]), we infer that, for each  $n \in \mathbb{N}$ , with  $n \geq \bar{n}$ ,  $I_n$  admits a critical point  $u_n \in W_0^{1,2}(\Omega)$  such that

$$I_n(u_n) = \inf_{\psi \in \Gamma} \sup_{\tau \in [0,1]} I_n(\psi(\tau))$$

where  $\Gamma \stackrel{\text{def}}{=} \{\psi \in C([0, 1], W_0^{1,2}(\Omega)) : \psi(0) = 0, \quad \psi(1) = v\}$ .

In particular, by (8), we get

$$I_n(u_n) \geq \left(\frac{1}{2} - \frac{1}{q}\right) R^2. \tag{10}$$

Of course,  $u_n$  is a nonzero weak solution to the problem

$$(P_n) \quad \begin{cases} -\Delta u = (\lambda_1 - \frac{1}{n}) u + u^{s-1} - \mu u^{r-1} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Our next step is showing that the sequence  $\{u_n\}_{n \geq \bar{n}}$  is bounded in  $W_0^{1,2}(\Omega)$ . To this end, let  $n \in \mathbb{N}$  with  $n \geq \bar{n}$ . Using the identity

$$I'_n(u_n)(u_n) = \|u_n\|^2 - \left(\lambda_1 - \frac{1}{n}\right) \|u_n\|_2^2 - \|u_n\|_s^s + \mu \|u_n\|_r^r = 0.$$

in (10), we get

$$\begin{aligned} 0 &< \left(\frac{1}{2} - \frac{1}{q}\right) R^2 \leq \\ &\leq I_n(u_n) = \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \left(\lambda_1 - \frac{1}{n}\right) \|u_n\|_2^2 - \frac{1}{s} \|u_n\|_s^s + \frac{\mu}{r} \|u_n\|_r^r = \\ &= \mu \left(\frac{1}{r} - \frac{1}{2}\right) \|u_n\|_r^r - \left(\frac{1}{s} - \frac{1}{2}\right) \|u_n\|_s^s, \end{aligned} \tag{11}$$

which implies

$$\|u_n\|_s \leq \mu \left(\frac{1}{r} - \frac{1}{2}\right)^{\frac{1}{s-r}} \left(\frac{1}{s} - \frac{1}{2}\right)^{-\frac{1}{s-r}} m(\Omega)^{\frac{1}{sr}}.$$

Therefore, the sequence  $\{u_n\}_{n \geq \bar{n}}$  is bounded in  $L^s(\Omega)$ . So, if we put

$$C = \sup_{n \in \mathbb{N}, n \geq \bar{n}} \|u_n\|_s^s,$$

by  $I'_n(u_n)(u_n) = 0$ , we get

$$\|u_n\|^2 - \left(\lambda_1 - \frac{1}{n}\right) \|u_n\|_2^2 = \|u_n\|_s^s - \mu \|u_n\|_r^r \leq C, \tag{12}$$

for all  $n \in \mathbb{N}$ , with  $n \geq \bar{n}$ . At this point, let  $H \subset W_0^{1,2}(\Omega)$  be the orthogonal subspace to  $\langle \phi_1 \rangle$  in  $W_0^{1,2}(\Omega)$  and let  $n \in \mathbb{N}$ , with  $n \geq \bar{n}$ . Then, there exist  $t_n \in \mathbb{R}$  and  $w_n \in H$  such that

$$u_n = t_n \phi_1 + w_n. \tag{13}$$

Using the orthogonal decomposition (13) and inequality (12), one obtains

$$\begin{aligned} \left(1 - \frac{\lambda_1 - \frac{1}{n}}{\lambda_2}\right) \|w_n\|^2 &\leq \|w_n\|^2 - \left(\lambda_1 - \frac{1}{n}\right) \|w_n\|_2^2 \leq \\ &\leq t_n^2 \|\phi_1\|^2 + \|w_n\|^2 - \left(\lambda_1 - \frac{1}{n}\right) (t_n^2 \|\phi_1\|_2^2 + \|w_n\|_2^2) = \\ &= \|u_n\|^2 - \left(\lambda_1 - \frac{1}{n}\right) \|u_n\|_2^2 \leq C, \end{aligned}$$

where  $\lambda_2$  is the second positive eigenvalue of the Laplacian on  $\Omega$ , whose variational characterization is

$$\lambda_2 = \inf_{v \in H} \frac{\|v\|^2}{\|v\|_2^2}.$$

Since  $\lambda_2 > \lambda_1$ , from the previous inequality we infer the boundedness of the sequence  $\{w_n\}_{n \geq \bar{n}}$  in  $W_0^{1,2}(\Omega)$ . Moreover, for each  $n \in \mathbb{N}$ , with  $n \geq \bar{n}$ , one has

$$C \geq \|u_n\|_s^s = \int_{\Omega} |t_n \phi_1 + w_n|^s dx \geq \int_{\Omega} (|t_n \phi_1| - |w_n| - 1) dx.$$

Exploiting this last inequality and the boundedness of the sequence  $\{w_n\}_{n \geq \bar{n}}$  in  $W_0^{1,2}(\Omega)$ , one promptly gets the boundedness of the sequence  $\{t_n\}_{n \geq \bar{n}}$  in  $\mathbb{R}$  which, jointly with (13), yields the boundedness of  $\{u_n\}_{n \geq \bar{n}}$  in  $W_0^{1,2}(\Omega)$ . Consequently, there exists  $u^* \in W_0^{1,2}(\Omega)$  such that, up to a subsequence,  $u_n \rightarrow u^*$  weakly in  $W_0^{1,2}(\Omega)$  and strongly in  $L^p(\Omega)$ , for each  $p \in [1, 2^*[$ . At this point, let  $v \in W_0^{1,2}(\Omega)$  be arbitrarily fixed. Since  $u_n$  is a weak solution to problem  $(P_n)$ , passing to the limit as  $n \rightarrow +\infty$  in the identity

$$I'_n(u_n)(v) = \int_{\Omega} \nabla u_n \nabla v dx - \int_{\Omega} \left[ \left(\lambda_1 - \frac{1}{n}\right) u_n - u_n^{s-1} + \mu u_n^{r-1} \right] v dx = 0,$$

one obtains  $I'_\mu(u^*)(v) = 0$ . Therefore,  $u^*$  is a solution to problem  $(P_\mu)$ . Finally, passing to the limit as  $n \rightarrow \infty$  in (11), one has

$$0 < \left(\frac{1}{2} - \frac{1}{q}\right) R^2 \leq \mu \left(\frac{1}{r} - \frac{1}{2}\right) \|u^*\|_r^r - \left(\frac{1}{s} - \frac{1}{2}\right) \|u^*\|_s^s,$$

which, in turn, implies  $u^* \neq 0$  as well. The proof is now complete.  $\square$

Here and in the sequel, for each  $\mu > 0$ ,  $S_\mu$  will denote the set of all nonzero solutions of problem  $(P_\mu)$ .

Our second main result concerns the validity of property (iii) for problem  $(P_\mu)$  provided that  $\mu$  is large enough. Recall that  $\mathcal{P}$  denotes the interior of the positive cone of  $C_0^1(\bar{\Omega})$  introduced in (7).

**Theorem 2.2.** *There exists  $\mu^* > 0$  such that, for each  $\mu > \mu^*$ , one has  $S_\mu \subset \mathcal{P}$ .*

**Proof.** Since the domain  $\Omega$  is of class  $C^2$ , then, by standard regularity theory, for each  $\mu > 0$  and for each  $u_\mu \in S_\mu$  one has  $u_\mu \in W^{2,q}(\Omega)$  for any  $q > 1$  (see [1] or Appendix B of [8], for instance). In particular,  $u_\mu \in C^{1,\alpha}(\bar{\Omega})$ , for some  $\alpha \in (0, 1)$ , by Sobolev embeddings.

Moreover,  $u_\mu$  must satisfy the inequality

$$\sup_{\Omega} u_\mu \geq \mu^{\frac{1}{s-r}}. \tag{14}$$

Indeed, if (14) was not true, then  $u_\mu^s - \mu u_\mu^r < 0$  in  $\Omega \setminus u_\mu^{-1}(0)$ , and  $\Omega \setminus u_\mu^{-1}(0) \neq \emptyset$  being  $u_\mu \neq 0$ . Consequently, from the identity

$$0 = I'_\mu(u)(u) = \|u_\mu\|^2 - \lambda_1 \|u_\mu\|_2^2 - \|u_\mu\|_s^s + \mu \|u_\mu\|_r^r \tag{15}$$

we should have  $0 > \|u_\mu\|^2 - \lambda_1 \|u_\mu\|_2^2$ , which contradicts (6).

Now, we claim that there exist two constants  $C$  and  $\mu_0 > 0$ , such that for each  $\mu \geq \mu_0$  and for each  $u_\mu \in S_\mu$ , one has

$$\|u_\mu\| \geq C \mu^{\frac{1}{s-r}}. \tag{16}$$

To prove our claim, let  $u_\mu \in S_\mu$ , with  $\mu > 0$ . Thanks to (15), one has

$$\|u_\mu\|_s^s - \mu \|u_\mu\|_r^r = \|u_\mu\|^2 - \lambda_1 \|u_\mu\|_2^2 \geq 0. \tag{17}$$

Fix any  $q > \frac{N}{2}$ . By Theorem 8.2 of [1] and the embedding  $W^{2,q}(\Omega) \hookrightarrow C^1(\bar{\Omega})$ , we can find two constants  $C_0, C_1 > 0$  (independent of  $\mu$ ) such that

$$\|u_\mu\|_{C^1(\bar{\Omega})} \leq C_0 \|u_\mu\|_{W^{2,q}(\Omega)} \leq C_1 (\lambda_1 \|u_\mu\|_q + \|u_\mu\|_{(s-1)q}^{s-1} + \mu \|u_\mu\|_{(r-1)q}^{r-1}). \tag{18}$$

Then, in view of (17), we get

$$\begin{aligned} \|u_\mu\|_\infty &\leq C_1 \left[ \lambda_1 \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_2^{\frac{2}{q}} + \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \|u_\mu\|_s^{\frac{s}{q}} + \mu \|u_\mu\|_\infty^{r-1-\frac{r}{q}} \|u_\mu\|_r^{\frac{r}{q}} \right] = \\ &= C_1 \left[ \lambda_1 \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_2^{\frac{2}{q}} + \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \left( \|u_\mu\|_s^{\frac{s}{q}} + \mu \|u_\mu\|_\infty^{(r-s)\frac{q-1}{q}} \|u_\mu\|_r^{\frac{r}{q}} \right) \right] \leq \\ &\leq C_1 \left[ \lambda_1 \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_2^{\frac{2}{q}} + \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \left( \|u_\mu\|_s^{\frac{s}{q}} + \mu \cdot \mu^{-\frac{q-1}{q}} \|u_\mu\|_r^{\frac{r}{q}} \right) \right] \leq \\ &\leq C_1 \left[ \lambda_1 \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_2^{\frac{2}{q}} + 2 \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \|u_\mu\|_s^{\frac{s}{q}} \right] \leq \\ &\leq C_2 \left[ \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_2^{\frac{2}{q}} + \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \|u_\mu\|_s^{\frac{s}{q}} \right] \end{aligned}$$

for some constant  $C_2 > 0$  independent of  $\mu$ . Dividing side to side the inequality

$$\|u_\mu\|_\infty \leq C_2 \left[ \|u_\mu\|_\infty^{\frac{q-2}{q}} \|u_\mu\|_\infty^{\frac{2}{q}} + \|u_\mu\|_\infty^{s-1-\frac{s}{q}} \|u_\mu\|_\infty^{\frac{s}{q}} \right]$$

by  $\|u_\mu\|_\infty^{\frac{q-2}{q}}$  and using inequality (14), one obtains

$$\begin{aligned} \mu^{\frac{2}{q(s-r)}} &\leq \|u_\mu\|_\infty^{\frac{2}{q}} \leq C_2 \left[ \|u_\mu\|_\infty^{\frac{2}{q}} + \|u_\mu\|_\infty^{(s-2)\frac{q-1}{q}} \|u_\mu\|_\infty^{\frac{s}{q}} \right] \\ &\leq C_2 \left[ \|u_\mu\|_\infty^{\frac{2}{q}} + \mu^{\frac{s-2}{s-r}\frac{q-1}{q}} \|u_\mu\|_\infty^{\frac{s}{q}} \right]. \end{aligned}$$

In particular, putting  $\mu_0 = \max \left\{ 1, (2C_2)^{\frac{q(s-r)}{2}} \right\}$ , we get, for  $\mu \geq \mu_0$ ,

$$\mu^{\frac{2}{q(s-r)}} \leq \|u_\mu\|_\infty^{\frac{2}{q}} \leq C_2 [\|u_\mu\|_\infty^{\frac{2}{q}} + \|u_\mu\|_\infty^{\frac{s}{q}}] \leq C_2 [2\|u_\mu\|_\infty^{\frac{2}{q}} + 1]. \tag{19}$$

Consequently,  $\|u_\mu\|_\infty^{\frac{2}{q}} \geq \frac{1}{2C_2} \mu^{\frac{2}{q(s-r)}} - \frac{1}{2} \geq \frac{1}{4C_2} \mu^{\frac{2}{q(s-r)}}$ , which proves (16).

Note that, from (16), one has  $\|u_\mu\| \rightarrow +\infty$ , as  $\mu \rightarrow +\infty$ . Then, from (19), we can find  $\mu_1 \in [\mu_0, +\infty[$  and a constant  $C_4 > 0$  such that

$$\|u_\mu\|_\infty \leq C_4 \|u_\mu\|, \quad \text{for all } \mu \in [\mu_1, +\infty[. \tag{20}$$

From now on, we always assume  $\mu \in [\mu_1, +\infty[$ .

Let  $H \subset W_0^{1,2}(\Omega)$  be the orthogonal subspace to  $\langle \phi_1 \rangle$  in  $W_0^{1,2}(\Omega)$  and let  $u_\mu \in S_\mu$ . Then, there exist a unique  $t_\mu \in \mathbb{R}$  and a unique  $w_\mu \in H$  such that

$$u_\mu = t_\mu \phi_1 + w_\mu. \tag{21}$$

Since  $u_\mu$  is nonzero and nonnegative, multiplying by  $\phi_1$  both sides of the identity  $u_\mu = t_\mu \phi_1 + w_\mu$  and integrating over  $\Omega$ , we can see that  $t_\mu > 0$ . Moreover, it is easy to check that  $w_\mu$  is a solution to the problem

$$\begin{cases} -\Delta u = \lambda_1 u + u_\mu^{s-1}(x) - \mu u_\mu^{r-1}(x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

In particular,  $w_\mu$  satisfies the identity

$$\int_\Omega |\nabla w_\mu(x)|^2 dx - \lambda_1 \int_\Omega |w_\mu(x)|^2 dx = \int_\Omega (u_\mu^{s-1} - \mu u_\mu^{r-1}) w_\mu dx \tag{22}$$

Now, testing the equation  $I'_\mu(u_\mu)(v) = 0$  with  $v = \phi_1$ , we get

$$\int_\Omega (u_\mu^{s-1} - \mu u_\mu^{r-1}) \phi_1 dx = \int_\Omega (\nabla u_\mu \nabla \phi_1 - \lambda_1 u_\mu \phi_1) dx = 0.$$

Consequently,

$$\int_{\Omega} (u_{\mu}^{s-1} - \mu u_{\mu}^{r-1}) w_{\mu} dx = \int_{\Omega} (u_{\mu}^s(x) - \mu u_{\mu}^r(x)) dx.$$

Thus, by (17) and (22), we obtain

$$\|w_{\mu}\|^2 - \lambda_1 \|w_{\mu}\|_2^2 = \int_{\Omega} (u_{\mu}^s(x) - \mu u_{\mu}^r(x)) dx \leq 2 \int_{\Omega} u_{\mu}^s(x) dx \leq 2c_s^s \|u_{\mu}\|^s. \quad (23)$$

Furthermore, note that, being  $w_{\mu} \in H$ , then  $\|w_{\mu}\|_2^2 \leq \lambda_2^{-1} \|w_{\mu}\|^2$ . Plugging this inequality in (23), we get

$$\|w_{\mu}\|^2 \leq \gamma \|u_{\mu}\|^s, \quad (24)$$

where  $\gamma = \frac{2\lambda_2 c_s^s}{\lambda_2 - \lambda_1}$ . Using again Theorem 8.2 of [1], we get for  $\|w_{\mu}\|_{C^1(\bar{\Omega})}$  an upper estimate as (18). More precisely, if we fix any  $q > \frac{N}{2}$ , we can find a constant  $C_5 > 0$  (independent of  $\mu$ ) such that

$$\|w_{\mu}\|_{C^1(\bar{\Omega})} \leq C_5 (\lambda_1 \|w_{\mu}\|_q + \|u_{\mu}\|_{(s-1)q}^{s-1} + \mu \|u_{\mu}\|_{(r-1)q}^{r-1}).$$

Then, following the same argument after (18), we arrive to

$$\|w_{\mu}\|_{C^1(\bar{\Omega})} \leq C_6 \left( \|w_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{q-2}{q}} \|w_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{q-2}{q}} + \|u_{\mu}\|_{\infty}^{s-1-\frac{s}{q}} \|u_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{s}{q}} \right).$$

Using (20) and (24), we also find a constant  $C_7$  (independent of  $\mu$ ) such that

$$\|w_{\mu}\|_{C^1(\bar{\Omega})} \leq C_7 \left( \|u_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{s}{q}} \|w_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{q-2}{q}} + \|u_{\mu}\|^{s-1} \right).$$

From the previous estimate, we infer that either

$$\|w_{\mu}\|_{C^1(\bar{\Omega})} \leq 2C_7 \|u_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{s}{q}} \|w_{\mu}\|_{C^1(\bar{\Omega})}^{\frac{q-2}{q}} \quad \text{or} \quad \|w_{\mu}\|_{C^1(\bar{\Omega})} \leq 2C_7 \|u_{\mu}\|^{s-1}.$$

Therefore,

$$\|w_{\mu}\|_{C^1(\bar{\Omega})} \leq C_8 \max\{\|u_{\mu}\|^{s-1}, \|u_{\mu}\|^{\frac{s}{2}}\} \quad (25)$$

where  $C_8 = \max\{2C_7, (2C_7)^{\frac{q}{2}}\}$ ,

At this point, note that by the orthogonal decomposition (21) and (24), one has

$$\|u_{\mu}\|^2 = t_{\mu}^2 \|\phi_1\|^2 + \|w_{\mu}\|^2 \leq t_{\mu}^2 \|\phi_1\|^2 + \gamma \|u_{\mu}\|^s.$$

Then, in view of (16), we obtain

$$t_{\mu}^2 = \|u_{\mu}\|^2 \left( \frac{1}{\|\phi_1\|^2} - \gamma \|u_{\mu}\|^{s-2} \right) \geq \|u_{\mu}\|^2 \left( \frac{1}{\|\phi_1\|^2} - \gamma C^{s-2} \mu^{\frac{s-2}{s-r}} \right).$$

So, if we fix  $\mu_2 \in [\mu_1, +\infty[$  such that

$$\gamma C^{s-2} \mu_2^{\frac{s-2}{s-r}} < \frac{3}{4\|\phi_1\|^2},$$

for  $\mu \geq \mu_2$  we get the following lower estimate of  $t_\mu$  (recall that  $t_\mu > 0$ )

$$t_\mu \geq \sigma \|u_\mu\|, \quad (26)$$

where  $\sigma = \frac{C}{2\|\phi_1\|} > 0$ .

Now, suppose  $\mu \geq \mu_2$  and let  $u_\mu \in S_\mu$ . By (16), (21), (25) and (26), we infer

$$\begin{aligned} \|t_\mu^{-1} u_\mu - \phi_1\|_{C^1(\bar{\Omega})} &= t_\mu^{-1} \|w_\mu\|_{C^1(\bar{\Omega})} \leq \\ &\leq \sigma^{-1} C_8 \max\{\|u_\mu\|^{s-2}, \|u_\mu\|^{\frac{s-2}{2}}\} \leq C_9 \max\{\mu^{\frac{s-2}{s-r}}, \mu^{\frac{s-2}{2(s-r)}}\} \end{aligned}$$

for some constant  $C_9$  (independent of  $\mu$ ).

Since  $\phi_1 \in \mathcal{P}$ , and  $\mathcal{P}$  is open in  $C^1(\bar{\Omega})$ , there exists  $\delta > 0$  such that

$$\{\psi \in C^1(\bar{\Omega}) : \|\psi - \phi_1\| < \delta\} \subset \mathcal{P}.$$

Therefore, if we fix  $\mu^* > \mu_2$  such that

$$C_9 \max\{(\mu^*)^{\frac{s-2}{s-r}}, (\mu^*)^{\frac{s-2}{2(s-r)}}\} < \delta,$$

we have  $t_\mu^{-1} u_\mu \in \mathcal{P}$ , and so  $u_\mu \in \mathcal{P}$  for each  $\mu \geq \mu^*$  and each  $u_\mu \in S_\mu$ . Hence, the proof is complete.  $\square$

**Remark 2.3.** Theorem 1 and Theorem 2 extend, in the semilinear case, Theorem 2.3 of [4] to higher dimension. This extension is not full. Indeed, for  $\Omega = ]0, 1[$ , Theorem 2.3 of [4] gives the following more precise conclusion:

*There exists  $\mu_1 > 0$  such that problem  $(P_\mu)$  admits:*

- a unique positive solution  $u_\mu$  which satisfies  $u'_\mu(0) > 0$ ,  $u'_\mu(1) < 0$ , if  $\mu > \mu_1$ ,
- a unique positive solution  $u_\mu$  which satisfies  $u'_\mu(0) = u'_\mu(1) = 0$  if  $\mu = \mu_1$ ,
- a continuum of nonzero solutions compactly supported in  $]0, 1[$ , if  $\mu \in ]0, \mu_1[$ .

So, it would be interesting to investigate, in higher dimension, the uniqueness of the positive solution as well as the existence of nonzero solutions compactly supported in  $\Omega$ .

**Remark 2.4.** The substitutions  $v = \mu^{\frac{1}{s-r}} u$  and  $\lambda = \mu^{\frac{s-2}{s-r}}$  transform problem  $(P_\mu)$  into the equivalent problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda(u^{s-1} - u^{r-1}), & \text{in } \Omega \\ u \geq 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where the nonlinearity  $t \in ]0, +\infty[ \rightarrow t^{s-1} - t^{r-1}$  is negative in  $]0, 1[$  and positive in  $]1, +\infty[$ . This suggests to investigate the more general problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda g(u), & \text{in } \Omega \\ u \geq 0, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where  $\lambda > 0$ , and  $g: ]0, +\infty[ \rightarrow \mathbb{R}$  is a continuous function, possibly diverging as  $t \rightarrow 0^+$  and satisfying some growth condition at infinity, such that for some  $a > 0$  we have  $g^{-1}(] - \infty, 0]) = ]0, a[$ , and  $\sup_{]0, +\infty[} g > 0$ . In particular, we think that it would be of interest to see whether for this more general problem, Theorem 1 and Theorem 2 still hold respectively for all  $\lambda$ , and for  $\lambda$  sufficiently small.

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