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From Convergence of Dynamical Equilibrium Systems to Bilevel Hierarchical Ky Fan Minimax Inequalities and Applications

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Inspired by the variational formulation of continuous Cauchy-Lipschitz systems and forward (descent) or backward (proximal) methods, and motivated by the solvability of bilevel equilibrium problems, we introduce first-order continuous Evolution Dynamical Equilibrium Systems, (EDES) for short. Then, our primary goal is to study the existence and uniqueness of solutions to (EDES). Secondly, we study the asymptotic behaviour of trajectories of Dynamical Ky Fan Minimax Inequalities (NDEMI) with nonautonomous equilibrium bifunctions defined in Hilbert spaces under monotonicity conditions. In this way, we provide conditions guaranteeing the weak ergodic convergence, i.e., convergence in average, of trajectories to an equilibrium point of an appropriate limit monotone bifunction. In the process of doing so, we consider Fitzpatrick's transforms introduced by M. H. Alizadeh and N. Hadjisavvas [4] and their related Brézis-Haraux transforms for time-dependant equilibrium bifunctions, which prove to be a key tool in our convergence analysis. Afterwards, by means of a first-order linear sum approach of two real-valued bifunctions, where the penalization term is a positive measurable function, we present several results concerning the weak convergence in average, weak and strong convergence of trajectories to solutions to the bilevel equilibrium problem subject to our treatment. Some applications, supported by numerical illustrations implemented in Scilab version 5.5.2, are thereafter discussed with respect to bilevel hierarchical minimization problem as well as dynamical systems for saddle convex-concave bifunctions. Our results generalize and extend some of those obtained in J. B. Baillon and H. Brézis [9], and H. Attouch and M. O. Czarnecki [7]. We end the paper by concluding remarks and suggestions for research perspectives.

Keywords: Ky Fan minimax inequality, bilevel hierarchical Ky Fan minimax inequality, strongly monotone, asymptotic behavior, equilibrium Brézis-Haraux transform, equilibrium Fitzpatrick transform.

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1. Introduction

Let H be a real Hilbert space endowed with the scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$, K be a closed convex subset of H and $f\colon K\times K\to\mathbb{R}$ be an equilibrium bifunction. As usual, we call f an *equilibrium bifunction* if $f(x,x)=0$ and $f(x,\cdot)$ is convex and lower semicontinuous for every $x\in K$. For a closed and convex subset $C\subset K$, the corresponding Ky Fan minimax inequality over C is defined as follows:

$$\text{(FMI)} \qquad \text{Find } \bar{x}\in C \text{ such that } f(\bar{x},y)\geq 0,\ \ \forall y\in C. \tag{1}$$

We denote by $S_f(C)$ the solution set of (1) on C, and when $C=K$, we simply write S_f.

We point out that existence of equilibrium points in (1) stems from the well known KKM Lemma (1929) that has been extended by Ky Fan (1961) to general infinite dimensional framework and has henceforth led to the well known Ky Fan minimax inequality [30] when the equilibrium bifunction f is upper semicontinuous relatively to the first variable. The existence of equilibrium point has been developed in [10, 19, 38] by replacing upper semicontinuity condition by monotonicity and upper hemicontinuity. In [13] Blum and Oettli introduced their very known mixed version of (1), which has been appeared subsequently in the scope of many research articles including also the vector case, quote for example [13, 23, 33, 34, 51], wherein the authors clarified how the theory of Ky Fan minimax inequalities unifies a large number of problems arising in optimization, variational inequalities, Panagiotopoulos inequalities, fixed-point, games, mathematical models of noncooperative games and Nash equilibrium concepts.

Thereby, a great interest has been brought to the study of (1) by means of splitting iterative methods that include gradient-descent (or forward) methods and Proximal point (or backward) methods; one can consult [5, 21, 39, 41, 45] and references therein.

Chbani and Riahi [26] introduced a class of demipositive bifunctions and used it to study the asymptotic behaviour of solutions for the corresponding general dynamical Ky Fan minimax system

$$x\in\mathcal{C}^0([0,+\infty];K) \text{ and } f(x(t),y)+\langle\dot{x}(t),y-x(t)\rangle\geq 0,\ \ \forall y\in K,\ \text{for a.e. } t\geq 0.$$

More precisely, they obtained the weak convergence of the trajectories to a solution of (1).

In the present paper, we consider the following bilevel hierarchical Ky Fan minimax inequality:

$$\text{(BFMI)} \qquad \text{Find } \bar{x}\in S_f \text{ such that } g(\bar{x},y)\geq 0,\ \ \forall y\in S_f, \tag{2}$$

where $g\colon K\times K\to\mathbb{R}$ is another equilibrium bifunction.

From now on, we assume that the solution set $S:=S_g(S_f)$ of problem (2) is nonempty.

Our approach provides solutions to (2) by analyzing the asymptotic behaviour of the following continuous evolution dynamical equilibrium system:
Find $x \in \mathcal{C}^0([0, +\infty]; K)$ such that

$$\text{(EDES)} \qquad \beta(t) f(x(t), y) + g(x(t), y) + \langle \dot{x}(t), y - x(t)\rangle \geq 0, \tag{3}$$

for all $y \in K$ and for a.e. $t \geq 0$. Let us mention that iterative methods for solving the problem (2) are discrete counterparts of the dynamical equilibrium system (3). The simplest ones among these are backward-splitting, double-backward, double-forward and forward-backward methods (see, e.g., [25, 27, 47]).

The nearest existing model in the literature to our problem (3) deals with dynamical monotone variational systems for which the existence of solutions is usually treated via two essential approaches. The first approach uses time discretization and solves an auxiliary problem in each step, then a solution to the proposed system is obtained via a convergence scheme under an appropriate control. The second one is based on Cauchy-Picard theorems and the theory of Yosida regularization technique. For our nonsmooth equilibrium system (3), we rather follow-up the second method and approximate it by Lipschitz continuous ordinary differential equations $\dot{x}_\lambda(t) + A_\lambda^{\Gamma_t} x_\lambda(t) = 0$, where $A_\lambda^{\Gamma_t}$ is the associated Yosida λ-approximation to the bifunction $\Gamma_t = \beta(t) f + g$. More specifically, we show in Theorem 2.9 that, under a suitable uniform condition on $A_\lambda^{\Gamma_t}$ and β, the regularized solutions x_λ converge to the unique absolutely continuous solution of the problem (3) with initial condition $x(0) = x_0$. In Corollary 2.10, we justify that the uniform condition on $A_\lambda^{\Gamma_t}$ and β is satisfied whenever the bifunction f is the following quadratic form $f(x, y) = \langle x, y - x\rangle$.

In the next step of work we introduce the equilibrium Brézis-Haraux transform of equilibrium bifunctions which is important for complexity analysis of the asymptotic behavior for hierarchical Ky Fan minimax inequalities (2). For $u \in H$ and $x \in K$, the equilibrium Brézis-Haraux transform $\mathcal{G}_f$ of the bifunction $f \colon K \times K \to \mathbb{R}$ is given by $\mathcal{G}_f(x, u) = \mathcal{F}_f(x, u) - \langle x, u\rangle$, where $\mathcal{F}_f$ is the equilibrium Fitzpatrick transform of f, introduced in [4, 16], and defined by $\mathcal{F}_f(x, u) = \sup_{y \in K}\{\langle u, y\rangle + f(y, x)\}$.

In the sequel we discuss the hierarchical minimization problem $\min_{\operatorname{argmin} \psi} \varphi$, where φ, $\psi \colon H \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semicontinuous functions. For this problem, Attouch and Czarnecki [7] obtained solutions by means of the asymptotic behavior of trajectories of the following first order dynamical system $0 \in \dot{x}(t) + \partial\varphi(x(t)) + \beta(t)\partial\psi(x(t))$ to a solution of $\min_{\operatorname{argmin} \psi} \varphi$, where the penalization term $\beta(t)$ is positive and measurable. Several ergodic and nonergodic convergence results have been justified for $x(t)$ under the integral key assumption (41). Motivated and inspired by [7], a number of research articles was devoted to this subject, including variational inequalities associated to maximal monotone operators in the constraints set. For more general formulation, [6] provide conditions involving the Brézis-Haraux transform associated to A_t and guaranteeing the weak ergodic convergence of each trajectory $x(\cdot)$ of the nonautonomous monotone evolution inclusion $0 \in \dot{x}(t) + A_t(x(t))$ to a zero of a maximal monotone operator A_∞ (which is the graph-limit of maximal monotone operators $A_t\colon H \rightrightarrows H$), as t tends to $+\infty$.

For our purpose we consider the following first-order nonautonomous continuous dynamical Ky Fan minimax inequalities: Find $x \in \mathcal{C}^0([0,+\infty];H)$ such that

$$\text{(NDFMI)} \qquad \Gamma_t(x(t),y) + \langle \dot{x}(t), y - x(t)\rangle \geq 0, \ \ \forall y \in K, \ \text{for a.e.} \ \ t \geq 0, \tag{4}$$

where $\{\Gamma_t\colon K \times K \to \mathbb{R},\ t \geq 0\}$ denotes a suitable family of monotone bifunctions.

Our time-dependent Ky Fan minimax inequality is nowadays considered one of the fashion research themes. Indeed, if we remove the derivative term in our problem (4), and include instead a control term we obtain an abstract evolution problem into which we can convert the time dependent variational inequalities arising in the constraints of differential variational inequalities, qualified recently as a new modeling paradigm of variational analysis to treat many applied problems in engineering, operations research, and physical sciences, see for example [35, 43] and references therein.

In order to prove the weak ergodic convergence of trajectories of (4), we need some general condition involving the equilibrium Brézis Haraux transform $\mathcal{G}_{\Gamma_t}$ of the bifunction Γ_t. If there exists a monotone and maximal bifunction $\Gamma_\infty\colon K \times K \to \mathbb{R}$ such that $S_{\Gamma_\infty} \neq \emptyset$ and

$$(\mathcal{C}_1) \qquad \int_0^{+\infty} \mathcal{G}_{\Gamma_t}(z,p)dt < +\infty, \ \forall (z,p) \in A^{\Gamma_\infty}, \tag{5}$$

where A^{Γ_∞} is the associate monotone operator to Γ_∞, we show in Theorem 4.4 that every global solution of (4) converges weakly in average, as $t \to +\infty$, toward an element of S_{Γ_∞}. Afterwards, in Theorem 4.6, we illustrate this general ergodic weak convergence result with the first order linear sum $\Gamma_t = \beta(t)f + g$, where $f, g\colon K \times K \to \mathbb{R}$ are monotone bifunctions and the penalization term $\beta(t)$ is positive and measurable.

In particular, we recover the weak ergodic convergence of a coupled dynamical systems with multiscale aspects presented in Theorem 2.1 due to Attouch and Czarnecki [7]. We then improve this convergence theorem (see Theorem 4.11), by using the weak convergence for the trajectory $x(t)$ of (3) to a solution of (2) under the additionally assumption $\lim\limits_{t\to+\infty} \beta(t) = +\infty$.

Corollary 5.2 confirms the interest of the result in our Theorem 4.11, since the weak convergence of $x(\cdot)$ to a solution of (2) was only proved for hierarchical minimization problems. Then, under an additional strong monotonicity condition on g, we obtain a strong convergence result in Theorem 4.15.

Following these ideas, some applications are thereafter proposed. The first one concerns asymptotic behavior theorems for a Ky Fan minimax inequality over the solution set of a convex minimization problem. As a consequence, we deduce the weak ergodic convergence result in [7] (see Corollary 5.1), and in addition, we obtain Corollary 5.2 which confirms the interest of the result in our Theorem 4.11. Furthermore, we apply our asymptotic behavior theorems to a convex minimization problem over the solution set of a Ky Fan minimax inequality for which we obtain a strong convergence result.

As a main last application, we consider the nonautonomous continuous dynamical systems which are linked to saddle point problems. For this purpose, we summarize the weak convergence Theorems 4.6, 4.11 and 4.15 in Corollary 5.6, and thereafter we give an example where the geometric condition that ensures weak convergence is satisfied. The results obtained analytically are illustrated by numerical examples for optimization problems and a non-variational one. The latter shows the impact of various types of dissipation on the spectrum of the generator as well as the dynamic behavior of the solution on a rectangular domain.

The paper is organized as follows. In section 2, we present some preliminary results and tools that will be useful throughout the paper. In section 3, we recall the notion of Fitzpatrick and equilibrium Brézis-Haraux transforms, and we present different properties and auxiliary results on these functions. In section 4, we prove weak-ergodic convergence of trajectories of the general parametric problem (4) to an equilibrium point of a monotone and maximal bifunction Γ_∞ under some general condition involving the equilibrium Brézis-Haraux transform $\mathcal{G}_{\Gamma_t}$. A particular attention is devoted to the case $\Gamma_t = \beta(t)f + g$. In this framework, we show several results of convergence, firstly the weak ergodic convergence, thus making more precise the preceding result. The key condition that implies weak convergence of the trajectories is $\lim\limits_{t\to+\infty} \beta(t) = +\infty$.

When g is assumed to be strongly monotone, we can even show strong (nonergodic) convergence of the trajectories to the unique solution of (2). Finally, in the last section, we present applications to bilevel hierarchical minimization problem and variational inequalities constrained by saddle-points for convex-concave bifunctions, and add some examples supported by numerical illustrations implemented in Scilab version 5.5.2.

2. Preliminaries and existence result for (EDES)

Throughout the paper, H is a real Hilbert space which is endowed with a scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let K be a nonempty closed convex subset of H whose interior with respect to the norm-topology is denoted by $\operatorname{int}(K)$.

The following notations and conditions for a bifunction $f\colon K \times K \to \mathbb{R}$ will be needed in the sequel:

(H_1) $f(x,x) = 0$ for each $x \in K$;

(H_2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for each $x, y \in K$;

(H_3) f is upper hemicontinuous, i.e., $\limsup_{t\searrow 0} f(tz + (1-t)x, y) \leq f(x,y)$ for any $x, y, z \in K$;

(H_4) for each $x \in K$, $y \longmapsto f(x,y)$ is convex and lower semicontinuous.

The bifunction f is called *maximal* if for all $(x,u) \in K \times H$, one has

$$f(y,x) + \langle u, y-x\rangle \leq 0, \forall y \in K \text{ implies } \langle u, y-x\rangle \leq f(x,y),\ \forall y \in K. \qquad (6)$$

Note that each bifunction f satisfying (H_1)–(H_4) is monotone and maximal (see [23, Lemma 2.1]). Oettli and Riahi [42] have established the relation between monotonicity and maximality of an operator A and those of its associated bifunction f defined by $f(x,y) = \sup_{v\in A(x)}\langle v, y-x\rangle$.

2.1. Convex analysis elements and equivalent formulations for problem (1)

To obtain solutions to (2) by means of convergence analysis of (3), we need a further background which we fix as follows.

For a proper lower semicontinuous convex function $h\colon H \to \mathbb{R}\cup\{+\infty\}$ and $x \in H$, let us denote by $h^*\colon H \to \mathbb{R}\cup\{+\infty\}$ its *Fenchel conjugate function*, which is defined by $h^*(x) := \sup_{y\in H}\{\langle x,y\rangle - h(y)\}$. The *subdifferential* of h at $x \in H$ is characterized by

$$\partial h(x) := \{x^* \in H : h^*(x^*) + h(x) = \langle x^*, x\rangle\}.$$

If $h = \delta_K$ is the indicator function of $K \subset H$, i.e., $\delta_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise, its Fenchel conjugate function at $x^* \in H$ is the support function of K at x^*, i.e., $\delta_K^*(x^*) = \sigma_K(x^*) = \sup_{y\in K}\langle x^*, y\rangle$. The normal cone to K at $x \in H$ is

$$N_K(x) = \begin{cases} \{x^* \in H \mid \ \langle x^*, u-x\rangle \leq 0,\ \forall u \in K\} & \text{if } x \in K \\ \emptyset & \text{otherwise.}\end{cases}$$

Observe that $N_K = \partial\delta_K$, and $x^* \in N_K(x)$ if, and only if, $\sigma_K(x^*) = \langle x^*, x\rangle$.

An *operator* T on H is a set-valued mapping for which $T(x) \subset H$ for each $x \in H$. The *domain*, the *range* and the *graph* of T are respectively defined by

$$\mathrm{dom}(T) = \{x \in H : T(x) \neq \emptyset\}, \quad \mathcal{R}(T) = T(H),$$
$$graph(T) = \{(x,x^*) \in H \times H : x^* \in T(x)\}.$$

We identify the operator T with its graph and write equivalently $x^* \in T(x)$ or $(x,x^*) \in T$. The operator T is said to be *monotone* if $\langle x^* - y^*, x-y\rangle \geq 0$ whenever $(x,x^*),(y,y^*) \in T$. If T is monotone, we call it *maximal monotone* if there is no other monotone operator including properly T. The subdifferential of a proper convex lower semicontinuous function is a maximal monotone operator.

Lemma 2.1. [8, Corollaire 1] *and* [46, Theorem 4.2]. *Let S and T be two maximal monotone operators on H such that $dom(T) \cap dom(S) \neq \emptyset$ and*

$$\mathbb{R}_+(dom(S) - dom(T)) = \bigcup_{a>0} a(dom(S) - dom(T))$$

is a closed linear subspace of H. Then $S+T$ is maximal monotone.

The assumption in this lemma is weaker than Rockafellar's [48] condition

$$(\text{int dom}(S)) \cap \text{dom}(T) \neq \emptyset.$$

Consider now a bifunction $f\colon K \times K \to \mathbb{R}$. Then, for every $u \in K$, the function f_u^* will denote the Fenchel conjugate of the function f_u defined on H by $f_u(x) = f(u, x)$ if $x \in K$ and $f_u(x) = +\infty$ otherwise. For a bifunction $f\colon K \times K \to \mathbb{R}$, we associate the operator A^f defined by:

$$A^f(x) := \partial f_x(x) = \begin{cases} \{z \in H : f(x, y) + \langle z, x - y\rangle \geq 0, \quad \forall y \in K\} & \text{if } x \in K, \\ \emptyset \quad \text{otherwise.} \end{cases}$$

Note that, under condition (H$_4$), the values of A^f are convex and closed in H and an easy computation shows that $\bar{x} \in S_f$ iff $0 \in A^f(\bar{x})$; and if f is monotone, A^f is also monotone. The monotonicity on f is a sufficient but not a necessary condition. Take for instance $f\colon H \times H \to \mathbb{R}$ defined by $f(x, y) = \|x - y\|^2$, then A^f is the identity of H, which is monotone, however f is nonmonotone.

The operator A^f is maximal monotone whenever the bifunction f satisfies conditions (H$_1$)–(H$_4$).

We also have for $f, g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = y^2 - x^2$, $g(x, y) = 2x(y - x)$, the bifunctions f, g satisfy (H$_1$)–(H$_4$) and $A^f(x) = A^g(x) = \{2x\}$; hence the maximal monotone operator A^f may represent many bifunctions satisfying (H$_1$)–(H$_4$), since $g(x, y) < f(x, y)$ unless $x = y$, (see [1, 36] for a detailed presentation).

Suppose in the sequel that $\text{dom}(A^g) = K$, and a condition that ensures the validity of $\partial g_x + N_{S_f} = \partial(g_x + \delta_{S_f})$, which is realized when we assume that $K \cap S_f \neq \emptyset$ and $\mathbb{R}_+(K - S_f)$ is a closed linear subspace of H.

The next Lemma introduce the notion of resolvent associated to bifunctions. This concept is crucial in the proof of existence for (3).

Lemma 2.2. [24] *Suppose that* $f\colon K \times K \to \mathbb{R}$ *satisfies* (H$_1$), (H$_2$), (H$_4$). *Then,* f *is maximal if, and only if, for each* $x \in H$ *and* $\lambda > 0$, *there exists a unique* $z_\lambda = J_\lambda^f(x) \in K$, *called the resolvent of* f *at* x, *such that*

$$\lambda f(z_\lambda, y) + \langle y - z_\lambda, z_\lambda - x\rangle \geq 0, \quad \forall y \in K. \tag{7}$$

Moreover, x *is an equilibrium point of* f *if, and only if* $x = J_\lambda^f(x)$ *for every* $\lambda > 0$ *if, and only if* $x = J_\lambda^f(x)$ *for some* $\lambda > 0$.

Proof. The proof of the first step (see [23]) is based on the generalized KKM-Fan's lemma. The second step relies on (7). □

Given that f is maximal monotone and satisfies the conditions (H$_1$)–(H$_4$), we have the existence and the uniqueness of $J_\lambda^f(x)$, for each $x \in H$ and $\lambda > 0$.

Definition 2.3. Let $f\colon K\times K\to\mathbb{R}$.

(1) For each $\lambda>0$, the associated Yosida λ-approximate to f over K is defined by $A_\lambda^f := \frac{1}{\lambda}(I-J_\lambda^f)$.

(2) The minimal section operator A° associated to f is defined by the condition $A^\circ x := P_{A^f x}0 \in A^f(x)$, where $P_{A^f x}$ is the metric projection of the origin on the set $A^f x$.

Remark 2.4. [23, 26, 40] (i) For each $\lambda>0$, the resolvent J_λ^f is firmly nonexpansive, namely

$$\langle J_\lambda^f x - J_\lambda^f y, x-y\rangle \geq \|J_\lambda^f x - J_\lambda^f y\|^2, \quad \forall\, x,y\in K. \tag{8}$$

As consequence, the Yosida λ-approximate is $\frac{1}{\lambda}$-Lipschitz continuous, that is

$$\|A_\lambda^f x - A_\lambda^f y\| \leq \frac{1}{\lambda}\|x-y\|, \quad \forall x,y\in K.$$

(ii) From [24, Lemma 2.5], we also have for every $x,y\in K,\lambda>0$,

$$\lambda\|A_\lambda^f x - A_\lambda^f y\|^2 \leq \langle A_\lambda^f x - A_\lambda^f y, y-x\rangle.$$

For $f\colon K\times K\to\mathbb{R}$, we also need the following Minty Ky Fan minimax inequality

$$\text{(MFMI)} \qquad \text{Find } \bar{x}\in K \text{ such that } f(y,\bar{x})\leq 0,\ \ \forall y\in K. \tag{9}$$

Lemma 2.5. (Minty's Lemma, [13])

(i) *Whenever* (H_2) *is satisfied, every solution to* (1) *is a solution to* (9).

(ii) *Conversely, if f satisfies* (H_1), (H_3) *and* (H_4), *then each solution of* (9) *is a solution of* (1).

Remark 2.6. (i) We conclude that under conditions (H_1)–(H_4), the problems (1) and (9) are equivalent, i.e., their sets of solutions coincide.

(ii) Observe that the assertion 2.5(i) remains true when we only assume f to be pseudomonotone, i.e., for each $x,y\in K$, $f(x,y)\geq 0$ implies $f(y,x)\leq 0$.

(iii) The link between (9) and standard ones is investigated in [22, Theorem 2] and [2, Proposition 5.32] in the framework of upper sign properties, see also [12].

2.2. Existence and uniqueness for the evolution dynamical equilibrium system (EDES)

Following Brézis [17, Definition 3.1], we say that $x\colon[0;+\infty)\to H$ is a solution of (4) (respectively (3)), if x is absolutely continuous (and then almost everywhere differentiable) on any bounded interval $[0,T]$ with $0<T<+\infty$, and (4) (respectively (3)) holds for almost every $t>0$.

As mentioned before, to solve the problem (3), our approach is essentially based on the following Cauchy-Lipschitz well known existence result.

Lemma 2.7. [See [17], Theorem I.4] *Assume that $K \subset H$ is nonempty, closed, convex and $J\colon [0,+\infty[\times K \to K$ satisfies the Lipschitz condition:*

$$\|J(t)x - J(t)y\| \leq \|x-y\|, \ \forall x,y \in K, \ t \geq 0.$$

Then, for each $\lambda > 0, u_0 \in K$, there exists a unique solution $u_\lambda\colon [0,+\infty[\to H$ to the following differential equation:

$$\begin{cases} \dot{u}_\lambda(t) + \dfrac{1}{\lambda}(u_\lambda(t) - J(t)u_\lambda(t)) = 0, \\ u_\lambda(0) = u_0. \end{cases} \tag{10}$$

To prove our main existence result, we also need the following useful dissipative property:

Lemma 2.8. *Assume that f and g are monotone. Then, for any solutions x_1 and x_2 to (3), the function $t \mapsto \|x_1(t) - x_2(t)\|$ is nonincreasing.*

Proof. It suffices to show that the real-valued function $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined for every $t \geq 0$ by $\theta(t) = \frac{1}{2}\|x_1(t) - x_2(t)\|^2$, is nonincreasing. Since x_1 and x_2 are solutions to (3), then respectively

$$\beta(t)f(x_1(t), x_2(t)) + g(x_1(t), x_2(t)) + \langle \dot{x}_1(t), x_2(t) - x_1(t)\rangle \geq 0,$$

and

$$\beta(t)f(x_2(t), x_1(t)) + g(x_2(t), x_1(t)) + \langle \dot{x}_2(t), x_1(t) - x_2(t)\rangle \geq 0.$$

By summing and using monotone condition (H_2) on f and g, we obtain

$$\dot{\theta}(t) = \langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \leq 0,$$

and so the function θ is nonincreasing. □

Theorem 2.9. *Suppose that the bifunctions f and g satisfy conditions* (H_1)–(H_4) *and the penalization function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+^\star$ is measurable with $\dot{\beta} \in L^\infty_{loc}(\mathbb{R})$. Assume moreover that for all $\lambda > 0$,*

$$\|A^{\Gamma_t}_\lambda u - A^{\Gamma_s}_\lambda u\| \leq |\beta(t) - \beta(s)| \left(\lambda \min\left(\|A^{\Gamma_t}_\lambda u\|, \|A^{\Gamma_s}_\lambda u\|\right) + \|u\|\right), \tag{11}$$

for all $u \in K$ and for all $s,t \geq 0$, where $\Gamma_t(u,v) := \beta(t)f(u,v) + g(u,v)$, for $t \geq 0$ and $u,v \in K$. Then, with initial value $u_0 \in K$, problem (3) *admits a unique solution.*

Proof. Fix $\lambda > 0$ and $u_0 \in K$, we have $J^{\Gamma_t}_\lambda$ is Lipschitz continuous (see (8)), and according to Lemma 2.7, there exists a unique solution denoted by u_λ to the following differential equation

$$\dot{u}_\lambda(t) + \frac{1}{\lambda}(u_\lambda(t) - J^{\Gamma_t}_\lambda u_\lambda(t)) = \dot{u}_\lambda(t) + A^{\Gamma_t}_\lambda u_\lambda(t) = 0 \quad \text{with} \quad u_\lambda(0) = u_0 \in K. \tag{12}$$

Let $T > 0$, for $t \in (0, T)$, if $h > 0$ is sufficiently small, we have $t, t+h \in [0, T]$, and therefore $\dot{u}_\lambda(t) = -A_\lambda^{\Gamma_t} u_\lambda(t)$ and $\dot{u}_\lambda(t+h) = -A_\lambda^{\Gamma_{t+h}} u_\lambda(t+h)$.

Using condition (11), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_\lambda(t+h) - u_\lambda(t)\|^2 = \langle \dot{u}_\lambda(t+h) - \dot{u}_\lambda(t), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&= \langle A_\lambda^{\Gamma_t} u_\lambda(t) - A_\lambda^{\Gamma_{t+h}} u_\lambda(t+h), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&= \langle A_\lambda^{\Gamma_t} u_\lambda(t) - A_\lambda^{\Gamma_{t+h}} u_\lambda(t), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&\qquad + \langle A_\lambda^{\Gamma_{t+h}} u_\lambda(t) - A_\lambda^{\Gamma_{t+h}} u_\lambda(t+h), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&\le \langle A_\lambda^{\Gamma_t} u_\lambda(t) - A_\lambda^{\Gamma_{t+h}} u_\lambda(t), u_\lambda(t+h) - u_\lambda(t)\rangle \\
&\qquad - \lambda\|A_\lambda^{\Gamma_{t+h}} u_\lambda(t) - A_\lambda^{\Gamma t+h} u_\lambda(t+h)\|^2 \\
&\le \|A_\lambda^{\Gamma_t} u_\lambda(t) - A_\lambda^{\Gamma_{t+h}} u_\lambda(t)\|\|u_\lambda(t+h) - u_\lambda(t)\| \\
&\le |\beta(t) - \beta(t+h)| \left(\lambda\|A_\lambda^{\Gamma_t} u_\lambda(t)\| + \|u_\lambda(t)\|\right) \|u_\lambda(t+h) - u_\lambda(t)\|.
\end{aligned}$$

By integrating from 0 to t we have

$$\frac{1}{2}\|u_\lambda(t+h) - u_\lambda(t)\|^2 \le \frac{1}{2}\|u_\lambda(h) - u_\lambda(0)\|^2 +$$
$$+ \int_0^t |\beta(s) - \beta(s+h)| \left(\lambda\|A_\lambda^{\Gamma_s} u_\lambda(s)\| + \|u_\lambda(s)\|\right) \|u_\lambda(s+h) - u_\lambda(s)\| ds.$$

By [17, Lemma A.5], we conclude

$$\|u_\lambda(t+h) - u_\lambda(t)\| \le \|u_\lambda(h) - u_0\| + \int_0^t |\beta(s) - \beta(s+h)| \left(\lambda\|A_\lambda^{\Gamma_s} u_\lambda(s)\| + \|u_\lambda(s)\|\right) ds,$$

and then multiplying by $\frac{1}{h}$, replacing $A_\lambda^{\Gamma_s} u_\lambda(s)$ by $-\dot{u}_\lambda(s)$ and letting $h \to 0$, we obtain: for a.e. $t \in [0, T]$

$$\begin{aligned}
\|\dot{u}_\lambda(t)\| &\le \|\dot{u}_\lambda(0)\| + \int_0^t |\dot{\beta}(s)| \left(\lambda\|\dot{u}_\lambda(s)\| + \|u_\lambda(s)\|\right) ds \\
&\le \|\dot{u}_\lambda(0)\| + \|\dot{\beta}\|_{T,\infty} \int_0^t \left(\lambda\|\dot{u}_\lambda(s)\| + \|u_\lambda(s)\|\right) ds.
\end{aligned} \tag{13}$$

Using u_λ of (12), which satisfies for each $t \in (0, T)$, $u_\lambda(t) = u_0 + \int_0^t \dot{u}_\lambda(s) ds$, and integrating on $[0, t]$ the norm of this relation, we obtain

$$\int_0^t \|u_\lambda(s)\| ds \le \|u_0\| T + \int_0^t s\|\dot{u}_\lambda(s)\| ds. \tag{14}$$

Combining (13) and (14), we get for a.e. $t \in [0, T]$

$$\|\dot{u}_\lambda(t)\| \le \left(\|\dot{u}_\lambda(0)\| + \|\dot{\beta}\|_{T,\infty}\|u_0\| T\right) + \int_0^t \left(\lambda\|\dot{\beta}\|_{T,\infty} + s\right) \|\dot{u}_\lambda(s)\| ds; \tag{15}$$

and Gronwall's lemma allows us to write, for a.e. $t \in [0, T]$,

$$\|\dot{u}_\lambda(t)\| \leq \Big(\|\dot{u}_\lambda(0)\| + \|\dot{\beta}\|_{T,\infty}\|u_0\|T\Big) \exp\Big((\lambda\|\dot{\beta}\|_{T,\infty} + T)T\Big).$$

According to [26, Lemma 2.5 (i)], we have $\|\dot{u}_\lambda(0)\| = \|A_\lambda^{\Gamma_0}u_0\| \leq \|(A^{\Gamma_0})^\circ u_0\|$, which leads, for all $0 < \lambda \leq 1$ and for a.e. $t \in [0, T]$, to

$$\|\dot{u}_\lambda(t)\| = \|A_\lambda^{\Gamma_t}u_\lambda(t)\| \leq C, \quad \text{where}$$
$$C = \Big(\|(A^{\Gamma_0})^\circ u_0\| + \|\dot{\beta}\|_{T,\infty}\|u_0\|T\Big) \exp\Big((\|\dot{\beta}\|_{T,\infty} + T)T\Big). \tag{16}$$

To show that $\{u_\lambda\}$ is a Cauchy family in the Banach space $\mathcal{C}([0, T], H)$ (as $\lambda \to 0$), let us fix $\lambda, \mu > 0$, and set, for a.e. $t \in [0, T]$, $a(t) := \frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2$. We have

$$a(t) = \frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 = -2\langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), u_\lambda(t) - u_\mu(t)\rangle.$$

Now, for every $t \in [0, T]$ and every $\theta \in \{\lambda, \mu\}$, write $u_\theta(t) = \theta A_\theta^{\Gamma_t}u_\theta(t) + J_\theta^{\Gamma_t}u_\theta(t)$ and obtain

$$\begin{aligned}&\langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), u_\lambda(t) - u_\mu(t)\rangle\\ &\quad= \langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), \lambda A_\lambda^{\Gamma_t}u_\lambda(t) - \mu A_\mu^{\Gamma_t}u_\mu(t)\rangle +\\ &\qquad+ \langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), J_\lambda^{\Gamma_t}u_\lambda(t) - J_\mu^{\Gamma_t}u_\mu(t)\rangle.\end{aligned}$$

Let us examine the term $\langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), J_\lambda^{\Gamma_t}u_\lambda(t) - J_\mu^{\Gamma_t}u_\mu(t)\rangle$.

We have $\Gamma_t(J_\lambda^{\Gamma_t}u_\lambda(t), J_\mu^{\Gamma_t}u_\mu(t)) - \langle A_\lambda^{\Gamma_t}u_\lambda(t), J_\mu^{\Gamma_t}u_\mu(t) - J_\lambda^{\Gamma_t}u_\lambda(t)\rangle \geq 0$,

and $\Gamma_t(J_\mu^{\Gamma_t}u_\mu(t), J_\lambda^{\Gamma_t}u_\lambda(t)) - \langle A_\mu^{\Gamma_t}u_\mu(t), J_\lambda^{\Gamma_t}u_\lambda(t) - J_\mu^{\Gamma_t}u_\mu(t)\rangle \geq 0$.

By summing these two inequalities and using the monotonicity of Γ_t, we obtain

$$\langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), J_\lambda^{\Gamma_t}u_\lambda(t) - J_\mu^{\Gamma_t}u_\mu(t)\rangle \geq 0.$$

Thus $a(t) \leq -2\langle A_\lambda^{\Gamma_t}u_\lambda(t) - A_\mu^{\Gamma_t}u_\mu(t), \lambda A_\lambda^{\Gamma_t}u_\lambda(t) - \mu A_\mu^{\Gamma_t}u_\mu(t)\rangle$,

hence $a(t) \leq 2\lambda\langle A_\lambda^{\Gamma_t}u_\lambda(t), A_\mu^{\Gamma_t}u_\mu(t)\rangle + 2\mu\langle A_\mu^{\Gamma_t}u_\mu(t), A_\lambda^{\Gamma_t}u_\lambda(t)\rangle -$

$$-2(\lambda\|A_\lambda^{\Gamma_t}u_\lambda(t)\|^2 + \mu\|A_\mu^{\Gamma_t}u_\mu(t)\|^2).$$

On the other hand, for $\theta, \theta' \in \{\lambda, \mu\}$

$$\begin{aligned}\theta\langle A_\theta^{\Gamma_t}u_\theta(t), A_{\theta'}^{\Gamma_t}u_{\theta'}(t)\rangle &\leq \theta\|A_\theta^{\Gamma_t}u_\theta(t)\|\|A_{\theta'}^{\Gamma_t}u_{\theta'}(t)\|\\ &\leq \theta\|A_\theta^{\Gamma_t}u_\theta(t)\|^2 + \frac{\theta}{4}\|A_{\theta'}^{\Gamma_t}u_{\theta'}(t)\|^2.\end{aligned}$$

Therefore, from (16) it follows that

$$\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 = a(t) \leq \frac{1}{2}(\lambda\|A_\lambda^{\Gamma_t}u_\lambda(t)\|^2 + \mu\|A_\mu^{\Gamma_t}u_\mu(t)\|^2) \leq \frac{\lambda+\mu}{2}C^2,$$

which in turns leads to $\|u_\lambda(t) - u_\mu(t)\| \leq \left(\frac{\lambda+\mu}{2}t\right)^{\frac{1}{2}} C.$

Thus, $\{u_\lambda\}$ is a Cauchy sequence uniformly in $t \in [0,T]$. Accordingly, $\{u_\lambda\}$ converges to some limit $u \in \mathcal{C}([0,T], H)$ with

$$\|u_\lambda(t) - u(t)\| \leq \left(\frac{\lambda T}{2}\right)^{\frac{1}{2}} C \text{ for every } 0 \leq t \leq T \text{ and } \lambda > 0.$$

Now, from the estimate $\|J_\lambda^{\Gamma_t}u_\lambda(t) - u_\lambda(t)\| = \lambda\|A_\lambda^{\Gamma_t}u_\lambda(t)\| \leq \lambda C$, it results that $J_\lambda^{\Gamma_t}u_\lambda$ converges in $\mathcal{C}([0,T]; H)$ to u. Observe also that (16) implies that $\{\dot{u}_\lambda\}$ is also bounded in the Hilbert space $L^2(0,T,H)$, so it admits a subsequence weakly converging to some $w \in L^2(0,T,H)$.

Passing to the limit in $u_\lambda(t) = u_0 + \int_0^t \dot{u}_\lambda(s)ds$ we find that the limit is $\dot{u}$ and take into account the uniqueness of the weak limit we get that $w = \dot{u}$.

Next, using the monotonicity of Γ_t and the convexity of $\Gamma_t(v,.)$, at the limit in the following inequality

$$\Gamma_t(J_\lambda^{\Gamma_t}u_\lambda(t), v) + \langle \dot{u}_\lambda(t), v - J_\lambda^{\Gamma_t}u_\lambda\rangle \geq 0, \quad \forall v \in K,$$

we obtain $\Gamma_t(v, u(t)) \leq \liminf_{\lambda\to 0} \Gamma_t(v, J_\lambda^{\Gamma_t}u_\lambda(t)) \leq \langle \dot{u}(t), v - u(t)\rangle.$

Finally, thanks to Lemma 2.5, for a.e. $t > 0$, we get

$$\Gamma_t(u(t), v) + \langle \dot{u}(t), v - u(t)\rangle \geq 0, \quad \forall v \in K.$$

Consequently, u is a solution to (3).

For uniqueness, suppose that u, v are solutions of the differential equation (3), and that u and v satisfy $u(0) = v(0) = u_0$, then according to Lemma 2.8, the function $t \mapsto \|v(t) - u(t)\|$ is nonincreasing, which means that, for every $t > 0$, we have $\|v(t) - u(t)\| \leq \|v(0) - u(0)\| = 0$. Thus $u = v$ on $\mathbb{R}_+$, and then the solution of (3) is unique. □

The above condition (11) appears natural in the proof of Theorem 2.9. In the next corollary, we give a particular case of bifunctions satisfying this condition. The existence of solutions for the system (4) requires more effort.

Corollary 2.10. *Suppose g satisfies* (H_1)–(H_4) *and let $f(u,v) = \langle u, v-u\rangle$, for all $u, v \in K$. Then, for each $u_0 \in K$, problem* (3) *with the initial condition $u(0) = u_0$ admits a unique (absolutely continuous) solution.*

Proof. We only need to verify condition (11). For $t \geq 0$ and $u, v \in K$ we have $\Gamma_t(u,v) := \beta(t)\langle u, v-u\rangle + g(u,v)$. So, by definition of the associated λ-Yosida approximate to Γ_t, we get, for each $t, s \geq 0$ and $u \in K$

$$\Gamma_t(J_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u) - \langle A_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle \geq 0,$$

and similarly $$\Gamma_s(J_\lambda^{\Gamma_s}u, J_\lambda^{\Gamma_t}u) - \langle A_\lambda^{\Gamma_s}u, J_\lambda^{\Gamma_t}u - J_\lambda^{\Gamma_s}u\rangle \geq 0.$$

By summing these two inequalities and using monotonicity of g we obtain

$$\beta(t)\langle J_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle + \beta(s)\langle J_\lambda^{\Gamma_s}u, J_\lambda^{\Gamma_t}u - J_\lambda^{\Gamma_s}u\rangle + \langle A_\lambda^{\Gamma_s}u - A_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle \geq 0.$$

Since $J_\lambda^{\Gamma_s} - J_\lambda^{\Gamma_t} = -\lambda(A_\lambda^{\Gamma_s} - A_\lambda^{\Gamma_t})$, then

$$\begin{aligned}
\lambda\|A_\lambda^{\Gamma_t}u - A_\lambda^{\Gamma_s}u\|^2 &\leq \beta(t)\langle J_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle + \beta(s)\langle J_\lambda^{\Gamma_s}u, J_\lambda^{\Gamma_t}u - J_\lambda^{\Gamma_s}u\rangle \\
&= \langle \beta(t)J_\lambda^{\Gamma_t}u - \beta(s)J_\lambda^{\Gamma_s}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle \\
&= (\beta(t)-\beta(s))\langle J_\lambda^{\Gamma_t}u, J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\rangle - \beta(s)\|J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\|^2 \\
&\leq |\beta(t)-\beta(s)|.\|J_\lambda^{\Gamma_t}u\|\|J_\lambda^{\Gamma_s}u - J_\lambda^{\Gamma_t}u\| \\
&\leq |\beta(t)-\beta(s)|\left(\lambda\|A_\lambda^{\Gamma_t}u\| + \|u\|\right)\lambda(\|A_\lambda^{\Gamma_t}u - A_\lambda^{\Gamma_s}u\|).
\end{aligned}$$

Suppose $A_\lambda^{\Gamma_t}u \neq A_\lambda^{\Gamma_s}u$, otherwise the required estimate is trivially satisfied, then

$$\|A_\lambda^{\Gamma_t}u - A_\lambda^{\Gamma_s}u\| \leq |\beta(t)-\beta(s)|\left(\lambda\|A_\lambda^{\Gamma_t}u\| + \|u\|\right).$$

By the same way we also prove

$$\|A_\lambda^{\Gamma_t}u - A_\lambda^{\Gamma_s}u\| \leq |\beta(t)-\beta(s)|\left(\lambda\|A_\lambda^{\Gamma_s}u\| + \|u\|\right).$$

From the last two inequalities, we deduce that assumption (11) is satisfied. □

3. Equilibrium Fitzpatrick transform

The notion of Fitzpatrick transform for equilibrium bifunctions was introduced by Boţ and Grad [16] and developed by Alizadeh and Hadjisavvas in [4] by adapting the original Fitzpatrick transform on monotone operators.

Given a monotone operator $T \subset H \times H$, the Brézis-Haraux transform (see [18]) $\mathcal{G}_T\colon H \times H \to \mathbb{R} \cup \{+\infty\}$ associated to the operator T is defined by

$$\mathcal{G}_T(x,u) = \sup_{(y,v)\in T} \langle x-y, v-u\rangle,$$

and the Fitzpatrick transform (see [31]) $\mathcal{F}_T\colon H \times H \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{F}_T(x,u) = \sup_{(y,v)\in T} \{\langle x,v\rangle + \langle y,u\rangle - \langle y,v\rangle\} = \mathcal{G}_T(x,u) + \langle x,u\rangle.$$

As a supremum of continuous affine functions, $\mathcal{F}_T$ is convex and lower semi-continuous with respect to the couple (x,u). This property turns out to be an important tool through which convex analysis methods have been widely applied to problems governed by maximal monotone operators (see for example [11, 50] and the references therein).

For every $(x, u) \in H \times H$, one has $\mathcal{F}_T(x, u) \geq \langle x, u\rangle$, and equality holds if, and only if $(x, u) \in T$; and thus the function $\mathcal{G}_T$ is nonnegative and takes the zero value on the graph of T. In many active researches, see for example [6, 14, 15], several conditions have been formulated by means of the Fitzpatrick and the Brézis-Haraux transforms in order to study the asymptotic convergence for monotone inclusion problems.

Based on these notions of real transforms associated with monotone operators, the authors of [4, 16] introduced the following definitions of Fitzpatrick and Brézis-Haraux transforms associated with real equilibrium bifunctions.

Definition 3.1. Let $f\colon K \times K \to \mathbb{R}$ be a bifunction. The *equilibrium Fitzpatrick transform* associated to f is the function $\mathcal{F}_f\colon K \times H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{F}_f(x, u) = \sup_{y \in K}\{\langle u, y\rangle + f(y, x)\}.$$

Similarly, the equilibrium Brézis-Haraux transform $\mathcal{G}_f$ is given by

$$\mathcal{G}_f(x, u) = \mathcal{F}_f(x, u) - \langle u, x\rangle.$$

Whenever $f(y, .)$ is convex lower semicontinuous for all $y \in K$, then $\mathcal{F}_f$ is also convex lower semicontinuous. By setting $y = x$ in the definition of $\mathcal{F}_f$, we first remark that $\mathcal{F}_f(x, u) \geq \langle u, x\rangle$ for all $(u, x) \in H \times K$, and then $\mathcal{G}_f$ is a nonnegative function. As shown in [4, Theorem 3.2], if moreover the bifunction f is monotone and maximal, then equality holds if and only if $u \in A^f x$. The link between monotonicity of f and the geometric property $\mathcal{F}_f(x, u) = \langle x, u\rangle$ is investigated in the following propositions.

Proposition 3.2. *Let $f\colon K \times K \to \mathbb{R}$ and $K' = dom(A^f) \subset K$.*

(1) *If f is monotone then $\mathcal{F}_f(x, u) = \langle u, x\rangle$ for all $x \in K'$ and all $u \in A^f x$.*

(2) *Conversely, if $\mathcal{F}_f(x, u) = \langle u, x\rangle$ for all $x \in K'$ and all $u \in A^f x$, then the operator A^f is monotone on K' and the bifunction f is relaxed monotone on K', i.e., there is a nonnegative symmetric bifunction $\gamma\colon K' \times K' \to \mathbb{R}_+$ such that $\gamma(x, x) = 0$ for each $x \in K'$ and*

$$f(x, y) + f(y, x) \leq \gamma(x, y) \ \ \forall x, y \in K'. \tag{17}$$

(3) *When, we suppose existence of some $u \in H$ such that $\mathcal{F}_f(x, u) = \langle u, x\rangle$ for all $x \in K = dom(A^f)$, we get f is monotone.*

Proof. (1) We first remark that $\mathcal{F}_f(x, u) \geq \langle u, x\rangle$ for all $(u, x) \in H \times K$. If $x \in K'$, $u \in A^f x$ and $y \in K$, we have $f(x, y) + \langle u, x - y\rangle \geq 0$, and then $\langle u, x\rangle \geq \langle u, y\rangle - f(x, y)$. Monotonicity of f implies $\langle u, x\rangle \geq \langle u, y\rangle + f(y, x)$, and thus $\langle u, x\rangle \geq \sup_{y\in K}\{\langle u, y\rangle + f(y, x)\} = \mathcal{F}_f(x, u)$.

(2) To prove that the operator A^f is monotone on K', fix $x_1, x_2 \in K'$ and let $u_1 \in A^f(x_1)$, $u_2 \in A^f(x_2)$. We have from the definition of the operator A^f

$$f(x_1, x_2) + \langle u_1, x_1 - x_2 \rangle \geq 0 \text{ and } f(x_2, x_1) + \langle u_2, x_2 - x_1 \rangle \geq 0.$$

By summing these two inequalities we obtain

$$\langle u_1 - u_2, x_1 - x_2 \rangle + f(x_1, x_2) + f(x_2, x_1) \geq 0. \tag{18}$$

On the other hand, by hypothesis we have

$$\mathcal{F}_f(x_1, u_1) = \langle x_1, u_1 \rangle \text{ and } \mathcal{F}_f(x_2, u_2) = \langle x_2, u_2 \rangle,$$

which implies

$$\langle u_1, x_2 \rangle + f(x_2, x_1) \leq \langle x_1, u_1 \rangle \quad \text{and} \quad \langle u_2, x_1 \rangle + f(x_1, x_2) \leq \langle x_2, u_2 \rangle. \tag{19}$$

Combining (18) and (19), we get $\langle u_1 - u_2, x_1 - x_2 \rangle \geq 0$, and then the operator A^f is monotone on K'.

Let $x_1, x_2 \in K'$, $u_1 \in A^f x_1$ and $u_2 \in A^f x_2$, then $\mathcal{F}_f(x_1, u_1) = \langle x_1, u_1 \rangle$ and $\mathcal{F}_f(x_2, u_2) = \langle x_2, u_2 \rangle$. By summing and using the definition of the function $\mathcal{F}_f$ we obtain

$$f(x_1, x_2) + f(x_2, x_1) \leq \langle u_1 - u_2, x_1 - x_2 \rangle. \tag{20}$$

According to A^f is monotone, the right hand side of the last inequality is nonnegative, and then by setting $\gamma(x, y) = \inf_{u \in A^f x,\ v \in A^f y} \langle u - v, x - y \rangle$, we get

$$f(x_1, x_2) + f(x_2, x_1) \leq \gamma(x_1, x_2),$$

where γ is nonnegative symmetric on $K' \times K'$ and equal to zero on the diagonal; whence the result.

(3) Returning to (20) for $x_1, x_2 \in K$ and noticing that $u_1 = u_2 = u$, we obtain $f(x_1, x_2) + f(x_2, x_1) \leq 0$, which concludes the proof. □

Remark 3.3. From maximality of f, see (6), we have $x \in K$ and $u \in A^f x$ imply $\mathcal{F}_f(x, u) = \langle x, u \rangle$; and then for a monotone and maximal bifunction f, we have for every $x \in \mathrm{dom}(A^f)$

$$u \in A^f x \text{ is equivalent to } \mathcal{F}_f(x, u) = \langle x, u \rangle.$$

Remark also that if x is in the domain of A^f, then the monotonicity on f entails $f(x, x) = 0$; in particular if $\mathrm{dom}(A^f) = K$ then $(\mathrm{H}_2) \Rightarrow (\mathrm{H}_1)$.

Remark 3.4. Let $J \colon K \to \mathbb{R}$ be a locally Lipschitz function, and consider f_J the bifunction defined by $f_J(x, y) = J^0(x; y - x)$, where J^0 is the generalized Clark's derivative of J. If J is quasiconvex (nonconvex), then A^{f_J} is quasimonotone, i.e., for all $x, y \in K$, $\sup_{x^* \in A^{f_J}(x)} \langle x^*, y - x \rangle > 0$ implies $\sup_{y^* \in A^{f_J}(y)} \langle y^*, x - y \rangle \leq 0$.

Indeed, let $x, y \in K$ and $x^* \in A^{f_J}(x)$ such that $\langle x^*, y - x\rangle > 0$.

Since $x^* \in A^{f_J}(x)$, we have, for all $v \in K$, $J^0(x; v - x) \geq \langle x^*, v - x\rangle$, and hence $f_J(x, y) = J^0(x; y - x) \geq \langle x^*, y - x\rangle > 0$.

Since J is quasiconvex, following [2], then the subdifferential $\partial^c J$ is quasimonotone, and therefore the bifunction $f_J(x, y) = J^0(x; y - x)$ is quasimonotone. We deduce, for all $y^* \in A^{f_J}(y)$, $\langle y^*, x - y\rangle \leq J^0(y; x - y) = f_J(y, x) \leq 0$, implying that A^{f_J} is quasimonotone. □

Remark 3.5. (1) For any nonconvex function $J\colon K \to \mathbb{R}$ which is quasiconvex and locally Lipschitz, the result of Proposition 3.2 is not satisfied for the bifunction $f_J(x, y) = J^0(x; y - x)$ since in this case f_J is only quasimonotone.

(2) One can also consult [3, Lemma 3.3, Remark 3.4] and [2, Example 3.4, Example 3.5] for the case where the bifunction f_J satisfies all the conditions (H_1), (H_3), (H_4) except (H_2) which can be replaced by the relaxed γ-monotonicity condition (17), where $\gamma(x, y) = c\|y - x\|^\delta$ with $c, \delta > 0$.

Proposition 3.6. *Let $f\colon K \times K \to \mathbb{R}$ be a monotone bifunction. Then, for each $(x, u) \in K \times H$, $\mathcal{F}_f(x, u) \leq f_x^*(u)$.*

Proof. Suppose that the bifunction f is monotone, i.e., $f(y, x) \leq -f(x, y)$ for all $y \in K$, then

$$\begin{aligned}\mathcal{F}_f(x, u) &= \sup_{y\in K}\{\langle u, y\rangle + f(y, x)\} \leq \sup_{y\in K}\{\langle u, y\rangle - f(x, y)\}\\ &= \sup_{y\in H}\{\langle u, y\rangle - f_x(y)\} = f_x^*(u).\end{aligned}$$ □

Example 3.7. To illustrate Propositions 3.2 and 3.6, we provide the following example. Take $K = \mathbb{R}$ and f defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} (x - y)y^2 & \text{if } y < 0\\ 0 & \text{if } y \geq 0.\end{cases}$$

We have $f(x, x) = 0$ for all $x \in \mathbb{R}$, and $f(x, y) + f(y, x) > 0$ if either $x, y < 0$ and $x \neq y$, or $x \geq 0$ and $y < 0$, or $x < 0$ and $y \geq 0$. We conclude that f is not monotone on $\mathbb{R}^2$. We also get $A^f(x) = \{0\}$ if $x \geq 0$ and $A^f(x) = \{-x^2\}$ if $x < 0$; then we have

$$\mathcal{F}_f(x, u) = \sup_{y\in\mathbb{R}}\{uy + f(y, x)\} = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } u = 0\\ -x^3 & \text{if } x < 0 \text{ and } u = -x^2,\end{cases}$$

and

$$f_x^*(u) = \sup_{y\in\mathbb{R}}\{uy - f(x, y)\} = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } u = 0\\ -x^3 & \text{if } x < 0 \text{ and } u = -x^2.\end{cases}$$

We conclude that $\mathcal{F}_f(x, u) = f_x^*(u)$ for each (x, u) in the graph of A^f, but f is not monotone, which means that monotonicity is a sufficient condition but not necessary in Proposition 3.6.

This example illustrates also that the converse in Proposition 3.2(1) is false, since $\mathcal{F}_f(x,u) = \langle u, x\rangle$ for all $x \in \mathbb{R}$ and $u \in A^f(x)$, and f is not monotone; while the assertion (2) is satisfied when taking $\gamma(x,y) = (x-y)^2|x+y|$ for each $x, y \in \mathbb{R}$. □

Proposition 3.8. *If $f(x,y) = \phi(y) - \phi(x)$ where $\phi\colon H \to \mathbb{R}\cup\{+\infty\}$ is convex and lower semicontinuous with dom$(\phi) \subset K$, then for $(x,u) \in K \times H$*

$$\mathcal{F}_f(x,u) = \phi(x) + \phi^*(u).$$

Proof. The required conclusion results from

$$\mathcal{F}_f(x,u) = \sup_{y\in K}\{\langle u, y\rangle - \phi(y)\} + \phi(x) = \phi^*(u) + \phi(x).$$ □

Proposition 3.9. *Let $f, g\colon K \times K \to \mathbb{R}$ be two bifunctions, then for every $\lambda > 0$ and every $(x,u) \in K \times H$*

(i) $\mathcal{F}_{\lambda f}(x,u) = \lambda \mathcal{F}_f\left(x, \frac{u}{\lambda}\right)$ *and* $\mathcal{G}_{\lambda f}(x,u) = \lambda \mathcal{G}_f\left(x, \frac{u}{\lambda}\right)$*;*

(ii) $\mathcal{F}_{f+g}(x,u) \leq \inf_{z\in H}\{\mathcal{F}_f(x,z) + \mathcal{F}_g(x,u-z)\}$ *and similarly for the function* $\mathcal{G}_{f+g}$.

Proof. (i) Let $(x,u) \in K \times H$ and $\lambda > 0$, then

$$\begin{aligned}\mathcal{F}_{\lambda f}(x,u) &= \sup_{y\in K}\{\langle u, y\rangle + \lambda f(y,x)\} = \lambda \sup_{y\in K}\left\{\left\langle \frac{u}{\lambda}, y\right\rangle + f(y,x)\right\} \\ &= \lambda \mathcal{F}_f\left(x, \frac{u}{\lambda}\right).\end{aligned}$$

Similarly

$$\mathcal{G}_{\lambda f}(x,u) = \mathcal{F}_{\lambda f}(x,u) - \langle x, u\rangle = \lambda\left[\mathcal{F}_f\left(x, \frac{u}{\lambda}\right) - \left\langle x, \frac{u}{\lambda}\right\rangle\right] = \lambda \mathcal{G}_f\left(x, \frac{u}{\lambda}\right).$$

(ii) Given $u \in H$ and $x \in K$, we have

$$\begin{aligned}\mathcal{F}_{f+g}(x,u) &= \sup_{y\in K}\{\langle u, y\rangle + f(y,x) + g(y,x)\} \\ &= \sup_{y\in K}\{\langle z, y\rangle + \langle u-z, y\rangle + f(y,x) + g(y,x)\}, \ \ \forall z \in H \\ &\leq \sup_{y\in K}\{\langle z, y\rangle + f(y,x)\} + \sup_{y\in K}\{\langle u-z, y\rangle + g(y,x)\}, \ \ \forall z \in H \\ &= \mathcal{F}_f(x,z) + \mathcal{F}_g(x,u-z), \ \ \forall z \in H.\end{aligned}$$

Hence $$\mathcal{F}_{f+g}(x,u) \leq \inf_{z\in H}\{\mathcal{F}_f(x,z) + \mathcal{F}_g(x,u-z)\},$$

and by subtracting $\langle x, u\rangle$ to each member, we find the inequality for $\mathcal{G}_{f+g}$. □

Proposition 3.10. *Let $A \subset H\times H$ be a monotone operator such that $K \subset$ dom(A), and f defined on $K \times K$ by $f(x,y) = \sup_{v\in Ax}\langle v, y-x\rangle$.*

Then, for each $(u,x) \in H \times K$, we have $\mathcal{F}_f(x,u) = \mathcal{F}_{\bar{A}}(x,u) \leq \mathcal{F}_A(x,u)$, where $\mathcal{F}_{\bar{A}}$ is the Fitzpatrick transform associated to the operator $\bar{A}x = Ax$ if $x \in K$ and $\bar{A}x = \emptyset$ otherwise.

Proof. Let $(u,x) \in H \times K$, we have

$$\begin{aligned}\mathcal{F}_f(x,u) &= \sup_{y\in K}\{\langle u,y\rangle + \sup_{v\in Ay}\langle v,x-y\rangle\} = \sup_{y\in K}\sup_{v\in Ay}\{\langle u,y\rangle + \langle v,x\rangle - \langle v,y\rangle\} \\ &= \sup_{(y,v)\in\bar{A}}\{\langle u,y\rangle + \langle v,x\rangle - \langle v,y\rangle\} = \mathcal{F}_{\bar{A}}(x,u),\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_A(x,u) = &\sup_{y\in \mathrm{dom}(A)}\{\langle u,y\rangle + \sup_{v\in Ay}\langle v,x-y\rangle\} \\ \geq &\sup_{y\in K}\{\langle u,y\rangle + \sup_{v\in Ay}\langle v,x-y\rangle\} = \mathcal{F}_{\bar{A}}(x,u). \qquad \square\end{aligned}$$

Remark 3.11. The function $f(x,y) = \sup_{v\in Ax}\langle v, y-x\rangle$ as defined can reach infinite values, so to make this proposition adequate we assume that A is locally bounded on each point $x \in K$.

We conclude this section by a link between equilibrium Brézis-Haraux transform and the resolvent of the associated bifunction.

Proposition 3.12. *Suppose that* $f\colon K\times K \to \mathbb{R}$ *satisfies* (H_1)–(H_4). *Then for each* $(x,u) \in K \times H$

$$\mathcal{G}_f(x,u) \geq \|x - J_1^f(x+u)\|^2. \tag{21}$$

Proof. Given $(x,u) \in K \times H$. There exists a unique $y = J_1^f(x+u) \in K$ by Lemma 2.2 such that

$$f(y,\xi) + \langle x+u-y, y-\xi\rangle \geq 0, \quad \forall \xi \in K.$$

By taking $\xi = x$, we obtain $f(y,x) + \langle u, y-x\rangle \geq \|x-y\|^2$, and then

$$\begin{aligned}\mathcal{G}_f(x,u) &= \sup_{\xi\in K}\{f(\xi,x) + \langle u,\xi - x\rangle\} \geq f(y,x) + \langle u, y-x\rangle \\ &\geq \|x-y\|^2 = \|x - J_1^f(x+u)\|^2. \qquad \square\end{aligned}$$

4. Weak and strong convergence for asymptotic behavior

Throughout this section we assume that the solution set S of the problem (2) is nonempty.

4.1. Dynamic Opial and Passty Lemmas

While usually the existence and uniqueness of such trajectories is guaranteed in the framework of the Cauchy-Lipschitz theorem, their convergence (in a sense to be specified) towards a solution of an adequate problem relies on Lyapunov analysis.

Recall now, the following

Lemma 4.1 (Opial Lemma). *Let B be a nonempty subset of a Hilbert space H and $x:[0,+\infty)\to H$. Suppose that:*

(i) $\lim_{t\to+\infty}\|x(t)-u\|$ *exists for each $u\in B$,*

(ii) *each sequential weak cluster points of $\{x(t)\}$ belongs to B.*

Then the weakly limit of $x(t)$, when t goes to $+\infty$, exists and belongs to B.

Lemma 4.2 (Opial-Passty Lemma, [44]). *Let B be a nonempty subset of a Hilbert space H and $x\colon[0,+\infty)\to H$. Set $X(t)=\frac{1}{t}\int_0^t x(s)\,ds$ and suppose that:*

(i) $\lim_{t\to+\infty}\|x(t)-u\|$ *exists for all $u\in B$,*

(ii) *each sequential weak cluster points of $\{X(t)\}$ belongs to B.*

Then, $\{x(t)\}$ satisfies the weak ergodic convergence in B, i.e., there exists $\bar{x}\in B$ such that $w-\lim_{t\to+\infty}X(t)=\bar{x}$.

4.2. Weak ergodic convergence

In order to arrange the study of weak ergodic asymptotic behavior of the trajectories in (3), we first analyze the convergence of the following general parametric problem (4): Find $x\in\mathcal{C}^0([0,+\infty];K)$ such that

$$\Gamma_t(x(t),y)+\langle\dot{x}(t),y-x(t)\rangle\geq 0,\ \ \forall y\in K,\ \text{for } a.e.\ t\geq 0,$$

where $\{\Gamma_t\colon K\times K\to\mathbb{R}, t\geq 0\}$ is a parametric family of monotone bifunctions. This parametric family of bifunctions is said to be v-convergent to Γ_∞, as $t\to+\infty$, and we write $\Gamma_\infty=v-\lim_{t\to+\infty}\Gamma_t$, if

(1) for every $(x,u)\in K\times H$ such that $\Gamma_\infty(x,y)+\langle u,x-y\rangle\geq 0$, for all $y\in K$, there exist $(x(t))\subset K$ and $(u(t))\subset H$ with $x(t)\to x$ and $u(t)\to u$ and $\Gamma_t(x(t),y)+\langle u(t),y-x(t)\rangle\geq 0$ for all $y\in K$;

(2) if there exist $t_n\to+\infty$, $x_n\to x$ and $u_n\rightharpoonup u$ such that, for all $y\in K$, we have $\Gamma_{t_n}(x(t_n),y)+\langle u(t_n),y-x(t_n)\rangle\geq 0$, then $\Gamma_\infty(x,y)+\langle u,y-x\rangle\geq 0$ for all $y\in K$.

Remark that, see [1, Proposition 4.1], the v-convergence of a parametric family of monotone bifunctions is equivalent to the strong convergence of the associate resolvents, i.e.,

$$\begin{aligned}\Gamma_\infty=v-\lim_{t\to+\infty}\Gamma_t &\iff \forall\lambda>0,\ J_\lambda^{\Gamma_t}y\longrightarrow J_\lambda^{\Gamma_\infty}y,\ \forall y\in H \\ &\iff \exists\lambda_0>0 \text{ such that } J_{\lambda_0}^{\Gamma_t}y\longrightarrow J_{\lambda_0}^{\Gamma_\infty}y,\forall y\in H.\end{aligned}\tag{22}$$

In the next proposition, we show that the v-convergence of a parametric family of monotone and maximal bifunctions can be formulated in terms of the equilibrium Brézis-Haraux transform.

Proposition 4.3. *Let $\{\Gamma_t \colon K \times K \to \mathbb{R}, t \geq 0\}$ be a parametric family of monotone bifunctions. Let $\Gamma_\infty \colon K \times K \to \mathbb{R}$ be a monotone and maximal bifunction such that*

$$\lim_{t\to\infty} \mathcal{G}_{\Gamma_t}(z,p) = 0, \quad \forall (z,p) \in A^{\Gamma_\infty}. \tag{23}$$

Then, the parametric family $\{\Gamma_t\}$ v-converges to Γ_∞ as $t \to +\infty$.

Proof. The proof of this proposition makes use of the equivalence (22), and then it suffices to show that

$$s - \lim_{t\to\infty} J_1^{\Gamma_t} y = J_1^{\Gamma_\infty} y, \ \text{ for each } y \in H.$$

Take $y \in H$ arbitrary. By maximality of Γ_∞ there exists a unique $z_1 = J_1^{\Gamma_\infty} y \in K$ such that

$$\Gamma_\infty(z_1, u) + \langle y - z_1, z_1 - u \rangle \geq 0, \ \ \forall u \in K, \ \text{ that is } (z_1, y - z_1) \in A^{\Gamma_\infty}.$$

Set $p = y - z_1$, we have $(z_1, p) \in A^{\Gamma_\infty}$, and according to Proposition 3.12,

$$\mathcal{G}_{\Gamma_t}(z_1, p) \ \ \geq \ \ \|z_1 - J_1^{\Gamma_t}(z_1 + p)\|^2 = \|J_1^{\Gamma_\infty} y - J_1^{\Gamma_t} y\|^2;$$

which implies, by using (23), the strong convergence of the resolvents $J_1^{\Gamma_t} y$ to $J_1^{\Gamma_\infty} y$ when t tends to infinity. □

Now, we state our general weak ergodic convergence theorem for the first-order nonautonomous continuous dynamical Ky Fan minimax inequality (4).

Theorem 4.4. *Let $\Gamma_\infty \colon K \times K \to \mathbb{R}$ be a bifunction that satisfies conditions* (H$_1$)–(H$_4$) *such that $S_{\Gamma_\infty} \neq \emptyset$. Let us assume that*

$$(\mathcal{C}_1) \qquad \int_0^{+\infty} \mathcal{G}_{\Gamma_t}(z,p)dt < +\infty, \ \forall (z,p) \in A^{\Gamma_\infty}. \tag{24}$$

Then, for any solution $x(.)$ of (4) there exists $\bar{x} \in S_{\Gamma_\infty}$ such that

$$w - \lim_{t\to+\infty} \frac{1}{t} \int_0^t x(s)ds = \bar{x}.$$

Proof. The proof relies on the Opial-Passty Lemma 4.2 applied with $B = S_{\Gamma_\infty}$.

We first show that for every $z \in S_{\Gamma_\infty}$, $\lim_{t\to+\infty} \|x(t) - z\|$ exists.

Set $h_z(t) = \frac{1}{2}\|x(t) - z\|^2$, then for almost every $t > 0$

$$\frac{dh_z}{dt}(t) = \langle \dot{x}(t), x(t) - z \rangle \leq \Gamma_t(x(t), z) \leq \sup_{y\in K} \Gamma_t(y,z) = \mathcal{G}_{\Gamma_t}(z, 0).$$

From this inequality and assumption (24) at the point $(z, 0)$, it follows that $(\dot{h}_z)_+ \in L^1(0, +\infty)$, which implies that $\lim_{t\to+\infty} h_z(t)$ exists in $\mathbb{R}$. This proves condition (i) of the Opial-Passty Lemma 4.2. For condition (ii), let us prove that if $X(t_n) = (1/t_n) \int_0^{t_n} x(s)ds$ and $X(t_n) \rightharpoonup \bar{z}$, for a sequence $t_n \to +\infty$, then $\bar{z}$ belongs to S_{Γ_∞}. Fix $(z,p) \in A^{\Gamma_\infty}$, then for almost every $t > 0$

$$\dot{h}_z(t) + \langle p, x(t) - z\rangle \leq \langle p, x(t)\rangle + \Gamma_t(x(t), z) - \langle z, p\rangle \leq \mathcal{G}_{\Gamma_t}(z,p).$$

After integrating on $[0, t_n]$ and dividing by t_n, since $h_z(t_n) \geq 0$, we deduce

$$\langle p, X(t_n) - z\rangle \leq \frac{h_z(0)}{t_n} + \frac{1}{t_n}\int_0^{+\infty} \mathcal{G}_{\Gamma_s}(z,p)ds \leq \frac{c}{t_n}, \tag{25}$$

where $c = h_z(0) + \int_0^{+\infty} \mathcal{G}_{\Gamma_s}(z,p)\, ds$ which is, by condition (24), a positive number. Passing to the limit in (25), as $t_n \to +\infty$, we immediately obtain $\langle p, \bar{z} - z\rangle \leq 0$. This being true for all $(z,p) \in A^{\Gamma_\infty}$, the maximal monotonicity of A^{Γ_∞} allows us to infer that $(\bar{z}, 0) \in A^{\Gamma_\infty}$; that is $\bar{z}$ belongs to S_{Γ_∞}. By Lemma 4.2, we conclude the weak ergodic convergence of the trajectories of (4). □

Remark 4.5. From the proof of Proposition 4.3, we deduce that Condition (24) implies

$$\int_0^{+\infty} \|J_1^{\Gamma_t} y - J_1^{\Gamma_\infty} y\|^2 dt < +\infty, \ \ \forall y \in H.$$

Hence, for all $y \in H$: $\text{ess-}\liminf_{t\to+\infty} \|J_1^{\Gamma_t} y - J_1^{\Gamma_\infty} y\| = 0$.

Now we specify our general ergodic weak convergence result to the case of a structured bifunction of the form $\Gamma_t = \beta(t) f + g$ where $\beta \colon \mathbb{R}_+ \to \mathbb{R}_+^\star$ is a measurable function.

Theorem 4.6. *Suppose f and g satisfy conditions* (H_1)–(H_4). *Suppose additionally that for all $u \in S_f$ and for all $p \in N_{S_f}(u)$*

$$(\mathcal{C}_2) \qquad \int_0^{+\infty} \beta(t)\left[\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty. \tag{26}$$

Then for every solution $x(.)$ of (3) *there is $\bar{x} \in S$ such that*

$$w - \lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)ds = \bar{x}.$$

Proof. To apply Theorem 4.4, it suffices to prove assumption (24) with

$$\Gamma_t = \beta(t) f + g \ \text{ and } \ A^{\Gamma_\infty}(u) := A^g(u) + N_{S_f}(u) = \partial g_u(u) + N_{S_f}(u)$$

for each $u \in H$. The conditions (H_1)–(H_4) for g guarantee maximal monotonicity of the operator A^g, while the criteria $\mathbb{R}_+(K - S_f)$ is a closed linear subspace of H realizes A^{Γ_∞} is maximal monotone.

Take $u \in S_f$ and $w \in A^{\Gamma_\infty}(u) = (A^g + N_{S_f})(u)$. Let $p \in N_{S_f}(u)$ be such that $w - p \in A^g(u)$. In view of Proposition 3.9 and $w - p \in A^g(u)$, we get

$$\mathcal{G}_{g+\beta(t)f}(u,w) \leq \mathcal{G}_g(u, w-p) + \mathcal{G}_{\beta(t)f}(u,p) = \mathcal{G}_{\beta(t)f}(u,p) = \beta(t)\mathcal{G}_f\left(u, \frac{p}{\beta(t)}\right).$$

We have for every $u \in S_f$ and $p \in N_{S_f}(u)$, $\sigma_{S_f}(p) = \langle p, u\rangle$. Since σ_{S_f} is positively homogeneous and $\beta(t) \geq 0$ we get $\sigma_{S_f}\left(\frac{p}{\beta(t)}\right) = \left\langle \frac{p}{\beta(t)}, u\right\rangle$, and then

$$\mathcal{G}_f\left(u, \frac{p}{\beta(t)}\right) = \mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \left\langle u, \frac{p}{\beta(t)}\right\rangle = \mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right).$$

This equality, together with (26), yields Condition (24) for $\Gamma_t = \beta(t)f + g$, i.e.,

$$\int_0^{+\infty} \mathcal{G}_{g+\beta(t)f}(u,w)dt < +\infty.$$

Applying Theorem 4.4, we conclude that $x(t)$, the solution of (3), converges weakly in average to some $\bar{x}$ such that $0 \in A^{\Gamma_\infty}(\bar{x}) = A^g(\bar{x}) + N_{S_f}(\bar{x})$, i.e.,

$$\exists\, q \in N_{S_f}(\bar{x}) \text{ such that } g(\bar{x}, y) \geq \langle -q, y - \bar{x}\rangle, \quad \forall y \in K.$$

Since $q \in N_{S_f}(\bar{x})$, for each $y \in S_f$, we have $\langle -q, y - \bar{x}\rangle \geq 0$, and then $g(\bar{x}, y) \geq 0$, $\forall y \in S_f$, that is $\bar{x} \in S$. □

Remark 4.7. In view of Proposition 3.6, we have

$$\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) \leq f_u^*\left(\frac{p}{\beta(t)}\right).$$

Then (26) is assured under the condition that for all $u \in S_f$ and for all $p \in N_{S_f}(u)$

$$(\mathcal{C}_3) \qquad \int_0^{+\infty} \beta(t)\left[f_u^*\left(\frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty. \qquad (27)$$

This condition can be seen as a continuous version of condition (A) introduced in [25].

4.3. Weak convergence

In this subsection, we show that when the parameter $\beta(t)$ is assumed to tend to $+\infty$ and the bifunction g is demipositive, we obtain the weak convergence of the trajectories of (3) to a solution of (2).

Using Opial's Lemma 4.1, in order to prove weak convergence of $\{x(t)\}$ to $x \in S$, when $t \to \infty$, one of two main conditions is that each weak cluster point lies in S. The key tool to prove this condition is demipositivity which was first introduced in Bruck [20] for monotone operators and developed to bifunctions as follows.

Definition 4.8. [26] A bifunction $f\colon K \times K \to \mathbb{R}$ is called *demipositive* on $C \subset K$ if $S_f(C) \neq \emptyset$ and the conditions $z \in S_f(C)$ (i.e., $z \in C$ and $f(z,u) \geq 0$ for all $u \in C$), $\{y_n\} \subset K$ converging weakly to $y \in C$ and $f(y_n, z) \to 0$ imply $y \in S_f(C)$.

The following proposition gives examples for bifunctions with the demipositive property.

Proposition 4.9. [26, Proposition 1] *Consider the bifunction $f\colon K \times K \to \mathbb{R}$ with $S_f \neq \emptyset$. Then each of the following conditions is sufficient for $f\colon K \times K \to \mathbb{R}$ to be demipositive on K:*

(i) *f satisfies* (H_1)–(H_4) *and* $\operatorname{int} S_f \neq \emptyset$*;*

(ii) *f satisfies* (H_1)–(H_4) *and is* 3*-monotone, i.e., $f(x,y) + f(y,z) + f(z,x) \leq 0$ for all $x, y, z \in K$;*

(iii) *f is α-strongly monotone for some $\alpha > 0$, i.e., $f(x,y) + f(y,x) \leq -\alpha\|x-y\|^2$ for each $x, y \in K$;*

(iv) *f verifies* (H_1)–(H_4) *and f is μ-cocoercive for some $\mu > 0$, i.e., for each $x \in K$, $f(x,\cdot)$ is differentiable and $f(x,y) + f(y,x) \leq -\mu\|\nabla_2 f(x,x) - \nabla_2 f(y,y)\|^2$ for all $x, y \in K$.*

We begin by the following nonlinear generalization of Gronwall's inequality that was given in [28, Theorem 23] and [37, p. 361].

Lemma 4.10. *If u, f, ϕ are continuous and nonnegative functions on $[0,T]$, $0 < \gamma < 1$, $c > 0$, and for all $t \in [0,T]$*

$$u(t) \leq f(t) + c\int_0^t \phi(s) u^{\gamma}(s) ds.$$

Then, there exists $\xi_0 > 0$ such that, for each $t \in [0,T]$

$$u(t) \leq f(t) + c\xi_0 \left(\int_0^t \phi^{\frac{1}{1-\gamma}}(s) ds\right)^{1-\gamma}.$$

Theorem 4.11. *Suppose f and g verify conditions* (H_1)–(H_4)*, β is absolutely continuous and consider $x(.)$ a solution of* (3)*.*

(1) *Then condition* (26) *on S (the solution set of* (2)*) implies $\lim_{t\to+\infty}\|x(t)-u\|$ exists for every $u \in S$.*

(2) *If moreover $\lim_{t\to+\infty}\beta(t) = +\infty$, $\dot\beta \in L^2(0;+\infty)$ and that the operator A^f is bounded on bounded sets, then each sequential weak cluster point of $\{x(t)\}$ belongs to S_f.*

(3) *If, in addition to the above conditions, g is demipositive on S_f then there is a solution $\bar{x} \in S$ such that the whole trajectory $\{x(t)\}$ weakly converges to $\bar{x}$ as $t \to +\infty$.*

Proof. Step 1. We first show that $\lim_{t\to+\infty} \|x(t)-u\|$ exists for all $u \in S$.

Fix $u \in S$ and set $h_u(t) = \frac{1}{2}\|x(t)-u\|^2$, we have $\dot{h}_u(t) = \langle \dot{x}(t), x(t)-u\rangle$ and $0 \in (A^g + N_{S_f})(u)$. Let $p \in N_{S_f}(u)$ be such that $-p \in A^g(u)$, then

$$g(u, x(t)) + \langle -p, u - x(t)\rangle \geq 0. \tag{28}$$

If we set $y = u$ in (3), we also have, for a.e. $t > 0$,

$$\beta(t)f(x(t), u) + g(x(t), u) + \langle \dot{x}(t), u - x(t)\rangle \geq 0. \tag{29}$$

Thanks to the monotonicity of g, summing (28) and (29), we get

$$\beta(t)f(x(t), u) + \langle \dot{x}(t), u - x(t)\rangle - \langle p, u - x(t)\rangle \geq 0,$$

and then for a.e. $t > 0$

$$\begin{aligned}
\dot{h}_u(t) &\leq \beta(t)f(x(t), u) - \langle p, u - x(t)\rangle \\
&\leq \beta(t)\left[\left\langle \frac{p}{\beta(t)}, x(t)\right\rangle + f(x(t), u) - \left\langle \frac{p}{\beta(t)}, u\right\rangle\right] \\
&\leq \beta(t)\left[\sup_{x\in K}\left\{\left\langle \frac{p}{\beta(t)}, x\right\rangle + f(x, u)\right\} - \left\langle \frac{p}{\beta(t)}, u\right\rangle\right] \\
&\leq \beta(t)\left[\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \left\langle \frac{p}{\beta(t)}, u\right\rangle\right] \\
&= \beta(t)\left[\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right].
\end{aligned}$$

Hence, for a.e. $t > 0$,

$$\gamma(t) := \dot{h}_u(t) - \beta(t)\left[\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right] \leq 0. \tag{30}$$

Now, the fact that x is absolutely continuous implies that h_u is absolutely continuous; then, we have

$$\begin{aligned}
\int_0^{+\infty} |\gamma(t)|dt &= -\int_0^{+\infty} \gamma(t)dt \\
&\leq \frac{1}{2}\|x(0)-u\|^2 + \int_0^{+\infty} \beta(t)\left[\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty,
\end{aligned}$$

which means $\gamma \in L^1(0, +\infty)$. Since

$$\mathcal{F}_f\left(u, \frac{p}{\beta(t)}\right) - \sigma_{S_f}\left(\frac{p}{\beta(t)}\right) \geq 0,$$

it follows from (30) and assumption (26) that $\dot{h}_u \in L^1(0, +\infty)$. We conclude also that $\lim_{t\to+\infty} h_u(t)$ exists in $\mathbb{R}$.

Step 2. We define, for $t, s > 0$, $\theta(t,s) = \frac{1}{2}\|x(t+s) - x(t)\|^2$. Consequently, $\frac{\partial\theta}{\partial t}(t,s) = \langle \dot{x}(t+s) - \dot{x}(t), x(t+s) - x(t)\rangle$. Thanks to monotonicity of f and g,

$$\begin{aligned}\frac{\partial\theta}{\partial t}(t,s) &\leq \beta(t+s)f(x(t+s),x(t)) + \beta(t)f(x(t),x(t+s))\\ &\leq (\beta(t) - \beta(t+s))f(x(t),x(t+s)).\end{aligned}$$

Integrating on $(0,t)$ leads to

$$\theta(t,s) \leq \theta(0,s) + \int_0^t (\beta(\tau) - \beta(\tau+s))f(x(\tau),x(\tau+s))d\tau. \tag{31}$$

For each $\tau \in (0,t)$ and $s \neq 0$ sufficiently small, we get for some $\xi(\tau) \in A^f(x(\tau))$,

$$f(x(\tau),x(\tau+s)) \geq \langle \xi(\tau), x(\tau+s) - x(\tau)\rangle.$$

Consider values of s approaching 0 from both the positive and the negative side.

Then $$\lim_{s\to 0} \frac{1}{s} f(x(\tau),x(\tau+s)) = \langle \xi(\tau), \dot{x}(\tau)\rangle.$$

Multiplying (31) by $\frac{2}{s^2}$ and tending s to 0, we obtain

$$\|\dot{x}(t)\|^2 \leq \|\dot{x}(0)\|^2 - 2\int_0^t \dot{\beta}(\tau)\langle \xi(\tau), \dot{x}(\tau)\rangle d\tau \leq \|\dot{x}(0)\|^2 + 2M\int_0^t \dot{\beta}(\tau)\|\dot{x}(\tau)\|d\tau,$$

where $M = \sup_{t\geq 0}\|\xi(t)\| < +\infty$ since from the first step $\{x(t)\}$ is bounded and A^f is bounded on bounded sets. We then apply Lemma 4.10 to conclude that for some $\xi_0 > 0$

$$\|\dot{x}(t)\|^2 \leq \|\dot{x}(0)\|^2 + 2M\xi_0^{\frac{1}{2}}\left(\int_0^t (\dot{\beta}(\tau))^2 d\tau\right)^{\frac{1}{2}}, \tag{32}$$

and then $\dot{\beta}(\cdot) \in L^2(0;+\infty)$ leads to $\{\dot{x}(t)\}$ is bounded.

Step 3. Now we show that each sequential weak cluster points of $\{x(t)\}$ belongs to S_f. Consider $t_n \to +\infty$ such that $x(t_n) \rightharpoonup x$. We have for every $y \in K$ and for all n large enough

$$\beta(t_n)f(x(t_n),y) + g(x(t_n),y) + \langle \dot{x}(t_n), y - x(t_n)\rangle \geq 0.$$

By using monotonicity of f and g, we get

$$\beta(t_n)f(y,x(t_n)) + g(y,x(t_n)) \leq \langle \dot{x}(t_n), y - x(t_n)\rangle, \tag{33}$$

and then

$$f(y,x(t_n)) \leq \frac{-1}{\beta(t_n)}g(y,x(t_n)) + \frac{1}{\beta(t_n)}\langle \dot{x}(t_n), y - x(t_n)\rangle. \tag{34}$$

Since $\partial g_y(y) \neq \emptyset$, one can find $x^*(y) \in H$ such that for every $u \in K$

$$g(y,u) \geq \langle x^*(y), u-y\rangle \geq -\|x^*(y)\| \cdot \|y-u\|.$$

Thus there exists $\gamma(y) := \|x^*(y)\| > 0$ such that

$$-g(y, x(t_n)) \leq \gamma(y).\|y - x(t_n)\|, \quad \forall n \in \mathbb{N}.$$

Returning to (34), we can write

$$f(y, x(t_n)) \leq \frac{\gamma(y)}{\beta(t_n)}\|y - x(t_n)\| + \frac{1}{\beta(t_n)}\|\dot{x}(t_n)\|.\|y - x(t_n)\|. \tag{35}$$

Using the fact that $\{\dot{x}(t_n)\}$ and $\{x(t_n)\}$ are bounded and $\beta(t_n) \to +\infty$, we deduce that $f(y,x) \leq 0$ for all $y \in K$, and Lemma 2.5 leads to $x \in S_f$.

Step 4. Now we appeal to Lemma 4.1 applied to $B = S$ to conclude that the trajectory $\{x(t)\}$ converges weakly to some $\bar{x} \in S$. Since item (i) of Lemma 4.1 results from Step 1, we only need to show that each sequential weak cluster point x belongs to S. Consider $t_n \to +\infty$ such that $x(t_n) \rightharpoonup x$, then our previous step ensures $x \in S_f$.

Let us set a point $z \in S$. Since $\dot{h}_z \in L^1(0,+\infty)$, $\dot{x}(\cdot)$ is bounded a.e. and $x(\cdot)$ is absolutely continuous, then following lines of the proof of [20, Theorem 1], we construct another subsequence $\{s_{n_k}\}$ such that both $|s_{n_k} - t_{n_k}|$, $\|x(s_{n_k}) - x(t_{n_k})\|$ and $\dot{h}_z(s_{n_k}) = \langle \dot{x}(s_{n_k}), x(s_{n_k}) - z\rangle$ converge to 0.

Thus $\{x(s_{n_k})\}$ converges weakly to $x \in S_f$, and taking into account $z \in S$, $x(\cdot)$ is a solution of (3), g is monotone and $\dot{h}_z(s_{n_k}) \to 0$, we conclude that

$$\begin{aligned} 0 = \lim_{k\to+\infty} \dot{h}_z(s_{n_k}) &\leq \liminf_{k\to+\infty} g(x(s_{n_k}), z) \leq \limsup_{k\to+\infty} g(x(s_{n_k}), z) \\ &\leq -\liminf_{k\to+\infty} g(z, x(s_{n_k})) \leq -g(z,x) \leq 0. \end{aligned}$$

Observe also that g is supposed demipositive on S_f, then $x \in S = S_g(S_f)$. We conclude from Lemma 4.1, the whole trajectory $\{x(t)\}$ converges weakly to some $\bar{x} \in S$, which completes the proof. □

4.4. Strong convergence

To ensure strong convergence of the solution $x(\cdot)$ of (3), we need to strengthen the monotonicity condition on the objective bifunction g.

Definition 4.12. The bifunction $g\colon K \times K \to \mathbb{R}$ is said to be *γ-strongly monotone* if there exists a continuous function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(s) > 0$ whenever $s > 0$, and $\gamma(s) \to 0$ implies $s \to 0$, such that for all $u, v \in K$,

$$g(u,v) + g(v,u) \leq -\gamma\left(\|u - v\|\right). \tag{36}$$

The inequality (36) is exactly φ-strong monotonicity with $\varphi(u,v) = \gamma\left(\|u-v\|\right)$ in the sense of [2, Definition 3.1].

Remark 4.13. Let us first notice that every γ-strongly monotone bifunction is also strictly monotone, i.e., $g(u,v)+g(v,u)<0$ for every $u,v\in K$ with $u\neq v$. Also, every function $\gamma\colon \mathbb{R}_+\to\mathbb{R}_+$ which is increasing and continuous at the origin with $\gamma(0)=0$ must satisfy all conditions for γ in Definition 4.12.

Proposition 4.14. *Suppose f and g satisfy conditions* (H_1)–(H_4)*, and in addition $S_f\neq\emptyset$ and g is γ-strongly monotone. Then* (2) *admits a unique solution $\bar{x}$.*

Proof. The uniqueness results from the strict monotonicity of g. For existence it is enough to use [23, Theorem 4.3]. □

Theorem 4.15. *Suppose f and g satisfy conditions* (H_1)–(H_4)*, hypothesis* (26) *is fulfilled by f, $S_f\neq\emptyset$ and in addition g is γ-strongly monotone. Then $x(t)$, a solution of* (3)*, strongly converges to the unique solution $\bar{x}$ of* (2).

Proof. Because of the γ-strong monotonicity of g, the monotone operator A^g is γ-strongly monotone, and the operator $A^g+N_{S_f}$ has the same property. As a consequence, there exists a unique solution $\bar{x}$ to the inclusion $A^g(\bar{x})+N_{S_f}(\bar{x})\ni 0$, and then there is some $p\in N_{S_f}(\bar{x})$ such that $-p\in A^g(\bar{x})=\partial g_{\bar{x}}(\bar{x})$, i.e.,

$$g(\bar{x},y)+\langle -p,\bar{x}-y\rangle\geq 0,\quad \forall y\in K. \tag{37}$$

By taking $y=x(t)$ in (37), we obtain

$$g(\bar{x},x(t))+\langle -p,\bar{x}-x(t)\rangle\geq 0. \tag{38}$$

If we set $y=\bar{x}$ in (3), we also have

$$\beta(t)f(x(t),\bar{x})+g(x(t),\bar{x})+\langle \dot{x}(t),\bar{x}-x(t)\rangle\geq 0. \tag{39}$$

The first statement of Theorem 4.11 ensures the convergence of $\|x(t)-\bar{x}\|$ as $t\to+\infty$. Summing (38) and (39) and using the γ-strong monotonicity of g, there exists $\gamma\colon\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\beta(t)f(x(t),\bar{x})-\gamma(\|\bar{x}-x(t)\|)+\langle \dot{x}(t),\bar{x}-x(t)\rangle+\langle -p,\bar{x}-x(t)\rangle\geq 0,$$

and then, for $h_{\bar{x}}(t)=\dfrac{1}{2}\|x(t)-\bar{x}\|^2$,

$$\dot{h}_{\bar{x}}(t)+\gamma(\|\bar{x}-x(t)\|)\leq \beta(t)f(x(t),\bar{x})-\langle p,\bar{x}-x(t)\rangle.$$

By a device similar to the first step in the proof of Theorem 4.11, we obtain

$$\dot{h}_{\bar{x}}(t)+\gamma(\|\bar{x}-x(t)\|)\leq \beta(t)\left[\mathcal{F}_f\left(\bar{x},\frac{p}{\beta(t)}\right)-\sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right].$$

After integrating from 0 to $+\infty$ the above inequality, and using assumption (26), one obtains

$$\int_0^{+\infty}\gamma(\|\bar{x}-x(t)\|)\,dt\leq h_{\bar{x}}(0)+\int_0^{+\infty}\beta(t)\left[\mathcal{F}_f\left(\bar{x},\frac{p}{\beta(t)}\right)-\sigma_{S_f}\left(\frac{p}{\beta(t)}\right)\right]dt<+\infty,$$

and since γ is continuous, we conclude $\liminf_{t\to+\infty} \gamma(\|\bar{x} - x(t)\|) = 0$. In view of the convergence of the hull family $\|x(t) - \bar{x}\|$ as $t \to +\infty$ and condition on γ, we deduce that $\lim_{t\to+\infty} \|\bar{x} - x(t)\| = 0$, hence $\{x(t)\}$ converges strongly to the unique solution $\bar{x}$, which concludes the proof. □

5. Applications

5.1. Optimization in the lower-level problem.

We first apply our asymptotic behavior theorems to a Ky Fan minimax inequality over the solution set of a convex minimization problem: $f(x, y) := \psi(y) - \psi(x)$, where $\psi\colon K \to \mathbb{R}$ is a convex lower semicontinuous function, then f satisfies (H_1)–(H_4), $S_f = \operatorname{argmin}_K\psi$, the minimum set of ψ on K, and (2) becomes:

$$\text{Find } \bar{x} \in \operatorname{argmin}_K\psi \;\text{ such that }\; g(\bar{x}, y) \geq 0, \;\; \forall y \in \operatorname{argmin}_K\psi. \tag{40}$$

Without loss of generality we assume $\min_K \psi = 0$. Set $\overline{\psi}(x) = \psi(x)$ if $x \in K$, and $\overline{\psi}(x) = +\infty$ if $x \notin K$, then $\operatorname{argmin}(\overline{\psi}) = \operatorname{argmin}(\psi)$, and $\overline{\psi}(x) - \delta_{\operatorname{argmin}_K\psi}(x) \leq 0$ for all $x \in H$.

Using the reverse inequality for their Fenchel conjugates, we deduce immediately $\overline{\psi}^*(p) - \sigma_{\operatorname{argmin}_K\psi}(p) \geq 0$ for all $p \in H$, and in view of Lemma 3.8, the condition (26) becomes, for all $u \in \operatorname{argmin}_K\psi$ and for all $p \in N_{\operatorname{argmin}_K\psi}(u)$:

$$(\mathcal{C}_4) \qquad \int_0^{+\infty} \beta(t) \left[\overline{\psi}^*\left(\frac{p}{\beta(t)}\right) - \sigma_{\operatorname{argmin}_K\psi}\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty. \tag{41}$$

As a consequence of Theorems 4.6 and 4.15, we deduce the weak ergodic convergence and strong convergence results of Attouch and Czarnecki [7]. Consider the following Multiscale Asymptotic Monotone Inclusion:

$$\text{(MAMI)} \qquad \dot{x}(t) + Ax(t) + \beta(t)\partial\psi(x(t)) \ni 0, \tag{42}$$

where $\psi\colon H \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function such that $K = \operatorname{dom}(\psi)$ is closed and A is a single valued monotone operator such that $K \subset \operatorname{dom}(A)$, $A + N_{\operatorname{argmin}_K\psi}$ is a maximal monotone operator (see [48, 49]) and $S = (A + N_{\operatorname{argmin}_K\psi})^{-1}(0) \neq \emptyset$. We have (42) is equivalent to

$$\beta(t)\left[\psi(y) - \psi(x(t))\right] + \langle Ax(t), y - x(t)\rangle + \langle \dot{x}(t), y - x(t)\rangle \geq 0, \;\; \forall y \in K.$$

This is exactly (3) for $f(x, y) = \psi(y) - \psi(x)$ and $g(x, y) = \langle Ax, y - x\rangle$.

Corollary 5.1. [7, Theorem 2.1 and Theorem 2.2]

(i) *Under the conditions above and assumption* (41), *if* $x(.)$ *is a solution of* (42), *then* $w - \lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\, ds$ *exists and belongs to* S.

(ii) *If moreover* A *is strongly monotone, we confirm strong convergence of* $x(.)$ *to the unique solution in* S.

By taking $\beta = 0$, we obtain the Baillon-Brézis result, see [9], on the ergodic convergence of generated semi-groups of contractions by maximal monotone operators in Hilbert spaces.

Corollary 5.2. *In addition to condition* (41)*, where u is only taken on S, assume that* (a) *β is absolutely continuous,* (b) $\lim_{t\to+\infty}\beta(t) = +\infty$*,* (c) $\dot\beta \in L^2(0;+\infty)$*,* (d) *the operator $\nabla\psi$ is bounded on bounded sets and* (e) *the operator A is demipositive on* $\operatorname{argmin}_K\psi$*, i.e., $z \in S \neq \emptyset$, $\{y_n\} \subset K$ converging weakly to $y \in$* $\operatorname{argmin}_K\psi$ *and $\langle Ay_n, z - y_n\rangle \to 0$ imply $y \in S$.*

Then the solution $x(.)$ of (42) *weakly converges to a solution $\bar x \in S$.*

Corollary 5.2 confirms the importance of Theorem 4.11, since the weak convergence of $x(\cdot)$ to a solution of (2) is only proved for hierarchical minimization problems, see [7] for more details.

Consider, for $C \subset K$ a nonempty closed convex set, the particular case

$$\psi(x) = \frac{1}{2}d(x,C)^2 = \frac{1}{2}\inf_{y\in C}\|y-x\|^2,\ \forall x \in K,\quad \text{then } \operatorname{argmin}(\psi) = C.$$

We conclude that for every $p \in H$, $\overline{\psi}^*(p) - \sigma_C(p) = \frac{1}{2}\|p\|^2$, and then condition (41) is equivalent to $\int_0^{+\infty}\frac{dt}{\beta(t)} < +\infty$; which is satisfied for example with $\beta(t) = (1+t)^p$ for every $p > 1$.

When ψ is convex lower semicontinuous and only $\psi(x) \geq \frac{1}{2}d(x,C)^2$, $\forall x \in K$, we have $\overline{\psi}^*(p) - \sigma_C(p) \leq \frac{1}{2}\|p\|^2$ for every $p \in \mathcal{R}(N_C)$, and then $\int_0^{+\infty}\frac{dt}{\beta(t)} < +\infty$ ensures (41).

Example 5.3. Consider the case where C is the closed unit ball of the space $H = \mathbb{R}^n$, endowed with the euclidean norm $\|\cdot\|$, and $\psi(x) = 0$ if $x \in C$ and $\psi(x) = \frac{1}{2}(\|x\|-1)^2$ if $x \notin C$. One has $\psi(x) = \frac{1}{2}d(x,C)^2$, for every $x \in \mathbb{R}^n$, which ensures equivalence between $\int_0^{+\infty}\frac{dt}{\beta(t)} < +\infty$ and (41). □

Indeed, the relation $\psi(x) = \frac{1}{2}d(x,C)^2$ is trivial whenever $x \in C$; otherwise $\frac{1}{2}d(x,C)^2$ can be seen as constrained minimization convex problem:

$$\min_{v(y)\leq 0} u(y) \text{ where } u(y) = \frac{1}{2}\sum_{i=1}^n (y_i - x_i)^2 \text{ and } v(y) = \sum_{i=1}^n y_i^2 - 1. \tag{43}$$

Considering the first-order optimality conditions for constrained convex optimization, we have $\overline{y}$ realizes the minimum for (43) iff there exists $\mu \geq 0$ such that $\nabla u(\overline{y}) + \mu\nabla v(\overline{y}) = 0$ and $\mu v(\overline{y}) = 0$. One concludes that $\mu = \|x\| - 1 > 0$ and $\overline{y} = \frac{1}{\|x\|}x$, and therefore

$$\frac{1}{2}d(x,C)^2 = u(\overline{y}) = \frac{1}{2}\|\overline{y} - x\|^2 = \frac{1}{2}(\|x\| - 1)^2 = \psi(x).$$

This proves $\psi = \frac{1}{2}d(\cdot, C)^2$ and the convexity of ψ. Also, argmin$(\psi) = C$ and the function ψ is continuously differentiable on $\mathbb{R}^n$ with

$$\nabla\psi(x) = \begin{cases} 0 & \text{if} \quad x \in C \\ \left(1 - \frac{1}{\|x\|}\right) x & \text{if} \quad x \notin C. \end{cases}$$

5.2. Optimization in the upper-level problem.

Here we apply our asymptotic behavior theorems to a convex minimization problem over the solution set of a Ky Fan minimax inequality:

$$\text{Find } \bar{x} \in S_f \text{ such that } \varphi(\bar{x}) \le \varphi(y), \quad \forall y \in S_f, \tag{44}$$

where $\varphi\colon K \to \mathbb{R}$ is a convex lower semicontinuous function. Setting $g(x, y) = \varphi(y) - \varphi(x)$ for $\varphi\colon H \to \mathbb{R} \cup \{+\infty\}$ a convex lower semicontinuous function such that $K = \mathrm{dom}(\varphi)$ is closed, then g is 3-monotone and thus, see Prop. 4.9 (ii), g is demipositive whenever the solution set S is nonempty. As a consequence of Theorem 4.11, we obtain weak convergence of the solution $x(.)$ of (3) to a solution of (44) under conditions $\beta(t) \to +\infty$ and (26) on S. We therefore omit the assumption

$$0 \le \dot{\beta}(t) \le c\beta(t) \text{ on } [t_0, +\infty) \text{ for some } t_0 > 0 \text{ and } c > 0,$$

used in [7, Theorem 3.1].

To ensure the strong convergence of the path $x(t)$ to a solution of (44), we must look for another function g, because this choice $g(x, y) = \varphi(y) - \varphi(x)$ may never be strongly monotone.

Set $g(x, y) = \varphi'(x; y - x) = \liminf_{t\to 0^+} \frac{1}{t}(\varphi(x + t(y - x)) - \varphi(x))$, the directional derivative. Let us note that when the convex function φ is continuous at x then

$$g(x, y) = \varphi'(x; y - x) = \sup_{\xi\in\partial\varphi(x)} \langle \xi, y - x\rangle$$

where $\partial\varphi$ is the convex subdifferential of φ (we identify φ with $\overline{\varphi}(x) = \varphi(x)$ if $x \in K$, and $\overline{\varphi}(x) = +\infty$ otherwise).

Lemma 5.4. *Suppose φ is a strongly convex-like real-valued function on K, i.e., for some $\alpha > 0$, for any $x, y \in K$, and any $t \in [0, 1]$,*

$$\varphi(ty + (1 - t)x) \le t\varphi(y) + (1 - t)\varphi(x) - \alpha \min\{t, 1 - t\}\|x - y\|. \tag{45}$$

Then g is γ-strongly monotone (see Definition 4.12) with $\gamma(t) = 2\alpha t$.

Proof. Take $x, y \in K$. Since for all $t \in [0, 1]$, $t(1-t) \le \min\{t, 1-t\}$, we conclude from (45)

$$\begin{aligned} g(x, y) = \varphi'(x; y - x) &= \liminf_{t\to 0^+} \frac{1}{t}(\varphi(x + t(y - x)) - \varphi(x)) \\ &\le \liminf_{t\to 0^+} (\varphi(y) - \varphi(x) - \alpha(1 - t)\|x - y\|) = \varphi(y) - \varphi(x) - \alpha\|x - y\|. \end{aligned}$$

Similarly, we have $g(y,x) = \varphi'(y; x-y) \leq \varphi(x) - \varphi(y) - \alpha\|x-y\|$, and therefore by summing these two inequalities, we deduce that the bifunction g is γ-strongly monotone, where $\gamma(t) = 2\alpha t$. □

Using this lemma, we conclude that Proposition 4.14 and Theorem 4.15 are valid for the function φ.

Corollary 5.5. *Suppose $S_f \neq \emptyset$, condition (26) for f, and φ is strongly convex-like. Then, the unique solution $\bar{x}$ of (44) exists, i.e., $S = \{\bar{x}\}$, and every solution $x(.)$ of (3) strongly converges to $\bar{x}$.*

5.3. Ky Fan minimax inequality under a saddle point constraint

Let H_1, H_2 be two real Hilbert spaces, $U \subset H_1$ and $V \subset H_2$ be nonempty closed convex sets, and let $L\colon U \times V \to \mathbb{R}$ be closed and convex-concave, i.e., for each $(u,v) \in U \times V$ the real functions $L(.,v)$ and $-L(u,.)$ are convex and lower semicontinuous.

Consider the saddle-point problem: Find $(\bar{u},\bar{v}) \in U \times V$ such that

$$\text{(SP)} \qquad L(\bar{u},v) \leq L(\bar{u},\bar{v}) \leq L(u,\bar{v}) \text{ for every } (u,v) \in U \times V, \tag{46}$$

which is equivalent, see [29], to

$$\max_{v\in V}\inf_{u\in U} L(u,v) = \min_{u\in U}\sup_{v\in V} L(u,v) = L(\bar{u},\bar{v}).$$

By defining $H = H_1 \times H_2$, $K = U \times V$ and $f\colon K \times K \to \mathbb{R}$ by

$$f((u,v),(u',v')) := L(u',v) - L(u,v'), \text{ for each } (u,v),(u',v') \in K, \tag{47}$$

we see that problems (46) and (47) are equivalent; we denote the solution set of (46) by S_L. By Definition 3.1, we have for all $(u,v),(u',v') \in K$

$$\begin{aligned}
\mathcal{F}_f((u,v),(u',v')) &= \sup_{(x,y)\in K} \{\langle u',x\rangle + \langle v',y\rangle + f((x,y),(u,v))\} \\
&= \sup_{(x,y)\in K} \{\langle v',y\rangle + L(u,y) - L(x,v) + \langle u',x\rangle\} \\
&= \sup_{y\in V}\{\langle v',y\rangle - (-L((u,y))\} + \sup_{x\in U}\{\langle u',x\rangle - L(x,v)\} \\
&= (-L(u,.))^*(v') + (L(.,v))^*(u'),
\end{aligned}$$

and therefore the condition (26) is satisfied when $\forall (u,v) \in S_f, \forall (p,q) \in N_{S_f}(u,v)$,

$$\begin{aligned}
(\mathcal{C}_5) \qquad \int_0^{+\infty} \beta(t)\Big[(-L(u,.))^*\Big(\frac{q}{\beta(t)}\Big) + (L(.,v))^*\Big(\frac{p}{\beta(t)}\Big) - \\
- \sigma_{S_f}\Big(\frac{p}{\beta(t)}, \frac{q}{\beta(t)}\Big)\Big] dt < +\infty.
\end{aligned} \tag{48}$$

Of course, we consider two single valued monotone operators A and B such that $K \subset \mathrm{dom}(A)\times\mathrm{dom}(B)$, $A\times B+N_{S_L}$ is a maximal monotone operator (see [48, 49]) and S_{VL} the solution set of $0 \in A\bar{x} \times B\bar{y} + N_{S_L}(\bar{x}, \bar{y})$ is nonempty. By $A \times B$, we mean the operator defined for $(u, v) \in H = H_1 \times H_2$ by $A \times B(u, v) = Au \times Bv$. When the monotone operator $A \times B + N_{S_L}$ is again maximal, then $(\bar{x}, \bar{y})$ is in S_{VL}, if and only if,

$$(\bar{x}, \bar{y}) \in S_L \text{ and } \langle A\bar{x}, u - \bar{x}\rangle + \langle B\bar{y}, v - \bar{y}\rangle \geq 0,\ \forall (u, v) \in S_L.$$

By setting $g((u, v), (u', v')) := \langle Au, u' - u\rangle + \langle Bv, v' - v\rangle$, for all $(u, v), (u', v') \in K$, we see that (3) becomes

$$\text{(EDSP)} \qquad \begin{cases} \text{for all } (u, v) \in U \times V, \text{ for a.e. } t > 0 \\ \langle \dot{x}(t), u - x(t)\rangle + \langle \dot{y}(t), v - y(t)\rangle + \langle Ax(t), u - x(t)\rangle + \\ + \langle By(t), v - y(t)\rangle + \beta(t)(L(u, y(t)) - L(x(t), v)) \geq 0. \end{cases} \tag{49}$$

In this case Theorems 4.6, 4.11 and 4.15 can be summarized as follows

Corollary 5.6. *Suppose $(x(t), y(t))$ is a solution of (49) and (48) is satisfied, then for $t \to +\infty$ the weak limit of $\frac{1}{t}\left(\int_0^t u(s)\,ds, \int_0^t v(s)\,ds\right)$ exists and realizes a solution in S_{VL}. The weak convergence is ensured when moreover β is absolutely continuous, $\lim_{t\to+\infty} \beta(t) = +\infty$, $\dot{\beta} \in L^2(0; +\infty)$ and $A \times B$ is demipositive on S_L. Also $(x(t), y(t))$ strongly converges to the unique element of S_{VL} if in addition $A \times B$ is strongly monotone on K.*

We note that this result extends Theorem 2.1 and Theorem 2.2 in [7], from monotone variational inequalities over the set of solutions for convex minimization problem, to variational problems constrained by the saddle points of a convex-convave function.

Here is an example where the condition (48) is verified for a non bilinear form.

Example 5.7. Take $K = [0, 1] \times [0, 1]$ and L the closed convex-concave function defined on K by $L(u, v) = u^2(1 + v)$. Then the set of saddle points of L, which is also the solution set S_f, is $S_f = \{0\} \times [0, 1]$. We also have

$$\begin{aligned} (p, q) \in N_{(\{0\}\times[0,1])}(0, v) &\Leftrightarrow (p, q)(s, t - v) \leq 0,\ \forall\, (s, t) \in (\{0\} \times [0, 1]) \\ &\Leftrightarrow q(t - v) \leq 0,\ \ \forall t \in [0, 1]; \end{aligned}$$

and then $\qquad N_{S_f}(0, 0) = \mathbb{R} \times \mathbb{R}_-, \quad N_{S_f}(0, 1) = \mathbb{R} \times \mathbb{R}_+$

and $\qquad N_{S_f}(0, v) = \mathbb{R} \times \{0\}$ for every $v \in]0, 1[$.

To ensure (48), we verify

$$\sigma_{S_f}\left(\frac{p}{\beta(t)}, \frac{q}{\beta(t)}\right) = \sup_{v\in[0,1]}\left\{v\frac{q}{\beta(t)}\right\} = \begin{cases} \dfrac{q}{\beta(t)} & \text{if } q > 0 \\ 0 & \text{if } q \leq 0, \end{cases}$$

$$(-L(0,.))^*\left(\tfrac{q}{\beta(t)}\right) = \sup_{0\le s\le 1}\left\{\frac{q}{\beta(t)}s\right\} = \begin{cases} \dfrac{q}{\beta(t)} & \text{if } q>0 \\ 0 & \text{if } q\le 0, \end{cases}$$

and $$(L(.,v))^*\left(\tfrac{p}{\beta(t)}\right) = \sup_{0\le s\le 1}\left\{\frac{p}{\beta(t)}s-(1+v)s^2\right\} = \frac{p^2}{4(1+v)\beta(t)^2}.$$ Thus

$$\int_0^{+\infty}\beta(t)\left[(-L(u,.))^*\left(\frac{q}{\beta(t)}\right)+(L(.,v))^*\left(\frac{p}{\beta(t)}\right)-\sigma_{S_f}\left(\frac{p}{\beta(t)},\frac{q}{\beta(t)}\right)\right]dt$$
$$=\frac{p^2}{4(1+v)}\int_0^{+\infty}\frac{dt}{\beta(t)};$$

and then (48) is satisfied, whenever $\displaystyle\int_0^{+\infty}\frac{dt}{\beta(t)}<+\infty.$ □

5.4. Numerical examples

All our examples were implemented in Scilab version 5.5.2 as an open source software.

5.4.1. Example 1. (Unique global solution)

Let us first consider a hierarchical convex minimization problem:

$$(\mathcal{M}) \qquad \text{minimize } \varphi(x) \text{ under constraint } x\in \operatorname{argmin}(\psi), \tag{50}$$

Set $K=\mathbb{R}^2$, $\varphi(x)=\frac{1}{2}x^TBx$ where $B=\begin{bmatrix}4&1\\1&2\end{bmatrix}$ and $\psi(x)=\frac{1}{4}(x_1+x_2-4)^2$.

The minimum set of ψ on $\mathbb{R}^2$ is $C=\nabla\psi^{-1}(0,0)=\{(x_1,x_2)\in\mathbb{R}^2 : x_2=4-x_1\}$ since ψ is convex. To check the condition (26), we calculate $\frac{1}{2}d(x,C)^2$ where $d(x,C)=\inf_{y\in C}\|y-x\|_2$ and $\|(x_1,x_2)\|_2 := \sqrt{x_1^2+x_2^2}$. For $x=(x_1,x_2)\in\mathbb{R}^2$, we have

$$d(x,C)^2=\inf_{y_1\in\mathbb{R}}\left((y_1-x_1)^2+(y_1+x_2-4)^2\right)=\inf_{y_1\in\mathbb{R}}\alpha(y_1).$$

Since $\alpha(t)=(t-x_1)^2+(t+x_2-4)^2$ is strongly convex and

$$\alpha'(\bar y_1)=2(2\bar y_1+x_2-4-x_1)=0 \;\Leftrightarrow\; \bar y_1=\frac{1}{2}(x_1-x_2+4),$$

we get $d(x,C)^2=\alpha(\bar y_1)=(\bar y_1-x_1)^2+(\bar y_1+x_2-4)^2=2\psi(x)$, which yields $\psi(x)=\frac{1}{2}d(x,C)^2$; and then the condition (26) is equivalent to $\int_0^{+\infty}\dfrac{dt}{\beta(t)}<+\infty$.

In the particular case $\beta(t)=(1+t)^a\ln^b(1+t)$ where ($a>1$ and $b\in\mathbb{R}$) or ($a=1$ and $b>1$), we have (26) is satisfied and $\lim_{t\to+\infty}\beta(t)=+\infty$.

The associate (3) to problem $(\mathcal{M})$ can be expressed as: $x(0) = (x_1^0, x_2^0)$ and for a.e. $t \geq 0$

$$\begin{cases} \dot{x}_1(t) &= -(4x_1(t) + x_2(t)) - \frac{\beta(t)}{2}(x_1(t) + x_2(t) - 4) \\ \dot{x}_2(t) &= -(x_1(t) + 2x_2(t)) - \frac{\beta(t)}{2}(x_1(t) + x_2(t) - 4). \end{cases}$$

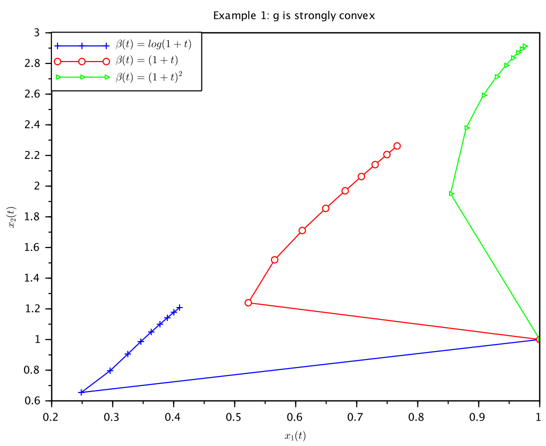
Note that the objective function φ is strongly convex, since its Hessian is positive, because the corresponding two eigenvalues $3+\sqrt{2}$ and $3-\sqrt{2}$ are positive. So it is more convenient to take $g(x,y) = \langle \nabla\varphi(x), y - x\rangle$ in order to get the strong monotonicity of g. In this case, when $\int_0^{+\infty}(1/\beta(t))\,dt < +\infty$, we use Theorem 4.15 to conclude the convergence of $x(t)$ to the unique solution $\bar{x} = (1,3)$ of $(\mathcal{M})$.

t	$x_1(t)$	$x_2(t)$	$\|x(t)-\bar{x}\|_2$
1	1	1	2.
2	0.2485553	0.6547854	2.4626613
3	0.2957831	0.7965406	2.3132563
4	0.3248042	0.9054741	2.2006654
5	0.3460741	0.9865067	2.1170202
6	0.3631922	1.0492516	2.0520584
7	0.3774884	1.0998038	1.9995665
8	0.3896911	1.1418189	1.9558409
9	0.4002801	1.1775742	1.9185671
10	0.4095936	1.2085649	1.8862183

$\beta(t) = \ln(1+t)$

t	$x_1(t)$	$x_2(t)$	$\|x(t)-\bar{x}\|_2$
1	1	1	2.
2	0.5222591	1.2396184	1.8240558
3	0.5654713	1.5198361	1.5426278
4	0.6108087	1.7107616	1.3467018
5	0.6492347	1.8551104	1.1974174
6	0.6810814	1.9695277	1.0786947
7	0.7076983	2.0628006	0.9817245
8	0.7302337	2.1404019	0.9009344
9	0.7495494	2.2060201	0.8325440
10	0.7662861	2.2622559	0.7738789

$\beta(t) = 1+t$



t	$x_1(t)$	$x_2(t)$	$\|x(t)-\bar{x}\|_2$
1	1	1	3.1622777
2	0.8539118	1.9507792	2.
3	0.8794178	2.379629	1.0593422
4	0.9075807	2.5925882	0.4177628
5	0.9290292	2.712801	0.2958379
6	0.9444542	2.7870605	0.2200650
7	0.9555965	2.8360313	0.1698746
8	0.9638034	2.8699681	0.1349759
9	0.9699831	2.8944221	0.1097620
10	0.9747348	2.9126086	0.0909703

$\beta(t) = (1+t)^2$

$\bar{x} = (1,3)$ is the unique solution of $(\mathcal{M})$
$\varphi(x) = 2x_1^2 + x_1x_2 + x_2^2$
$\psi(x) = \frac{1}{4}(x_1 + x_2 - 4)^2$
For a two levels hierarchical minimization problem the convergence is faster when asymptotic exterior penalization coefficient $\beta(t)$ is stronger at infinity.

5.4.2. Example 2. (Non variational Ky Fan minimax inequality)

Set $K = \mathbb{R}^2$ and consider the bifunctions: for each $x = (x_1, x_2), y = (y_1, y_2) \in K$, $f(x,y) = \psi(y) - \psi(x)$, for $\psi(x) = \frac{1}{2}d(x,C)^2$ and C is the closed unit ball (see Example 5.3), and $g(x,y) = -\left(x_1^2 - x_1y_1 - x_2y_2 + x_2^2\right) - x_1^4 + x_1y_2 - x_2y_1 + y_1^4$. Then g is strongly monotone, since $g(x,y) + g(y,x) = -\|y-x\|^2$, $\forall x,y \in K$. Moreover, for every $x \in K$, $g(x,x) = 0$ and $g(x,\cdot)$ is convex and continuously differentiable, since for every $x, y \in K$,

$$\nabla_y g(x,y) = \begin{pmatrix} x_1 - x_2 + 4y_1^3 \\ x_1 + x_2 \end{pmatrix} \text{ and } \nabla_y^2 g(x,y) = \begin{pmatrix} 12y_1^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, we are going to attain the unique solution $\overline{x}$ of the Ky Fan minimax inequality:

$$\text{Find } x \in C \text{ such that } g(x,y) \geq 0 \text{ for every } y \in C,$$

by studing the asymptotic behavior of the trajectories of the system:

$$\dot{x}(t) + \nabla_y g(x(t), x(t)) + \beta(t)\nabla\psi(x(t)) = 0,$$

which is equivalent to

$$\begin{cases} \dot{x}_1(t) & = -\beta(t)\max\left(0, 1 - \frac{1}{\|x(t)\|}\right) x_1(t) - (x_1(t) + 4x_1^3(t) - x_2(t)) \\ \dot{x}_2(t) & = -\beta(t)\max\left(0, 1 - \frac{1}{\|x(t)\|}\right) x_2(t) - (x_1(t) + x_2(t)). \end{cases}$$

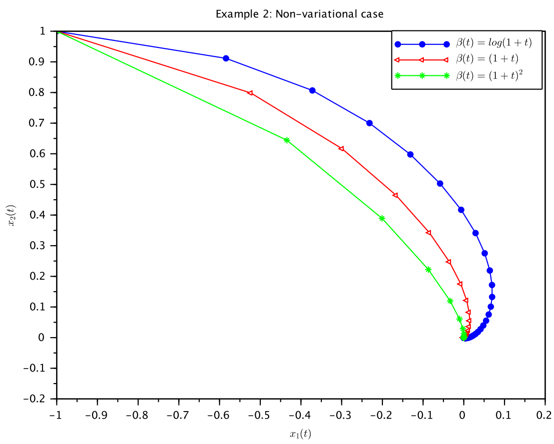
Since all conditions are satisfied, Theorem 4.11 ensures, for every starting point x_0, convergence of $x(t)$ (when $t \to +\infty$) to the solution of the problem $(BFMI)$.

t	$x_1(t)$	$x_2(t)$	$\|x(t)-\overline{x}\|_2$
1	-1	1	1.4142136
2	0.0693421	0.1717814	0.1852490
3	0.0190993	0.0037046	0.0194553
4	0.0011004	- 0.0011523	0.0015933
5	- 0.0000251	- 0.0001037	0.0001067
6	- 0.0000057	- 0.0000020	0.0000060
7	- 0.0000002	0.0000002	0.0000003
8	1.390D-09	1.309D-08	1.316D-08
9	4.537D-10	2.221D-10	5.051D-10
10	- 2.812D-11	- 3.150D-11	4.222D-11

$\beta(t) = \ln(1+t)$

t	$x_1(t)$	$x_2(t)$	$\|x(t)-\overline{x}\|_2$
1	-1	1	1.4142136
2	0.0134318	0.0358715	0.0383038
3	0.0004159	0.0000898	0.0004255
4	0.0000012	0.0000012	0.0000017
5	- 5.953D-10	- 2.498D-09	2.568D-09
6	1.124D-11	2.518D-11	2.757D-11
7	5.798D-12	8.187D-12	1.003D-11
8	4.344D-12	6.127D-12	7.511D-12
9	2.890D-12	4.068D-12	4.989D-12
10	1.436D-12	2.008D-12	2.468D-12

$\beta(t) = 1+t$



t	$x_1(t)$	$x_2(t)$	$\|x(t)-\overline{x}\|_2$
1	-1	1	1.4142136
2	- 0.0000467	0.0003831	0.0003859
3	6.129D-10	8.835D-10	1.075D-09
4	- 6.853D-12	- 8.991D-12	1.130D-11
5	- 2.264D-12	- 2.969D-12	3.734D-12
6	- 1.488D-12	- 1.929D-12	2.436D-12
7	- 4.085D-12	- 5.295D-12	6.687D-12
8	- 4.272D-12	- 5.537D-12	6.993D-12
9	- 3.987D-12	- 5.168D-12	6.527D-12
10	- 3.702D-12	- 4.799D-12	6.061D-12

$\beta(t) = (1+t)^2$

$\overline{x} = (0,0)$ is the limit point of $x(t)$

$f(x,y) = \frac{1}{2}\left(d(y,B)^2 - d(x,B)^2\right)$

$g(x,y) = -\left(x_1^2 - x_1y_1 - x_2y_2 + x_2^2\right) - \left(x_1^4 - x_1y_2 + x_2y_1 - y_1^4\right)$

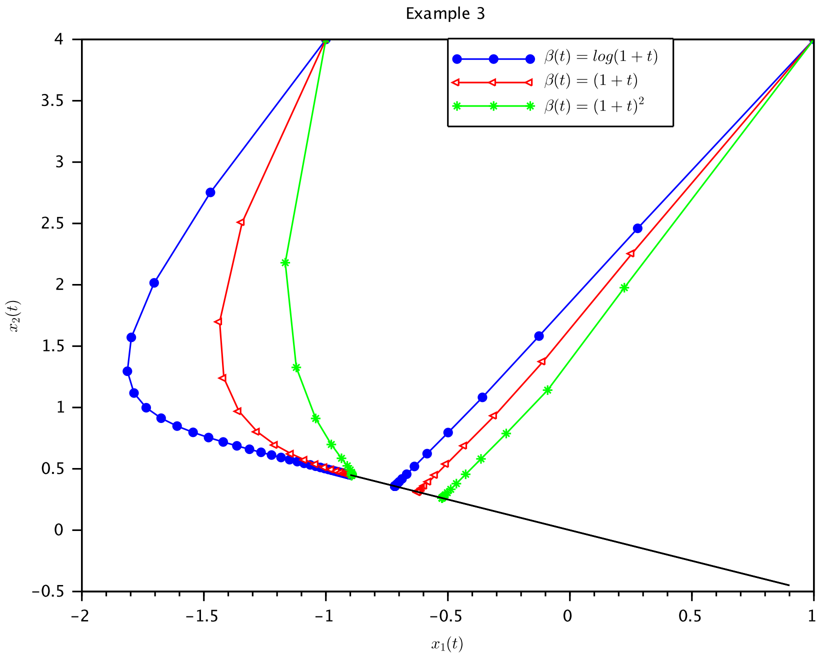
Here, for a nonvariational Ky Fan minimax inequality, convergence is faster and similar when $\beta(t) = \ln(1+t), (1+t)$ and $(1+t)^2$.

5.4.3. Example 3. (Non unique global solution)

When choosing ψ as in Example 1 and the convex function

$$\varphi(x) = (x_1 + x_2)^2 = \frac{1}{2}x^T Bx \quad \text{with} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

we then have that φ is not strongly convex, since the lines of B are collinear, and the minimum set of ψ is $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 4 - x_1\}$.



Note that φ is constant on C, and then the solution set of the hierarchical minimization problem $(\mathcal{M})$ is the whole line $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 4 - x_1\}$. In this case, we obtain the convergence of (3) with a similar quickness to that of Example 1. However, each solution limit $\bar{x}$ depends on $x(0) = (x_1^0, x_2^0)$, the starting point, and further even if the starting point $(1, 4)$ is the same for different functions β we also attain different solutions in C. Hence the limit points of $x(t)$ depends on the starting point and the choice of the penalization function $\beta(t)$.

6. Conclusion and perspective

This paper studied the asymptotic behaviour of trajectories of a general dynamical equilibrium system (4) where the associate monotone equilibrium bifunctions are nonautonomous. As t goes to $+\infty$, we only proved the weak ergodic convergence of such trajectories to an equilibrium point of an appropriate limit bifunction Γ_∞. This system in the particular model (3), where $\Gamma_t(x, y) = \beta(t)f(x, y) + g(x, y)$ is an asymptotic exterior penalization with coefficient $\beta(t)$, allowed us to prove under one of the conditions $(\mathcal{C}_2) - (\mathcal{C}_5)$ (equations (26), (27), (41), (48)), the weak and strong convergences of trajectory to one solution of (2).

We observe that results obtained in Section 4 can be applied to reach the solution of different hierarchical problems. We treated some particular cases in Section 5, where we applied the asymptotic behavior theorems to a Ky Fan minimax inequality over the solution set of a convex minimization problem and also over the solution set of a convex-concave saddle point problem. Conversely, we considered a convex minimization problem over the solution set of a Ky Fan minimax inequality. We also justified, for the case where $f(x, y) = \frac{1}{2}d(y, C)^2 - \frac{1}{2}d(x, C)^2$, that condition (41) is equivalent to $\int_0^{+\infty} \frac{dt}{\beta(t)} < +\infty$, and depends only on $\beta(t)$.

Numerical results illustrated that with a suitable choice of the parameters, the convergence conditions are satisfied and the proposed trajectories succeeded in approximating a solution of the proposed bilevel hierarchical Ky Fan minimax problem.

Finally, we note that several extensions of our main results may be analyzed. An interesting direction of future research is to see if the existence of solutions for (3), remains valid for (4). In Section 3, we initiated an analysis of the relationship between the monotonicity of a real bifunction f and its associate equilibrium Fitzpatrick transform, as well of some properties of these transforms in Propositions 3.8, 3.9 and 3.10. In Remarks 3.4 and 3.5, we also started a direction for future research works related to the relationship between the nonmonotone bifunctions and the associate operators, see the work of [2, 3, 4, 16, 31, 32]. Another interesting question is whether convergences of trajectories for the asymptotic behavior of the proposed dynamic systems don't require one of the conditions $(\mathcal{C}_2) - (\mathcal{C}_5)$?

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References

[1] M. Ait Mansour, Z. Chbani, H. Riahi: *Recession bifunction and solvability of noncoercive equilibrium problems*, Comm. Appl. Analysis 7 (2003) 369–377.

[2] M. Ait Mansour, R.-A. Elakri, M. Laghdir: *Equilibrium and quasi-equilibrium problems under ϕ-quasimonotonicity and ϕ-quasiconvexity*, Minimax Theory Appl. 2(2) (2017) 175–229.

[3] M. Ait Mansour, H. Riahi: *On the eigenvalues problem for hemivariational inequalities: Existence and stability*, J. Convex Analysis 9 (2002) 309–325.

[4] M. H. Alizadeh, N. Hadjisavvas: *On the Fitzpatrick transform of a monotone bifunction*, Optimization 62 (2013) 693–701.

[5] A. S. Antipin: *The convergence of proximal methods to fixed points of extremal mappings and estimates of their rate of convergence*, Comput. Maths. Math. Phys. 35(5) (1995) 539–551.

[6] H. Attouch, A. Cabot, M. O. Czarnecki: *Asymptotic behavior of nonautonomous monotone and subgradient evolution equations*, Trans. Amer. Math. Soc. 370 (2018) 755–790.

[7] H. Attouch, M. O. Czarnecki: *Asymptotic behavior of coupled dynamical systems with multiscale aspects*, J. Differential Equations 248(6) (2010) 1315–1344.

[8] H. Attouch, H. Riahi, M. Théra: *Somme ponctuelle d'opérateurs maximaux monotones*, Serdica Math. J. 22 (1996) 267–292.

[9] J. B. Baillon, H. Brézis: *Une remarque sur le comportement asymptotique des semi-groupes non linéaires*, Houston J. Math. 2 (1976) 5–7.

[10] C. Baiocchi, A. Capelo: *Variational and Quasivariational Inequalities. Applications to Free Boundary Problems*, John Wiley and Sons, New York (1984).

[11] H. H. Bauschke, P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, Springer, New York (2011).

[12] M. Bianchi, R. Pini: *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl. 124 (2005) 79–92.

[13] E. Blum, W. Oettli: *From optimisation and variational inequalities to Ky Fan minimax inequalities*, Math. Student 63 (1994) 123–145.

[14] R. I. Boţ, E. R. Csetnek: *Forward-Backward and Tseng's type penalty schemes for monotone inclusion problems*, Set-Valued Var. Anal. 22 (2014) 313–331.

[15] R. I. Boţ, E. R. Csetnek: *Approaching the solving of constrained variational inequalities via penalty term-based dynamical systems*, J. Math. Anal. Appl. 435(2) (2016) 1688–1700.

[16] R. I. Boţ, S. M. Grad: *Approaching the maximal monotonicity of bifunctions via representative functions*, J. Convex Analysis 19(3) (2012) 713–724.

[17] H. Brézis: *Opérateurs Maximaux Monotones dans les Espaces de Hilbert et Équations d'Évolution*, Lecture Notes 5, North-Holland, Amsterdam (1972).

[18] H. Brézis: *Quelques propriétés des opérateurs monotones et des semi-groupes non linéaires*, in: Nonlinear Operators and the Calculus of Variations", Summer School, Université Libre de Bruxelles, Brussels (1975), Lecture Notes in Mathematics 543, Springer, Berlin et al. (1976) 56–82.

[19] H. Brézis, L. Nirenberg, G. Stampacchia: *A remark on Ky Fan's minimax principle*, Boll. Unione Mat. Ital., Ser. III-VI (1972) 129–132.

[20] R. E. Bruck: *Asymptotic convergence of nonlinear contraction semigroups in Hilbert space*, J. Funct. Analysis 18 (1975) 15–26.

[21] R. Burachik, G. Kassay: *On a generalized proximal point method for solving equilibrium problems in Banach spaces*, Nonlinear Analysis 75 (2012) 6456–6464.

[22] M. Castellani, M. Giuli: *Refinements of existence results for relaxed quasimonotone equilibrium problems*, J. Global Optim. 57 (2013) 1213–1227.

[23] O. Chadli, Z. Chbani, H. Riahi: *Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities*, J. Optim. Theory Appl. 105(2) (2000) 299–323.

[24] Z. Chbani, H. Riahi: *Variational principle for monotone and maximal bifunctions*, Serdica Math. J. 29 (2003) 159–166.

[25] Z. Chbani, H. Riahi: *Weak and strong convergence of prox-penalization and splitting algorithms for bilevel equilibrium problems*, Numer. Algebra Control Optim. 3(2) (2013) 353–366.

[26] Z. Chbani, H. Riahi: *Existence and asymptotic behavior for solutions of dynamical equilibrium systems*, Evol. Equ. Control Theory 3(1) (2014) 1–14.

[27] Z. Chbani, H. Riahi: *Weak and strong convergence of proximal penalization and proximal splitting algorithms for two-levels hierarchical Ky Fan minimax inequalities*, Optimization 64 (2015) 1285–1303.

[28] S. S. Dragomir: *Some Gronwall Type Inequalities and Applications*, Nova Science Publishers, http://rgmia.vu.edu.au/ SSDragomirWeb.html (2003).

[29] I. Ekeland, R. Temam: *Convex Analysis and Variational Problems*, Dunod, Paris (1974).

[30] K. Fan: *A Minimax Inequality and Applications*, in: Inequalities III, O. Shisha (ed.), Academic Press, New York (1972) 103–113.

[31] S. Fitzpatrick: *Representing monotone operators by convex functions*, in: Workshop Miniconference on Functional Analysis and Optimization (Canberra, 1988), Proc. of the Centre for Mathematical Analysis vol. 20, Australian National University, Canberra (1988) 59–65.

[32] F. Flores-Bazàn: *Existence theorems for generalized noncoercive equilibrium problems: the quasiconvex case*, SIAM J. Optimization 11 (2000) 675–690.

[33] F. Giannessi, A. Maugeri, P. M. Pardalos: *Equilibrium problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic Publishers, Doordrecht (2002).

[34] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu: *Variational Methods in Partially Ordered Spaces*, Springer, New York et al. (2003).

[35] J. Gwinner: *On a new class of differential variational inequalities and a stability result*, Math. Program. 139 (2013) 205–221.

[36] N. Hadjisavvas, H. Khatibzadeh: *Maximal monotonicity of bifunctions*, Optimization 59 (2010) 147–160.

[37] D. S. Mitrinovic, J. E. Pecaric, A. M. Fink: *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Doordrecht (1994).

[38] U. Mosco: *Implicit variational problems and quasivariational inequalities*, in: Lecture Notes in Mathematics 543, Springer, Berlin et al. (1976) 83–156.

[39] A. Moudafi: *Proximal point methods extended to equilibrium problems*, J. Natur. Geom. 15 (1999) 91–100.

[40] A. Moudafi: *A recession notion for a class of monotone bivariate functions*, Serdica Math. J. 26 (2000) 207–220.

[41] A. Moudafi, M. Théra: *Proximal and dynamical approaches to equilibrium problems*, in: Ill-Posed Variational Problems and Regularization Techniques, Lecture Notes in Economics and Mathematical Systems 477, Springer, Berlin et al. (1999) 187–201.

[42] W. Oettli, H. Riahi: *On maximal ψ-monotonicity of sums of operators*, Comm. Appl. Nonlinear Analysis 5 (1998) 1–17.

[43] J.-S. Pang, D. E. Stewart: *Differential variational inequalities*, Math. Program. 113 (2008) 345–424.

[44] G. B. Passty: *Ergodic convergence to a zero of the sum of monotone operators in Hilbert spaces*, J. Math. Anal. Appl. 72 (1979) 383–390.

[45] J. Peypouquet, S. Sorin: *Evolution equations for maximal monotone operators: asymptotic analysis in continuous and discrete time*, J. Convex Analysis 1 (2010) 1113–1163.

[46] H. Riahi: *On the maximality of the sum of two maximal monotone operators*, Publ. Mat. 34 (1990) 269–271.

[47] H. Riahi, Z. Chbani, M. T. Loumi: *Weak and strong convergences of the generalized penalty forward-forward and forward-backward splitting algorithms for solving bilevel hierarchical pseudomonotone equilibrium problems*, Optimization 67(10) (2018) 1745–1767.

[48] R. T. Rockafellar: *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. 33 (1970) 209–216.

[49] R. T. Rockafellar: *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970) 75–88.

[50] S. Simons, C. Zalinescu: *Fenchel duality, Fitzpatrick functions and maximal monotonicity*, J. Nonlinear Convex Analysis 6 (2005) 1–22.

[51] G. X. Z. Yuan: *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York (1999).