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On an Open Question of Ricceri Concerning a Kirchhoff-Type Problem

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We prove a multiplicity result for a Kirchhoff type problem involving a critical term, giving a partial positive answer to a problem raised by Ricceri.

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1. Introduction

Nonlocal boundary value problems of the type

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f(x,u), & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega \end{cases}$$

are related to the stationary version of the Kirchhoff equation

$$\frac{\partial^2 u}{\partial t^2} - \left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f_1(t,x,u),$$

first proposed by Kirchhoff to describe the transversal oscillations of a stretched string. Here Ω is a bounded domain of $\mathbb{R}^N$, u denotes the displacement, f_1 is the external force, b is the initial tension and a is related to the intrinsic properties of the string.

Note that, this type of nonlocal equations appears in other fields like biological systems, where u describes a process depending on the average of itself, like population density (see for instance [4]).

The first attempt to find solutions for subcritical nonlinearities, by means of variational methods, is due to Ma and Rivera [13] and Alves, Corrêa and Ma [2] who

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combined minimization arguments with truncation techniques and a priori estimates. Using Yang index and critical group arguments or the theory of invariant sets of descent flows, Perera and Zhang (see [17, 24]) proved existence results for the above problem. Multiplicity theorems can be found for instance in [3, 14, 19].

The existence or multiplicity of solutions of the Kirchhoff type problem with critical exponents in a bounded domain (or even in the whole space) has been studied by using different techniques as variational methods, genus theory, the Nehari manifold, the Ljusternik–Schnirelmann category theory (see for instance [5, 6, 7, 8]). It is worth mentioning that Mountain Pass arguments combined with the Lions' Concentration Compactness principle [12] are still the most popular tools to deal with such problems in the presence of a critical term. Applications to the lower dimensional case ($N < 4$) can be found in [1, 11, 16], while for higher dimensions ($N \geq 4$) we refer to [9, 10, 15, 23]. Notice that in order to employ the Concentration Compactness principle, a and b need to satisfy suitable constraints.

In order to state our main result we introduce the following notations: we endow the Sobolev space $H_0^1(\Omega)$ with the classical norm $\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}$ and denote by $\|u\|_q$ the Lebesgue norm in $L^q(\Omega)$ for $1 \leq q \leq 2^\star$, i.e. $\|u\|_q = \left(\int_\Omega |u|^q\, dx\right)^{\frac{1}{q}}$. Let S_N be the embedding constant of $H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega)$, i.e.

$$\|u\|_{2^\star}^2 \leq S_N^{-1}\|u\|^2 \qquad \text{for every} \quad u \in H_0^1(\Omega).$$

Let us recall that (see Talenti [22] and Hebey [9] for an explicit espression)

$$S_N = \frac{N(N-2)}{4}\,\omega_N^{\frac{2}{N}}, \tag{1}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. For $N \geq 4$ denote by $C_1(N)$ and $C_2(N)$ the constants

$$C_1(N) = \begin{cases} \dfrac{4(N-4)^{\frac{N-4}{2}}}{N^{\frac{N-2}{2}} S_N^{\frac{N}{2}}} & N > 4 \\ 1/S_4^2, & N = 4, \end{cases} \quad \text{and} \quad C_2(N) = \begin{cases} \dfrac{2(N-4)^{\frac{N-4}{2}}}{(N-2)^{\frac{N-2}{2}} S_N^{\frac{N}{2}}} & N > 4 \\ 1/S_4^2, & N = 4. \end{cases}.$$

Notice that $C_1(N) \leq C_2(N)$. Our result reads as follows:

Theorem 1.1. *Let a, b be positive numbers, $N \geq 4$.*

(a) *If $a^{\frac{N-4}{2}} b \geq C_1(N)$, then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H_0^1(\Omega)$, there exists $v^* \in C$ such that the functional*

$$u \to \frac{a}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - \frac{1}{2^\star}\int_\Omega |u|^{2^\star} dx - \frac{\lambda}{2}\int_\Omega |u(x) - v^*(x)|^2 dx$$

has two global minima.

(b) *If* $a^{\frac{N-4}{2}}b > C_2(N)$*, then, for each* $\lambda > 0$ *large enough and for each convex set* $C \subseteq L^2(\Omega)$ *whose closure in* $L^2(\Omega)$ *contains* $H_0^1(\Omega)$*, there exists* $v^* \in C$ *such that the problem*

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^{2^\star-2}u + \lambda(u - v^*(x)), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ *of the energy functional defined in* (a)*.*

The paper is motivated by a recent work of Ricceri where the author studied problem $(\mathcal{P}_\lambda)$ in the subcritical case, i.e. when $|u|^{2^\star-2}u$ is replaced by $|u|^{p-2}u$ with $p < 2^\star$. In [21, Proposition 1], the existence of two global minima for the energy functional (and three solutions for the associated Kirchhoff problem) is obtained for every $a \geq 0$ and $b > 0$. In the same paper, the following challenging question was raised (see [21, Problem 1]):

Question 1.2. Does the conclusion of Proposition 1 hold if $N > 4$ and $p = 2^\star$?

Notice that, for $N > 4$ and $p = 2^\star$ the energy functional associated to $(\mathcal{P}_\lambda)$ is bounded from below while if $N = 4 (= 2^\star)$ this is not true any more for arbitrary b. Moreover, when $p = 2^\star$ the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ fails to be compact and one can not apply directly the abstract tool which leads to [21, Theorem 1 and Proposition 1].

The main result of the present note gives a partial positive answer to the above question and prove that Proposition 1 of [21] holds for $p = 2^\star$ and $N \geq 4$ provided that a and b satisfies a suitable crucial inequality. Namely, we prove that the interaction between the Kirchhoff type operator and the critical nonlinearity ensures the sequentially weakly lower semicontinuity of the energy functional, a key property which allows to apply the minimax theory developed in [20, Theorem 3.2] (see also Theorem 2.3 below).

2. Proofs

The proof of Theorem 1.1 relies on the following key lemma.

Lemma 2.1. *Let* $N \geq 4$ *and* a, b *be positive numbers such that* $a^{\frac{N-4}{2}}b \geq C_1(N)$*. Denote by* $\mathcal{F}\colon H_0^1(\Omega) \to \mathbb{R}$ *the functional*

$$\mathcal{F}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^\star}\|u\|_{2^\star}^{2^\star} \qquad \text{for every } \ u \in H_0^1(\Omega).$$

Then, $\mathcal{F}$ *is sequentially weakly lower semicontinuous in* $H_0^1(\Omega)$*.*

Proof. Fix $u \in H_0^1(\Omega)$ and let $\{u_n\} \subset H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Thus,

$$\mathcal{F}(u_n) - \mathcal{F}(u) = \frac{a}{2}(\|u_n\|^2 - \|u\|^2) + \frac{b}{4}(\|u_n\|^4 - \|u\|^4) - \frac{1}{2^\star}\left(\|u_n\|_{2^\star}^{2^\star} - \|u\|_{2^\star}^{2^\star}\right).$$

It is clear that

$$\|u_n\|^2 - \|u\|^2 = \|u_n - u\|^2 + 2\int_\Omega \nabla(u_n - u)\nabla u = \|u_n - u\|^2 + o(1),$$

and

$$\begin{aligned}\|u_n\|^4 - \|u\|^4 &= \left(\|u_n - u\|^2 + o(1)\right)\left(\|u_n - u\|^2 + 2\int_\Omega \nabla u_n \nabla u\right)\\ &= \left(\|u_n - u\|^2 + o(1)\right)\left(\|u_n - u\|^2 + 2\int_\Omega \nabla(u_n - u)\nabla u + 2\|u\|^2\right)\\ &= \left(\|u_n - u\|^2 + o(1)\right)\left(\|u_n - u\|^2 + 2\|u\|^2 + o(1)\right).\end{aligned}$$

Moreover, from the Brézis-Lieb lemma, one has

$$\|u_n\|_{2^\star}^{2^\star} - \|u\|_{2^\star}^{2^\star} = \|u_n - u\|_{2^\star}^{2^\star} + o(1).$$

Putting together the above outcomes,

$$\begin{aligned}\mathcal{F}(u_n) - \mathcal{F}(u) &= \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\left(\|u_n - u\|^4 + 2\|u\|^2\|u_n - u\|^2\right) -\\ &\qquad - \frac{1}{2^\star}\|u_n - u\|_{2^\star}^{2^\star} + o(1)\\ &\geq \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\left(\|u_n - u\|^4 + 2\|u\|^2\|u_n - u\|^2\right) -\\ &\qquad - \frac{S_N^{-\frac{2^\star}{2}}}{2^\star}\|u_n - u\|^{2^\star} + o(1)\\ &\geq \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\|u_n - u\|^4 - \frac{S_N^{-\frac{2^\star}{2}}}{2^\star}\|u_n - u\|^{2^\star} + o(1)\\ &= \|u_n - u\|^2\left(\frac{a}{2} + \frac{b}{4}\|u_n - u\|^2 - \frac{S_N^{-\frac{2^\star}{2}}}{2^\star}\|u_n - u\|^{2^\star - 2}\right) + o(1).\end{aligned}$$

Denote by $f\colon [0, +\infty[\to \mathbb{R}$ the function $f(x) = \frac{a}{2} + \frac{b}{4}x^2 - \frac{S_N^{-\frac{2^\star}{2}}}{2^\star}x^{2^\star - 2}$.

We claim that $f(x) \geq 0$ for all $x \geq 0$.

Indeed, when $N = 4$, and $bS_4^2 \geq 1$,

$$f(x) = \frac{a}{2} + \frac{b}{4}x^2 - \frac{S_4^{-2}}{4}x^2 = \frac{a}{2} + \frac{1}{4}\left(b - \frac{1}{S_4^2}\right)x^2 \geq \frac{a}{2}.$$

If $N > 4$, it is immediately seen that f attains its minimum at

$$x_0 = \left(\frac{2^\star}{2(2^\star - 2)}S_N^{\frac{2^\star}{2}}b\right)^{\frac{1}{2^\star - 4}}$$

and the claim is a consequence of the assumption $a^{\frac{N-4}{2}} b \geq C_1(N)$. Thus

$$\liminf_{n\to\infty}(\mathcal{F}(u_n) - \mathcal{F}(u)) \geq \liminf_{n\to\infty} \|u_n - u\|^2 f(\|u_n - u\|) \geq 0,$$

and the thesis follows. □

In the next lemma we prove the Palais Smale property for our energy functional. Notice that the same constraints on a and b appear in [9] where such property was investigated for the critical Kirchhoff equation on closed manifolds by employing the H^1 (which is the underlying Sobolev space) decomposition.

Lemma 2.2. *Let $N \geq 4$ and a, b be positive numbers such that $a^{\frac{N-4}{2}} b > C_2(N)$. For $\lambda > 0, v^* \in H_0^1(\Omega)$ denote by $\mathcal{E}\colon H_0^1(\Omega) \to \mathbb{R}$ the functional defined by*

$$\mathcal{E}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^\star}\|u\|_{2^\star}^{2^\star} - \frac{\lambda}{2}\|u - v^\star\|_2^2 \qquad \text{for every } \ u \in H_0^1(\Omega).$$

Then, $\mathcal{E}$ satisfies the Palais-Smale (shortly (PS)) condition.

Proof. Let $\{u_n\}$ be a (PS) sequence for $\mathcal{E}$, that is

$$\begin{cases} \mathcal{E}(u_n) \to c \\ \mathcal{E}'(u_n) \to 0 \end{cases} \quad \text{as } n \to \infty.$$

Since $\mathcal{E}$ is coercive, $\{u_n\}$ is bounded and there exists $u \in H_0^1(\Omega)$ such that (up to a subsequence)

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \quad u_n \to u \text{ in } L^p(\Omega),\ p \in [1, 2^\star), \quad u_n \to u \text{ a.e. in } \Omega.$$

Using the second concentration compactness lemma of Lions [12], there exist an at most countable index set J, a set of points $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and two families of positive numbers $\{\eta_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ such that

$$|\nabla u_n|^2 \rightharpoonup d\eta \geq |\nabla u|^2 + \sum_{j\in J} \eta_j \delta_{x_j}, \quad \text{and} \quad |u_n|^{2^\star} \rightharpoonup d\nu = |u|^{2^\star} + \sum_{j\in J} \nu_j \delta_{x_j},$$

(weak star convergence in the sense of measures), where δ_{x_j} is the Dirac mass concentrated at x_j and such that

$$S_N \nu_j^{\frac{2}{2^\star}} \leq \eta_j \qquad \text{for every } j \in J.$$

Next, we will prove that the index set J is empty. Arguing by contradiction, we may assume that there exists a j_0 such that $\nu_{j_0} \neq 0$. Consider now, for $\varepsilon > 0$ a non negative cut-off function ϕ_ε such that

$$\phi_\varepsilon = 1 \text{ on } B(x_0, \varepsilon), \quad \phi_\varepsilon = 0 \text{ on } \Omega \setminus B(x_0, 2\varepsilon), \quad |\nabla \phi_\varepsilon| \leq \frac{2}{\varepsilon}.$$

It is clear that the sequence $\{u_n \phi_\varepsilon\}_n$ is bounded in $H_0^1(\Omega)$, so that

$$\lim_{n\to\infty} \mathcal{E}'(u_n)(u_n \phi_\varepsilon) = 0.$$

Thus
$$\begin{aligned} o(1) &= (a+b\|u_n\|^2)\int_\Omega \nabla u_n \nabla(u_n\phi_\varepsilon) - \int_\Omega |u_n|^{2^\star}\phi_\varepsilon - \\ &\quad - \lambda \int_\Omega (u_n - v^*)(u_n\phi_\varepsilon) \\ &= (a+b\|u_n\|^2)\left(\int_\Omega |\nabla u_n|^2\phi_\varepsilon + \int_\Omega u_n\nabla u_n\nabla\phi_\varepsilon\right) - \\ &\quad - \int_\Omega |u_n|^{2^\star}\phi_\varepsilon - \lambda\int_\Omega (u_n - v^*)(u_n\phi_\varepsilon). \qquad (2) \end{aligned}$$

Moreover, using Hölder inequality, one has

$$\left|\int_\Omega (u_n - v^*)(u_n\phi_\varepsilon)\right| \le \left(\int_{B(x_0,2\varepsilon)} (u_n - v^*)^2\right)^{\frac12}\left(\int_{B(x_0,2\varepsilon)} u_n^2\right)^{\frac12},$$

so that
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (u_n - v^*)(u_n\phi_\varepsilon) = 0.$$

Also
$$\left|\int_\Omega u_n\nabla u_n\nabla\phi_\varepsilon\right| = \left|\int_{B(x_0,2\varepsilon)} u_n\nabla u_n\nabla\phi_\varepsilon\right|$$
$$\le \left(\int_{B(x_0,2\varepsilon)} |\nabla u_n|^2\right)^{\frac12}\left(\int_{B(x_0,2\varepsilon)} |u_n\nabla\phi_\varepsilon|^2\right)^{\frac12} \le C\left(\int_{B(x_0,2\varepsilon)} |u_n\nabla\phi_\varepsilon|^2\right)^{\frac12}.$$

Since
$$\lim_{n\to\infty}\int_{B(x_0,2\varepsilon)} |u_n\nabla\phi_\varepsilon|^2 = \int_{B(x_0,2\varepsilon)} |u\nabla\phi_\varepsilon|^2,$$

and
$$\begin{aligned} \left(\int_{B(x_0,2\varepsilon)} |u\nabla\phi_\varepsilon|^2\right)^{\frac12} &\le \left(\int_{B(x_0,2\varepsilon)} |u|^{2^\star}\right)^{\frac{1}{2^\star}}\left(\int_{B(x_0,2\varepsilon)} |\nabla\phi_\varepsilon|^N\right)^{\frac1N} \\ &\le C\left(\int_{B(x_0,2\varepsilon)} |u|^{2^\star}\right)^{\frac{1}{2^\star}} \end{aligned}$$

we get
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^2)\left|\int_\Omega u_n\nabla u_n\nabla\phi_\varepsilon\right| = 0.$$

Moreover, as $0 \le \phi_\varepsilon \le 1$,

$$\begin{aligned} \lim_{n\to\infty}(a+b\|u_n\|^2)\int_\Omega |\nabla u_n|^2\phi_\varepsilon &\ge \lim_{n\to\infty}\left[a\int_{B(x_0,2\varepsilon)} |\nabla u_n|^2\phi_\varepsilon + b\left(\int_\Omega |\nabla u_n|^2\phi_\varepsilon\right)^2\right] \\ &\ge a\int_{B(x_0,2\varepsilon)} |\nabla u|^2\phi_\varepsilon + b\left(\int_\Omega |\nabla u|^2\phi_\varepsilon\right)^2 + a\eta_{j_0} + b\eta_{j_0}^2. \end{aligned}$$

So, as $\int_{B(x_0,2\varepsilon)} |\nabla u|^2\phi_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^2)\int_\Omega |\nabla u_n|^2\phi_\varepsilon \ge a\eta_{j_0} + b\eta_{j_0}^2.$$

Finally,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^{2^\star}\phi_\varepsilon = \lim_{\varepsilon\to 0}\int_\Omega |u|^{2^\star}\phi_\varepsilon + \nu_{j_0} = \lim_{\varepsilon\to 0}\int_{B(x_0,2\varepsilon)} |u|^{2^\star}\phi_\varepsilon + \nu_{j_0} = \nu_{j_0}.$$

Summing up the above outcomes, from (2) one obtains

$$\begin{aligned} 0 &\geq a\eta_{j_0} + b\eta_{j_0}^2 - \nu_{j_0} \geq a\eta_{j_0} + b\eta_{j_0}^2 - S_N^{-\frac{2^\star}{2}}\eta_{j_0}^{\frac{2^\star}{2}} \\ &= \eta_{j_0}\left(a + b\eta_{j_0} - S_N^{-\frac{2^\star}{2}}\eta_{j_0}^{\frac{2^\star-2}{2}}\right). \end{aligned}$$

Denote by $f_1\colon [0,+\infty[\to \mathbb{R}$ the function $f_1(x) = a + bx - S_N^{-\frac{2^\star}{2}} x^{\frac{2^\star-2}{2}}$. As before, assumptions on a and b imply that $f_1(x) > 0$ for all $x \geq 0$. Thus

$$a + b\eta_{j_0} - S_N^{-\frac{2^\star}{2}}\eta_{j_0}^{\frac{2^\star-2}{2}} > 0,$$

therefore $\eta_{j_0} = 0$, which is a contradiction. Such conclusion implies that J is empty, that is

$$\lim_{n\to\infty}\int_\Omega |u_n|^{2^\star} = \int_\Omega |u|^{2^\star}$$

and the uniform convexity of $L^{2^\star}(\Omega)$ implies that $\quad u_n \to u$ in $L^{2^\star}(\Omega)$.

Now, recalling that the derivative of the function

$$u \to \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4$$

satisfies the (S_+) property, in a standard way one can see that $u_n \to u$ in $H_0^1(\Omega)$, which proves our lemma. □

In the proof of our result, the main tool is the following theorem:

Theorem 2.3. (Ricceri [20], Theorem 3.2) *Let X be a topological space, E a real Hausdorff topological vector space, $C \subseteq E$ a convex set, $f\colon X\times C \to \mathbb{R}$ a function which is lower semicontinuous, inf–compact in X, and upper semicontinuous and concave in C. Assume also that*

$$\sup_{v\in C}\inf_{x\in X} f(x,v) < \inf_{x\in X}\sup_{v\in C} f(x,v). \tag{3}$$

Then, there exists $v^* \in C$ such that the function $f(\cdot, v^*)$ has at least two global minima.

Proof of Theorem 1.1 We apply Theorem 2.3 with $X = H_0^1(\Omega)$ endowed with the weak topology, $E = L^2(\Omega)$ with the strong topology, C as in the assumptions. Let $\mathcal{F}$ as in Lemma 2.1, i.e.

$$\mathcal{F}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^\star}\|u\|_{2^\star}^{2^\star} \qquad \text{for every } \ u \in H_0^1(\Omega).$$

From Lemma 2.1, $\mathcal{F}$ is sequentially weakly lower semicontinuous, and coercive, thus, the set $M_{\mathcal{F}}$ of its global minima is non empty. Denote by

$$\lambda^\star = \inf\left\{\frac{\mathcal{F}(u)-\mathcal{F}(v)}{\|u-v\|_2^2} \ : \ (v,u)\in M_{\mathcal{F}}\times H_0^1(\Omega),\ v\neq u\right\} \tag{4}$$

and fix $\lambda > 2\lambda^\star$. Let $f\colon H_0^1(\Omega)\times C\to\mathbb{R}$ be the function

$$f(u,v) = \mathcal{F}(u) - \frac{\lambda}{2}\|u-v\|_2^2.$$

From the Eberlein-Smulyan theorem it follows that $f(\cdot,v)$ has weakly compact sublevel sets in $H_0^1(\Omega)$. It is also clear that $f(u,\cdot)$ is continuous and concave in $L^2(\Omega)$. Let us prove (3).

Recalling that the closure of C in $L^2(\Omega)$ (denoted by $\overline{C}$) contains $H_0^1(\Omega)$, one has

$$\inf_{u\in H_0^1(\Omega)}\sup_{v\in C} f(u,v) = \inf_{u\in H_0^1(\Omega)}\sup_{v\in\overline{C}} f(u,v) \geq \inf_{u\in H_0^1(\Omega)}\sup_{v\in H_0^1(\Omega)} f(u,v) \tag{5}$$

$$= \inf_{u\in H_0^1(\Omega)}\sup_{v\in H_0^1(\Omega)}\left(\mathcal{F}(u)-\frac{\lambda}{2}\|u-v\|_2^2\right) = \inf_{u\in H_0^1(\Omega)}\left(\mathcal{F}(u)-\frac{\lambda}{2}\inf_{v\in H_0^1(\Omega)}\|u-v\|_2^2\right) = \min_{H_0^1(\Omega)}\mathcal{F}.$$

Since $\lambda > 2\lambda^\star$, there exist $u_0, v_0\in H_0^1(\Omega), u_0\neq v_0$ and $\varepsilon>0$ such that

$$\mathcal{F}(u_0) - \frac{\lambda}{2}\|u_0-v_0\|_2^2 < \mathcal{F}(v_0)-\varepsilon, \quad \text{and} \quad \mathcal{F}(v_0) = \min_{H_0^1(\Omega)}\mathcal{F}.$$

Thus, if $h\colon L^2(\Omega)\to\mathbb{R}$ is the map defined by $h(v) = \inf_{u\in H_0^1(\Omega)}(\mathcal{F}(u)-\frac{\lambda}{2}\|u-v\|_2^2)$, then h is upper semicontinuous in $L^2(\Omega)$ and

$$h(v_0) \leq \mathcal{F}(u_0) - \frac{\lambda}{2}\|u_0-v_0\|_2^2 < \mathcal{F}(v_0)-\varepsilon.$$

So, there exists $\delta>0$ such that $h(v)<\mathcal{F}(v_0)-\varepsilon$ for all $\|v-v_0\|_2\leq\delta$. Therefore,

$$\sup_{\|v-v_0\|_2\leq\delta}\inf_{u\in H_0^1(\Omega)}\left(\mathcal{F}(u)-\frac{\lambda}{2}\|u-v\|_2^2\right) \leq \mathcal{F}(v_0)-\varepsilon.$$

On the other hand,

$$\sup_{\|v-v_0\|_2\geq\delta}\inf_{u\in H_0^1(\Omega)}\left(\mathcal{F}(u)-\frac{\lambda}{2}\|u-v\|_2^2\right) \leq \sup_{\|v-v_0\|_2\geq\delta}\left(\mathcal{F}(v_0)-\frac{\lambda}{2}\|v_0-v\|_2^2\right) \leq \mathcal{F}(v_0)-\frac{\lambda}{2}\delta^2.$$

Summing up the above outcomes, we obtain

$$\sup_{v\in C}\inf_{u\in H_0^1(\Omega)} f(u,v) \leq \sup_{v\in L^2(\Omega)}\inf_{u\in H_0^1(\Omega)} f(u,v)$$

$$= \sup_{v\in L^2(\Omega)}\inf_{u\in H_0^1(\Omega)}\left(\mathcal{F}(u)-\frac{\lambda}{2}\|u-v\|_2^2\right) < \mathcal{F}(v_0) = \min_{H_0^1(\Omega)}\mathcal{F}. \tag{6}$$

From (5) and (2), claim (3) follows. Applying Theorem 2.3, we deduce the existence of $v^* \in C$ such that the energy functional

$$\mathcal{E}(u) = \mathcal{F}(u) - \frac{\lambda}{2}\|u - v^*\|_2^2$$

associated to our problem has two global minima, which is claim (a). In order to prove (b) we observe that, since the functional is of class C^1, such global minima turns out to be weak solutions of our problem. The third solution follows by Lemma 2.2 (recall that $C_2(N) \geq C_1(N)$) and a classical version of the Mountain Pass theorem by Pucci and Serrin [18]. □

Remark 2.4. For sake of clarity, we calculate the approximate values of the constants $C_1(N)$ and $C_2(N)$ for some N:

N	$C_1(N)$	$C_2(N)$
5	0.002495906672	0.002685168050
6	0.0001990835458	0.0002239689890
7	0.00001712333233	0.00001985538802
9	$1.269275934 \cdot 10^{-7}$	$1.529437355 \cdot 10^{-7}$

Question 2.5. Notice that if $N = 4$ then, for $bS_N^2 < 1$, $\mathcal{E}$ is unbounded from below. Indeed, if $\{u_n\}$ is such that $\frac{\|u_n\|^2}{\|u_n\|_4^2} \to S_N$, then we can fix c and $\bar{n}$ such that $\frac{\|u_{\bar{n}}\|^2}{\|u_{\bar{n}}\|_4^2} < c < b^{-\frac{1}{2}}$. Thus

$$\mathcal{E}(\tau u_{\bar{n}}) < \frac{a\tau^2}{2}\|u_{\bar{n}}\|^2 + \frac{\tau^4}{4}\left(b - \frac{1}{c^2}\right)\|u_{\bar{n}}\|^4 - \frac{\lambda}{2}\|\tau u_{\bar{n}} - v^*\|_2^2 \to -\infty, \text{as } \tau \to +\infty.$$

It remains an open question if, when $N > 4$, Theorem 1.1 holds for every $a \geq 0, b > 0$ with $a^{\frac{N-4}{2}} b < C_1(N)$.

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