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Robust phase-compatible strong-neighbor Roman domination in complex bipolar fuzzy graphs

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Abstract
This paper introduces a robust phase-compatible strong-neighbor Roman domination framework for complex bipolar fuzzy graphs with uncertain amplitude and phase param-eters. In a nominal phase-compatible model, protection feasibility is evaluated using one fixed realization of the graph parameters. Such a solution may become infeasible when membership amplitudes, phase values or phase tolerances vary. To address this limitation, an interval-based uncertain complex bipolar fuzzy graph model is formulated. Possible and robust strong edges are introduced through lower and upper connectedness strengths, while worst-case circular phase discrepancies are used to characterize robust phase com-patibility. Under independent rectangular edge-score uncertainty, an exact criterion for identifying robust phase-compatible strong edges is established. These edges generate a de-terministic robust backbone on which the proposed Roman domination problem is defined. A backbone-reduction theorem shows that the resulting parameter equals the weighted Ro-man domination number of the robust backbone. Fundamental bounds, comparison results and monotonicity properties are derived, and exact values are obtained for complete graphs, empty graphs, stars, paths and cycles. Critical phase tolerances, phase-uncertainty radii and robust-edge stability intervals are also determined. Finally, a cyclic network example illustrates how amplitude and phase uncertainty can alter the reliable edge structure and the minimum Roman protection cost.

Keywords: Complex bipolar fuzzy graph, Roman domination, strong neighborhood, phase compatibility, parameter uncertainty, robust optimization

Introduction

The theory of fuzzy sets was introduced by Zadeh [20] to represent uncertainty arising from in-formation that cannot be described adequately by ordinary binary membership. Rosenfeld [17] subsequently extended this framework to graphs by assigning membership values to vertices and edges. Fuzzy graphs have since provided useful mathematical models for communication sys-tems, transportation networks, facility-location problems, social networks and decision-support systems in which the strength of a connection cannot always be represented by a crisp value.

In several practical systems, uncertainty possesses both favourable and unfavourable aspects. Zhang [21] introduced bipolar fuzzy sets to describe these two aspects simultaneously. The positive membership degree represents the satisfaction of a property, whereas the negative membership degree represents the satisfaction of a related counter-property. Bipolar fuzzy graphs extend this idea to network structures and provide a framework for describing support and opposition, benefit and risk, or reliability and vulnerability within the same mathematical model [16, 7].

Ordinary fuzzy and bipolar fuzzy memberships principally describe the magnitude of un-certain information. However, many network systems also involve periodic, temporal or syn-chronization information. Ramot et al. [15] introduced complex fuzzy sets, in which every membership value contains an amplitude component and a phase component. The amplitude represents the degree of membership, while the phase may describe periodicity, timing, delay or synchronization. Complex fuzzy sets and their applications have subsequently been investigated from theoretical and applied perspectives [19, 9]. By combining complex fuzzy information

with bipolar membership, a bipolar complex fuzzy model can represent both dual-valued uncertainty and phase-dependent behaviour [10].

Domination is a fundamental concept in graph theory with applications to facility location, communication, surveillance, resource allocation and network protection. The principal con-cepts and classical results of domination theory are presented by Haynes et al. [8]. A dominating set identifies a collection of vertices capable of covering or controlling the remaining vertices of a graph. In fuzzy graph structures, the effectiveness of domination can depend not only on adja-cency but also on the strength of the corresponding edges. Strong neighborhoods, connectivity and domination in fuzzy graphs have been investigated in [13, 11, 12]. Consequently, strong neighborhoods are more appropriate than ordinary neighborhoods when domination must occur through sufficiently reliable connections.

Roman domination was introduced by Cockayne et al. [6] as a mathematical model for the strategic allocation of defensive resources. A Roman dominating function on a graph $G = (V, E)$ is a function

$f : V \longrightarrow \{0, 1, 2\}$

such that every vertex assigned the value 0 is adjacent to at least one vertex assigned the value 2. A vertex assigned 2 represents a reinforced location capable of supporting an adjacent unprotected location, while a vertex assigned 1 possesses only local protection. The minimum total weight of such a function is called the Roman domination number.

Several variants of Roman domination have subsequently been studied, including double Roman domination [3], total Roman domination [1] and strong Roman domination [2]. Roman domination and strong-neighbor Roman protection have also been extended to fuzzy graph structures [14, 18].

In a complex bipolar fuzzy network, the transfer of protection may depend on both the am-plitude strength of a connection and the phase compatibility of its endpoints. A zero-labelled vertex should therefore receive support only through an edge that is sufficiently strong in am-plitude and whose endpoints are sufficiently close in both positive and negative phases. This requirement leads naturally to phase-compatible strong-neighbor Roman domination.

A nominal phase-compatible model evaluates every edge using a fixed collection of ampli-tude values, phase values and phase tolerances. In practice, however, these quantities may be affected by measurement errors, temporal variation, estimation uncertainty or incomplete in-formation. An edge classified as strong and phase compatible under nominal values may lose either property after a small parameter variation. A Roman protection assignment constructed from the nominal graph may consequently fail to provide support under another admissible realization.

Robust optimization provides a systematic approach for constructing solutions that remain feasible under a prescribed collection of parameter realizations [4, 5]. Instead of optimizing with respect to a single nominal realization, a robust model requires one solution to satisfy the relevant constraints for every realization in an uncertainty set. This approach is appropriate for phase-sensitive network protection because it distinguishes connections that are merely reliable under nominal information from those that remain reliable under all admissible amplitude and phase variations.

Motivated by these observations, this paper introduces robust phase-compatible strong-neighbor Roman domination in complex bipolar fuzzy graphs. Positive and negative amplitudes are allowed to vary within prescribed intervals, while phase uncertainty is represented by circu-lar arcs centred at nominal phase values. The phase tolerances may also vary within specified intervals. An edge is classified as robust phase-compatible strong when it remains both strong and phase compatible under every admissible realization.

The robust edges generate a deterministic spanning graph, called the robust phase-compatible backbone. This backbone provides a direct connection between the uncertain complex bipolar fuzzy model and weighted Roman domination in ordinary graphs. It also makes it possible to derive structural properties, comparison results, exact values and parameter-stability results without repeatedly examining every realization of the uncertain graph.

Although robust optimization, fuzzy graph domination and Roman domination have each been studied extensively, their combination raises several unresolved questions. In particular, it is necessary to determine how an edge can be certified as strong under all admissible ampli-tude realizations, how circular phase uncertainty changes phase compatibility, and how these two requirements jointly influence the minimum Roman protection cost. The identification of critical tolerance and uncertainty values at which the reliable network structure changes is also important. The present work addresses these questions within a unified theoretical framework.

The principal contributions of this paper are summarized as follows:

(i) An interval-based uncertain complex bipolar fuzzy graph model is formulated by incorpo-rating uncertainty in positive and negative amplitudes, vertex phases and phase tolerances.

(ii) Possible and robust strong edges are introduced using lower edge-support scores and upper alternative-path connectedness strengths.

(iii) Exact expressions are obtained for the minimum and maximum circular phase discrepan-cies generated by phase-uncertainty arcs.

(iv) Necessary and sufficient conditions are established for robust phase compatibility. Under independent

rectangular edge-score uncertainty, an exact criterion is also obtained for robust strong edges.

(v) Robust phase-compatible strong-neighbor Roman dominating functions and their mini-mum scalar weights are defined using the resulting robust neighborhoods.

(vi) A backbone-reduction theorem is established, showing that the proposed parameter is equal to the weighted Roman domination number of the robust phase-compatible back-bone.

(vii) Fundamental bounds, nominal–robust comparisons, uncertainty-set monotonicity, toler-ance monotonicity and component properties are derived.

(viii) Exact values are obtained when the robust backbone is a complete graph, an empty graph, a star, a path, a cycle or a disjoint union of standard components.

(ix) Critical phase tolerances, phase-uncertainty radii, robust-strength margins and stability intervals are investigated.

(x) A cyclic network example is presented to demonstrate how both amplitude uncertainty and phase uncertainty can modify the robust backbone and the associated minimum Roman protection cost.

The proposed model is deliberately restricted to parameter uncertainty. Edge-failure tol-erance and computational protection models constitute a separate research problem and are therefore not included in the present theoretical framework.

The remainder of the paper is organized as follows. Section 2 recalls the required concepts from fuzzy graph theory, complex bipolar fuzzy graphs and Roman domination. Section 3 formu-lates the uncertain complex bipolar fuzzy graph model. Section 4 introduces possible and robust phase-compatible strong edges and defines the robust backbone. Section 5 presents robust phase-compatible strong-neighbor Roman domination. Section 6 establishes the backbone-reduction theorem, fundamental bounds and monotonicity properties. Section 7 determines exact values for selected classes of robust backbones. Section 8 investigates critical parameter thresholds and stability intervals. Section 9 presents an illustrative cyclic network. Finally, Section 10 summarizes the principal findings and outlines directions for future research.

Preliminaries

This section recalls the concepts from fuzzy graph theory, bipolar fuzzy graphs, complex bipolar fuzzy graphs and Roman domination that are required in the subsequent development. Through-out the paper, all underlying crisp graphs are assumed to be finite, simple and undirected.

Fuzzy Graphs and Strong Edges

Let

$$G^* = (V, E)$$

be a crisp graph. A fuzzy graph on G^* is an ordered pair

$$G = (\mu_V, \mu_E),$$

where

$$\mu_V : V \longrightarrow [0, 1]$$

is the vertex-membership function and

$$\mu_E : E \longrightarrow [0, 1]$$

is the edge-membership function satisfying

$$\mu_E(uv) \leq \min\{\mu_V(u), \mu_V(v)\}$$

for every $uv \in E$.

The value $\mu_E(uv)$ represents the membership strength of the edge uv.

Definition 2.1 (Path strength)**.** Let

$$P : u = v_0, v_1, \ldots, v_k = v$$

be a path in a fuzzy graph G. The strength of P is

$$s(P) = \min_{1 \leq j \leq k} \mu_E(v_{j-1} v_j).$$

Definition 2.2 (Connectedness strength)**.** The connectedness strength between two vertices $u, v \in V$ is

$$\mathrm{CONN}_G(u, v) = \max_{P \in \mathrm{P}_{u,v}(G)} s(P),$$

where $\mathrm{P}_{u,v}(G)$ denotes the collection of all u–v paths in G.

If there is no u–v path, then

$\mathrm{CONN}_G(u, v) = 0$.

For an edge $uv \in E$, let $G - uv$ denote the fuzzy graph obtained by deleting the edge uv while retaining all vertices and all other edges.

Definition 2.3 (Strong edge)**.** An edge $uv \in E$ is called a strong edge if

$\mu_E(uv) \geq \mathrm{CONN}_{G-uv}(u, v)$.

Thus, an edge is strong when its membership strength is at least as large as the strength of every alternative connection between its endpoints.

Definition 2.4 (Strong neighborhood)**.** The strong neighborhood of a vertex $u \in V$ is $N_s(u) = \{v \in V : uv \in E \text{ is a strong edge}\}$.

Bipolar Fuzzy Graphs

A bipolar fuzzy set assigns a positive membership degree and a negative membership degree to every element. The positive degree describes the satisfaction of a property, while the negative degree describes the satisfaction of an associated counter-property.

To work with nonnegative amplitudes, the negative membership value will be written as the negative of a number in [0, 1].

Definition 2.5 (Bipolar fuzzy graph)**.** A bipolar fuzzy graph on $G^* = (V, E)$ is an ordered pair

$$G_b = (\mu_{V_b}, \mu_{E_b}),$$

where

$$\mu_{V_b}(v) = \underset{V}{r^+}(v), -\underset{V}{r^-}(v), \qquad v \in V,$$

and

$$\mu_{E_b}(uv) = \underset{E}{r^+}(uv), -\underset{E}{r^-}(uv), \qquad uv \in E,$$

with

$$\underset{V}{r^+}(v), \underset{V}{r^-}(v), \underset{E}{r^+}(uv), \underset{E}{r^-}(uv) \in [0, 1].$$

The edge amplitudes satisfy

$$\underset{E}{r^+}(uv) \leq \min\{\underset{V}{r^+}(u), \underset{V}{r^+}(v)\}$$

and

$$\underset{E}{r^-}(uv) \leq \max\{\underset{V}{r^-}(u), \underset{V}{r^-}(v)\}$$

for every $uv \in E$.

Remark 2.6. The displayed negative-amplitude restriction follows the nonnegative-amplitude convention used in this paper. If negative memberships are written directly as values in $[-1, 0]$, the equivalent restriction must be expressed using the corresponding signed quantities.

Complex Bipolar Fuzzy Graphs

A complex fuzzy membership combines an amplitude with a phase. Accordingly, the mem-bership value can describe both the strength of an association and its periodic, temporal or synchronization characteristic.

Definition 2.7 (Complex bipolar fuzzy vertex set)**.** A complex bipolar fuzzy vertex set on V is a function

$$\mu_{V_c} : V \longrightarrow C \times C$$

defined by

$$\underset{V_c}{\mu}(v) = \underset{V}{r^+}(v)e^{i\underset{V}{\vartheta^+}(v)}, -\underset{V}{r^-}(v)e^{i\underset{V}{\vartheta^-}(v)},$$

where

$$\underset{V}{r^+}(v), \underset{V}{r^-}(v) \in [0, 1]$$

and

$$\underset{V}{\vartheta^+}(v), \underset{V}{\vartheta^-}(v) \in [0, 2\pi).$$

Definition 2.8 (Complex bipolar fuzzy edge set)**.** A complex bipolar fuzzy edge set on E is a function

$$\mu_{E_c} : E \longrightarrow \mathbf{C} \times \mathbf{C}$$

defined by

$$\mu_{E_c}(uv) = \left(r^{+}_{E}(uv) e^{i\vartheta^{+}_{E}(uv)}, -r^{-}_{E}(uv) e^{i\vartheta^{-}_{E}(uv)} \right),$$

where

$$r^{+}_{E}(uv),\ r^{-}_{E}(uv) \in [0,\ 1]$$

and

$$\vartheta^{+}_{E}(uv),\ \vartheta^{-}_{E}(uv) \in [0,\ 2\pi).$$

The edge amplitudes are assumed to satisfy

$$r^{+}_{E}(uv) \leq \min\{r^{+}_{V}(u),\ r^{+}_{V}(v)\}$$

and

$$r^{-}_{E}(uv) \leq \max\{r^{-}_{V}(u),\ r^{-}_{V}(v)\}.$$

The edge phases are assumed to satisfy the phase restrictions adopted for the complex bipolar fuzzy graph under consideration.

Definition 2.9 (Complex bipolar fuzzy graph)**.** A complex bipolar fuzzy graph is an ordered pair $G_c = (\mu_{VC}, \mu_{EC})$ defined on an underlying crisp graph $G^{*} = (V, E)$, where μ_{V} and μ_{E} are, respectively, a c c complex bipolar fuzzy vertex set on V and a complex bipolar fuzzy edge set on E, satisfying the prescribed amplitude and phase restrictions.

Uncertain Complex Bipolar Fuzzy Graph Model

A nominal complex bipolar fuzzy graph assumes that all vertex and edge amplitudes, phase components and phase tolerances are known exactly. In practice, these quantities may contain measurement errors or may vary over time. This section introduces an interval-based uncertainty model that represents such variations.

Amplitude Uncertainty

For every vertex $v \in V_c$ and every $\sigma \in \{+, -\}$, let

$$\underline{r}^{\sigma}_{V}(v) \quad \text{and} \quad \overline{r}^{\sigma}_{V}(v)$$

denote the lower and upper admissible amplitudes, respectively, where

$$0 \leq \underline{r}^{\sigma}_{V}(v) \leq \overline{r}^{\sigma}_{V}(v) \leq 1.$$

The uncertain vertex amplitude therefore satisfies

$$r^{\sigma}_{V}(v) \in [\underline{r}^{\sigma}_{V}(v), \overline{r}^{\sigma}_{V}(v)].$$

Similarly, for every edge $uv \in E_c$, let

$$\underline{r}^{\sigma}_{E}(uv) \quad \text{and} \quad \overline{r}^{\sigma}_{E}(uv)$$

be its lower and upper admissible amplitudes, with

$$0 \leq \underline{r}^{\sigma}_{E}(uv) \leq \overline{r}^{\sigma}_{E}(uv) \leq 1.$$

Thus,

$$r^{\sigma}_{E}(uv) \in [\underline{r}^{\sigma}_{E}(uv), \overline{r}^{\sigma}_{E}(uv)].$$

The positive edge-amplitude restriction must hold for every admissible realization. Consequently,

$$\overline{r}^{+}_{E}(uv) \leq \min\{\overline{r}^{+}_{V}(u), \overline{r}^{+}_{V}(v)\}.$$

Using the nonnegative magnitude convention for the negative membership component, the corresponding negative-amplitude restriction is

$$\bar{r}_E(uv) \le \min\left\{\bar{r}_V(u), \bar{r}_V(v)\right\}.$$

These conditions guarantee that the complex bipolar fuzzy edge-amplitude restrictions re-main valid for all admissible choices of the uncertain amplitudes.

Remark 3.1. The negative membership of a vertex is written as

$$-\bar{r}_V(v)e^{i\bar{\vartheta}_V(v)},$$

where $\bar{r}(v) \ge 0$ is its magnitude. Therefore, inequalities written in terms of negative member-ship values reverse direction when expressed using their nonnegative magnitudes.

Phase Uncertainty

Phase values belong to the unit circle and must therefore be represented using circular uncer-tainty sets.

Definition 3.2 (Circular uncertainty arc)**.** For a nominal phase

$$\hat{\vartheta} \in [0, 2\pi)$$

and an uncertainty radius

$$\delta \in [0, \pi],$$

define

$$A(\hat{\vartheta}, \delta) = \left\{\vartheta \in [0, 2\pi) : d_{S^1}(\vartheta, \hat{\vartheta}) \le \delta\right\}.$$

For every vertex $v \in V$, and $\sigma \in \{+, -\}$, let

$$\hat{\vartheta}_V^{\sigma}(v)$$

be the nominal phase and let

$$\delta_V^{\sigma}(v) \in [0, \pi]$$

be its phase-uncertainty radius. The uncertain phase satisfies

$$\vartheta_V^{\sigma}(v) \in A\left(\hat{\vartheta}_V^{\sigma}(v), \delta_V^{\sigma}(v)\right).$$

Similarly, an uncertain edge phase may be represented by

$$\vartheta_E^{\sigma}(uv) \in A\left(\hat{\vartheta}_E^{\sigma}(uv), \delta_E^{\sigma}(uv)\right).$$

In the present domination model, phase compatibility is determined by the vertex phases at the endpoints of an edge. Edge-phase uncertainty is retained as part of the general uncertain graph model but does not enter the vertex phase-compatibility criterion directly.

Uncertain Phase Tolerances

The positive and negative phase tolerances may also be uncertain. For $\sigma \in \{+, -\}$, suppose that

$$\tau^{\sigma} \in [\underline{\tau}^{\sigma}, \overline{\tau}^{\sigma}],$$

where

$$0 \le \underline{\tau}^{\sigma} \le \overline{\tau}^{\sigma} \le \pi.$$

The lower tolerance

$$\underline{\tau}^{\sigma}$$

represents the most restrictive admissible tolerance, whereas

$$\overline{\tau}^{\sigma}$$

Remark 3.3. Robust phase compatibility must hold even when the tolerance takes its smallest admissible value. Consequently, the lower tolerance $\underline{\tau}^{\sigma}$, rather than the upper tolerance, appears in the robust phase-compatibility condition.

Uncertain Complex Bipolar Fuzzy Graph

Definition 3.4 (Uncertain complex bipolar fuzzy graph)**.** An uncertain complex bipolar fuzzy graph is an ordered pair
(G_c, S),
where $G_c = (V_c, E_c)$ specifies the underlying complex bipolar fuzzy graph structure and S is a nonempty collection of admissible amplitude, phase and tolerance realizations.
For every vertex $v \in V_c$, an admissible vertex membership has the form

$$\mu_{V_c}^{\xi}(v) = r_{V,\xi}^{+}(v)e^{i\vartheta_{V,\xi}^{+}(v)}, -r_{V,\xi}^{-}(v)e^{i\vartheta_{V,\xi}^{-}(v)},$$

where

$$r_{V,\xi}^{\sigma}(v) \in [\underline{r}_{V}^{\sigma}(v), \overline{r}_{V}^{\sigma}(v)]$$

and

$$\vartheta_{V,\xi}^{\sigma}(v) \in A\ \vartheta^{\wedge\sigma}{}_{V}(v), \delta_{V}^{\sigma}(v)\ .$$

For every edge $uv \in E_c$, an admissible edge membership is

$$\mu_{E_c}^{\xi}(uv) = r_{E,\xi}^{+}(uv)e^{i\vartheta_{E,\xi}^{+}(uv)}, -r_{E,\xi}^{-}(uv)e^{i\vartheta_{E,\xi}^{-}(uv)}$$

The phase tolerances in realization ξ satisfy

$$\tau_{\xi}^{\sigma} \in [\underline{\tau}^{\sigma}, \overline{\tau}^{\sigma}]\ .$$

Admissible Realizations

Definition 3.5 (Admissible realization)**.** An admissible realization

$$\xi \in S$$

is a simultaneous selection of all uncertain vertex amplitudes, edge amplitudes, vertex phases, edge phases and phase tolerances from their prescribed uncertainty sets such that the complex bipolar fuzzy graph restrictions are satisfied.
The complex bipolar fuzzy graph corresponding to a realization $\xi \in S$ is denoted by
$G_c(\xi)$.
The nominal realization is denoted by
$\hat{\xi}$.
It is obtained by selecting the nominal amplitude and phase values together with the pre-scribed nominal tolerances.
Remark 3.6. The graph $G_c(\xi)$ is deterministic for every fixed $\xi \in S$ The robust model, however, requires the relevant edge and domination conditions to hold simultaneously for every realization in S.

Rectangular and Correlated Uncertainty

The structure of the uncertainty set affects the characterization of robust edges.
Definition 3.7 (Rectangular uncertainty set)**.** The uncertainty set S is called rectangular if every uncertain parameter may be selected independently from its prescribed interval or circular arc. Equivalently, S is the Cartesian product of the individual parameter uncertainty sets.
Under rectangular uncertainty, the amplitude of one edge may attain its lowest admissible value while the amplitudes of the remaining edges simultaneously attain their highest admissible values.
Definition 3.8 (Correlated uncertainty set)**.** An uncertainty set is called correlated if the admissible value of one or more parameters depends on the values selected for other parameters.
For correlated uncertainty, the simultaneous occurrence of all individual extreme values need not be admissible. Consequently, conditions based on independent lower and upper bounds may remain sufficient without being necessary.

Lower and Upper Edge-Support Scores

For a realization $\xi \in S$, the scalar support score of an edge $uv \in E_c$ is

$$s_c^{\xi}(uv) = \frac{r_{E,\xi}^{+}(uv) + 1 - r_{E,\xi}^{-}(uv)}{2}.$$

Because this expression is increasing in r_E^+ and decreasing in r_E^-, its smallest admissible value is

$$\frac{r_F(uv) + 1 - r_F(uv)}{2}$$

while its largest admissible value is

$$\bar{s}_c(uv) = \frac{r_E^+(uv) + 1 - r_E^-(uv)}{2}.$$

Therefore, for every realization $\xi \in S$,

$$\underline{s}_c(uv) \le s_c^{\xi}(uv) \le \bar{s}_c(uv).$$

Proposition 3.9. *For every edge $uv \in E_c$,*

$$0 \le \underline{s}_c(uv) \le \bar{s}_c(uv) \le 1.$$

Proof. The result follows from

$$0 \le r_E^+(uv) \le r_E^+(uv) \le 1$$

and

$$0 \le r_E^-(uv) \le r_E^-(uv) \le 1.$$

Since the scalar score is increasing in its positive amplitude and decreasing in its negative amplitude, the stated ordering follows.

Lower and Upper Path Strengths

Let

$$P : u = v_0, v_1, \ldots, v_\ell = v$$

be a path in the underlying graph of G_c.

The lower strength of P is

$$\underline{s}_c(P) = \min_{0 \le j < \ell} \underline{s}_c(v_j v_{j+1}),$$

and its upper strength is

$$\bar{s}_c(P) = \min_{0 \le j < \ell} \bar{s}_c(v_j v_{j+1}).$$

For every realization $\xi \in S$,

$$\underline{s}_c(P) \le s_c^{\xi}(P) \le \bar{s}_c(P).$$

Lower and Upper Connectedness Strengths

The lower connectedness strength between u and v is

$$\underline{\mathrm{CONN}}_{G}(u, v) = \max_{P \in \mathcal{P}_{u,v}(G_c)} \underline{s}_c(P),$$

where

$$\mathcal{P}_{u,v}(G_c)$$

denotes the collection of all u–v paths in the underlying graph of G_c.

Similarly, the upper connectedness strength is

$$\overline{\mathrm{CONN}}_{G_c}(u, v) = \max_{P \in \mathcal{P}_{u,v}(G_c)} \overline{s}_c(P).$$

If no u–v path exists, both quantities are defined to be zero.

Proposition 3.10. *For every pair $u, v \in V_c$ and every realization $\xi \in S$,*

$$\underline{\mathrm{CONN}}_{G_c}(u, v) \le \underline{\mathrm{CONN}}_{G_c(\xi)}(u, v) \le \overline{\mathrm{CONN}}_{G_c}(u, v).$$

Proof. For every edge $xy \in E_c$,

$$\underline{s}_c(xy) \le s_c^{\xi}(xy) \le \overline{s}_c(xy).$$

Taking the minimum over the edges of any fixed path preserves these inequalities. Taking the maximum over all u–v paths then gives the required bounds.

Assumptions of the Robust Model

The following assumptions are used throughout the theoretical development:

(A1) The underlying crisp graph $G^* = (V, E)$ is finite, simple and undirected.

(A2) Every amplitude interval is nonempty and contained in $[0, 1]$.

(A3) The complex bipolar fuzzy edge-amplitude restrictions hold for every admissible realiza-tion.

(A4) Every nominal phase belongs to $[0, 2\pi)$.

(A5) Every phase-uncertainty radius belongs to $[0, \pi]$. (A6) Every phase tolerance belongs to $[0, \pi]$.

(A7) The uncertainty set S is nonempty and compact.

(A8) Unless stated otherwise, the vertex-cost function is fixed, positive and real valued.

The next section uses this uncertainty model to distinguish edges that are strong or phase compatible in only some realizations from those that retain these properties under every admis-sible realization.

Robust Phase-Compatible Strong Edges

This section distinguishes edges that satisfy the required conditions under at least one admissible realization from those that remain valid under every admissible realization.

Possible and Robust Strong Edges

For every realization $\xi \in S$, let

$$E_s\ G_c(\xi)$$

denote the strong-edge set of the realized complex bipolar fuzzy graph $G_c(\xi)$.

Definition 4.1 (Possible strong edge). An edge $uv \in E_c$ is called a possible strong edge if there exists at least one realization $\xi \in S$ such that

$$uv \in E_s\ G_c(\xi).$$

The possible strong-edge set is

$$E_s^{\mathrm{pos}}(G_c, S) = \bigcup_{\xi \in C} E_s\ G_c(\xi).$$

Definition 4.2 (Robust strong edge). An edge $uv \in E_c$ is called a robust strong edge if

$$uv \in E_s\ G_c(\xi)$$

for every realization $\xi \in S$.

The robust strong-edge set is

$$E_s^{\mathrm{rob}}(G_c, S) = \bigcap_{\xi \in C} E_s\ G_c(\xi).$$

It follows immediately that

$$E_s^{\mathrm{rob}}(G_c, S) \subseteq E_s^{\mathrm{pos}}(G_c, S) \subseteq E_c.$$

Remark 4.3. A possible strong edge is reliable in at least one admissible state, whereas a robust strong edge is reliable in every admissible state. Thus, possible strength describes optimistic behaviour, while robust strength describes worst-case behaviour.

Alternative-Path Connectedness

For an edge $uv \in E_c$, let

$$P_{u,v}(G_c - uv)$$

be the collection of all u–v paths in the underlying graph after deleting uv.

The lower alternative connectedness strength is

$$\underline{\mathrm{CONN}}_{G_c - uv}(u, v) = \max_{P \in P_{u,v}(G_c - uv)} \min_{xy \in E(P)} \underline{s}_c(xy).$$

The upper alternative connectedness strength is

$$\overline{\mathrm{CONN}}_{G_c - uv}(u, v) = \max_{P \in P_{u,v}(G_c - uv)} \min_{xy \in E(P)} \overline{s}_c(xy).$$

If u and v are disconnected in $G_c - uv$, we adopt the convention

$$\underline{\mathrm{CONN}}_{G_c - uv}(u, v) = \overline{\mathrm{CONN}}_{G_c - uv}(u, v) = 0.$$

For every realization $\xi \in S$,

$$\underline{\mathrm{CONN}}_{G_c - uv}(u, v) \le \mathrm{CONN}_{G_c(\xi) - uv}(u, v) \le \overline{\mathrm{CONN}}_{G_c - uv}(u, v).$$

Exact Robust Strong-Edge Criterion

We first establish a criterion valid for an arbitrary uncertainty set.

Theorem 4.4 (Sufficient robust strong-edge criterion)**.** *Let $uv \in E_c$. If*

$$\underline{s}_c(uv) \ge \overline{\mathrm{CONN}}_{G_c - uv}(u, v),$$

then uv is a robust strong edge.

Proof. Let $\xi \in S$ be arbitrary. From the definitions of the lower edge score and upper alternative connectedness strength,

$$s_\xi(uv) \ge \underline{s}_c(uv)$$

and

$$\mathrm{CONN}_{G_c(\xi) - uv}(u, v) \le \overline{\mathrm{CONN}}_{G_c - uv}(u, v).$$

Therefore,

$$s_\xi(uv) \ge \underline{s}_c(uv) \ge \underline{\mathrm{CONN}}_{G_c - uv}(u, v) \ge \mathrm{CONN}_{G_c(\xi) - uv}(u, v).$$

Hence, uv is strong in $G_c(\xi)$. Since ξ was arbitrary, the edge uv is strong under every admissible realization. Therefore,

$$uv \in E_\xi^{\mathrm{rob}}(G_c, S).$$

Under independent rectangular edge-score uncertainty, the preceding condition is also nec-essary.

Theorem 4.5 (Exact robust strong-edge criterion)**.** *Assume that the edge-score uncertainty set is independent and rectangular. Then an edge $uv \in E_c$ is robust strong if and only if*

$$\underline{s}_c(uv) \geq \overline{\mathrm{CONN}}_{G_c-uv}(u, v).$$

Proof. The sufficiency follows from Theorem 4.4.

Conversely, suppose that

$$\underline{s}_c(uv) < \overline{\mathrm{CONN}}_{G_c-uv}(u, v).$$

Because the graph is finite, there exists a u–v path

$$P^* \in \mathcal{P}_{u,v}(G_c - uv)$$

such that

$$\min_{xy \in E(P^*)} \bar{s}_c(xy) = \overline{\mathrm{CONN}}_{G_c-uv}(u, v).$$

Under independent rectangular uncertainty, an admissible realization $\xi^* \in S$ can be selected in which

$$s^{\xi^*}_c(uv) = \underline{s}_c(uv)$$

and

$$s^{\xi^*}_c(xy) = \underline{s}_c(xy)$$

for every $xy \in E(P^*)$. Consequently,

$$\mathrm{CONN}_{G_c(\xi^*)-uv}(u, v) \geq \min_{xy \in E(P^*)} s^{\xi^*}_c(xy) = \underline{\mathrm{CONN}}_{G_c-uv}(u, v).$$

It follows that

$$s^{\xi^*}_c(uv) = \underline{s}_c(uv) < \underline{\mathrm{CONN}}_{G_c-uv}(u, v) \leq \mathrm{CONN}_{G_c(\xi^*)-uv}(u, v).$$

Thus, uv is not strong in the admissible realization ξ^*, contradicting robust strength. There-fore, every robust strong edge must satisfy
$\underline{s}_C(uv) \geq \mathrm{CONN}_{G_C-uv}(u, v)$.

□

Remark 4.6. The rectangularity assumption is needed only for the necessity direction. Under correlated uncertainty, the lowest score of uv and the highest scores of all edges on an alternative path may not occur simultaneously. In that case, Theorem 4.4 remains valid, but the condition need not be necessary.

Corollary 4.7. *Assume independent rectangular edge-score uncertainty. If uv is a bridge of the underlying crisp graph, then uv is robust strong.*

Proof. If uv is a bridge, then there is no u–v path in $G_c - uv$. Hence,

$$\overline{\mathrm{CONN}}_{G_C-uv}(u, v) = 0.$$

Since
the result follows from Theorem 4.5.
$\underline{s}_C(uv) \geq 0$, Worst-Case Phase Discrepancy

For every pair $u, v \in V_c$ and $\sigma \in \{+, -\}$, define the nominal phase discrepancy by

$$\hat{\Delta}^{\sigma}(u, v) = d_{S^1}\left(\hat{\vartheta}^{\sigma}_{V}(u), \hat{\vartheta}^{\sigma}_{V}(v)\right).$$

The maximum admissible phase discrepancy is

$$\overline{\Delta}^{\sigma}(u, v) = \max\left\{ d_{S^1}\left(\vartheta^{\sigma}_{V}(u), \vartheta^{\sigma}_{V}(v)\right) : \vartheta^{\sigma}_{V}(u) \in A\left(\hat{\vartheta}^{\sigma}_{V}(u), \delta^{\sigma}_{V}(u)\right), \vartheta^{\sigma}_{V}(v) \in A\left(\hat{\vartheta}^{\sigma}_{V}(v), \delta^{\sigma}_{V}(v)\right) \right\}.$$

Similarly, the minimum admissible phase discrepancy is

$$\underline{\Delta}^{\sigma}(u, v) = \min\left\{ d_{S^1}\left(\vartheta^{\sigma}_{V}(u), \vartheta^{\sigma}_{V}(v)\right) : \vartheta^{\sigma}_{V}(u) \in A\left(\hat{\vartheta}^{\sigma}_{V}(u), \delta^{\sigma}_{V}(u)\right), \vartheta^{\sigma}_{V}(v) \in A\left(\hat{\vartheta}^{\sigma}_{V}(v), \delta^{\sigma}_{V}(v)\right) \right\}.$$

Proposition 4.8 (Extremal phase discrepancies). *For every* $\sigma \in \{+, -\}$,

$$\overline{\Delta}^{\sigma}(u, v) = \min\left(\pi,\ \hat{\Delta}^{\sigma}(u, v) + \delta^{\sigma}_{V}(u) + \delta^{\sigma}_{V}(v)\right),$$

and

$$\underline{\Delta}^{\sigma}(u, v) = \max\left(0,\ \hat{\Delta}^{\sigma}(u, v) - \delta^{\sigma}_{V}(u) - \delta^{\sigma}_{V}(v)\right).$$

Proof. By the triangle inequality for the circular metric,

$$d_{S^1}\left(\vartheta^{\sigma}_{V}(u), \vartheta^{\sigma}_{V}(v)\right) \le \hat{\Delta}^{\sigma}(u, v) + \delta^{\sigma}_{V}(u) + \delta^{\sigma}_{V}(v).$$

Since the maximum possible circular distance is π, it follows that

$$\overline{\Delta}^{\sigma}(u, v) = \min\left(\pi,\ \hat{\Delta}^{\sigma}(u, v) + \delta^{\sigma}_{V}(u) + \delta^{\sigma}_{V}(v)\right).$$

Moving the two phases toward one another decreases their separation by at most

$$\delta^{\sigma}_{V}(u) + \delta^{\sigma}_{V}(v).$$

The separation cannot become negative. Therefore,

$$\underline{\Delta}^{\sigma}(u, v) = \max\left(0,\ \hat{\Delta}^{\sigma}(u, v) - \delta^{\sigma}_{V}(u) - \delta^{\sigma}_{V}(v)\right).$$

Robust Phase Compatibility

Definition 4.9 (Robustly phase-compatible vertices)**.** Two vertices $u, v \in V_c$ are called robustly phase compatible if they are phase compatible for every admissible realization $\xi \in S$.

Theorem 4.10 (Robust phase-compatibility criterion)**.** *Two vertices* $u, v \in V_c$ *are robustly phase compatible if and only if*

and

Proof. Suppose first that and

$$\overline{\Delta}^{+}(u, v) \le \underline{\tau}^{+}$$

and

$$\overline{\Delta}^{-}(u, v) \le \underline{\tau}^{-}.$$

Proof. Suppose first that

$$\overline{\Delta}^{+}(u, v) \le \underline{\tau}^{+}$$

and

$$\overline{\Delta}^{-}(u, v) \le \underline{\tau}^{-}.$$

For every $\xi \in S$ and every $\sigma \in \{+, -\}$,

$$d_{S^1}\left(\vartheta_{\xi}^{\sigma}(u), \vartheta_{\xi}^{\sigma}(v)\right) \le \overline{\Delta}^{\sigma}(u, v) \le \underline{\tau}^{\sigma} \le \tau_{\xi}^{\sigma}.$$

Hence, u and v are phase compatible in every admissible realization.
Conversely, suppose that one of the inequalities fails. Without loss of generality, let

$$\overline{\Delta}^{+}(u, v) > \underline{\tau}^{+}.$$

By compactness of the phase-uncertainty arcs, the maximum discrepancy $\overline{\Delta}^{+}(u, v)$ is attained. Select the corresponding positive phases and choose the tolerance

$$\tau^{+} = \underline{\tau}^{+}.$$

This produces an admissible realization in which

$$d_{S^1}\left(\vartheta_{v}^{+}(u), \vartheta_{v}^{+}(v)\right) > \tau^{+}.$$

Thus, the vertices fail to be phase compatible in that realization, contradicting robust phase compatibility. Therefore, both stated inequalities are necessary.

Robust Phase-Compatible Strong Edges

Definition 4.11 (Possible phase-compatible strong edge)**.** An edge $uv \in E_c$ is called a possible phase-compatible strong edge if there exists at least one realization $\xi \in S$ in which uv is both strong and phase compatible.
The collection of all possible phase-compatible strong edges is denoted by

$$E_{\mathrm{pcs}}^{\mathrm{pos}}(G_c, S).$$

Definition 4.12 (Robust phase-compatible strong edge)**.** An edge $uv \in E_c$ is called a robust phase-compatible strong edge if it remains both strong and phase compatible for every realiza-tion $\xi \in S$.
The robust phase-compatible strong-edge set is

$$E_{\mathrm{rpcs}}(G_c, S) = \bigcap_{\xi \in C} E_{\mathrm{pcs}}\left(G_c(\xi)\right).$$

Theorem 4.13 (Robust phase-compatible strong-edge certificate)**.** *Let $uv \in E_c$. If*

$$s_c(uv) \ge \overline{\mathrm{CONN}}_{G_c - uv}(u, v),$$

$$\overline{\Delta}^{+}(u, v) \le \underline{\tau}^{+},$$

and

$$\overline{\Delta}^{-}(u, v) \le \underline{\tau}^{-},$$

then

$$uv \in E_{\mathrm{rpcs}}(G_c, S).$$

Proof. The first inequality guarantees, by Theorem 4.4, that uv remains strong under every admissible realization. The second and third inequalities guarantee, by Theorem 4.10, that its endpoints remain phase compatible under every admissible realization. Therefore, uv is a robust phase-compatible strong edge.

Theorem 4.14 (Exact robust phase-compatible strong-edge criterion)**.** *Assume independent rectangular uncertainty. Then*

$$uv \in E_{rpcs}(G_c, S)$$

if and only if

$$s_c(uv) \geq \overline{\mathrm{CONN}}_{G_c - uv}(u, v),$$

$$\overline{\Delta}^{+}(u, v) \leq \underline{\tau}^{+},$$

and

$$\overline{\Delta}^{-}(u, v) \leq \underline{\tau}^{-}.$$

Proof. Under independent rectangular edge-score uncertainty, Theorem 4.5 characterizes robust strength. Theorem 4.10 characterizes robust phase compatibility. An edge is robust phase-compatible strong precisely when both properties hold simultaneously. Therefore, the three inequalities are necessary and sufficient.
The following inclusions hold:

$$E_{rpcs}(G_c, S) \subseteq \underset{s}{E^{rob}}(G_c, S) \subseteq \underset{s}{E^{pos}}(G_c, S) \subseteq E_c,$$

and

$$E_{rpcs}(G_c, S) \subseteq E^{pos}_{pcs}(G_c, S).$$

Robust Phase-Compatible Strong Neighborhoods
Definition 4.15 (Robust phase-compatible strong neighborhood)**.** For a vertex $u \in V_c$, its robust phase-compatible strong neighborhood is

$$N_{rpcs}(u) = \{v \in V_c : uv \in E_{rpcs}(G_c, S)\}.$$

The corresponding closed neighborhood is

$$N_{rpcs}[u] = N_{rpcs}(u) \cup \{u\}.$$

The robust phase-compatible strong degree of u is

$$d_{rpcs}(u) = |N_{rpcs}(u)|.$$

The minimum and maximum robust degrees are

$$\delta_{rpcs}(G_c, S) = \min_{u \in V_c} d_{rpcs}(u)$$

and

$$\Delta_{rpcs}(G_c, S) = \max_{u \in V_c} d_{rpcs}(u).$$

Robust Phase-Compatible Backbone
Definition 4.16 (Robust phase-compatible backbone)**.** The robust phase-compatible backbone of (G_c, S) is the deterministic crisp graph

$$B_{rpcs}(G_c, S) = (V_c, E_{rpcs}(G_c, S)).$$

Proposition 4.17. *For every vertex $u \in V_c$, its neighborhood in the robust backbone satisfies*

$$N_{B_{rpcs}}(u) = N_{rpcs}(u).$$

Proof. By definition, v is adjacent to u in B_{rpcs} exactly when

$$uv \in E_{rpcs}(G_c, S).$$

This is precisely the condition

$$v \in N_{rpcs}(u).$$

Remark 4.18. The robust backbone contains exactly those links that remain available for phase-compatible strong-neighbor protection under every admissible parameter realization. It converts the uncertain complex bipolar fuzzy network into a deterministic graph while preserving all robustly reliable protection relationships. The next section defines Roman domination using the neighborhoods of this robust back-bone.

Robust Phase-Compatible Strong-Neighbor Roman Domina-tion

This section introduces Roman protection assignments that remain feasible under every admis-sible realization of the uncertain complex bipolar fuzzy graph.

Robust Phase-Compatible Strong-Neighbor Roman Dominating Func-tions

Definition 5.1 (RPCSNR dominating function)**.** A function

$f: V_c \longrightarrow \{0, 1, 2\}$

is called a robust phase-compatible strong-neighbor Roman dominating function, abbreviated as an RPCSNR dominating function, if every vertex $u \in V_c$ satisfying $f(u) = 0$ has at least one vertex

$$v \in N_{rpcs}(u)$$

such that

$$f(v) = 2.$$

Thus, every zero-labelled vertex must receive protection from a vertex labelled 2 through an edge that remains both strong and phase compatible under every admissible realization.

Remark 5.2. The label 0 represents a facility without locally allocated protection, the label 1 represents a facility possessing only local protection, and the label 2 represents a reinforced facility capable of supporting robustly adjacent zero-labelled facilities.

Roman Partition

Let f be an RPCSNR dominating function. For $i \in \{0, 1, 2\}$, define

$$V_i = \{v \in V_c : f(v) = i\}.$$

The ordered triple

$$(V_0, V_1, V_2)$$

is called the Roman partition induced by f.

The function may therefore be written as

$$f = (V_0, V_1, V_2).$$

Since every vertex receives exactly one label,

$$V_0 \cup V_1 \cup V_2 = V_c$$

and

$$V_i \cap V_j = \emptyset \qquad \text{whenever } i \neq j.$$

Feasibility Characterization

Theorem 5.3 (Neighborhood characterization). *A function*

$$f = (V_0, V_1, V_2)$$

is an RPCSNR dominating function if and only if

$$V_0 \subseteq \bigcap_{v \in V_2} N_{rpcs}(v).$$

Proof. Suppose first that f is an RPCSNR dominating function. For every $u \in V_0$, there exists a vertex $v \in V_2$ such that

$$v \in N_{rpcs}(u).$$

Since the underlying graph is undirected,

$$v \in N_{rpcs}(u) \iff u \in N_{rpcs}(v).$$

Therefore,

$$u \in \bigcap_{v \in V_2} N_{rpcs}(v).$$

Since $u \in V_0$ was arbitrary,

$$V_0 \subseteq \bigcap_{v \in V_2} N_{rpcs}(v).$$

Conversely, suppose that

$$V_0 \subseteq \bigcap_{v \in V_2} N_{rpcs}(v).$$

For every $u \in V_0$, there exists $v \in V_2$ such that

$$u \in N_{rpcs}(v).$$

By symmetry of adjacency,

$$v \in N_{rpcs}(u).$$

Thus, every zero-labelled vertex has a robust phase-compatible strong neighbor labelled 2. Hence, f is an RPCSNR dominating function.

Corollary 5.4. *A function $f = (V_0, V_1, V_2)$ is an RPCSNR dominating function if and only if* $N_{rpcs}(u) \cap V_2 \neq \emptyset$

for every $u \in V_0$.

Proof. The condition states directly that every zero-labelled vertex has at least one robust phase-compatible strong neighbor in V_2.

Existence

Proposition 5.5 (Existence)**.** *Every uncertain complex bipolar fuzzy graph* (G_c, S) *admits an RPCSNR dominating function.*

Proof. Define

$f(v) = 1$ for every $v \in V_c$.

This function contains no zero-labelled vertex. Therefore, the RPCSNR domination condi-tion is satisfied vacuously, and f is an RPCSNR dominating function.

The preceding proposition ensures that the collection of RPCSNR dominating functions is always nonempty, even when the robust backbone has no edges.

Scalar Weight

Let

$$\omega_c : V_c \dashrightarrow R_{>0}$$

be a positive vertex-cost function.

Definition 5.6 (Scalar weight)**.** The scalar weight of an RPCSNR dominating function $f : V_c \to \{0, 1, 2\}$ is

$$w_c(f) = \sum_{v \in V_c} f(v)\omega_c(v).$$

If $f = (V_0, V_1, V_2)$, then

$$w_c(f) = \sum_{v \in V_1} \omega_c(v) + 2 \sum_{v \in V_2} \omega_c(v).$$

Under unit vertex costs,

$$\omega_c(v) = 1 \quad \text{for every } v \in V_c$$

and hence

$$w_c(f) = |V_1| + 2|V_2|.$$

Robust Phase-Compatible Strong-Neighbor Roman Domination Num-ber

Definition 5.7 (RPCSNR domination number)**.** The robust phase-compatible strong-neighbor Roman domination number of (G_c, S) is

$\gamma_{rpcsR}(G_c, S) = \min \{w_c(f) : f$ is an RPCSNR dominating function$\}$.

An RPCSNR dominating function f satisfying

$w_c(f) = \gamma_{rpcsR}(G_c, S)$ is called a minimum RPCSNR dominating function.

When all vertex costs are equal to one, the notation

γrpcsR(G_c, S) also denotes the corresponding unit-cost parameter.

Comparison with Nominal Feasibility

Let

$$\hat{\xi} \in S$$

be the nominal realization.

Proposition 5.8. *Every RPCSNR dominating function of* (G_c, S) *is a nominal phase-compatible strong-neighbor Roman dominating function of* $G_c(\hat{\xi})$.

Proof. Every robust phase-compatible strong edge belongs to the nominal phase-compatible strong-edge set because

$$\hat{\xi} \in S.$$

Thus,

$$E_{rpcs}(G_c, S) \subseteq E_{pcs}\, G_c(\hat{\xi}) .$$

Consequently,

$$N_{rpcs}(u) \subseteq N^{\hat{\xi}}_{pcs}(u)$$

for every $u \in V_c$. Hence, every zero-labelled vertex protected through a robust phase-compatible strong edge is also protected through a nominal phase-compatible strong edge.

Corollary 5.9. *For the same positive vertex-cost function,*
$\gamma_{pcsR}\ G_c(\xi) \leq \gamma_{rpcsR}(G_c, S)$.

Proof. By Proposition 5.8, the robust feasible set is contained in the nominal feasible set. Mini-mization over the larger nominal feasible set cannot produce a value greater than minimization over the robust feasible set.

Remark 5.10. The inequality in Corollary 5.9 may be strict. Parameter uncertainty can remove nominal phase-compatible strong edges, thereby reducing the available protection neighbor-hoods and increasing the minimum Roman protection cost.

The following section shows that RPCSNR domination is equivalent to weighted Roman domination on the robust phase-compatible backbone. This reduction will be used to derive structural properties, bounds and exact values.

Backbone Reduction and Fundamental Properties

This section establishes the fundamental relationship between the proposed robust parameter and weighted Roman domination in an ordinary graph. This relationship allows classical graph-theoretic methods to be applied after constructing the robust phase-compatible backbone.

Backbone-Reduction Theorem

Recall that

$$B_{rpcs}(G_c, S) = (V_c, E_{rpcs}(G_c, S))$$

is the robust phase-compatible backbone of (G_c, S).

Theorem 6.1 (Backbone-reduction theorem). *A function*

$$f : V_c \longrightarrow \{0, 1, 2\}$$

is an RPCSNR dominating function of (G_c, S) *if and only if it is a Roman dominating function of*

$$B_{rpcs}(G_c, S).$$

Consequently,

$$\gamma_{rpcsR}(G_c, S) = \gamma_R^{\omega}(B_{rpcs}(G_c, S)) .$$

Proof. Suppose first that f is an RPCSNR dominating function of (G_c, S). Let $u \in V_c$ satisfy $f(u) = 0$. By definition, there exists a vertex

Remark 6.2. The theorem does not imply that the uncertain amplitude and phase information is irrelevant. That information determines the robust edge set and hence the structure of B_{rpcs}. Once the robust backbone has been constructed, however, the remaining optimization problem is a weighted Roman domination problem on a deterministic graph.

Unit-Cost Bounds

Throughout this subsection, assume

$$\omega_c(v) = 1 \qquad \text{for every } v \in V_c.$$

Let

$$n = |V_c|$$

and let

$$\Delta_{rpcs} = \Delta\, B_{rpcs}(G_c, S)$$

be the maximum degree of the robust backbone.

Theorem 6.3 (Order bounds)**.** *For every nonempty uncertain complex bipolar fuzzy graph,*

$$1 \le \gamma_{rpcsR}(G_c, S) \le n.$$

Proof. The lower bound follows because every Roman dominating function of a nonempty graph has positive weight. The all-one function is feasible and has weight n. Therefore,

$$\gamma_{rpcsR}(G_c, S) \le n.$$

Theorem 6.4 (Maximum-degree lower bound)**.** *If*

$$\Delta_{rpcs} \ge 1,$$

then

$$\gamma_{rpcsR}(G_c, S) \ge \left\lceil \frac{2n}{\Delta_{rpcs} + 1} \right\rceil.$$

Proof. Let

$$f = (V_0, V_1, V_2)$$

be a minimum RPCSNR dominating function. Every vertex in V_0 has a neighbor in V_2. Since each vertex in V_2 has at most Δ_{rpcs} neighbors,

$$|V_0| \le \Delta_{rpcs}|V_2|.$$

Hence,

$$n = |V_0| + |V_1| + |V_2| \le |V_1| + (\Delta_{rpcs} + 1)|V_2|.$$

Because $\Delta_{rpcs} \ge 1$,

$$|V_1| \le \frac{\Delta_{rpcs} + 1}{2}|V_1|.$$

Therefore,

$$n \le \frac{\Delta_{rpcs} + 1}{2}(|V_1| + 2|V_2|).$$

Since

$$|V_1| + 2|V_2| = \gamma_{rpcsR}(G_c, S),$$

we obtain

$$\gamma_{rpcsR}(G_c, S) \ge \frac{2n}{\Delta_{rpcs} + 1}.$$

The domination number is an integer, and hence

$$\gamma_{rpcsR}(G_c, S) \ge \left\lceil \frac{2n}{\Delta_{rpcs} + 1} \right\rceil.$$

Corollary 6.5. *If then*

$$\Delta_{\mathrm{rpcs}} = 0,$$

then

$$\gamma_{\mathrm{rpcsR}}(G_c, S) = n.$$

Proof. If $\Delta_{\mathrm{rpcs}} = 0$, every vertex is isolated in the robust backbone. Therefore, every vertex must receive a positive label. The all-one assignment is optimal and has weight n. □

6.3 Weighted Bounds

Let $\omega_{\min} = \min_{v \in V_c} \omega_c(v)$ and $\omega_{\max} = \max_{v \in V_c} \omega_c(v)$. Since

$$\omega_{\min} \sum_{v \in V_c} f(v) \le w_c(f) \le \omega_{\max} \sum_{v \in V_c} f(v),$$

the backbone-reduction theorem yields

$$\omega_{\min} \gamma_R(B_{\mathrm{rpcs}}) \le \gamma_{\mathrm{rpcsR}}(G_c, S) \le \omega_{\max} \gamma_R(B_{\mathrm{rpcs}}).$$

In addition, if D is a dominating set of B_{rpcs}, assigning label 2 to D and label 0 elsewhere gives

$$\gamma_{\mathrm{rpcsR}}(G_c, S) \le \min \left(\Omega(G_c),\ 2 \sum_{v \in D} \omega_c(v) \right).$$

Uncertainty-Set Monotonicity

Theorem 6.6 (Uncertainty-set monotonicity). *Let*

$$S_1 \subseteq S_2.$$

Then

$$E_{\mathrm{rpcs}}(G_c, S_2) \subseteq E_{\mathrm{rpcs}}(G_c, S_1)$$

and

$$\gamma_{\mathrm{rpcsR}}(G_c, S_1) \le \gamma_{\mathrm{rpcsR}}(G_c, S_2).$$

Proof. An edge belonging to

$$E_{\mathrm{rpcs}}(G_c, S_2)$$

remains strong and phase compatible for every realization in S_2. Since

$$S_1 \subseteq S_2,$$

the edge also remains strong and phase compatible for every realization in S_1. Therefore,

$$E_{\mathrm{rpcs}}(G_c, S_2) \subseteq E_{\mathrm{rpcs}}(G_c, S_1).$$

Thus, the robust backbone associated with S_2 is a spanning subgraph of the robust backbone associated with S_1. Deleting edges cannot create additional feasible Roman dominating functions. Consequently,

$$\gamma_{\mathrm{rpcsR}}(G_c, S_1) \le \gamma_{\mathrm{rpcsR}}(G_c, S_2).$$

Remark 6.7. Increasing parameter uncertainty can remove robust edges but cannot create them. Hence, greater uncertainty cannot decrease the minimum robust Roman protection cost.

Phase-Tolerance Monotonicity

Let

$$\boldsymbol{\tau}_1 = (\tau_1^+, \tau_1^-)$$

and

$$\boldsymbol{\tau}_2 = (\tau_2^+, \tau_2^-)$$

be two fixed pairs of minimum phase tolerances.

Theorem 6.8 (Tolerance monotonicity). *Suppose that*

$$\tau_1^+ \le \tau_2^+ \quad \text{and} \quad \tau_1^- \le \tau_2^-.$$

Then

$$F_{\text{rpts}}^{\boldsymbol{\tau}_1} \subseteq E_{\text{rpcs}}^{\boldsymbol{\tau}_2}$$

and

$$\gamma_{\text{rpcsR}}^{\boldsymbol{\tau}_2}(G, S) \le \gamma_{\text{rpcsR}}^{\boldsymbol{\tau}_1}(G, S).$$

Proof. If an edge is robustly phase compatible under $\boldsymbol{\tau}_1$, then

$$\Delta^+(u, v) \le \tau_1^+ \le \tau_2^+$$

and

$$\Delta^-(u, v) \le \tau_1^- \le \tau_2^-.$$

Therefore, every robust phase-compatible strong edge under $\boldsymbol{\tau}_1$ remains admissible under $\boldsymbol{\tau}_2$. Hence,

$$F_{\text{rpts}}^{\boldsymbol{\tau}_1} \subseteq E_{\text{rpcs}}^{\boldsymbol{\tau}_2}.$$

The second robust backbone contains every edge of the first. Therefore, every RPCSNR dominating function feasible under $\boldsymbol{\tau}_1$ is also feasible under $\boldsymbol{\tau}_2$. Minimization over this larger feasible set gives

$$\gamma_{\text{rpcsR}}^{\boldsymbol{\tau}_2}(G, S) \le \gamma_{\text{rpcsR}}^{\boldsymbol{\tau}_1}(G, S).$$

Component Additivity

Theorem 6.9 (Component additivity). *Suppose that the robust backbone has connected components*

$$B_1, B_2, \ldots, B_t.$$

Then

$$\gamma_{\text{rpcsR}}(G, S) = \sum_{j=1}^{t} \gamma_R^{\omega}(B_j).$$

Proof. No vertex in one connected component can dominate a vertex in another component. Therefore, the restriction of an RPCSNR dominating function to each B_j must be a Roman dominating function of B_j.

Consequently, every RPCSNR dominating function f satisfies

$$w_c(f) \ge \sum_{j=1}^{t} \gamma_R^{\omega}(B_j).$$

Conversely, select a minimum weighted Roman dominating function f_j on each component B_j. Since the components are pairwise disjoint, the functions $f_1, \ldots, f_t$ can be combined to form a Roman dominating function of the complete robust backbone. Its weight is

$$\sum_{j=1}^{t} \gamma_R^{\omega}(B_j).$$

Applying the backbone-reduction theorem gives

$$\gamma_{\mathrm{rpcsR}}(G_c, S) = \sum_{j=1}^{t} \gamma_R^{\omega}(B_j).$$

Corollary 6.10. *If the robust backbone contains q isolated vertices $u_1, u_2, \ldots, u_q$ and nontrivial connected components*

$$B_1, B_2, \ldots, B_t$$

then

$$\gamma_{\mathrm{rpcsR}}(G_c, S) = \sum_{\ell=1}^{q} \omega_c(u_\ell) + \sum_{j=1}^{t} \gamma_R^{\omega}(B_j).$$

Proof. The weighted Roman domination number of an isolated vertex u_ℓ is $\omega_c(u_\ell)$, because the optimal label is 1. The result now follows from Theorem 6.9. □

Exact Values for Selected Robust Backbones

This section records consequences of Theorem 6.1. Unit vertex costs are assumed unless stated otherwise.

Theorem 7.1. *If the robust backbone is a standard graph, then*

$$\gamma_{\mathrm{rpcsR}}^{c}(G, S) = \begin{cases} 1, & B_{\mathrm{rpcs}} \cong K_1, \\ 2, & B_{\mathrm{rpcs}} \cong K_n, \ n \geq 2, \\ 2, & B_{\mathrm{rpcs}} \cong K_{1,n-1}, \ n \geq 2, \\ \lceil 2n/3 \rceil, & B_{\mathrm{rpcs}} \cong P_n, \\ \lceil 2n/3 \rceil, & B_{\mathrm{rpcs}} \cong C_n, \ n \geq 3. \end{cases}$$

Proof. By Theorem 6.1, the required values equal the ordinary Roman domination numbers of the displayed graphs. A single vertex labelled 2 dominates K_n and the centre dominates a star, while every vertex of K_n must receive label 1. For paths and cycles, repeating the pattern (0, 2, 0), with the appropriate terminal adjustment, gives weight $\lceil 2n/3 \rceil$; the standard three-vertex block argument gives the matching lower bound [6].

For positive costs, two useful exact formulas are

$$\gamma_{\mathrm{rpcsR}}(K_n) = \min\{\Omega(G_c), 2\omega_{\min}\}, \qquad n \geq 2,$$

and, for a star with centre c and leaf set L,

$$\gamma_{\mathrm{rpcsR}}(K_{1,m}) = \min\left\{\Omega(G_c), 2\omega_c(c), \min_{\ell \in L}\left\{2\omega_c(\ell) + \sum_{x \in L \setminus \{\ell\}} \omega(x)\right\}\right\}.$$

The first formula follows by comparing the all-one function with a minimum-cost universal vertex labelled 2. For the star, an optimal function is either all-one, centre-supported, or supported by one leaf while the remaining leaves receive label 1.

Corollary 7.2. *If B_{rpcs} consists of paths $P_{n1}, \ldots, P_{nt}$, then*

$$\gamma_{\mathrm{rpcsR}}(G_c, S) = \sum_{j=1}^{t} \lceil 3^j \rceil.$$

More generally, the parameter is additive over the connected components of the robust backbone.

Critical Thresholds and Stability Analysis

For an edge $uv \in E_c$, define

$$m_s(uv) = s_c(uv) - \text{CONN}_{G_c - uv}(u, v), \qquad m_\vartheta^\sigma(uv) = \tau^\sigma - \Delta^\sigma(u, v),$$

and

$$m_{\text{rpcs}}(uv) = \min\{m_s(uv), m_\vartheta^+(uv), m_\vartheta^-(uv)\}.$$

Theorem 8.1 (Margin characterization). *Under independent rectangular uncertainty,*

$$uv \in E_{\text{rpcs}}(G_c, S) \iff m_{\text{rpcs}}(uv) \geq 0.$$

Proof. The three margins are nonnegative exactly when the robust strength condition and both robust phase-compatibility conditions hold. □

Phase-Tolerance Thresholds

For a robust strong edge, define

$$\tau_{uv}^{+,*} = \Delta^+(u, v), \qquad \tau_{uv}^{-,*} = \Delta^-(u, v).$$

The edge is robustly phase compatible exactly when $\tau^\sigma \geq \tau_{uv}^{\sigma,*}$ for both signs. Under a common tolerance τ, this becomes

$$\tau \geq \tau_{uv}^*, \qquad \tau_{uv}^* = \max\{\Delta^+(u, v), \Delta^-(u, v)\}.$$

The endpoint is included because compatibility is defined using $\leq$.

Theorem 8.2 (Tolerance stability). *Let $t_1 < \cdots < t_q$ be the distinct values τ_{uv}^*. Then the robust backbone, and hence $\gamma^\tau{}_{\text{rpcsR}}$, is constant on each interval $[t_j, t_{j+1})$. Moreover, $w^\tau{}_{\text{rpcsR}}$ is non-increasing in τ.*

Proof. An edge enters precisely when τ reaches its threshold. No edge changes status be-tween consecutive thresholds. The result follows from the backbone-reduction and tolerance-monotonicity theorems.

Phase-Uncertainty Thresholds

Suppose $\delta_v^\sigma(v; \eta) = \eta a_v^\sigma(v)$, where $a_v^\sigma(v) \geq 0$. Then

$$\overline{\Delta}_\eta^\sigma(u, v) = \min_{\pi} \left\{ \hat{\Delta}^\sigma(u, v) + \eta\left(a_v^\sigma(u) + a_v^\sigma(v)\right) \right\}.$$

When the denominator is positive and the numerator is nonnegative, the componentwise critical radius is

$$\eta_{uv}^{\sigma,*} = \frac{\tau^\sigma - \widehat{\Delta}^\sigma(u, v)}{a_v^\sigma(u) + a_v^\sigma(v)}.$$

Theorem 8.3 (Phase-uncertainty threshold). *A robust strong edge remains robustly phase com-patible precisely for*

$$0 \leq \eta \leq \eta_{uv}^* := \min\{\eta_{uv}^{+,*}, \eta_{uv}^{-,*}\},$$

provided the nominal discrepancies do not exceed the corresponding minimum tolerances. It is removed only when $\eta > \eta_{uv}^*$.

Proof. Solving $\overline{\Delta}_\eta^\sigma(u, v) \leq \tau^\sigma$ for η gives the two componentwise inequalities; both must hold. □

For a uniform radius $\delta_v^\sigma(v) = \eta$,

$$\eta_{uv}^* = \frac{\tau - \max\{\hat{\Delta}^+(u, v), \hat{\Delta}^-(u, v)\}}{2}$$

under a common tolerance τ.

$(u, v)\}$

Amplitude Uncertainty and Piecewise Stability

Let $S_\Delta(\lambda)$ be a nested family of rectangular amplitude uncertainty sets and set

$$m_s(uv;\lambda) = \underline{s}_c(uv;\lambda) - \overline{\mathrm{CONN}}_{G_c-uv}(u, v;\lambda).$$

As λ increases, the first term cannot increase and the second cannot decrease. Hence $m_s(uv;\lambda)$ is non-increasing. Define

$$\lambda^*_{uv} = \sup\{\lambda \geq 0 : m_s(uv;\lambda) \geq 0\}.$$

If the margin is continuous and crosses zero at this value, the edge is robust strong for $\lambda \leq \lambda^*_{uv}$ and fails for $\lambda > \lambda^*_{uv}$.

Since the graph has finitely many edges, only finitely many edge thresholds occur on any parameter range satisfying these hypotheses. Consequently, the robust backbone and γ_{rpcsR} are piecewise constant between consecutive critical values. Expanding uncertainty makes the parameter non-decreasing, whereas increasing tolerance makes it non-increasing.

Illustrative Cyclic Network

This section illustrates the proposed framework using a six-vertex cyclic network. Unlike a tree, the network contains alternative paths between several pairs of vertices. Therefore, the robust strong-edge criterion is nontrivial.

Network Structure

Let

$V_c = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

and let the underlying edge set be

The edges

$E_c = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_4\}$.

$v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1$

form a cycle, while v_1v_4 is an additional communication link.

The underlying network is shown in Figure 1.

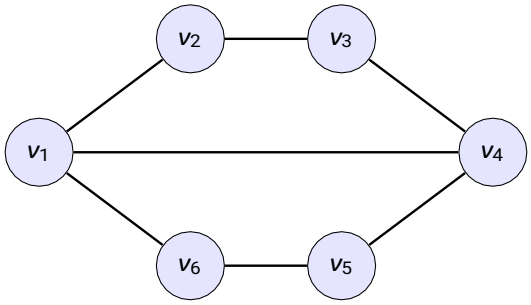

Figure 1: Underlying cyclic network

Vertex Amplitudes, Phases and Costs

For every vertex v_j, let

and

$$r^+_V(v_j) \in [0.96,\ 0.98]$$

and

$$r^-_V(v_j) \in [0.30,\ 0.32].$$

These intervals are sufficiently large to satisfy the required edge amplitude restrictions for every admissible realization.

For simplicity, the positive and negative nominal phases are taken to be equal:

$$\hat{\vartheta}^+_V(v_j) = \hat{\vartheta}^-_V(v_j).$$

The vertex information is presented in Table 1. All phase values are expressed as multiples of π.

Table 1: Vertex phase and cost information

Vertex	$\hat{\vartheta}_V/\pi$	$\hat{\vartheta}_V/\pi$	δ_V/π	δ_V/π	ω_c
v_1	0.00	0.00	0.01	0.01	1
v_2	0.04	0.04	0.01	0.01	1
v_3	0.08	0.08	0.01	0.01	1
v_4	0.12	0.12	0.01	0.01	1
v_5	0.18	0.18	0.01	0.01	1
v_6	0.22	0.22	0.01	0.01	1

The common minimum phase tolerance is initially fixed at
$\underline{\tau}^+ = \underline{\tau}^- = 0.07\pi$.
All vertex costs are equal to one. Therefore, the example uses the unit-cost RPCSNR domination number.

Edge-Amplitude Uncertainty

The positive and negative edge-amplitude intervals are given in Table 2.

Table 2: Edge-amplitude intervals and support-score bounds

Edge	$[\underline{r}^+, \overline{r}^+]$	$[\underline{r}^-, \overline{r}^-]$	$\underline{s}_c$	$\overline{s}_c$
v_1v_2	[0.92, 0.94]	[0.10, 0.12]	0.90	0.92
v_2v_3	[0.90, 0.92]	[0.12, 0.14]	0.88	0.90
v_3v_4	[0.88, 0.90]	[0.14, 0.16]	0.86	0.88
v_4v_5	[0.86, 0.88]	[0.16, 0.18]	0.84	0.86
v_5v_6	[0.84, 0.86]	[0.18, 0.20]	0.82	0.84
v_6v_1	[0.82, 0.84]	[0.20, 0.22]	0.80	0.82
v_1v_4	[0.80, 0.82]	[0.22, 0.24]	0.78	0.80

For example, the lower score of v_1v_2 is

$$\underline{s}_c(v_1v_2) = \frac{0.92 + 1 - 0.12}{2} = 0.90,$$

while its upper score is

$$\overline{s}_c(v_1v_2) = \frac{0.94 + 1 - 0.10}{2} = 0.92.$$

The other score bounds are obtained in the same way.

The other score bounds are obtained in the same way.

Robust Strong-Edge Determination

For every edge uv, the robust strong-edge condition is

$$\underline{s}_c(uv) \geq \overline{\mathrm{CONN}}_{G_c - uv}(u, v).$$

The relevant upper alternative connectedness strengths are shown in Table 3.

Table 3: Robust strong-edge calculations

Edge	$\underline{s}_c(uv)$	$\overline{\text{CONN}}_{G_c-uv}(u, v)$	Robust strong
v_1v_2	0.90	0.82	Yes
v_2v_3	0.88	0.82	Yes
v_3v_4	0.86	0.82	Yes
v_4v_5	0.84	0.82	Yes
v_5v_6	0.82	0.82	Yes
v_6v_1	0.80	0.84	No
v_1v_4	0.78	0.88	No

For the edge v_6v_1, the alternative path
$v6v5v4v3v2v1$
has upper path strength
$\min\{0.84, 0.86, 0.88, 0.90, 0.92\} = 0.84$.
Therefore,

$$\underline{s}_C(v_6v_1) = 0.80 < 0.84 = \overline{\text{CONN}}_{G_C-v_6v_1}(v_6, v_1).$$

Hence, v_6v_1 is not robust strong.
For the chord v_1v_4, the alternative path has upper path strength Since
$v_1v_2v_3v_4$
$\min\{0.92, 0.90, 0.88\} = 0.88$.
$\underline{s}_C(v_1v_4) = 0.78 < 0.88$,
the chord v_1v_4 is also not robust strong.
Consequently, the robust strong-edge set is
$E^{\text{rob}}(G_c, S) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6\}$.
Thus, amplitude uncertainty removes the edges
v_6v_1 and v_1v_4
from the robust strong-edge set.

Robust Phase-Compatibility Calculations

Since the positive and negative phase data are equal, both phase components have the same worst-case discrepancy.
For every edge uv,

$$\Delta^+(u, v) = \Delta^-(u, v) = \overline{\hat{\Delta}}(u, v) + 0.02\pi.$$

The phase calculations for the robust strong edges are presented in Table 4.

Table 4: Robust phase-compatibility calculations

Edge	Nominal discrepancy	Worst-case discrepancy	Robust phase compatible
v_1v_2	0.04π	0.06π	Yes
v_2v_3	0.04π	0.06π	Yes
v_3v_4	0.04π	0.06π	Yes
v_4v_5	0.06π	0.08π	No
v_5v_6	0.04π	0.06π	Yes

The edge v_4v_5 is phase compatible under the nominal phase values because

$$\hat{\Delta}(v_4, v_5) = 0.06\pi \leq 0.07\pi.$$

However, its worst-case discrepancy is

$$\overline{\hat{\Delta}}(v_4, v_5) = 0.06\pi + 0.01\pi + 0.01\pi = 0.08\pi.$$

Since

$$0.08\pi > 0.07\pi,$$

the edge v_4v_5 is not robustly phase compatible.
This demonstrates that an edge may be strong and nominally phase compatible but fail the robust phase-compatibility requirement.

Robust Phase-Compatible Backbone

Combining the robust strength and robust phase-compatibility conditions gives

$$E_{rpcs}(G_\sigma, S) = \{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_5v_6\}.$$

Thus, the robust backbone is

$$B_{rpcs}(G_\sigma, S) = P_4 \cup P_2,$$

where the first component is

$$v_1v_2v_3v_4$$

and the second component is

$$v_5v_6.$$

The complete edge classification is illustrated in Figure 2.

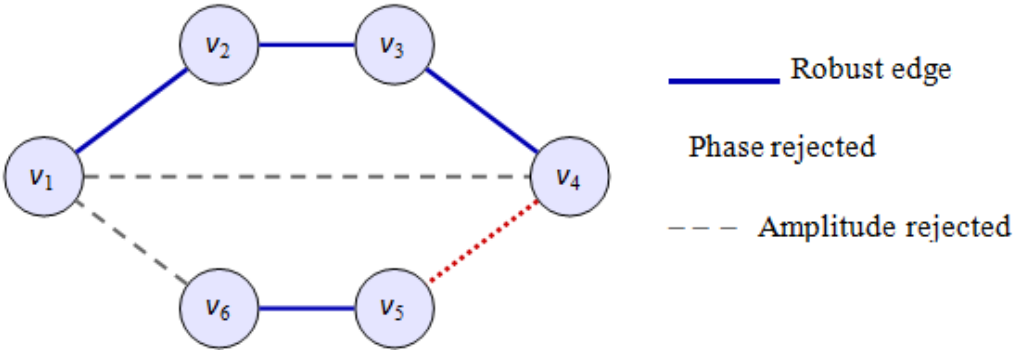

Figure 2: Classification of the uncertain network edges

Optimal Robust Roman Assignment

Since

$$B_{rpcs}(G_\sigma, S) = P_4 \cup P_2,$$

the component-additivity theorem gives

$$\gamma_{rpcsR}(G_\sigma, S) = \gamma_R(P_4) + \gamma_R(P_2).$$

Using

$$\gamma_R(P_n) = \frac{2n}{3},$$

we obtain

$$\gamma_{rpcsR}(G_\sigma, S) = \frac{8}{3} + \frac{4}{3} = 3 + 2 = 5.$$

One minimum RPCSNR dominating function is

$$f(v_1) = 0, \quad f(v_2) = 2, \quad f(v_3) = 0,$$
$$f(v_4) = 1, \quad f(v_5) = 0, \quad f(v_6) = 2.$$

The corresponding Roman partition is

$$V_0 = \{v_1, v_3, v_5\},$$
$$V_1 = \{v_4\},$$

and

$$V_2 = \{v_2, v_6\}.$$

Its weight is

$$w_f(f) = |V_1| + 2|V_2| = 1 + 2(2) = 5.$$

Therefore,

$$\boxed{\gamma_{rpcsR}(G_\sigma, S) = 5}.$$

Nominal and Robust Comparison

Under the nominal phases, the edge v_4v_5 is phase compatible. After removing only the two edges that fail the robust amplitude condition, the nominal phase-compatible backbone is

$$v_1v_2v_3v_4v_5v_6 \cong P_6.$$

Therefore, its nominal phase-compatible strong-neighbor Roman domination number is

$$\gamma_{pcsR}\, G_c(\hat{\xi}) = \gamma_R(P_6) = \frac{12}{3} = 4.$$

After incorporating phase uncertainty, v_4v_5 is removed and the minimum protection cost becomes

$$\gamma_{rpcsR}(G_\sigma, S) = 5.$$

Hence,

$$\gamma_{pcsR}\, G_c(\hat{\xi}) = 4 < 5 = \gamma_{rpcsR}(G_\sigma, S).$$

This strict inequality shows that the nominal model underestimates the protection required under parameter uncertainty.

Tolerance Sensitivity

The critical common tolerance of v_4v_5 is

$$\tau^*_{v_4v_5} = 0.08\pi.$$

For

$$0.06\pi \le \tau < 0.08\pi,$$

the robust phase-compatible backbone is

$$P_4 \cup P_2,$$

and

$$\gamma_{rpcsR}(G_\sigma, S) = 5.$$

At

$$\tau = 0.08\pi,$$

the edge v_4v_5 becomes robustly phase compatible because

$$\overline{\Delta}(v_4, v_5) = 0.08\pi \le \tau.$$

The robust backbone then becomes

$$P_6,$$

and the robust domination number decreases to

$$\gamma^{0.08\pi}_{rpcsR}(G_\sigma, S) = \gamma_R(P_6) = 4.$$

The result is summarized in Table 5.

Table 5: Effect of the common phase tolerance

Tolerance range	Robust backbone	γ_{rpcsR}
$[0.06\pi, 0.08\pi)$	$P_4 \cup P_2$	5
$[0.08\pi, \pi]$	P_6	4

This confirms that the robust domination number is piecewise constant and non-increasing with respect to the phase tolerance.

Conclusion

This paper developed robust phase-compatible strong-neighbor Roman domination for com-plex bipolar fuzzy graphs with uncertain amplitude, phase and tolerance parameters. Under independent rectangular edge-score uncertainty, the exact robust strong-edge condition is

$$s_c(uv) \geq \overline{\mathrm{CONN}}_{G_c-uv}(u, v),$$

while robust phase compatibility is characterized by

$$\overline{\Delta}^{\sigma}(u, v) \leq \underline{\tau}^{\sigma}, \qquad \sigma \in \{+, -\}.$$

These conditions generate the deterministic robust backbone $B_{\mathrm{rpcs}} = (V, E_{\mathrm{rpcs}})$.

The backbone-reduction theorem showed that

$$\gamma_{\mathrm{rpcsR}}(G, S) = \gamma_R^{\omega}(B_{\mathrm{rpcs}}),$$

which enabled bounds, monotonicity properties, component additivity and exact values for standard backbone classes. Critical tolerance, phase-uncertainty and amplitude-uncertainty levels were described by edge margins, and the parameter was proved piecewise constant between consecutive critical values.

The cyclic example demonstrated both mechanisms of edge loss. Two links failed the robust strength test because of competing paths, and one nominally compatible link failed under phase uncertainty. The resulting change from P_6 to $P_4 \cup P_2$ increased the Roman protection cost from 4 to 5.

The present model assumes a finite, static, simple and undirected network and primarily interval-based uncertainty. Future work may consider correlated or budgeted uncertainty, di-rected and temporal networks, large-scale algorithms, graph operations, and mixed vertex-edge disruptions. The framework provides a foundation for robust phase-sensitive protection in com-plex bipolar fuzzy networks.

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Author contributions

U. R. Ramakrishnan contributed to conceptualization, methodology, formal analysis, mathemat-ical investigation, development of the robust model and preparation of the original manuscript.

M. M. Shanmugapriya contributed to validation of the theoretical results, interpretation of the mathematical framework, review of the illustrative example and critical revision of the manuscript. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no conflict of interest.

Data availability and ethics statement

All numerical values used in the illustrative cyclic-network example are provided within the manuscript. No external dataset or software was used in this study. This study did not involve human participants, patient records, clinical trials or animal experiments; therefore, ethical approval and consent to participate were not required.

References

1. H. A. Ahangar, M. A. Henning, V. Samodivkin and I. G. Yero, Total Roman domination in graphs, *Applicable Analysis and Discrete Mathematics* **10** (2016), 501–517. https://doi.org/10.2298/AADM160802017A

2. M. P. Alvarez-Ruiz, T. W. Haynes, J. A. Rodriguez-Velazquez and L. Sigarreta, On the strong Roman domination number of graphs, *Discrete Applied Mathematics* **231** (2017), 44–59. https://doi.org/10.1016/j.dam.2016.12.013

3. R. A. Beeler, T. W. Haynes and S. T. Hedetniemi, Double Roman domination, *Discrete Applied Mathematics* **211** (2016), 23–29. https://doi.org/10.1016/j.dam.2016.03.017

4. A. Ben-Tal, L. El Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton University Press, Princeton, 2009. https://doi.org/10.1515/9781400831050

5. D. Bertsimas, D. B. Brown and C. Caramanis, Theory and applications of robust optimiza-tion, *SIAM Review* **53**(3) (2011), 464–501. https://doi.org/10.1137/080734510
6. E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, Roman domina-tion in graphs, *Discrete Mathematics* **278**(1–3) (2004), 11–22. https://doi.org/10.1016/ j.disc.2003.06.004
7. S. Gong, G. Wang and B. Davvaz, Domination of bipolar fuzzy graphs in various settings, *International Journal of Fuzzy Systems* **24** (2022), 2395–2409.
8. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998. https://doi.org/10.1201/9781482246582
9. J. Ma, G. Zhang and J. Lu, A method for multiple periodic factor prediction problems using complex fuzzy sets, *IEEE Transactions on Fuzzy Systems* **20**(1) (2012), 32–45. https://doi.org/10.1109/TFUZZ.2011.2164084
10. T. Mahmood and U. Ur Rehman, A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures, *International Journal of Intelligent Systems* **37**(1) (2022), 535–567. https://doi.org/10.1002/int.22639
11. O. T. Manjusha and M. S. Sunitha, Notes on domination in fuzzy graphs, *Journal of Intel-ligent & Fuzzy Systems* **27**(6) (2014), 3205–3212. https://doi.org/10.3233/IFS-141277
12. O. T. Manjusha and M. S. Sunitha, Strong domination in fuzzy graphs, *Fuzzy Information and Engineering* **7**(3) (2015), 369–377. https://doi.org/10.1016/j.fiae.2015.09.007
13. A. Mathew and M. S. Sunitha, Node connectivity and arc connectivity of a fuzzy graph, *In-formation Sciences* **180**(4) (2010), 519–531. https://doi.org/10.1016/j.ins.2009.10.006
14. M. Mojahedfar, A. A. Talebi and H. Rashmanlou, The novel concept of Roman domination in fuzzy graph, *Journal of Physical Sciences* **28** (2023), 1–9. https://doi.org/10.62424/ jps.2023.28.00.01
15. D. Ramot, R. Milo, M. Friedman and A. Kandel, Complex fuzzy sets, *IEEE Transactions on Fuzzy Systems* **10**(2) (2002), 171–186. https://doi.org/10.1109/91.995119
16. H. Rashmanlou, S. Samanta, M. Pal and R. A. Borzooei, Bipolar fuzzy graphs with categor-ical properties, *International Journal of Computational Intelligence Systems* **8**(5) (2015), 808–818. https://doi.org/10.1080/18756891.2015.1063243
17. A. Rosenfeld, Fuzzy graphs, in: L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shimura (Eds.), *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, Academic Press, New York, 1975, pp. 77–95. https://doi.org/10.1016/B978-0-12-775260-0.50008-6
18. J. C. Valenzuela-Tripodoro, P. Garcia-Vazquez and M. Cera Lopez, Strong edge-based Roman domination in fuzzy graphs, *Symmetry* **18**(9) (2026), Article 1452. https://doi. org/10.3390/sym18091452
19. O. Yazdanbakhsh and S. Dick, A systematic review of complex fuzzy sets and logic, *Fuzzy Sets and Systems* **338** (2018), 1–22. https://doi.org/10.1016/j.fss.2017.01.010
20. L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338–353. https://doi.org/ 10.1016/S0019-9958(65)90241-X
21. W.-R. Zhang, Bipolar fuzzy sets and relations: A computational framework for cogni-tive modeling and multiagent decision analysis, in: Proceedings of the IEEE International Conference on Fuzzy Systems, 1994, pp. 305–309. https://doi.org/10.1109/IJCF.1994. 375115