Minimax Theory and its Applications
Volume 11 (2026), No. 4s, 923 to 938

Formulation and Basic Properties of Fuzzy Fractional Delay Volterra–Fredholm Differential–Integral Equations

Dr. S. P. Geetha[1], S. Vidhya[2*], Dr. S. Sathya[3], R. Swathika[4], Dr. John Grasias S[5]

[1]Associate Professor, Department of Mathematics, Vellalar College for Women, Thindal, Erode, Tamil Nadu 638012, India
[2]Research Scholar, Department of Mathematics, Vellalar College for Women, Thindal, Erode, Tamil Nadu 638012, India
[3]Assistant Professor (Sl.G), Department of Mathematics, Kongu Engineering College, Erode, Tamil Nadu, India
[4]Assistant Professor, Department of Computer Applications, Vignan Foundation for Science and Technology and Research Vadlamudi, India
[5]Head and Assistant Professor, Department of Computer Science, AJK College of Arts and Science, Coimbatore, Tamil Nadu, India
Email: geetha_sams@rediffmail.com[1], vidhyakulandai@gmail.com[2], nagulpranav@gmail.com[3], rs_ca@vignan.ac.in[4], johngrasias@gmail.com[5]
*Corresponding author: S. Vidhya, vidhyakulandai@gmail.com

Abstract
Mathematical models incorporating uncertainty, memory, and hereditary effects play a crucial role in the realistic description of complex dynamical systems. In this article, analyze a new class of fuzzy fractional delay differential integral equa-tions involving both Volterra and Fredholm operators. The proposed model inte-grates fuzzy uncertainty through the α-cut framework, fractional dynamics via the Caputo derivative, and history-dependent behavior through explicit delay terms. By employing the α-cut representation of fuzzy-valued functions, the fuzzy system is transformed into a family of coupled fractional delay integro-differential equa-tions with interval-valued endpoints. Under suitable Lipschitz-type and continuity assumptions on the nonlinear terms and kernel functions, fundamental properties of the model are established, including well-posedness and existence–uniqueness of solutions using Banach fixed-point arguments. Two illustrative examples, including linear and nonlinear fuzzy fractional delay Volterra–Fredholm models, are presented to demonstrate the applicability of the theoretical results and to verify the consis-tency of the proposed formulation with the corresponding classical deterministic models.

Keywords*:* Fuzzy fractional differential equations; Caputo derivative; Delay systems; Volterra–Fredholm integral operators; α-cut representation; Existence and uniqueness.

Introduction

Mathematical models capable of representing uncertainty, memory, and hereditary effects are essential for the accurate description of many complex dynamical systems arising in engineering, biological sciences, control theory, and materials science. In practical appli-cations, system parameters, initial conditions, and external influences are often impre-cise or linguistically defined, making classical deterministic models inadequate. More-over, many physical and biological processes exhibit long-range dependence and history-sensitive behavior that cannot be captured by integer-order differential equations. These limitations motivate the development of hybrid mathematical frameworks that simulta-neously incorporate uncertainty modeling, fractional-order dynamics, delay effects, and integral operators.

Fuzzy set theory, introduced by Zadeh [1], provides a natural mathematical founda-tion for modeling imprecision. In the context of dynamical systems, fuzzy differential equations allow uncertainty to be embedded directly into the governing equations. The α-cut approach proposed by Seikkala [2] plays a central role in fuzzy calculus by reducing fuzzy-valued problems to families of interval-valued endpoint systems, thereby enabling the application of classical analytical techniques. Subsequent developments by Klir and Yuan [3] and Bede [4] refined fuzzy arithmetic, differentiability concepts, and continuity properties, forming the standard framework for fuzzy dynamical system analysis.

Fractional calculus has emerged as an effective mathematical tool for capturing mem-ory and hereditary characteristics in dynamical systems. Fractional derivatives generalize classical differentiation by incorporating nonlocal effects, which are essential for describing anomalous diffusion, viscoelastic behavior, and nonlocal transport phenomena. Among the various definitions of fractional derivatives, the Caputo derivative [5] is particularly suitable for physical and engineering applications due to its compatibility with classical initial conditions. Foundational works by Podlubny [6] and Kilbas et al. [7] established the analytical basis for fractional differential equations and demonstrated their superiority over integer-order models in many applications.

Integral equations of Volterra and Fredholm type provide a natural extension of differ-ential equations for modeling cumulative memory effects and global interactions. Volterra operators describe temporal accumulation, whereas Fredholm operators represent inter-actions over fixed domains. The functional analytic foundations of these equations are well documented in the classical literature [8,9]. When delay arguments are incorporated,such models explicitly account for history-dependent dynamics, further enhancing their descriptive capability but also increasing analytical complexity.

In recent years, increasing attention has been devoted to fuzzy fractional integro-differential equations. Allahviranloo [19] developed a rigorous framework for fuzzy frac-tional calculus consistent with the α-cut interpretation. Ahmad et al. [12] analyzed fuzzy fractional Volterra–Fredholm integro-differential equations, while Abbas et al. [10] estab-lished existence, uniqueness, and stability results for weighted fuzzy fractional Volterra–Fredholm models. Further contributions by Ghazouani et al. [11] addressed nonlinear fuzzy fractional Volterra–Fredholm systems under Ulam–Hyers stability criteria. In ad-dition, several numerical and semi-analytical approaches, including spectral and decom-position methods, have been proposed for solving related fuzzy fractional models [13, 14].

Despite these advances, the existing literature largely focuses on partial combinations of the modeling components. Most studies consider fuzzy fractional systems without explicit delay terms, or fractional delay equations without coupled Volterra and Fred-holm integral operators. Models that simultaneously integrate fuzzy uncertainty, Caputo fractional dynamics, explicit delay dependence, and both Volterra and Fredholm integral operators remain relatively unexplored. Moreover, when fuzzy delay Volterra–Fredholm integro-differential equations are studied, the emphasis is often placed on numerical ap-proximation techniques rather than on foundational analytical properties. For example, the recent work of Yookesh et. al. [16, 17] primarily addresses numerical decomposition methods, leaving the theoretical aspects of well-posedness insufficiently developed.

Motivated by these observations, the present article formulates and analyzes a gen-eral class of fuzzy fractional delay differential–integral equations involving both Volterra and Fredholm operators. By employing the α-cut representation, the proposed fuzzy problem is systematically reduced to a family of coupled real-valued fractional delay integro-differential equations. Under suitable continuity and Lipschitz-type assumptions, existence and uniqueness of solutions are established using Banach fixed-point arguments. The resulting theoretical framework unifies fuzzy uncertainty, fractional dynamics, delay effects, and integral operators within a single analytically tractable model.

Preliminaries

In this section, we recall basic definitions and fundamental results related to fuzzy num-bers, α-cut representations, fractional calculus, and delay functions, which are required for the formulation and analysis of fuzzy fractional delay differential–integral equations. Throughout this paper, R denotes the set of real numbers, and C([a, b], R) denotes the space of continuous real-valued functions on [a, b].

Fuzzy Numbers and α-Cut Representation

Definition 2.1. A fuzzy set $A\tilde{}$ on R is characterized by a membership function $\mu A\tilde{} : R \to [0, 1]$.

Definition 2.2. A fuzzy set $u\tilde{}$ on R is called a fuzzy number if it satisfies the following conditions:

$u\tilde{}$ is normal, i.e., $\sup_{x \in R} \mu u\tilde{}(x) = 1$;

$u\tilde{}$ is fuzzy convex, that is, $\mu u\tilde{}(\lambda x + (1 - \lambda)y) \geq \min\{\mu u\tilde{}(x), \mu u\tilde{}(y)\}$ for all $x, y \in R$ and $\lambda \in [0, 1]$;

$\mu u\tilde{}$ is upper semi-continuous;

The support $\{x \in R : \mu u\tilde{}(x) > 0\}$ is compact. The family of all fuzzy numbers is denoted by E.

Definition 2.3. [4] For a fuzzy number $u\tilde{} \in E$ and $\alpha \in [0, 1]$, the α-cut of $u\tilde{}$ is defined as

$[u\tilde{}]\alpha = \{x \in R : \mu u\tilde{}(x) \geq \alpha\} = [u^{\alpha}, u^{\alpha}]$,

where u^{α} and u^{α} denote the lower and upper endpoints, respectively.

It is well known that for each $\alpha \in [0, 1]$, the α-cut of a fuzzy number is a closed and bounded interval, and the mappings $\alpha \mapsto u\alpha$ and $\alpha \mapsto u\alpha$ are monotone functions with respect to α.

Fuzzy-Valued Functions and Seikkala Differentiability

Let $[0, T] \subset R$ and let $u\tilde{} : [0, T] \to E$ be a fuzzy-valued function. For each $t \in [0, T]$ and $\alpha \in [0, 1]$, the α-cut representation of $u\tilde{}(t)$ is given by

$$[\tilde{u}(t)]^{\alpha} = \underline{u}(t, \alpha), \overline{u}(t, \alpha) .$$

Definition 2.4. A fuzzy-valued function $u\tilde{}(t)$ is said to be Seikkala differentiable at $t \in [0, T]$ if, for every $\alpha \in [0, 1]$, the functions $u(t, \alpha)$ and $u(t, \alpha)$ are differentiable with respect to t and satisfy

$$\frac{d}{dt}\underline{u}(t,\alpha) \le \frac{d}{dt}\overline{u}(t,\alpha), \qquad \forall\, t \in [0,T].$$

Under Seikkala differentiability, the derivative of $\tilde{u}(t)$ is defined by

$$\left[\frac{d\tilde{u}(t)}{dt}\right]^{\alpha} = \left[\frac{d}{dt}\underline{u}(t,\alpha), \frac{d}{dt}\overline{u}(t,\alpha)\right], \qquad \alpha \in [0,1].$$

Fractional Calculus

Let $0 < \nu < 1$. The fractional integral and fractional derivative in the sense of Caputo are defined as follows.

Definition 2.5. The Riemann–Liouville fractional integral of order ν of a function $u \in L1([0, T])$ is defined by

$$(I^{\nu}u)(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)\,ds, \qquad t \in [0,T],$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.6. The Caputo fractional derivative of order ν of a function $u \in C1([0, T])$ is defined by

$$D^{\nu}_{C}(u)(t) = \frac{1}{\Gamma(1-\nu)}\int_0^t (t-s)^{-\nu}u'(s)\,ds, \qquad t \in [0,T].$$

For fuzzy-valued functions, the Caputo fractional derivative is interpreted in the α-cut sense. That is, for a fuzzy function $\tilde{u}(t)$,

$$[D^{\nu}_{C}\tilde{u}(t)]^{\alpha} = \left[D^{\nu}_{C}\underline{u}(t,\alpha), D^{\nu}_{C}\overline{u}(t,\alpha)\right], \qquad \alpha \in [0,1].$$

Delay Functions and Integral Operators

Let $\tau : [0, T] \to [0, T]$ be a continuous delay function satisfying $0 \le \tau(t) \le t$ for all $t \in [0, T]$. For a fuzzy-valued function $\tilde{u}(t)$, the delayed term $\tilde{u}(\tau(t))$ is defined pointwise via its α-cut representation.

Let $K_1(t, s)$ and $K_2(t, s)$ be continuous kernel functions defined on $[0, T] \times [0, T]$. The Volterra and Fredholm integral operators acting on a fuzzy-valued function $\tilde{u}(t)$ are defined, respectively, by

$$\int_0^t K_1(t,s)\tilde{u}(s)\,ds, \qquad \int_0^T K_2(t,s)\tilde{u}(s)\,ds,$$

where the integrals are understood in the α-cut sense.

Remark 2.1. The α-cut formulation allows fuzzy fractional delay differential–integral equations to be transformed into systems of coupled fractional delay integro-differential equations with real-valued endpoints. This representation forms the basis for the theoret-ical analysis carried out in subsequent sections.

Problem Formulation

In this section, the general class of fuzzy fractional delay differential–integral equations considered in this work and state the assumptions required for subsequent theoretical analysis.

Let $T > 0$ be a fixed real number and let $\tilde{u} : [0, T] \to E$ be an unknown fuzzy-valued function.Consider the following fuzzy fractional delay differential–integral equation:

$$D^{\nu}_{C}\tilde{u}(t) = \tilde{f}\left(t, \tilde{u}(t), \tilde{u}(\tau(t))\right) + \int_0^t K_1(t,s)\,\tilde{g}\left(s, \tilde{u}(s), \tilde{u}(\tau(s))\right)ds + \int_0^T K_2(t,s)\,\tilde{h}\left(s, \tilde{u}(s), \tilde{u}(\tau(s))\right)ds, \qquad t \in [0,T], \tag{1}$$

subject to the fuzzy initial condition

$$\tilde{u}(0) = \tilde{u}_0, \tag{2}$$

where $0 < \nu < 1$, $\tilde{u}_0 \in E$ is a given fuzzy number, and $\tau : [0, T] \to [0, T]$ is a continuous delay function satisfying $0 \le \tau(t) \le t$ for all $t \in [0, T]$.

Here, $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ are given fuzzy-valued functions defined on $[0, T] \times E \times E$, and

$K_1(t, s)$ and $K_2(t, s)$ are real-valued kernel functions corresponding to the Volterra and Fredholm integral operators, respectively.

Equivalent Integral Formulation

Using the properties of the Caputo fractional derivative, problem (1)–(2) is equivalent to the following fuzzy fractional integral equation:

$$\tilde{u}(t) = \tilde{u}_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \tilde{f}(s, \tilde{u}(s), \tilde{u}(\tau(s))) + \int_0^s K_1(s, \xi)\, \tilde{g}(\xi, \tilde{u}(\xi), \tilde{u}(\tau(\xi)))\, d\xi + \int_0^T K_2(s, \xi)\, \tilde{h}(\xi, \tilde{u}(\xi), \tilde{u}(\tau(\xi)))\, d\xi\, ds, \quad (3)$$

for all $t \in [0, T]$.

The integral formulation (3) will be used to establish the existence and uniqueness of solutions using fixed-point techniques.

Assumptions

To ensure the well-posedness of problem (1) impose the following assumptions:

(A1) The functions $\tilde{f}$, $[0, T] \times E \times E$.

$\tilde{g}$, and $\tilde{h}$

are continuous with respect to all their arguments on

(A2) There exist positive constants L_f, L_g, and L_h such that, for all $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \in E$ and $t \in [0, T]$,

where $d(\cdot, \cdot)$ denotes the Hausdorff metric

$$d\tilde{f}(t, \tilde{u}_1, \tilde{v}_1), \tilde{f}(t, \tilde{u}_2, \tilde{v}_2) \leq L_f\, d(\tilde{u}_1, \tilde{u}_2) + d(\tilde{v}_1, \tilde{v}_2),$$

$$d\tilde{g}(t, \tilde{u}_1, \tilde{v}_1), \tilde{g}(t, \tilde{u}_2, \tilde{v}_2) \leq L_g\, d(\tilde{u}_1, \tilde{u}_2) + d(\tilde{v}_1, \tilde{v}_2),$$

$$d\tilde{h}(t, \tilde{u}_1, \tilde{v}_1), \tilde{h}(t, \tilde{u}_2, \tilde{v}_2) \leq L_h\, d(\tilde{u}_1, \tilde{u}_2) + d(\tilde{v}_1, \tilde{v}_2),$$

o n E

(A3) The kernel functions K1(t, s) and K2(t, s) are continuous on $[0, T] \times [0, T]$ and satisfy

$$\sup_{t \in [0,T]} \int_0^t |K_1(t, s)|\, ds \leq M_1, \qquad \sup_{t \in [0,T]} \int_0^T |K_2(t, s)|\, ds \leq M_2,$$

for some positive constants M_1 and M_2

(A4) The delay function $\tau(t)$ is continuous on $[0, T]$ and satisfies $0 \leq \tau(t) \leq t$

Remark 3.1. Under assumptions (A1)–(A4), the fuzzy fractional delay differential–integral equation (1) admits a well-defined α-cut representation, and the corresponding endpoint system can be analyzed using classical fixed-point methods.

α-Cut Reduction and Equivalent Endpoint System

In this section, the α-cut technique applied to the fuzzy fractional delay differential–integral equation (1) and derive an equivalent family of real-valued fractional delay integro-differential equations governing the lower and upper endpoints of the fuzzy solu-tion.

Let $\tilde{u} : [0, T] \to E$ be a fuzzy-valued function satisfying (1)–(2). For each $t \in [0, T]$ and $\alpha \in [0, 1]$, the α-cut of $\tilde{u}(t)$ is represented as

$$[\tilde{u}(t)]^\alpha = \underline{u}(t, \alpha), \overline{u}(t, \alpha).$$

Similarly, the α-cut representations of the fuzzy-valued functions $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$ are defined, respectively, by $[\tilde{f}(t, \tilde{u}(t), \tilde{u}(\tau(t)))]\alpha = f(t, u(t, \alpha), u(t, \alpha), u(\tau(t), \alpha), u(\tau(t), \alpha))$, $f(t, u(t, \alpha), \underline{u}(t, \alpha), u(\tau(t), \alpha), u(\tau(t), \alpha$ with analogous definitions for $\tilde{g}$ and $\tilde{h}$.

Endpoint System

Applying the α-cut operator to both sides of (1) and using the properties of the Caputo fractional derivative,Obtain the following coupled system for each fixed $\alpha \in [0, 1]$:

$$
\begin{aligned}
D^{\nu}_{C}\underline{u}(t,\alpha) &= \underline{f}\Big(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), \underline{u}(\tau(t),\alpha), \overline{u}(\tau(t),\alpha) \\
&\quad + \int_0^t K_1(t,s)\,\underline{g}\Big(s, \underline{u}(s,\alpha), \overline{u}(s,\alpha), \underline{u}(\tau(s),\alpha), u(\tau(s),\alpha)\Big)\,ds \\
&\quad + \int_0^T K_2(t,s)\,\underline{h}\Big(s, \underline{u}(s,\alpha), \overline{u}(s,\alpha), \underline{u}(\tau(s),\alpha), \overline{u}(\tau(s),\alpha)\Big)\,ds,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
D^{\nu}_{C}\overline{u}(t,\alpha) &= \overline{f}\Big(t, \underline{u}(t,\alpha), \overline{u}(t,\alpha), \underline{u}(\tau(t),\alpha), \overline{u}(\tau(t),\alpha) \\
&\quad + \int_0^t K_1(t,s)\,\overline{g}\Big(s, \underline{u}(s,\alpha), \overline{u}(s,\alpha), \underline{u}(\tau(s),\alpha), \overline{u}(\tau(s),\alpha)\Big)\,ds \\
&\quad + \int_0^T K_2(t,s)\,\overline{h}\Big(s, \underline{u}(s,\alpha), \overline{u}(s,\alpha), \underline{u}(\tau(s),\alpha), \overline{u}(\tau(s),\alpha)\Big)\,ds,
\end{aligned}
$$

subject to the initial conditions

$$
\underline{u}(0,\alpha) = \underline{u}_0^{\alpha}, \qquad \overline{u}(0,\alpha) = \overline{u}_0^{\alpha}. \tag{5}
$$

Equivalent Integral Form

Using the integral representation of the Caputo fractional derivative, system (4)–(5) is equivalent to the following system of fractional integral equations:

$$
\begin{aligned}
\underline{u}(t,\alpha) &= \underline{u}_0^{\alpha} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\Big[\underline{f}(\cdot) + \int_0^s K_1(s,\xi)\,\underline{g}(\cdot)\,d\xi + \int_0^T K_2(s,\xi)\,\underline{h}(\cdot)\,d\xi\Big]\,ds, \\
\overline{u}(t,\alpha) &= \overline{u}_0^{\alpha} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\Big[\overline{f}(\cdot) + \int_0^s K_1(s,\xi)\,\overline{g}(\cdot)\,d\xi + \int_0^T K_2(s,\xi)\,\overline{h}(\cdot)\,d\xi\Big]\,ds,
\end{aligned}
\tag{6}
$$

where $(\cdot)$ denotes the corresponding endpoint functions defined in (4).

Remark 4.1. For each $\alpha \in [0, 1]$, system (4) constitutes a coupled system of real-valued fractional delay integro-differential equations. The fuzzy solution $\tilde{u}(t)$ exists if and only if
$\underline{u}(t, \alpha) \le \overline{u}(t, \alpha), \quad \forall\, t \in [0, T], \alpha \in [0, 1]$.

Remark 4.2. The α-cut reduction transforms the original fuzzy fractional delay differential–integral equation into a parameterized family of classical problems, enabling the application of standard fixed-point techniques for establishing existence and uniqueness of solutions.

Existence and Uniqueness of Solutions

In this section, discussed the existence and uniqueness of solutions to the fuzzy fractional delay differential–integral equation (1). The analysis is carried out by applying Banach's fixed-point theorem to the equivalent α-cut endpoint system derived in Section 4.

Functional Setting

For a fixed $\alpha \in [0, 1]$, let

$X = C([0, T], R) \times C([0, T], R)$,
equipped with the norm

$$\|(u_1, u_2)\|_X = \max_{t\in[0,T]} \{|u_1(t)|, |u_2(t)|\}.$$

Then $(X, \|\cdot\|_x)$ is a Banach space.

Define the operator $T : X \to X$ by

$$T(u_1, u_2) = T_1(u_1, u_2), T_2(u_1, u_2),$$

where

$$T_1(u_1, u_2)(t) = u_0^{\alpha} + \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} f(\cdot) + \int_0^s K_1(s, \xi)\, g(\cdot)\, d\xi + \int_0^T K_2(s, \xi)\, h(\cdot)\, d\xi \; ds, \tag{7}$$

$$T_2(u_1, u_2)(t) = \bar{u}_0^{\alpha} + \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} \bar{f}(\cdot) + \int_0^s K_1(s, \xi)\, \bar{g}(\cdot)\, d\xi + \int_0^T K_2(s, \xi)\, \bar{h}(\cdot)\, d\xi \; ds, \tag{8}$$

where
and $(\cdot)$ denotes the corresponding endpoint functions defined in Section 4.

Existence and Uniqueness at the α-Level

Theorem 5.1. Assume that conditions (A1)–(A4) hold. If

$$\frac{2(L_f + M_1 L_a + M_2 L_h)T^v}{\Gamma(v+1)} < 1, \tag{9}$$

then, for each $\alpha \in [0, 1]$, *the endpoint system* (4)–(5) *admits a unique solution*

$$(u(\cdot, \alpha), \bar{u}(\cdot, \alpha)) \in X.$$

Proof. Let $(u_1, u_2), (v_1, v_2) \in X$. Using assumptions (A2) and (A3), we estimate

$$\|T_1(u_1, u_2) - T_1(v_1, v_2)\|_X \le \frac{(L_f + M_1 L_g + M_2 L_h)T^v}{\Gamma(v+1)} \|u_1 - v_1\|_X + \|u_2 - v_2\|_X .$$

A similar inequality holds for T_2. Consequently,

$$\|T(u_1, u_2) - T(v_1, v_2)\|_X \le \frac{2(L_f + M_1 L_g + M_2 L_h)T^v}{\Gamma(v+1)} \|(u_1, u_2) - (v_1, v_2)\|_X.$$

Condition (9) ensures that T is a contraction on X. Hence, by Banach's fixed-point theorem, T has a unique fixed point in X, which corresponds to a unique solution of the endpoint system.

Existence and Uniqueness of the Fuzzy Solution

Theorem 5.2. Under the assumptions of Theorem 5.1, the fuzzy fractional delay differential–integral equation (1)–(2) admits a unique fuzzy solution

$$\tilde{u} \in C([0, T], E).$$

Proof. By Theorem 5.1, for each $\alpha \in [0, 1]$, there exists a unique solution $(u(t, \alpha), \bar{u}(t, \alpha))$ of the endpoint system. The monotonicity and continuity of the endpoint functions with respect to α follow from the properties of fuzzy arithmetic and the α-cut representation. Moreover,

$u(t, \alpha) \le \bar{u}(t, \alpha), \quad \forall\, t \in [0, T], \alpha \in [0, 1].$

Therefore, the family of α-cuts defines a unique fuzzy-valued function $\tilde{u}(t)$ satisfying (1). This completes the proof.

Remark 5.1. The contraction condition (9) provides an explicit sufficient criterion relat-ing the fractional order ν, the Lipschitz constants of the nonlinear terms, and the kernel bounds. This condition highlights the combined influence of fractional dynamics, delay effects, and integral operators on the well-posedness of the fuzzy system.

Illustrative Examples

In this section,Two illustrative examples are given to demonstrate the applicability and generality of the theoretical results established in the previous sections. The examples are intentionally kept simple and analytical in nature, with the objective of verifying the assumptions required for existence and uniqueness, rather than providing numerical approximations or simulations.

Example 6.1: Linear Fuzzy Fractional Delay Volterra–Fredholm Model

Consider the fuzzy fractional delay Volterra–Fredholm differential–integral equation

$$D^{\nu}_{C}\tilde{u}(t) = \lambda\tilde{u}(t) + \mu\tilde{u}(\tau(t)) + \int_0^t (t-s)\tilde{u}(s)\,ds + \int_0^T e^{-(t+s)}\tilde{u}(s)\,ds, \qquad t \in [0, T], \tag{10}$$

subject to the triangular fuzzy initial condition

$$\tilde{u}(0) = \tilde{u}0 = (1, 2, 3),$$

where $0 < \nu < 1$, $\lambda, \mu \in R$ are constants, and the delay function is given by

$$\tau(t) = \frac{t}{2},$$

which satisfies $0 \le \tau(t) \le t$ for every $t \in [0, T]$.

The nonlinear mappings are defined by

$$\tilde{f}(t, \tilde{u}, \tilde{v}) = \lambda\tilde{u} + \mu\tilde{v},$$

$$\tilde{g}(t, \tilde{u}, \tilde{v}) = \tilde{u},$$

and

$$\tilde{h}(t, \tilde{u}, \tilde{v}) = \tilde{u},$$

with kernel functions

$$K_1(t, s) = t - s, \qquad K_2(t, s) = e^{-(t+s)}.$$

α-cut representation: For the triangular fuzzy number

$\tilde{u}0 = (1, 2, 3)$,

the α-cut representation is given by

$[\tilde{u}0]\alpha = [1 + \alpha, 3 - \alpha], \quad 0 \le \alpha \le 1.$

Similarly, the fuzzy-valued solution is represented by

$[\tilde{u}(t)]\alpha = [u(t, \alpha), u(t, \alpha)]$,

where

$$\underline{u}(t, \alpha) \le \overline{u}(t, \alpha), \qquad t \in [0, T], \ \alpha \in [0, 1].$$

Applying the α-cut representation to (10), the fuzzy fractional delay differential–integral equation is transformed into an equivalent family of coupled endpoint systems.

Equivalent endpoint system: For each fixed $\alpha \in [0, 1]$, the lower endpoint satisfies

$$D^{\nu}_{C}\underline{u}(t, \alpha) = \lambda\underline{u}(t, \alpha) + \mu\,\underline{u}\left(\frac{t}{2}, \alpha\right) + \int_0^t (t-s)\underline{u}(s, \alpha)\,ds + \int_0^T e^{-(t+s)}\,\underline{u}(s, \alpha)\,ds, \tag{11}$$

subject to

$$\underline{u}(0, \alpha) = 1 + \alpha.$$

Similarly, the upper endpoint satisfies

$$D^{\nu}_{C} \overline{u}(t, \alpha) = \lambda \overline{u}(t, \alpha) + \mu \overline{u}\left(\frac{t}{2}, \alpha\right) + \int_0^t (t - s)\overline{u}(s, \alpha)\, ds + \int_0^T e^{-(t+s)}\, \overline{u}(s, \alpha)\, ds, \tag{12}$$

subject to

$$\overline{u}(0, \alpha) = 3 - \alpha.$$

Equations (11) and (12) constitute a coupled family of real-valued fractional delay Volterra–Fredholm integro-differential equations parameterized by α ∈ [0, 1]. Conse-quently, the original fuzzy problem is reduced to the corresponding endpoint system, thereby illustrating the α-cut methodology developed in Section 4.

Verification of the assumptions: To verify that the hypotheses of Theorem 5.2 are satisfied.

The mappings

$$\tilde{f}(t, \tilde{u}, \tilde{v}) = \lambda \tilde{u} + \mu \tilde{v}, \qquad \tilde{g}(t, \tilde{u}, \tilde{v}) = \tilde{u}, \qquad \tilde{h}(t, \tilde{u}, \tilde{v}) = \tilde{u},$$

are continuous with respect to all their arguments on $[0, T] \times E \times E$.

Moreover, for any fuzzy numbers $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \in E$, the Hausdorff metric satisfies

$$\begin{aligned} d\left(\tilde{f}(t, \tilde{u}_1, \tilde{v}_1), \tilde{f}(t, \tilde{u}_2, \tilde{v}_2)\right) &\le |\lambda|\, d(\tilde{u}_1, \tilde{u}_2) + |\mu|\, d(\tilde{v}_1, \tilde{v}_2) \\ &\le (|\lambda| + |\mu|)\left[d(\tilde{u}_1, \tilde{u}_2) + d(\tilde{v}_1, \tilde{v}_2)\right]. \end{aligned}$$

Hence,

$$L_f = |\lambda| + |\mu|.$$

Similarly,

$$d(\tilde{g}_1, \tilde{g}_2) \le d(\tilde{u}_1, \tilde{u}_2),$$

and

$$d(\tilde{h}_1, \tilde{h}_2) \le d(\tilde{u}_1, \tilde{u}_2),$$

which imply

$$L_g = L_h = 1.$$

Next, the Volterra kernel satisfies

$$\int_0^t |K_1(t, s)|\, ds = \int_0^t (t - s)\, ds = \frac{t^2}{2} \le \frac{T^2}{2}.$$

Therefore,

$$M_1 = \sup_{t\in[0,T]} \int_0^t |K_1(t,s)|\,ds = \frac{T^2}{2}.$$

Similarly,

$$\int_0^T |K_2(t,s)|\,ds = \int_0^T e^{-(t+s)}\,ds = e^{-t}(1-e^{-T}) \le 1-e^{-T},$$

and hence

$$M_2 = \sup_{t\in[0,T]} \int_0^T |K_2(t,s)|\,ds = 1-e^{-T}.$$

Finally, the delay function

$$\tau(t) = \frac{t}{2}$$

is continuous and satisfies

$$0 \le \tau(t) \le t, \qquad t \in [0,T].$$

Therefore, assumptions (A1)–(A4) are completely verified.

Verification of the contraction condition: To illustrate the applicability of Theorem 5.2, consider the parameter values

$$\lambda = 0.05, \qquad \mu = 0.02, \qquad \nu = 0.8, \qquad T = 0.2.$$

Then,

$$L_f = 0.07, \qquad L_g = L_h = 1,$$

and

$$M_1 = \frac{0.2^2}{2} = 0.02, \qquad M_2 = 1-e^{-0.2} \approx 0.1813.$$

Consequently,

$$q = \frac{2(L_f + M_1L_a + M_2L_h)T^{\nu}}{\Gamma(\nu+1)} = \frac{2(0.07+0.02+0.1813)(0.2)^{0.8}}{\Gamma(1.8)} \approx 0.20 < 1.$$

Hence, the contraction condition of Theorem 5.2 is satisfied. Therefore, the endpoint operator is a contraction on the associated Banach space and admits a unique fixed point.

Application of Theorem 5.2. Since the mappings f˜, g˜, and h˜ are continuous and satisfy the required Lipschitz conditions, the kernel functions satisfy the boundedness assumption, and the delay function is continuous with $0 \le \tau(t) \le t$, all assumptions (A1)–(A4) are fulfilled. Furthermore, the contraction constant satisfies

$q < 1$,

which guarantees that the associated operator is a contraction on the Banach space of continuous fuzzy-valued functions. Therefore, by Theorem 5.2, problem (10) admits a unique fuzzy solution

$\tilde{u} \in C([0, T], E)$.

Consequently, for every $\alpha \in [0, 1]$, the endpoint system (11)–(12) possesses a unique pair of continuous solutions $(u(t, \alpha), \overline{u}(t, \alpha))$,

which together determine the unique fuzzy solution of the original fractional delay Volterra–Fredholm differential–integral equation.

Comparison with the corresponding crisp model. To demonstrate that the pro-posed formulation generalizes the classical theory, consider the corresponding crisp frac-tional delay Volterra–Fredholm differential–integral equation obtained by replacing the fuzzy-valued function with a real-valued function u(t):

$$D^{\nu}_{c} u(t) = \lambda u(t) + \mu u\left(\frac{t}{2}\right) + \int_0^t (t - s)u(s)\, ds + \int_0^T e^{-(t+s)}u(s)\, ds, \qquad (13)$$

subject to the initial condition

$u(0) = 2$,

which corresponds to the crisp counterpart of the triangular fuzzy initial condition $\tilde{u}0 = (1, 2, 3)$.

Observe that if the uncertainty associated with the fuzzy initial condition vanishes, that is,

$[\tilde{u}0]_\alpha = [c, c], \quad 0 \le \alpha \le 1$,

for some constant $c \in R$, then

$u(t, \alpha) = \overline{u}(t, \alpha) = u(t)$,

and the fuzzy model reduces exactly to the classical fractional delay Volterra–Fredholm differential–integral equation (13). Hence, the proposed fuzzy formulation naturally ex-tends the corresponding deterministic model while preserving its mathematical structure. Furthermore, the fuzzy solution may be interpreted as an interval-valued extension of the crisp solution, where the lower and upper endpoint functions quantify the uncertainty propagated through the fractional dynamics, delay term, and Volterra–Fredholm integral operators. Thus, the deterministic solution is recovered as a special limiting case when the uncertainty width approaches zero.

Proposition 6.1 (Consistency with the Classical Model). Suppose that the fuzzy initial condition u˜0 degenerates to a singleton fuzzy number, i.e.,

$[\tilde{u}0]_\alpha = [c, c], \quad \forall\, \alpha \in [0, 1]$,

for some constant $c \in R$, and that the fuzzy-valued mappings f˜, g˜, and h˜ reduce to their corresponding real-valued functions f, g, and h. Then the fuzzy fractional delay Volterra–Fredholm differential–integral equation (10) reduces to the corresponding classi-cal fractional delay Volterra–Fredholm differential–integral equation (13). Consequently, the unique fuzzy solution collapses to the unique classical solution of the deterministic problem.

Proof. If the fuzzy initial condition is a singleton, then its lower and upper endpoints coincide for every $\alpha \in [0, 1]$, that is,

$u0(\alpha) = \overline{u0}(\alpha) = c$.

Similarly, when the fuzzy mappings f˜, g˜, and h˜ reduce to their crisp counterparts, the lower and upper endpoint equations obtained through the α-cut representation become identical. Hence,

$u(t, \alpha) = \overline{u}(t, \alpha) = u(t), \quad \forall\, t \in [0, T], \forall\, \alpha \in [0, 1]$,

where u(t) satisfies the classical fractional delay Volterra–Fredholm differential–integral equation (13). Therefore, the interval-valued fuzzy solution collapses to the unique de-terministic solution, establishing the consistency of the proposed fuzzy formulation with the corresponding classical model.

Remark 6.1. Example 6.1 confirms that the proposed fuzzy fractional delay Volterra–Fredholm model is a natural extension of its classical counterpart. The α-cut methodology preserves the fractional, delay, and integral structures while incorporating uncertainty through interval-valued endpoint systems. Consequently, the existence and uniqueness theory developed in Section 5 remains fully compatible with the deterministic formulation when the fuzzy uncertainty vanishes.

Discussion. This example illustrates the applicability of the proposed theoretical frame-work to linear fuzzy fractional delay Volterra–Fredholm differential–integral equations. The explicit computation of the Lipschitz constants, kernel bounds, and contraction con-stant confirms that all hypotheses of Theorem 5.2 are satisfied. Moreover, the comparison with the corresponding crisp model shows that the proposed fuzzy formulation is a genuine extension of the classical fractional delay Volterra–Fredholm equation. In particular, the incorporation of fuzzy uncertainty through the α-cut representation preserves the math-ematical structure of the deterministic problem while providing interval-valued solutions that effectively characterize uncertainty in the system parameters and initial conditions.

Example 6.2: Nonlinear Fuzzy Fractional Delay Volterra–Fredholm Model

Consider the nonlinear fuzzy fractional delay Volterra–Fredholm differential–integral equa-tion

$$D^{\nu}_{C}\tilde{u}(t) = 0.3\ \tanh(\tilde{u}(t)) + 0.2\ \tilde{u}(\tau(t)) + \int_0^t (t-s)^2 \sin(\tilde{u}(s))\ ds + \int_0^{\tau} \frac{\tilde{u}(s)}{1+s^2}\ ds, \qquad t \in [0, T], \tag{14}$$

subject to the triangular fuzzy initial condition

$$\tilde{u}(0) = \tilde{u}_0 = (0.5,\ 1,\ 1.5),$$

where $0 < \nu < 1$ and the delay function is defined by

$$\tau(t) = \frac{t}{3},$$

which satisfies

$$0 \le \tau(t) \le t, \qquad t \in [0, T].$$

The nonlinear mappings are defined as

$$\tilde{f}(t, \tilde{u},\ \tilde{v}) = 0.3\ \tanh(\tilde{u}) + 0.2\ \tilde{v},$$

$$\tilde{g}(t, \tilde{u},\ \tilde{v}) = \sin(\tilde{u}),$$

and

$$\tilde{h}(t, \tilde{u},\ \tilde{v}) = \tilde{u},$$

with kernel functions

$$K_1(t, s) = (t - s)^2, \qquad K_2(t, s) = \frac{1}{1+s^2}.$$

α-cut representation. Applying the α-cut methodology developed in Section 4 and illustrated in Example 6.1, the triangular fuzzy initial value

$\tilde{u}0 = (0.5, 1, 1.5)$

admits the α-cut representation

$[\tilde{u}0]_\alpha = [\ 0.5 + 0.5\alpha,\ 1.5 - 0.5\alpha\ ],\ 0 \le \alpha \le 1.$

Similarly, the fuzzy-valued solution is represented by

$$[\tilde{u}(t)]_\alpha = [\underline{u}(t, \alpha), \overline{u}(t, \alpha)],$$

where

$$\underline{u}(t, \alpha) \le \overline{u}(t, \alpha), \qquad t \in [0, T], \qquad \alpha \in [0, 1].$$

Applying the α-cut representation to (14), the nonlinear fuzzy fractional delay Volterra–Fredholm differential–integral equation is transformed into an equivalent family of coupled nonlinear endpoint systems.

Equivalent endpoint system. For each fixed $\alpha \in [0, 1]$, the lower endpoint satisfies

$$D_C^{\nu} \underline{u}(t, \alpha) = 0.3 \tanh(\underline{u}(t, \alpha)) + 0.2\, \underline{u}\left(\frac{t}{3}, \alpha\right) + \int_0^t (t-s)^2 \sin(\underline{u}(s, \alpha))\, ds + \int_0^T \frac{\underline{u}(s, \alpha)}{1+s^2}\, ds, \tag{15}$$

subject to

$$\underline{u}(0, \alpha) = 0.5 + 0.5\alpha.$$

Similarly, the upper endpoint satisfies

$$D_C^{\nu} \overline{u}(t, \alpha) = 0.3 \tanh(\overline{u}(t, \alpha)) + 0.2\, \overline{u}\left(\frac{t}{3}, \alpha\right) + \int_0^t (t-s)^2 \sin(\overline{u}(s, \alpha))\, ds + \int_0^T \frac{\overline{u}(s, \alpha)}{1+s^2}\, ds, \tag{16}$$

subject to

$$\overline{u}(0, \alpha) = 1.5 - 0.5\alpha.$$

Equations (15) and (16) constitute a coupled family of nonlinear real-valued frac-tional delay Volterra–Fredholm integro-differential equations parameterized by $\alpha \in [0, 1]$. Consequently, the original fuzzy problem is transformed into an equivalent endpoint for-mulation, thereby extending the analytical framework developed for the linear model in Example 6.1 to nonlinear fuzzy fractional systems.

Verification of the assumptions. We now verify that the nonlinear fuzzy fractional delay Volterra–Fredholm model satisfies the hypotheses of Theorem 5.2.

The nonlinear mappings

$$\tilde{f}(t, \tilde{u}, \tilde{v}) = 0.3 \tanh(\tilde{u}) + 0.2\, \tilde{v}, \qquad \tilde{g}(t, \tilde{u}, \tilde{v}) = \sin(\tilde{u}), \qquad \tilde{h}(t, \tilde{u}, \tilde{v}) = \tilde{u},$$

are continuous on

$$[0, T] \times \mathbb{E} \times \mathbb{E}.$$

Moreover, since

$$\frac{d}{du}\tanh(u) = \operatorname{sech}^2(u) \leq 1,$$

the hyperbolic tangent is globally Lipschitz. Therefore, for any $\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2 \in \mathsf{E}$,

$$\begin{aligned} d\left(\tilde{f}(t, \tilde{u}_1, \tilde{v}_1), \tilde{f}(t, \tilde{u}_2, \tilde{v}_2)\right) &\leq 0.3\, d(\tanh(\tilde{u}_1), \tanh(\tilde{u}_2)) + 0.2\, d(\tilde{v}_1, \tilde{v}_2) \\ &\leq 0.3\, d(\tilde{u}_1, \tilde{u}_2) + 0.2\, d(\tilde{v}_1, \tilde{v}_2) \\ &\leq 0.5 \left[d(\tilde{u}_1, \tilde{u}_2) + d(\tilde{v}_1, \tilde{v}_2) \right]. \end{aligned}$$

Hence,

$$L_f = 0.5.$$

Similarly, since

$$|\sin(x) - \sin(y)| \leq |x - y|,$$

the mapping $\tilde{g}$ is globally Lipschitz with

$$L_g = 1.$$

Furthermore,

$$d(\tilde{h}_1, \tilde{h}_2) = d(\tilde{u}_1, \tilde{u}_2),$$

which implies

$$L_h = 1.$$

Next, for the Volterra kernel,

$$\int_0^t |K_1(t, s)|\, ds = \int_0^t (t - s)^2\, ds = \frac{t^3}{3} \leq \frac{T^3}{3}.$$

$$\int_0^t |K_1(t, s)|\, ds = \int_0^t (t - s)^2\, ds = \frac{t^3}{3} \leq \frac{T^3}{3}.$$

Therefore,

$$M_1 = \sup_{t\in[0,T]} \int_0^t |K_1(t, s)|\, ds = \frac{T^3}{3}.$$

Similarly,

$$\int_0^T |K_2(t, s)|\, ds = \int_0^T \frac{1}{1 + s^2}\, ds = \arctan(T),$$

and hence

$$M_2 = \sup_{t\in[0,T]} \int_0^T |K_2(t, s)|\, ds = \arctan(T).$$

Finally, the delay function

$$\tau(t) = \frac{t}{3}$$

is continuous and satisfies

$$0 \le \tau(t) \le t, \qquad t \in [0, T].$$

Consequently, assumptions (A1)–(A4) are satisfied.

Verification of the contraction condition. To illustrate the applicability of Theorem 5.2, consider the parameter values

$$v = 0.8, \qquad T = 0.2.$$

Then

$$L_f = 0.5, \qquad L_a = 1, \qquad L_h = 1,$$

and

$$M_1 = \frac{0.2^3}{3} = 0.00267,$$

$$M_2 = \arctan(0.2) \approx 0.19740.$$

Consequently,

$$q = \frac{2(L_f + M_1 L_a + M_2 L_h)T^v}{\Gamma(v + 1)} = \frac{2(0.5 + 0.00267 + 0.19740)(0.2)^{0.8}}{\Gamma(1.8)} \approx 0.52 < 1.$$

Hence, the contraction condition of Theorem 5.2 is satisfied. Therefore, the associated nonlinear operator is a contraction on the corresponding Banach space and consequently admits a unique fixed point.
Remark 6.2. Example 6.2 demonstrates that the analytical framework developed in Sections 3–5 is equally applicable to nonlinear fuzzy fractional delay Volterra–Fredholm differential–integral equations. Unlike the linear model presented in Example 6.1, the present example involves nonlinear fuzzy mappings and nonlinear integral operators. Nev-ertheless, the α-cut transformation, together with the Lipschitz continuity assumptions, enables the direct application of Theorem 5.2. This confirms that the proposed theoretical framework extends naturally from linear to nonlinear fuzzy fractional systems.

Discussion

Examples 6.1 and 6.2 demonstrate that the proposed theoretical framework accommo-dates both linear and nonlinear fuzzy fractional delay systems with coupled Volterra and Fredholm integral operators. These examples confirm the generality and applicability of the established existence and uniqueness results without resorting to numerical or approximation techniques.

Conclusion

This paper has presented a unified theoretical framework for a general class of fuzzy frac-tional delay Volterra–Fredholm differential–integral equations by integrating fractional-order dynamics, delay effects, Volterra and Fredholm integral operators, and fuzzy un-certainty within a single mathematical formulation. The proposed model extends sev-eral existing fuzzy fractional differential and integral equation models by simultaneously incorporating memory effects, hereditary characteristics, and uncertainty arising from imprecise initial conditions and system parameters.
Using the α-cut representation, the original fuzzy problem was systematically trans-formed into an equivalent family of coupled real-valued fractional delay integro-differential equations. This transformation enabled the development of a rigorous existence and uniqueness theory based on the Banach Fixed Point Theorem under suitable continuity, Lipschitz, and kernel boundedness assumptions. The derived contraction criterion explic-itly characterizes the combined influence of the fractional order, delay term, nonlinear mappings, and Volterra–Fredholm kernels on the well-posedness of the proposed fuzzy system.
The theoretical developments were further validated through two illustrative exam-ples. The first example demonstrated the complete analytical procedure for a linear fuzzy fractional delay Volterra–Fredholm model, while the second established that the same analytical framework extends naturally to nonlinear fuzzy systems. Furthermore, comparisons with the corresponding crisp models confirmed that the proposed formula-tion is fully consistent with the classical fractional delay Volterra–Fredholm theory and naturally reduces to the deterministic model when the fuzzy uncertainty vanishes.
The proposed framework provides a mathematically rigorous foundation for the analy-sis of uncertain fractional dynamical systems involving delay and memory effects. Owing to its generality, the developed theory has the potential to support mathematical mod-elling in diverse application areas, including biological systems, control engineering, vis-coelasticity, epidemiology, and other complex systems where uncertainty and hereditary phenomena coexist.

Declarations

Funding
The authors received no financial support for the research, authorship, and/or publication of this article.

Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability

No datasets were generated or analysed during the current study. Data sharing is not applicable to this article.

Ethical Approval

This article does not contain any studies involving human participants or animals per-formed by any of the authors.

Consent for Publication

All authors have read and approved the final manuscript and consent to its publication.

Author Contributions

Vidhya S:Conceptualisation,Methodology, Formal Analysis,Writing–Original Draft.S.P.Geetha:Data Curation,Investigation,Writing–Review& Editing.

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