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A Theorem on Variational Inequalities for Affine Mappings

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We give a proof of a conjecture by B. Ricceri on variational inequalties which extends a partial result of N. D. Yen. We need an additional hypothesis of compactness for a convex set, and refute the conjecture if this additional hypothesis is removed.

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1. Introduction

In his paper [3], B. Ricceri shows the following theorem, in a slightly more general form:

Theorem 1.1. (B. Ricceri) *Let E be a topological linear Hausdorff space, X a closed convex subset of E with nonempty interior, $A\colon X \to E^*$ a continuous mapping to the dual of E equipped with the weak* topology, K a compact subset of X, and $K_1 \subset K$ a closed finite-dimensional set.*
Assume that for every $x \in X \setminus K$: $\sup_{y \in K_1} \langle A(x), x - y \rangle > 0$. Then there exists some $x^* \in K$ such that $\sup_{y \in X} \langle A(x^*), x^* - y \rangle \le 0$.

In particular this statement has the following corollary cited in [4] and [5]:

Theorem 1.2. *Let H be an infinite-dimensional Hilbert space, X a closed convex subset of H with nonempty weak interior, $A\colon H \to H$ a weakly continuous mapping, K a weakly compact subset of X, and $K_1 \subset K$ a closed finite-dimensional set. Assume that for every $x \in X \setminus K$: $\sup_{y \in K_1} \langle A(x), x - y \rangle > 0$. Then there exists some $x^* \in K$ such that $\sup_{y \in X} \langle A(x^*), x^* - y \rangle \le 0$.*

As shown in [1] the hypothesis of nonempty interior on X cannot be removed: Frasca and Villani construct an example of a continuous affine function A on a Hilbert space satisfying $\sup_{y \in B} \langle Ax, x - y \rangle > 0$ for all x in the unit ball B of H.

In [4] (Problem 1) and again in [5] (Problem 1.1) B. Ricceri asked whether Theo-

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rem 1.1 still holds without the hypothesis that K_1 be finite-dimensional. And he conjectured that the following statement would fail even with A affine.

Conjecture 1.3. *Let H be an infinite-dimensional Hilbert space, X a closed convex subset of H with nonempty weak interior, $A\colon H \to H$ a weakly continuous mapping, and K a weakly compact subset.*
Assume that for every $x \in X \setminus K$: $\sup_{y\in K}\langle A(x), x-y\rangle > 0$. Then there exists some $x^* \in K$ such that $\sup_{y\in X}\langle A(x^*), x^*-y\rangle \leq 0$.

In fact, as proved by Yen in [6], this is not the case if A is affine and X is the whole H. Indeed:

Theorem 1.4. (N. D. Yen) *Let H be an infinite-dimensional Hilbert space, $A\colon H \to H$ an affine mapping, and K a weakly compact subset of H.*
Assume that for every $x \in H \setminus K$: $\sup_{y\in K}\langle A(x), x-y\rangle > 0$. Then there exists some $x^* \in K$ such that $A(x^*) = 0$ (equivalently $\langle A(x^*), x^*-y\rangle = 0$ for all $y \in H$).

The proof of Yen is based on a theorem of [2] founded on the monotonicity of A, which shows that the hypothesis of continuity for A is not needed. In fact in the first part of his proof, Yen shows that A is monotone on H and the following is known (see [7], Proposition 26.4):

Theorem 1.5. *Let E be a Banach space, E^* be its dual and $\varphi\colon E \to E^*$ be a monotone linear operator. Then φ is continuous.*

So the continuity of the operator A follows from the hypotheses. In fact we can weaken slightly the hypotheses of Yen's theorem.

Theorem 1.6. *Let E be an infinite-dimensional Banach space, A an affine mapping from E to E^*, and K a weakly compact subset of E. Assume that for every $x \in E \setminus K$: $A(x) \neq 0$ and $\sup_{y\in K}\langle A(x), x-y\rangle \geq 0$. Then there exists some $x^* \in K$ such that $A(x^*) = 0$ (equivalently $\langle A(x^*), x^*-y\rangle \leq 0$ for all $y \in E$).*

In fact after this paper was written down B. Ricceri informed me that he could show by a simpler proof using Kneser's minimax theorem the following version of previous thorem:

Theorem 1.7 (B. Ricceri). *Let E be a real Hausdorff locally convex topological vector space, $K \subset E$ a compact set, and $A\colon E \to E^*$ an affine operator such that $\sup_{y\in K}\langle A(x), x-y\rangle \geq 0$ for all $x \in E\setminus K$. Assume also that $\overline{\mathrm{conv}}(K)$ is compact. Then there exists $x^* \in \overline{\mathrm{conv}}(K)$ such that $A(x^*) = 0$.*

We will also prove the following extension of Yen's result:

Theorem 1.8. *Let E be Banach space space, X a closed convex subset of E with nonempty weak interior, $A\colon E \to E^*$ a continuous affine mapping, and K a weakly compact subset of X. Assume that for every $x \in X\setminus K$: $\sup_{y\in K}\langle A(x), x-y\rangle > 0$. Then there exists some $x^* \in K$ such that $\langle A(x^*), x^*-y\rangle \leq 0$ for all $y \in X$.*

Theorem 1.8 will in fact appear as a corollary of the following theorem which answers Conjecture 1.3 with the additional hypothesis that $\overline{\mathrm{conv}}(K)$ is compact in E, and is the main positive result of this paper, aside a negative answer to Problem 1 of [4].

Theorem 1.9. *Let E be a locally convex Hausdorff vector space, X a closed convex subset of E with non-empty interior for $\sigma(E, E^*)$, $K \subset X$ a compact subset, and $A\colon E \to E^*$ a continuous affine mapping for the weak* topology. Assume moreover that there exists some compact convex subset $\tilde{K}$ of E containing K and that $\sup_{y \in K} \langle A(x), x - y\rangle > 0$ for all $x \in X \setminus K$. Then there exists a point $x^* \in K$ such that $\langle A(x^*), x^* - y\rangle \leq 0$ for all $y \in X$.*

Of course for deducing Theorem 1.8 from Theorem 1.9 we equip E with the weak topology and the completeness of E ensures that the weak compact set K has a relatively weakly compact convex hull in E.

In Section 2 we will prove our main result. Number of lemmas in this section are known but their proofs are quite simple and we will give them for sake of self-containedness.

In Section 3 we construct an example where the conclusion of Theorem 1.9 fails but the hypotheses of it all hold except for the existence of the compact convex set $\tilde{K}$ containing K. So it gives a negative answer to Problem 1 of [4].

In Section 4 we extend the result of Section 2 to the case where E is a Banach space and the strict inequality " $\sup_{y \in K} \langle Ax, x - y\rangle > 0$ " is replaced as in the statement of Theorem 1.6 by the weaker inequality " $\sup_{y \in K} \langle Ax, x - y\rangle \geq 0$ " for all $x \in X \setminus K$.

In Section 5 an auxiliary result is proved for the case where A has finite rank, and finally in Section 6 we prove that in Theorem 1.8 the continuity of the operator A does not follow from the hypothesis *"$\sup_{y \in K} \langle A(x), x - y\rangle > 0$ for every x in $X \setminus K$"* but the conclusion *"there exists some $x^* \in K$ such that $\langle Ax^*, x^* - y\rangle \leq 0$ for all $y \in X$"* still holds as in Theorem 1.8 without the hypothesis of continuity of A. Nevertheless in this frame we must keep the hypothesis of strict inequality " $\sup_{y \in K} \langle Ax, x - y\rangle > 0$ " as shown by a counterexample.

2. The main result

It is clear that the statement of Theorem 1.9 is invariant under translations in E. So without loss of generality we can and do assume that X is a weak neighborhood of 0. Furthermore we can replace if necessary the compact convex set $\tilde{K}$ by $X \cap \tilde{K}$ which is compact and convex too and contains K. So we will assume that $K \subset \tilde{K} \subset X$.

Lemma 2.1. *If X is a closed convex neighborhood of 0 for the weak topology $\sigma(E, E^*)$, there exists a closed cofinite-dimensional subspace W of E, a complement Z of W in E and a closed convex neighborhood Y of 0 in Z such that $E = W \oplus Z$ and $X = W + Y$. We will denote by P and Q the projections on W and Z, respectively; then P and Q are continuous mappings from E to itself.*

Proof. By definition of the weak topology $\sigma(E, E^*)$ there are finitely many f_j in E^* ($j = 1, 2, \dots d$) such that $X \supset \{x \in E : \sup_j |\langle f_j, x\rangle| \leq 1\}$. Define $W = \bigcap_j \ker(f_j)$ which is a closed subspace of E of finite codimension $k \leq d$ and choose any finite-dimensional subspace Z of dimension k such that $W \oplus Z = E$. If $x \in X$, $w \in W$ and n is any integer then $2^n w \in W$ and

$$x_n = (1 - 2^{-n})x + w = (1 - 2^{-n})x + 2^{-n}(2^n w) \in \operatorname{conv}(\{x\} \cup W) \subset X$$

And since X is closed then $x + w = \lim_n x_n \in X$. It follows that for every $w \in W$: $x \in X \iff x + w \in X$. Set $Y = X \cap Z$ which is a closed convex subset of Z and a neighborhood of 0 in Z. For each $x \in E$ there is a unique $w \in W$ such that $x - w \in Z$; so $x \in X \iff x - w \in Y$, and $X = Y + W$. □

In the sequel of this section we will always silently assume that X is a closed weak neighborhood of 0 in E and that W, Z, P and Q are as in previous lemma.

We assume now that $K \subset X$ is compact, that $A\colon E \to E^*$ is affine and that $\sup_{y \in K}\langle A(x), x - y\rangle > 0$ for all $x \in X \setminus K$. There exists a linear mapping $\varphi\colon E \to E^*$ and some $a = -A(0) \in E^*$ such that $A(x) = \varphi(x) - a$ for all $x \in E$.

Lemma 2.2. *The mapping $\varphi_{|W}$ is monotone on W, i.e. $\langle\varphi(w), w\rangle \geq 0$ for all $w \in W$.*

Proof. Let $w \in W \setminus \{0\}$. Since K is compact there is some positive real R such that $t.w \notin K$ for $t > R$. For $t > R$ there exists by hypothesis some $y_t \in K$ such that $\langle A(t.w), t.w - y_t\rangle > 0$. It follows that

$$\begin{aligned} 0 < \langle A(t.w), t.w - y_t\rangle &= \langle \varphi(t.w) - a, t.w - y_t\rangle \\ &= t^2\langle\varphi(w), w\rangle - t\langle a, w\rangle - t\langle\varphi(w), y_t\rangle + \langle a, y_t\rangle \\ &\leq t^2\langle\varphi(w), w\rangle + t|\langle a, w\rangle| + t|\langle\varphi(w), y_t\rangle| + |\langle a, y_t\rangle| \end{aligned}$$

Since a and $\varphi(w)$ belong to E^*, their absolute values are bounded on the compact set K by m_a et m_w respectively, and one gets

$$\langle\varphi(w), w\rangle \geq \sup_{t>R}\Big(-\frac{|\langle a, w\rangle|}{t} - \frac{m_w}{t} - \frac{m_a}{t^2}\Big) = 0$$

what is the expected result since $\langle\varphi(w), w\rangle = 0$ for $w = 0$. Notice that this lemma needs not φ to be continuous. If E is a Banach space and $X = E$ (hence $W = E$) this implies the continuity of A by Theorem 1.5. □

From now on in this section we assume that A is continuous, hence also the linear mapping $\varphi\colon x \mapsto A(x) - A(0)$.

Definition 2.3. An affine mapping $f\colon E \to E^*$ will be called *almost monotone* if there exists some continuous linear mapping $g\colon E \to E^*$ with finite rank such that $f - g$ is monotone.

Lemma 2.4. *Let $f\colon E \to E^*$ be a continuous affine mapping and W be a closed cofinite-dimensional subspace of E. If $f_{|W}$ is monotone then f is almost monotone.*

Proof. Denote by φ the continuous linear mapping $x \mapsto f(x) - f(0)$. Let Z be a complement of W in E, P and $Q\colon E \to E$ be the projections onto W and Z respectively (so $P + Q$ is the identity mapping of E).

For any linear subspace L of E, $L^\perp$ will denote the weak*-closed linear subspace $\{f \in E^* : \forall z \in L\ \langle f, z\rangle = 0\}$ of E^*. The continuous linear mappings P^* and Q^* from E^* to itself are respectively the projections on $Z^\perp$ and $W^\perp$. We have $\dim(W^\perp) = \dim(Z) = k$ and $P^* + Q^* = (P + Q)^* = Id_E^* = Id_{E^*}$. Hence

$$\varphi = \varphi(P + Q) = \varphi P + \varphi Q = (P^* + Q^*)\varphi P + \varphi Q = P^*\varphi P + Q^*\varphi P + \varphi Q\,.$$

Define the linear mapping $\psi = Q^*\varphi P + \varphi Q$ which has finite rang since so do Q and Q^* and is continuous since so is φ.

Then with $\tilde\varphi = \varphi - \psi = P^*\varphi P$, $\langle \tilde\varphi(x), x\rangle = \langle P^*\varphi(Px), x\rangle = \langle \varphi(Px), Px\rangle$ which is non-negative since $A_{|W}$ is monotone, hence also $\varphi_{|W}$; and this completes the proof, since $\tilde f = f - \psi$ is then affine and monotone. □

Corollary 2.5. *A is almost monotone.*

Proof. This follows immediately from Lemma 2.4. □

Lemma 2.6. *Let $f\colon E \to E^*$ be an almost monotone affine mapping and B be a bounded subset of E. Then the function $x \mapsto \langle f(x), x\rangle$ is weakly lower semi-continuous on B.*

Proof. Let $\psi\colon E \to E^*$ be a continuous linear mapping with finite rank such that $f - \psi$ is monotone and $\mathfrak{U}$ be a filter on B which converges weakly to $\hat x$. We have to prove that we have $\langle f(\hat x), \hat x\rangle \le \liminf_{x,\mathfrak{U}}\langle f(x), x\rangle$. Let $a = f(0)$ and $\varphi\colon x \mapsto f(x) - a$. Since we have $\lim_{x,\mathfrak{U}}\langle a, x\rangle = \langle a, \hat x\rangle$ we have to show that

$$\langle \varphi(\hat x), \hat x\rangle \le \liminf_{x,\mathfrak{U}}\langle \varphi(x), x\rangle\,.$$

And since the continuous function ψ has finite rank there are finitely many $(f_i)_{i\le n}$ and $(g_i)_{i\le n}$ in E^* such that $\psi(x) = \sum_{i\le n}\langle f_i, x\rangle g_i$. So we have

$$\begin{aligned}\lim_{x,\mathfrak{U}}\langle \psi(x), x\rangle &= \sum_{i\le n}\lim_{x,\mathfrak{U}}\langle f_i, x\rangle\langle g_i, x\rangle = \sum_{i\le n}\lim_{x,\mathfrak{U}}\langle f_i, x\rangle \lim_{x,\mathfrak{U}}\langle g_i, x\rangle\\ &= \sum_{i\le n}\langle f_i, \hat x\rangle\langle g_i, \hat x\rangle = \langle \psi(\hat x), \hat x\rangle\end{aligned}$$

So it remains only to prove that $\langle \tilde\varphi(\hat x), \hat x\rangle \le \liminf_{x,\mathfrak{U}}\langle \tilde\varphi(x), x\rangle$. Since $\tilde\varphi$ is monotone we have $\langle \tilde\varphi(x) - \tilde\varphi(\hat x), x - \hat x\rangle \ge 0$, hence

$$\langle \tilde\varphi(x), x\rangle \ge \langle \tilde\varphi(x), \hat x\rangle + \langle \tilde\varphi(\hat x),\ x\rangle - \langle \tilde\varphi(\hat x), \hat x\rangle$$

and since the right-hand side converges to

$$\langle\tilde\varphi(\hat x),\hat x\rangle+\langle\tilde\varphi(\hat x),\hat x\rangle-\langle\tilde\varphi(\hat x),\hat x\rangle=\langle\tilde\varphi(\hat x),\hat x\rangle\,,$$

we get $\liminf_{x,\mathfrak{U}}\langle\tilde\varphi(x),x\rangle\geq\langle\tilde\varphi(\hat x),\hat x\rangle$. □

Lemma 2.7. *Let B be a compact convex subset of E , V a finite-dimensional linear subspace of E meeting B, and $f\colon V\to E^*$ a continuous mapping. Then there is a point $x\in B\cap V$ such that $\langle f(x),x-y\rangle\leq 0$ for all $y\in B\cap V$.*

Proof. Assume the contrary: then the set $U_y=\{x:\langle f(x),x-y\rangle>0\}$ would be open in $B\cap V$ for all $y\in B\cap V$, and the family $(U_y)_{y\in B\cap V}$ would cover the compact set $B\cap V$. So there should be a finite continuous partition of the unity $(\chi_y)_{y\in J}$ such that for all $y\in J$: $\chi_y>0\Rightarrow x\in U_y$. Define then the continuous function $g:B\cap V\to B\cap V$ by

$$g(x)=\sum_{y\in J}\chi_y(x).y\ .$$

It follows that for all $x\in B\cap V:\ \langle f(x),x-g(x)\rangle=\sum_{y\in J}\psi_y(x)\langle f(x),x-y\rangle>0$. Thus the function g could not have any fixed point, and this would contradict Brouwer's theorem since $B\cap V$ is finite-dimensional, non-empty, compact and convex. □

Lemma 2.8. *Let B be a non-empty compact convex subset of E and $f\colon E\to E^*$ be an almost monotone affine mapping. Then there is a point $\hat x\in B$ such that $\langle f(\hat x),\hat x-y\rangle\leq 0$ for all $y\in B$. Moreover if $K\subset B$ and $\sup_{y\in K}\langle f(x),x-y\rangle>0$ for all $x\in B\setminus K$, this point $\hat x$ has to be in K.*

Proof. Let $\mathcal{V}$ be the set of all finite-dimensional linear subspaces of E meeting B and $\mathfrak{F}$ a filter on $\mathcal{V}$ such that for all $V\in\mathcal{V}$ the set $\{V'\in\mathcal{V}:V\subset V'\}$ belongs to $\mathfrak{F}$. For each $V\in\mathcal{V}$ there is by Lemma 2.7 some $x_V\in B\cap V$ such that $\langle f(x_V),x_V\rangle\leq\langle f(x_V),y\rangle$ for all $y\in B\cap V$. By compactness we can find a cluster point $\hat x$ of x_V along $\mathfrak{F}$.

For every $y\in B$ the set $\{V\in\mathcal{V}:y\in V\}$ is in $\mathfrak{F}$. By Lemma 2.6 we have

$$\langle f(\hat x),\hat x\rangle\leq\limsup_{V,\mathfrak{F}}\langle f(x),x\rangle\leq\limsup_{V,\mathfrak{F}}\langle f(x),y\rangle\leq\langle f(\hat x),y\rangle$$

what is the expected inequality.

If moreover $K\subset B$ it is impossible that such a $\hat x$ lies in $B\setminus K$: if not by hypothesis there should be an $y\in K\subset B$ such that $\langle f(\hat x),\hat x-y\rangle>0$, in contradiction with the choice of $\hat x$. □

Let F be any finite-dimensional linear subspace of W and F_1 be a generating symmetric convex compact subset of F, and choose a compact convex subset Y_0 of Y. Consider the compact convex subset B of X defined by

$$B=\operatorname{conv}(\tilde K\cup Y_0)+F_1 \tag{1}$$

which is clearly contained in $X+F\subset X+W=X$, and contains K.

Lemma 2.9. *There exists at least a point* $\hat{x} \in K$ *such that* $\langle A(\hat{x}), \hat{x} - y\rangle \leq 0$ *for all* $y \in Y_0 + F$.

Proof. Following Corollary 2.5 the mapping A is almost monotone. Applying Lemma 2.8 to the compact convex set B defined by (1) we get some $\hat{x}$ in K such that $\langle A(\hat{x}), \hat{x} - y\rangle \leq 0$ for all $y \in B$, a fortiori for all $y \in Y_0$. Moreover, for $z \in F_1$, the points $y_1 = \hat{x} + z$ and $y_2 = \hat{x} - z$ both belong to B. Thus for $j = 1, 2$:

$$\langle A(\hat{x}), \hat{x} - y_j\rangle = (-1)^j . \langle A(\hat{x}), z\rangle \leq 0$$

from what we deduce $\langle A(\hat{x}), z\rangle = 0$ for all $z \in F_1$ hence for all $z \in F$. So for $y_0 \in Y_0$ and $z \in F$:

$$\langle A(\hat{x}), \hat{x} - (y_0 + z)\rangle = \langle A(\hat{x}), \hat{x} - y_0\rangle - \langle A(\hat{x}), z\rangle = \langle A(\hat{x}), \hat{x} - y_0\rangle \leq 0$$

as claimed. □

We now can prove our main result (Theorem 1.9 in the introduction):

Theorem 2.10. *Let* E *be a locally convex Hausdorff vector space,* X *a closed convex subset of* E *with non-empty interior for* $\sigma(E, E^*)$, $K \subset X$ *a compact subset, and* $A \colon E \to E^*$ *a continuous affine mapping for the weak* topology. Assume moreover that there exists some compact convex subset* $\widetilde{K}$ *of* E *containing* K *and that* $\sup_{y \in K} \langle Ax, x - y\rangle > 0$ *for all* $x \in X \setminus K$.
Then there exists a point $x^* \in K$ *such that* $\langle Ax^*, x^* - y\rangle \leq 0$ *for all* $y \in X$.

Proof. For all finite-dimensional linear subspace F of W and every compact convex subset Y_0 of Y, previous lemma yields a $\hat{x}_{Y_0,F} \in K$ satisfying the inequality $\langle Ax^*, x^* - y\rangle \leq 0$ for all $y \in Y_0 + F$. Choose a filter $\mathfrak{F}$ on the set $\mathcal{P}$ of such pairs (Y_0, F), such that the set $\{(C, G) \in \mathcal{P} : Y_0 \subset C$ and $F \subset G\}$ belongs to $\mathfrak{F}$ for all (Y_0, F) then a cluster value $x^* \in K$ of $\hat{x}_{Y_0,F}$ along $\mathfrak{F}$ since K is compact. And refining $\mathfrak{F}$ into $\mathfrak{U}$ if necessary we can assume that $x^* = \lim_{(Y_0,F),\mathfrak{U}} \hat{x}_{Y_0,F}$.

For $y \in Y + W$ the set $\{(Y_0, F) \in \mathcal{P} : (Qy, Py) \in Y_0 \times F\}$ belongs to $\mathfrak{U}$ and it follows from Lemma 2.6 that

$$\begin{aligned} \langle A(x^*), x^* - y\rangle &= \langle A(x^*), x^*\rangle - \langle A(x^*), y\rangle \\ &\leq \limsup_{(Y_0,F),\mathfrak{U}} \langle A(\hat{x}_{Y_0,F}), \hat{x}_{Y_0,F}\rangle - \lim_{(Y_0,F),\mathfrak{U}} \langle A(\hat{x}_{Y_0,F}), y\rangle \\ &= \limsup_{(Y_0,F),\mathfrak{U}} \langle A(\hat{x}_{Y_0,F}), \hat{x}_{Y_0,F} - y\rangle \leq 0 \end{aligned}$$

and since $X = Y + W$ the proof is complete. □

Proof of Theorem 1.6. Recall that a Banach space E is said to be *weakly compactly generated* (w.c.g. for short) if it contains a weakly compact subset B which is total in E (i.e. the closed linear subspace generated by B is the whole E). It is easy to see that any reflexive or separable Banach space is w.c.g.

Assume first that E is w.c.g. Then there exists a weakly compact symmetric subset B of E which is total in E. For $\varepsilon > 0$ we set $K_\varepsilon = K + \varepsilon.B$. Then K_ε is weakly compact.

It follows that $p(\xi) = \sup_{u\in B}\langle \xi, u\rangle > 0$ for all $\xi \in E^* \setminus \{0\}$. For all $\varepsilon > 0$ and all $x \in E \setminus K_\varepsilon$:

$$\begin{aligned}\sup_{y\in K_\varepsilon}\langle A(x), x-y\rangle &= \sup_{y\in K}\langle A(x), x-y\rangle + \varepsilon.\sup_{u\in B}\langle A(x), u\rangle \\ &= \sup_{y\in K}\langle A(x), x-y\rangle + \varepsilon.p(A(x)) \geq \varepsilon.p(A(x)) > 0\end{aligned}$$

since $A(x) \neq 0$ for $x \notin K_\varepsilon$.

It follows then from Theorem 1.8 applied to $X = E$ that there exists some $x^*_\varepsilon \in K_\varepsilon$ such that $\langle A(x^*_\varepsilon), x^*_\varepsilon - y\rangle \leq 0$ for all $y \in E$, it is $A(x^*_\varepsilon) = 0$. And since we assumed that $A(x) \neq 0$ for all $x \in E \setminus K$ we conclude that $x^*_\varepsilon \in K$. Again since A is monotone on the Banach space E by Lemma 2.2 (notice that $W = E$) A is continuous. Then if x^* is any weak cluster point of x^*_ε, x^* lies in K and $A(x^*)$, cluster point of $A(x^*_\varepsilon)$, is 0.

In the general case denote by E_0 the closed linear subspace of E generated by K, by B the unit ball of E and by $\mathcal{F}$ the set of finite-dimensional subspaces of E ordered by inclusion. It is clear that if $F \in \mathcal{F}$ then $K + (B \cap F)$ is weakly compact and generates the Banach space $E_0 + F$. Thus $E_0 + F$ is w.c.g. and it follows from what precedes applied to $E_0 + F$ that there is some $x^*_F \in K$ such that $\langle A(x^*_F), y\rangle = 0$ for all $y \in E_0 + F$. Again if x^* is a cluster point of $(x^*_F)_{F\in\mathcal{F}}$ the continuity of the monotone mapping A implies that $\langle A(x^*), y\rangle = 0$ for all $y \in E$, it is $A(x^*) = 0$. □

3. The negative result

We now prove by constructing a counterexample that Theorem 1.9 fails when removing the hypothesis that K is contained in some compact convex subset $\tilde{K}$ of E. Of course such a space E cannot be complete because of Theorem 1.4.

Theorem 3.1. *There exists a prehilbertian space E, a continuous affine mapping A from E to E^* and a norm-compact subset K of E such that the following two inequalities hold:*

$$\sup_{y\in K}\langle A(x), x-y\rangle > 0 \quad \textit{for all} \;\; x \in E \setminus K \tag{2}$$

$$\sup_{y\in E}\langle A(x), x-y\rangle > 0 \quad \textit{for all} \;\; x \in K\,. \tag{3}$$

Let $\mathbb{T}$ be the unit circle of the complex plane, which is a compact group, and equip it with the normalized Haar measure λ, so that $\int_{\mathbb{T}} ds = 1$ and $\int_{\mathbb{T}} z^m \, d\lambda(z) = 0$ for $m \in \mathbb{Z} \setminus \{0\}$.

The space E will be a linear subspace of the Sobolev space $H^{-1}(\mathbb{T})$ of distributions f on $\mathbb{T}$ whose Fourier series $\hat{f}$ satisfies

$$\sum_{n\in\mathbb{Z}} \frac{|\hat{f}(n)|^2}{1+n^2} < \infty \quad \text{equipped with the norm} \quad \|f\|_{-1} = \Big(\sum_{n\in\mathbb{Z}} \frac{|\hat{f}(n)|^2}{1+n^2}\Big)^{1/2}.$$

This space is a Hilbert space whose dual can be identified with the space $H^1(\mathbb{T})$ of distributions g on $\mathbb{T}$ such that g and its derivative g' are both in $L^2(\mathbb{T})$. We will respectively denote by $\|.\|_{-1}$ and $\|.\|_1$ the norms on $H^{-1}(\mathbb{T})$ and $H^1(\mathbb{T})$.

Lemma 3.2. *For each $s \in \mathbb{T}$ the Dirac measure δ_s at s is in $H^{-1}(\mathbb{T})$. Moreover the mapping $s \mapsto \delta_s$ is continuous (and even Hölder) from $\mathbb{T}$ to $H^{-1}(\mathbb{T})$.*

Proof. Let $\mathbf{e}_n$ be the function $z \mapsto z^{-n}$. We have $\hat{\delta}_s(n) = \langle \mathbf{e}_n, \delta_s\rangle = s^{-n}$, and

$$\sum_{n\in\mathbb{Z}} \frac{|\hat{\delta}_s(n)|^2}{1+n^2} = \sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} < +\infty\,.$$

We have for any $m \in \mathbb{Z}$, $s \neq t \in \mathbb{T}$,

$$|s^m - t^m| = |s^{|m|} - t^{|m|}| = |(s-t)\big(\sum_{p+q=|m|-1} s^p t^q\big)| \le |m|.|s-t|$$

hence

$$\begin{aligned}
\|\delta_s - \delta_t\|_{-1}^2 &= \sum_{n\in\mathbb{Z}} \frac{|\hat{\delta}_s(n) - \hat{\delta}_t(n)|^2}{1+n^2} = \sum_{n\in\mathbb{Z}} \frac{|s^{-n} - t^{-n}|^2}{1+n^2} = \sum_{n\neq 0} \frac{|s^{-n} - t^{-n}|^2}{1+n^2} \\
&\le \sum_{0<|n|\le p} \frac{n^2|s-t|^2}{1+n^2} + \sum_{|n|>p} \frac{4}{1+n^2} \\
&\le 2p.|s-t|^2 + 4\sum_{n=p+1}^{\infty} \big(\frac{1}{n-1} - \frac{1}{n}\big) \le 2p.|s-t|^2 + \frac{4}{p}
\end{aligned}$$

Take p such that $1 \le p \le \theta := \dfrac{2}{|s-t|} < p+1$. We get

$$\|\delta_s - \delta_t\|_{-1}^2 \le |s-t|\big(2p\cdot\frac{2}{\theta} + \frac{4}{p}\cdot\frac{\theta}{2}\big) \le |s-t|\big(4 + 2\cdot\frac{p+1}{p}\big) \le 8|s-t|$$

hence $\|\delta_s - \delta_t\| \le \sqrt{8|s-t|}$. □

It follows from the previous lemma that $K = \{\delta_s : s \in \mathbb{T}\}$ is a norm-compact subset of $H^{-1}(\mathbb{T})$.

Let $(\alpha_m)_{m\ge 0}$ be a series such that $\sup_m (m^2+1)\alpha_m < +\infty$ and $\alpha_m > 0$ for all m (e.g. $\alpha_m = 2^{-m}$) then consider the continuous function

$$h\colon z \mapsto \alpha_0 + 2\sum_{m>0} \alpha_m.\Re(z^m) = \sum_{m\in\mathbb{Z}} \alpha_{|m|}.z^m$$

from $\mathbb{T}$ to $\mathbb{R}$. We want to define φ from $H^{-1}(\mathbb{T})$ to $H^1(\mathbb{T})$ by $\varphi(f) = f * h$.

Lemma 3.3. *The mapping φ is linear and continuous from $H^{-1}(\mathbb{T})$ to $H^1(\mathbb{T})$.*

Proof. The mapping φ is clearly linear. By Fourier transform it is enough to show that $\|f * h\|_1^2 \leq \theta^2 \|f\|_{-1}^2$ for some constant θ. We have

$$\|f\|_{-1}^2 = \sum_{m\in\mathbb{Z}} \frac{|\hat{f}(m)|^2}{m^2+1},$$

$\widehat{f*h}(m) = \hat{f}(m).\hat{h}(m)$, $\hat{h}(m) = \alpha_{|m|}$, $\|h * f\|_1^2 = \sum_{m\in\mathbb{Z}} (m^2+1)|\widehat{f*h}(m)|^2$, hence

$$\|h * f\|_1^2 = \sum_{m\in\mathbb{Z}} (m^2+1)|\widehat{f*h}(m)|^2 \leq \sum_{m\in Z} \alpha_{|m|}^2 (m^2+1)^2 \frac{|\hat{f}(m)|^2}{m^2+1} \leq \theta^2 \|f\|_{-1}^2$$

with $\theta = \sup_{m\geq 0}(m^2+1)\alpha_m$. □

Let us define $E \subset H^{-1}(\mathbb{T})$ as the linear space spanned by the $(\delta_s)_{s\in\mathbb{T}}$ and the map A by

$$A(f) = \varphi(f) - \alpha_0.\mathbb{1}$$

where $\mathbb{1}$ denotes the constant function on $\mathbb{T}$ with value 1, which belongs to $H^1(\mathbb{T})$. Thus K is a norm-compact subset of E and A is an affine continuous mapping.

Lemma 3.4. *There exists no $\delta_s \in K$ such that $\langle A(\delta_s), \delta_s - f\rangle \leq 0$ for all $f \in E$; so A satisfies* (3)*.*

Proof. If for some $s \in \mathbb{T}$ this inequality were true for all $f \in E$ we would have for all integer $n \in \mathbb{Z}$:

$$\langle A(\delta_s), \delta_s\rangle \leq \langle A(\delta_s), nf\rangle$$

hence necessarily $\langle A(\delta_s), f\rangle = 0$ for all $f \in E$ and in particular for $f = \delta_s - \delta_t$ for all $t \in \mathbb{T}$. Since $\langle A(\delta_s), \delta_s - \delta_t\rangle = \langle h * \delta_s, \delta_s - \delta_t\rangle = h(0) - h(s-t)$ we should have

$$0 = \int_{\mathbb{T}} \big(h(0) - h(s-t)\big)\, dt = h(0) - \int_{\mathbb{T}} h(t)\, dt = h(0) - \alpha_0 = 2\sum_{m>0} \alpha_m > 0$$

and this contradiction achieves the proof. □

Lemma 3.5. *Let $f = \sum_j \lambda_j \delta_{s_j} \in E$. Then $\langle \varphi(f), f\rangle = \sum_{m=0}^{\infty} \alpha_{|m|} |\hat{f}(m)|^2$, where $\hat{f}(m) = \langle \mathbf{e}_m, f\rangle$ is the m^{th} Fourier coefficient of f.*

Proof. Indeed:

$$\begin{aligned}
\langle \varphi(f), f\rangle &= \sum_{j,k} \lambda_j \lambda_k h(s_j.s_k^{-1}) = \sum_{m\in\mathbb{Z}} \alpha_{|m|} \sum_{j,k} \lambda_j \lambda_k s_j^{-m} s_k^m \\
&= \sum_{m\in\mathbb{Z}} \alpha_{|m|} \big(\sum_j \lambda_j s_j^{-m}\big)\big(\sum_k \lambda_k s_k^m\big) \\
&= \sum_{m\in\mathbb{Z}} \alpha_{|m|} \big(\sum_j \lambda_j s_j^{-m}\big)\big(\overline{\sum_k \lambda_k s_k^{-m}}\big) = \sum_{m\in\mathbb{Z}} \alpha_{|m|} \Big|\sum_j \lambda_j s_j^{-m}\Big|^2 \\
&= \sum_{m\in\mathbb{Z}} \alpha_{|m|} |\langle \mathbf{e}_m, f\rangle|^2 = \sum_{m\in\mathbb{Z}} \alpha_{|m|} |\hat{f}(m)|^2
\end{aligned}$$

which is the expected result. Notice that $\hat{f}(0) = \langle \mathbb{1}, f\rangle = \sum_j \lambda_j$. □

Lemma 3.6. *Let $f = \sum_j \lambda_j \delta_{s_j} \in E \setminus K$. Then there exists at least one $s \in \mathbb{T}$ such that $\langle A(f), f - \delta_s\rangle > 0$; so A satisfies* (2).

Proof. Since the function $s \mapsto \langle A(f), f - \delta_s\rangle$ is continuous on $\mathbb{T}$, it is enough to show that $\int_{\mathbb{T}} \langle A(f), f - \delta_s\rangle\, ds > 0$ for $f \in E \setminus K$. Indeed, following Lemma 3.5

$$\langle A(f), f\rangle = \langle \varphi(f), f\rangle - \alpha_0 \langle \mathbb{1}, f\rangle = \sum_{m\in\mathbb{Z}} \alpha_{|m|} |\hat{f}(m)|^2 - \alpha_0 \hat{f}(0)$$

and $\langle A(f), \delta_s\rangle = \langle \varphi(f), \delta_s\rangle - \alpha_0\langle \mathbb{1}, \delta_s\rangle = \langle \varphi(f), \delta_s\rangle - \alpha_0$. Thus

$$\int_{\mathbb{T}} \langle A(f), \delta_s\rangle\, ds = \sum_j \lambda_j \int h(s.s_j^{-1})\, ds - \alpha_0$$

$$= \Big(\sum_{m\in\mathbb{Z}} \sum_j \alpha_{|m|} \lambda_j \int_{\mathbb{T}} z^{-m} s_j^m\, dz\Big) - \alpha_0 = \alpha_0\big(\sum_j \lambda_j\big) - \alpha_0 = \alpha_0 \hat{f}(0) - \alpha_0.$$

It follows that

$$\begin{aligned}\int_{\mathbb{T}} \langle A(f), f - \delta_s\rangle\, ds &= \sum_{m\in\mathbb{Z}} \alpha_m |\hat{f}(m)|^2 - \alpha_0 \hat{f}(0) - \big(\alpha_0\hat{f}(0) - \alpha_0\big)\\ &= \sum_{m\in\mathbb{Z}, m\neq 0} \alpha_m |\hat{f}(m)|^2 + \alpha_0 \hat{f}(0)^2 - \alpha_0\hat{f}(0) - \big(\alpha_0 \hat{f}(0) - \alpha_0\big)\\ &= 2\sum_{m=1}^{\infty} \alpha_m |\hat{f}(m)|^2 + \alpha_0 |\hat{f}(0) - 1|^2 \geq 0\end{aligned}$$

Moreover if we had $\int_{\mathbb{T}} \langle A(f), f - \delta_s\rangle\, ds \leq 0$ we would necessarily have $\hat{f}(m) = 0$ for all $m \neq 0$ and $\hat{f}(0) = 1$. But the only element in $H^{-1}(\mathbb{T})$ having these Fourier coefficients is the normalized Haar measure which does not belong to E. It follows that

$$\sup_{s\in\mathbb{T}} \langle A(f), f - \delta_s\rangle\, ds \geq \int_{\mathbb{T}} \langle A(f), f - \delta_s\rangle\, ds > 0$$

for all $f \in E \setminus K$. □

4. The non-strict case

The aim of this section is to prove the following:

Theorem 4.1. *Let E be a Banach space, X a closed convex subset of E with non-empty interior, K a weakly compact convex subset of X and $A\colon E \to E^*$ an almost monotone affine function.*

Assume that for all $x \in X$, $\sup_{y\in K} \langle A(x), x - y\rangle \geq 0$. Then there exists some $x^* \in K$ such that $\langle A(x^*), x^* - y\rangle \leq 0$ for all $y \in X$.

Moreover if W is a closed cofinite-dimensional linear subspace of E such that $X + W = X$ and $K_0 = \{y \in K : A(y)_{|W} = 0\}$ then $K \subset K_0 + W$.

Notice that in this statement the strict inequality $\sup_{y\in K}\langle A(x^*), x^*-y\rangle > 0$ which was assumed in Theorem 1.9 has been replaced by the weak inequality $\sup_{y\in K}\langle A(x^*), x^*-y\rangle \geq 0$. In order to prove this last statement we need some preliminary results. The next lemma is probably well-known.

Lemma 4.2. *Let E be a Banach space, K' be a weakly compact subset of E, f be a non-zero continuous linear functional on E and $\mu = \sup_{x\in K}\langle f, x\rangle$. Then the intersection of the hyperplane $H = f^{-1}(\mu)$ and the closed convex hull K_1 of K' is the closed convex hull of $K'\cap H$.*

Proof. Since $K'\cap H$ is contained in H and in $\overline{\mathrm{conv}}(K')$ it is clear that the weakly compact convex set $K'' = \overline{\mathrm{conv}}(K'\cap H)$ is contained in $H\cap\overline{\mathrm{conv}}(K')$. Conversely if there were some point $a\in H\setminus K''$ there would be some $g\in E^*$ such that $g(a) > \sup_{z\in K'}\langle g, z\rangle$. Consider the weakly compact sets

$$L_n = \{x\in K' : 2^n\big(f(x)-\mu\big) + g(x) - g(a) \geq 0\}$$

Since $f(x)\leq\mu$ for $x\in K'$, we have for $x\in L_{n+1}$:

$$0\leq \mu - f(x) \leq 2^{-n-1}(g(x)-g(a))$$

hence $g(x)\geq g(a)$ and $2^{-n-1}(g(x)-g(a)) \leq 2^{-n}(g(x)-g(a))$. It follows that $L_{n+1}\subset L_n$. If $x_0\in\bigcap_n L_n$ we have necessarily $f(x_0)\leq\mu$ hence $f(x_0)=\mu$ and $x_0\in H\cap K'$ and also $g(x_0)\geq g(a) > \sup_{z\in K\cap H}\langle g, z\rangle$. This contradiction implies that $\bigcap_n L_n = \emptyset$ hence by compactness that $L_m=\emptyset$ for some integer m. Thus the linear functional $g + 2^m f$ separates a from K' and $a\notin H\cap\overline{\mathrm{conv}}(K')$, so $K'' = H\cap\overline{\mathrm{conv}}(K')$. □

Lemma 4.3. *Let E be a w.c.g. Banach space, $K\subset E$ a weakly convex compact set, $a\in E\setminus K$, B a weakly compact symmetric and total subset of E and $\varepsilon>0$ such that $(a+\varepsilon B)\cap K=\emptyset$. Then the set*

$$K' = \{(1-t)x + ta + \varepsilon t(1-t)u : t\in[0,1],\ x\in K,\ u\in B\}$$

is a weakly compact set contained in $C_\varepsilon = \mathrm{conv}(K\cup(a+\varepsilon B))$ containing $K\cup\{a\}$. Moreover if p denotes the norm $f\mapsto\sup_{u\in B}\langle f, u\rangle$ on E^* and

$$\gamma(f) := \sup_{0\leq t\leq 1}\Big((1-t)\sup_{z\in K}\langle f, z\rangle + tf(a) + \varepsilon t(1-t)p(f)\Big),\quad f\in E^*,$$

then the set $K_1 = \{x\in E : \forall f\in E^*\ f(x)\leq\gamma(f)\}$ is the closed convex hull of K'.

Proof. For $t\in[0,1]$ and $u\in B$ the point $a+\varepsilon(1-t)u$ belongs to $a+\varepsilon B$. Thus for $x\in K$: $(1-t)x + ta + \varepsilon t(1-t)u = (1-t)x + t\big(a+\varepsilon(1-t)u\big)\in C_\varepsilon$. And using $t=0$ or $t=1$ it is clear that $K\subset K'$ and $a\in K'$. Since $[0,1]$, K and B are weakly compact, so are C_ε and K'.

For $f\in E^*$ it is clear that

$$\sup_{z\in K'}\langle f, z\rangle = \sup_{0\leq t\leq 1}\Big((1-t)\sup_{z\in K}\langle f, z\rangle + tf(a) + \varepsilon t(1-t)p(f)\Big) = \gamma(f).$$

It follows that $K' \subset K_1$ and that K_1 is closed and convex. We now prove that K_1 is the closed convex hull of K': if $b \in K_1 \setminus \overline{\text{conv}}(K')$ there exists by Hahn-Banach an $f \in E^*$ such that $f(b) > \sup_{z\in \overline{\text{conv}}(K')}\langle f, z\rangle = \sup_{z\in K'}\langle f, z\rangle = \gamma(f)$. And $f(b) > \gamma(f)$ contradicts $b \in K_1$. In particular $K_1 \subset C_\varepsilon$ hence K_1 is weakly compact. □

Lemma 4.4. *If $f \in E^*$ is not 0, it attains its maximum on K_1 at some point x, and if $\sup_{z\in K}\langle f, z\rangle = \gamma(f)$ then $x \in K$.*

Proof. Since K' is compact and f continuous there is some point $x' \in K' \subset K_1$ such that $m := \langle f, x'\rangle = \sup_{z\in K'}\langle f, z\rangle = \gamma(f)$. And since $\{z \in E : \langle f, z\rangle \le m\}$ is closed convex and contains K', it contains K_1. So $\sup_{z\in K_1}\langle f, z\rangle = m = f(x')$.

Assume $f \neq 0$ attains its maximum on K_1 at x and denote $\mu = \sup_{z\in K}\langle f, z\rangle$. We have

$$f(x) = \gamma(f) = \sup_{0\le t\le 1}\Big((1-t)\mu + tf(a) + \varepsilon t(1-t)p(f)\Big)$$

Since $f \neq 0$ we have $p(f) > 0$ and the function

$$\rho\colon t \mapsto (1-t)\mu + tf(a) + \varepsilon t(1-t)p(f)$$

is strictly concave on $[0, 1]$: so it attains its maximum m on $[0, 1]$ at a unique point s. If $\gamma(f) = \mu = \rho(0)$ we necessarily have $s = 0$. Since $x' \in K'$ there are $\theta \in [0, 1]$, $y \in K$ and $u \in B$ such that $x' = (1-\theta)y + \theta a + \varepsilon\theta(1-\theta)u$. Therefore

$$\begin{aligned}\gamma(f) = f(x') &= (1-\theta)f(y) + \theta f(a) + \varepsilon\theta(1-\theta)f(u)\\ &\le (1-\theta)\mu + \theta f(a) + \varepsilon\theta(1-\theta)p(f) = \rho(\theta) \le \gamma(f) = \mu\end{aligned}$$

hence $\rho(\theta) = \mu$. This implies that $\theta = 0$ and $x' = y \in K$. So $K' \cap f^{-1}(\mu) \subset K$. And since $K_1 = \overline{\text{conv}}(K')$ it follows from Lemma 4.2 that

$$x \in K_1 \cap f^{-1}(\mu) \subset \overline{\text{conv}}(K' \cap f^{-1}(\mu)) \subset \overline{\text{conv}}(K) = K\,,$$

which is the desired conclusion. □

Theorem 4.5. *Let E be a w.c.g. Banach space, X be a closed convex subset of E, $K \subset X$ be a weakly compact convex set and $a \notin K$ an interior point of X. Then there exists a convex weakly compact set K_1 such that $K \subset K_1 \subset X$ and that for all $f \in E^*$ the set $M(f) := \{x \in K_1 : f(x) = \sup_{z\in K_1} f(z)\}$ satisfies $M(f) \subset K$ or $M(f) \cap K = \emptyset$.*

Proof. This follows immediately from Lemmas 4.3 and 4.4. □

Theorem 4.6. *Let E be a w.c.g. Banach space, X a closed convex subset of E with non-empty interior, K a weakly compact convex subset of X, and $A\colon E \to E^*$ an almost monotone affine function.*
Assume that for all $x \in X$, $\sup_{y\in K}\langle A(x), x - y\rangle \ge 0$. Then there exists some $x^* \in K$ such that $\langle A(x^*), x^* - y\rangle \le 0$ for all $y \in X$.

Proof. Assume first that $A(x) \neq 0$ for all $x \in X \setminus K$.

Claim 4.7. *Let $J = \{a_0, a_1, \ldots, a_{m-1}\}$ be a finite subset of the interior X_0 of X. Then there exists some $\hat{x}_J \in K$ such that $\langle A(\hat{x}_J, \hat{x}_J - y\rangle \leq 0$ for all $y \in K \cup J$.*

Proof. Using Theorem 4.5 we construct inductively a finite non-decreasing sequence $(K_n)_{n\leq m}$ of convex weakly compact subsets of X such that $K_0 = K$, $a_n \in K_{n+1}$ and for all $f \neq 0$ in E^* and $0 \leq n \leq m$

$$M_n(f) \subset K \quad \text{or} \quad M_n(f) \cap K = \emptyset \tag{4}$$

where $M_n(f)$ denotes the set $\{x \in K_n : \langle f, x\rangle = \gamma_n(f) := \sup_{z\in K_n}\langle f, z\rangle\}$. Indeed (4) holds for K_0. Assume $0 \leq n < m$, K_n is constructed and (4) holds for K_n.

If $a_n \in K_n$ set $K_{n+1} = K_n$. And if $a_n \notin K_n$ choose $\varepsilon_n > 0$ small enough for $a_n + \varepsilon_n B \subset X \setminus K_n$. Then the convex weakly compact set K_{n+1} given by Lemma 4.3 from K_n and $a_n + \varepsilon_n B$ is contained in $\operatorname{conv}(K_n \cup (a_n + \varepsilon_n B))$ hence in X. Moreover for $f \neq 0$ in E^*

- either $M_{n+1}(f) \cap K_n = \emptyset$ and a fortiori $M_{n+1}(f) \cap K = \emptyset$,
- or $M_{n+1}(f) \subset K_n$, so $\gamma_{n+1}(f) = \gamma_n(f)$ and

$$M_n(f) \subset M_{n+1}(f) \subset \{x \in K_n : \langle f, x\rangle = \gamma_n(f))\} = M_n(f)$$

And since (4) holds for K_n,

$$M_{n+1}(f) = M_n(f) \subset K \quad \text{or} \quad M_{n+1}(f) \cap K = M_n(f) \cap K = \emptyset$$

hence (4) holds for K_{n+1}. Since A is almost monotone it follows from Lemma 2.8 that one can find some $\hat{x}_J = \hat{x}_m \in K_m$ such that $\langle A(\hat{x}_J), \hat{x}_J - y\rangle \leq 0$ for all $y \in K_m$ and in particular $\langle A(\hat{x}_J), \hat{x}_J - y\rangle \leq 0$ for all $y \in K \cup J$. □

By hypothesis the linear functional $f = -A(\hat{x}_J)$ satisfies

$$\exists y_0 \in K \ \ \langle f, y_0\rangle \geq \langle f, \hat{x}_J\rangle \quad \text{and} \quad \forall y \in K_m \ \ \langle f, \hat{x}_J\rangle \geq \langle f, y\rangle$$

for $\hat{x}_J \in M_m(f)$ and $y_0 \in K \cap M_m(f)$. So by (4) we get $\hat{x}_J \in K$ or $f = 0$, hence $\hat{x}_J \in K$ since $A(x) \neq 0$ for $x \in X \setminus K$. And if x^* is a cluster value in K of the $\hat{x}_J$'s for finite $J \subset X \setminus K$ we get by Lemma 2.6: $\langle A(x^*), x^* - y\rangle \leq 0$ for all finite $J \subset X_0 \setminus K$ and all $y \in K \cup J$, hence for all $y \in X$ since X_0 is dense in X.

If there exists some $\hat{x} \in K$ such that $A(\hat{x}) = 0$, this $\hat{x}$ satisfies $\langle A(\hat{x}), \hat{x} - y\rangle = 0$ for all $y \in X$. Finally, if the closed convex set $N = \{x \in X : A(x) = 0\}$ is non-empty and disjoint from K, one can find by Hahn-Banach some $h \in E^*$ and some $\alpha \in \mathbb{R}$ such that $\sup_{x\in K}\langle h, x\rangle < \alpha < \inf_{y\in N}\langle h, x\rangle$.

Then replace X by $X_\alpha = \{y \in X : \langle h, y\rangle \leq \alpha\} \supset K$: we get $A(x) \neq 0$ for all $x \in X_\alpha$ and $\sup_{y\in K}\langle A(x), x - y\rangle \geq 0$ for all $x \in X_\alpha$; and it follows from what precedes that one can find some $x^* \in K$ such that $\langle A(x^*), x^* - y\rangle \leq 0$ for all $y \in X_\alpha$. Then for $y \in X \setminus X_\alpha$ put $y_t = x^* + t(y - x^*)$ for $0 < t \leq 1$.

Since $\langle h, y_t\rangle = \langle h, x^*\rangle + t\langle h, y - x^*\rangle$ we get $\langle h, y_t\rangle = \alpha$, hence $y_t \in X_\alpha$ for

$$t = \frac{\alpha - \langle h, x^*\rangle}{(\langle h, y\rangle - \alpha) + (\alpha - \langle h, x^*\rangle)} \in]0, 1[,$$

then $x^* - y_t = t(x^* - y)$ and $\langle A(x^*), x^* - y\rangle = \dfrac{1}{t}\langle A(x^*), x^* - y_t\rangle \le 0$, thus $\langle A(x^*), x^* - y\rangle \le 0$ for all $y \in X$. □

Proof of Theorem 4.1. If E is w.c.g. this is exactly the statement of Theorem 4.6. If not, as in the proof of Theorem 1.6, denote by E_0 the closed linear space generated by K, by $\mathcal{F}$ the set of finite-dimensional linear subspaces of E and by B_E the unit ball of E. For all $F \in \mathcal{F}$, $E_0 + F$ is a w.c.g. Banach space since $K + F \cap B_E$ is weakly compact and total in $E_0 + F$, and $X_F = X \cap (E_0 + F)$ is a closed convex neighborhood of K in $E_0 + F$. Moreover, $A_{|E_0+F}$ is almost monotone and $\sup_{y\in K}\langle A(x), x - y\rangle \ge 0$ for all $x \in X \cap (E_0 + F)$. It follows from Theorem 4.6 that one can find some $\hat{x}_F \in K$ such that $\langle A(\hat{x}_F), \hat{x}_F - y\rangle \le 0$ for all $y \in X_F$. Then by Lemma 2.6 any weak cluster value x^* in K of the $\hat{x}_F$'s will satisfy $\langle A(x^*), x^* - y\rangle \le 0$ for all $y \in X$.

Let Z be a complement of W in E, $Q\colon E \to Z$ be the linear projection with kernel W and W^o the polar set of W, i.e. the set $\{f \in E^* : \forall w \in W\ \langle f, w\rangle = 0\}$. Since A is affine and continuous, $K_0 = K \cap A^{-1}(W^o)$ is convex and compact, and so is $Q(K_0) \subset Z$.

Replace X by $X' = K + W$. For any linear functional ℓ on Z and any integer n consider $\alpha = \sup_{x\in K}\langle \ell, Q(x)\rangle = \sup_{x\in X'}\langle \ell, Q(x)\rangle$ and the affine mapping $A'\colon E \to E^*$ defined by

$$A'(x) = A(x) + 2^n\big(\langle Q^*(\ell), x\rangle - \alpha\big).Q^*(\ell)\,.$$

We claim that for all $x \in X' \setminus K$ we have $\sup_{y\in K}\langle A'(x), x - y\rangle \ge 0$. Indeed if $y_0 \in K$ satisfies $\langle A(x), x - y_0\rangle = \sup_{y\in K}\langle A(x), x - y\rangle \ge 0$ we have

$$\begin{aligned}\langle A'(x), x - y_0\rangle &= \langle A(x), x - y_0\rangle + 2^n\big(\langle \ell, Q(x) - \alpha\rangle\big).\langle \ell, Q(x) - Q(y_0)\rangle\\ &\ge 2^n\big(\langle \ell, Q(x)\rangle - \alpha\big).\langle \ell, Q(x) - Q(y_0)\rangle \ge 0\end{aligned}$$

since $\langle \ell, Q(x)\rangle \le \alpha$ and $\langle \ell, Q(x)\rangle \le \langle \ell, Q(y_0)\rangle$. So A' satisfies the conditions of Theorem 4.6 and there exists $x' \in K$ such that $\langle A'(x'), x' - y\rangle \le 0$ for all $y \in X'$ and in particular for all $y \in x' + W$. Hence $A'(x')_{|W} = 0$, and since $Q_{|W} = 0$ we have $A(x')_{|W} = A'(x')_{|W} = 0$, thus $x' \in K_0$. Moreover,

$$\begin{aligned}0 &\ge \langle A'(x'), x' - y_0\rangle\\ &= \langle A(x'), x' - y_0\rangle + 2^n\big(\langle \ell, Q(x')\rangle - \alpha\big).\langle \ell, Q(x') - Q(y_0)\rangle\\ &\ge -\|A(x')\|.\|x' - y_0\| + 2^n\big(\alpha - \langle \ell, Q(x')\rangle\big)^2.\end{aligned}$$

Since the set $\{A(x) : x \in K\}$ is weak*-compact in E^*, it is bounded and $M^2 = \sup_{x\in K}\|A(x)\|.\operatorname{diam}(K) < +\infty$. So $\alpha - \langle \ell, Q(x')\rangle \le 2^{-n/2}.M$ and

$$\sup_{y\in K_0}\langle \ell, Q(y)\rangle \ge \langle \ell, Q(x')\rangle \ge \alpha - 2^{-n/2}.M = \sup_{x\in K}\langle \ell, Q(x)\rangle - 2^{-n/2}.M$$

It follows that for all $\ell \in Z^*$ we get $\sup_{z\in Q(K_0)}\langle \ell, z\rangle = \sup_{z\in Q(K)}\langle \ell, z\rangle$. So $Q(K_0) = Q(K)$ by Hahn-Banach's theorem whence $K \subset K + W = K_0 + W$. □

5. The finite-rank case

We now examine the case where the (non-necessarily continuous) affine operator A on $E = W \times Z$ takes values in $W^o = \{f \in E^* : \forall w \in W \ \langle f, w\rangle = 0\}$, with W Banach space and Z finite-dimensional space.

Again we assume that K is a weakly compact subset of E, Y is a closed convex subset of Z, $K \subset X = W \times Y$ and that for all $x \in X \setminus K$ the inequality

$$\sup_{y \in K} \langle A(x), x - y\rangle > 0 \tag{5}$$

holds. And we want to prove that there exists some $x^* \in K$ such that

$$\sup_{y \in X} \langle A(x^*), x^* - y\rangle \leq 0\,. \tag{6}$$

As previously we denote by P and Q the projections on W and Z. We also denote by M the finite-dimensional compact convex set $Q(K) \subset Z$ that we look as a subset of E by identifying $z \in Z$ with $(z, 0) \in E$. Let $\varphi \colon W \to Z^*$ be the linear mapping $w \mapsto A(w) - A(0)$; in particular we have $A(x + w) = A(x) + \varphi(w)$ for all $x \in E$ and all $w \in W$.

Lemma 5.1. *If W is infinite-dimensional there exist points x_* in K and x in $X \setminus K$ such that $Q(x) = Q(x_*)$ and $\langle A(x), x_*\rangle = \sup_{y \in K}\langle A(x), y\rangle$.*

Proof. Since Z is finite-dimensional the affine mapping $h \colon Z \to Z^*$ defined by $h(z) = -A(z)$, is continuous. So applying Lemma 2.7 to M and h we can find $u^* \in M$ such that $\sup_{v \in M}\langle h(u^*), u^* - v\rangle \leq 0$, so in fact $\sup_{v \in M}\langle h(u^*), u^* - v\rangle = 0$ since $\sup_{v \in M}\langle h(u^*), u^* - v\rangle \geq \langle h(u^*), u^* - u^*\rangle = 0$. This means that

$$\begin{aligned} \sup_{v \in M}\langle A(u^*), v\rangle &= \langle A(u^*), u^*\rangle + \sup_{v \in M}\langle A(u^*), v - u^*\rangle \\ &= \langle A(u^*), u^*\rangle + \sup_{v \in M}\langle h(u^*), u^* - v\rangle = \langle A(u^*), u^*\rangle \end{aligned}$$

We claim that there are $x \in X \setminus K$ and $x_* \in K$ such that $Q(x) = Q(x_*) = u^*$. Indeed $u^* \in M = Q(K)$ means exactly that there exists $x_* \in K$ such that $u^* = Q(x_*)$. Moreover since Z is finite-dimensional and W is not, the linear mapping φ cannot be one-to-one, and there exists $w_0 \in \ker(\varphi) \setminus \{0\}$. Since K is bounded one can find some real number λ large enough for $u^* + \lambda w_0 \notin K$. Then $x = u^* + \lambda w_0 \in X \setminus K$.

For $y \in X$ and $v = Q(y) \in M$ we have $w_1 = x - u^* \in W$, $w_2 = y - v \in W$ and get since $A(x) \in W^o$:

$$\begin{aligned} \langle A(u^*), u^*\rangle &= \langle A(x - \lambda w_0), x - \lambda w_0\rangle = \langle A(x) - \lambda\varphi(w_0), x - \lambda w_0\rangle \\ &= \langle A(x), x - \lambda w_0\rangle = \langle A(x), x\rangle - \lambda\langle A(x), w_0\rangle = \langle A(x), x\rangle \end{aligned}$$

and similarly

$$\begin{aligned} \langle A(x), y\rangle &= \langle A(u^* + \lambda w_0), v + w_2\rangle = \langle A(u^*) + \lambda\varphi(w_0), v + w_2\rangle \\ &= \langle A(u^*), v + w_2\rangle = \langle A(u^*), v\rangle + \langle A(u^*), w_2\rangle = \langle A(u^*), v\rangle\,. \end{aligned}$$

It follows that

$$\sup_{y\in K}\langle A(x),y\rangle = \sup_{v\in M}\langle A(u^*),v\rangle = \langle A(u^*),u^*\rangle = \langle A(x),x\rangle$$

and this completes the proof. □

Theorem 5.2. *Under the above hypotheses there exists some $x^* \in K$ such that $\langle A(x^*), x^* - y\rangle \leq 0$ for all $y \in X$.*

Proof. It follows from Lemma 5.1 that if W were not finite-dimensional A could not satisfy (5). Thus W is finite-dimensional and $E = W \times Z$ too. Thus A is continuous and the result follows from Theorem 1.9. □

6. Getting rid of continuity

The aim of this section is to show that in Theorem 1.8 the hypothesis of continuity on A does not follow automatically from the hypothesis

$$\sup_{y\in K}\langle A(x),x-y\rangle > 0 \text{ for all } x \in X\setminus K \tag{7}$$

but that the conclusion still holds without this assumption.

As observed in the introduction Yen's theorem (Theorem 1.4) needs no continuity hypothesis on the affine mapping $A\colon H \to H$ because the hypothesis

$$\sup_{y\in K}\langle A(x),x-y\rangle \geq 0 \text{ for all } x \in H\setminus K$$

implies the monotonicity of A which in turn implies the continuity of A and

$$\exists x^* \in K \quad \sup_{y\in X}\langle A(x^*),x^*-y\rangle \leq 0 \tag{8}$$

follows from Theorem 1.9. Nevertheless, in general, when $X \neq E$ the hypothesis (7) does not imply the continuity of A as shown by the trivial following example.

Example 6.1. There exists a Hilbert space H, a weakly compact subset K of H, a closed convex subset X of H with non-empty weak interior containing K and an affine discontinuous mapping $A\colon H \to H^* \simeq H$ satisfying (7). □

Let W be the Hilbert space ℓ^2, $(\mathbf{e}_n)_n$ its canonic orthonormal basis, $H = W \times \mathbb{R}$, $X = W \times [-1,1]$, $K = \{0\} \times [-1,1]$.

Equip H with the norm: $(w,t) \mapsto \big(\|w\|^2 + t^2\big)^{1/2}$ for which it is a Hilbert space. Choose using the Axiom of Choice some linear functional $f\colon W \to \mathbb{R}$ satisfying $\langle f, \mathbf{e}_n\rangle = n$. Then f is discontinuous.

Define the linear operator $A\colon H \to H$ by: $\quad A(w,t) = \big(w, f(w) + t\big) \quad .$

If A were continuous so would be f since $f(w) = \langle A(w,0),(0,1)\rangle$. Thus A is discontinuous but for $x = (w,t) \in X \setminus K$ and $y = (0,t) \in K$ we have $w \neq 0$ hence

$$\langle Ax, x-y\rangle = \langle w, w-0\rangle + (f(w)+t).(t-t) = \|w\|^2 > 0$$

and this shows that (7) holds. Of course $x^* = (0,0) \in K$ satisfies for all element $y = (w,t)$ of X:

$$\langle Ax^*, x^* - y\rangle = \langle 0, -w\rangle - t.(f(0)+0) = 0,$$

hence (8) holds too. □

The following example shows that if we replace in (7) the strict inequality by a weak inequality as in Section 4 the conclusion does not longer hold without assuming the continuity of A.

Example 6.2. There exists a Hilbert space $\mathcal{H}$, a weakly compact subset K of $\mathcal{H}$, a closed convex subset X with non-empty weak interior of $\mathcal{H}$ containing K and an affine discontinuous mapping $A\colon \mathcal{H} \to \mathcal{H}^* \simeq \mathcal{H}$ satisfying $\sup_{y\in K}\langle A(x), x-y\rangle \geq 0$ for all $x \in \mathcal{H} \setminus K$ but $\sup_{y\in X}\langle A(x), x-y\rangle > 0$ for all $x \in K$.

Proof. We embed as in Section 3 the Radon measures on the unit circle $\mathbb{T} \subset \mathbb{C}$ in the Hilbert space $\mathcal{H} = H^{-1}(\mathbb{T})$ and take for K the set of probability Radon measures on $\mathbb{T}$. As in Section 3 we have for μ and ν in K:

$$\|\mu-\nu\|_{-1}^2 = \sum_{p\in\mathbb{Z}} \frac{|\hat\mu(p) - \hat\nu(p)|^2}{p^2+1}$$

and $|\hat\mu(p) - \hat\nu(p)| \leq 2$ for all integer p. It follows that the embedding of K into $H^{-1}(\mathbb{T})$ is continuous when K is equipped with the weak*-topology $\sigma(K, \mathcal{C}(\mathbb{T}))$ and $H^{-1}(\mathbb{T})$ with the norm topology, hence K is norm-compact in $H^{-1}(\mathbb{T})$.

The functions $\Re\colon x+iy \mapsto x$ and $\Im\colon x+iy \mapsto y$ from $\mathbb{T}$ to $\mathbb{R}$ are both in $H^1(\mathbb{T})$; so the linear mapping $Q\colon T \mapsto \langle T, \Re + i\Im\rangle$ is continuous on $H^{-1}(\mathbb{T})$ to $\mathbb{C} \simeq \mathbb{R}^2$ and it is easy to check that $Q(K)$ is the closed unit disk $\mathbb{D}$ since for $\mu \in K$: $|Q(\mu)| \leq \int_{\mathbb{T}} |z|\, d\mu(z) = 1$ and $Q(\delta_u) = u$ for $u \in \mathbb{T}$. Moreover if $\mu \in K$ and $Q(\mu) = u \in \mathbb{T}$ then

$$0 = \Re\big(1 - \bar u Q(\mu)\big) = \int_{\mathbb{T}} \Re(1 - \bar u.z)\, d\mu(z)$$

while the continuous function on $\mathbb{T}$: $z \mapsto \Re(1-\bar u.z)$ is non-negative and vanishes only at u: it follows that $\mu = \delta_u$ is the only solution in K of the equation $Q(\mu) = u$. Let $X = Q^{-1}(\mathbb{D})$.

We now consider the affine function χ_0 on K defined by $\chi_0(\mu) = \mu(\{1\})$ and extend it (using the Axiom of Choice) to a (necessarily discontinuous) real linear function χ on $H^{-1}(\mathbb{T})$.

Then for S and $T \in H^{-1}(\mathbb{T})$ we define $A(T) = \Re + \chi(T).\Im \in H^1(\mathbb{T})$, so:

$$\langle A(T), S\rangle = \langle S, \Re\rangle + \chi(T).\langle S, \Im\rangle = \langle S, \Re + \chi(T).\Im\rangle \in \mathbb{R}$$

This yields an affine function A from $\mathcal{H} = H^{-1}(\mathbb{T})$ to its dual $\mathcal{H}^* = H^1(\mathbb{T})$. Notice that whenever $Q(S) = 0$ then $\langle S, \Re\rangle = \langle S, \Im\rangle = 0$ hence $\langle A(T), S\rangle = 0$ for all $T \in \mathcal{H}$. So with $W = \ker(Q) \subset \mathcal{H}$, we get $A(T) \in W^o$. □

Claim 6.3. *We have*

$$\begin{cases} \sup\limits_{S\in K}\langle A(T), T-S\rangle \geq 0 & \textit{for all } T \in X\,, \\ \sup\limits_{S\in X}\langle A(T), T-S\rangle > 0 & \textit{for all } T \in K\,. \end{cases}$$

Proof. For $T \in X = Q^{-1}(\mathbb{D})$ we can choose some probability measure μ on $\mathbb{T}$ such that $\int_{\mathbb{T}} z\, d\mu(z) = Q(T) \in \mathbb{D}$. Then $\mu \in K$ satisfies $Q(T-\mu) = 0$ hence $\langle A(T), T-\mu\rangle = 0$, thus

$$\sup_{S\in K}\langle A(T), T-S\rangle \geq \langle A(T), T-\mu\rangle = 0\,.$$

Finally since $\langle A(T), \delta_1\rangle = \langle \delta_1, \Re + \chi(T).\Im\rangle = 1$ it is clear that $A(T) \neq 0$ for all $T \in \mathcal{H}$ and that if $T^* \in K$ satisfies $\langle A(T^*), T^*-S\rangle \leq 0$ for all $S \in X$ we have for $S = T^* + \alpha\Re + \beta\Im$

$$Q(S) = Q(T^*) + \alpha + i\beta \in \mathbb{D} \Rightarrow \langle A(T^*), \alpha\Re + \beta\Im\rangle \leq 0\,.$$

Thus if $w = u + iv = Q(T^*) \in \mathbb{D}$

$$|w + \alpha + i\beta| \leq 1 \Rightarrow \langle A(T^*), \alpha\Re + \beta\Im\rangle = \alpha + \beta\chi(T^*) \leq 0$$

This means that the linear function $(\alpha, \beta) \mapsto \alpha + \beta\chi(T^*)$ attains its maximum 0 on the disk $D(-w, 1)$ at 0.

For $\lambda \in \mathbb{R}$ and $\omega = 1 + i\lambda$ the function $u = \alpha + i\beta \mapsto \alpha + \lambda\beta = \Re\big(\bar{\omega}.u\big)$ attains its maximum on $D(-w, 1)$ at $-w + \dfrac{\omega}{|\omega|}$, and this is 0 iff $w = \dfrac{1+i\lambda}{|1+i\lambda|}$. It follows that $|w| = |Q(T^*)| = 1$ for $T^* \in K$ hence that $T^* = \delta_w$ and $\chi(\delta_w) = 1$ if $w = 1$, $\chi(\delta_w) = 0$ if $w \neq 1$. So

$$\frac{1 + i\chi(\delta_w)}{|1 + i\chi(\delta_w)|} \neq w$$

for all $w \in \mathbb{T}$. This completes the proof of the Claim. □

We conclude that $\sup_{S\in X}\langle A(T), T-S\rangle > 0$ for all $T \in K$, and that Theorem 4.1 does not hold without any hypothesis of continuity. □

Nevertheless we can show the following.

Theorem 6.4. *Let E be a Banach space, X a closed convex subset of E with non-empty weak interior and $A\colon E \to E^*$ a (non necessarily continuous) affine operator. Assume that A satisfies* (7) *with respect to K and X. Then there exists some point $x^* \in K$ such that $\langle A(x^*), x^*-y\rangle \leq 0$ for all $y \in X$.*

For the proof of this theorem we need the following lemmas.

Lemma 6.5. *With the notations of Section* 2*, Condition* (8) *for* $x^* \in K$ *is equivalent to*
(i) $P^*A(x^*) = 0$ *and* (ii) $\forall z \in Y \quad \langle A(x^*), Q(x^*) - z\rangle \le 0\,.$

Proof. If (8) holds for $x^* \in K$ then for all $w \in W$ and $y = x^* - w \in X$ we have $\langle A(x^*), x^* - y\rangle = \langle A(x^*), w\rangle \le 0$. And applying to $-w$ we get $\langle A(x^*), -w\rangle \le 0$ hence $\langle A(x^*), w\rangle = 0$. So for all $u \in E$, $w = P(u) \in W$ and

$$\langle P^*A(x^*), u\rangle = \langle A(x^*), P(u)\rangle = \langle A(x^*), w\rangle = 0$$

thus $P^*A(x^*) = 0$. Moreover, with $w = P(x^*) \in W$, $z \in Y$ and $y = z+P(x^*) \in X$

$$\langle A(x^*), Q(x^*) - z\rangle = \langle A(x^*), x^* - (P(x^*) + z)\rangle = \langle A(x^*), x^* - y\rangle \le 0\,.$$

Conversely, if (i) and (ii) hold then for any $y \in X$ there is some $z \in Y$ such that $w = z - y \in W$. So $P(w) = w$ and

$$\begin{aligned}\langle A(x^*), x^* - y\rangle &= \langle A(x^*), Q(x^*) + P(x^*) + w - z\rangle \\ &= \langle A(x^*), Q(x^*) - z\rangle + \langle P^*A(x^*), x^* + w\rangle = \langle A(x^*), Q(x^*) - z\rangle \le 0.\end{aligned}$$ □

Lemma 6.6. *Let F be a finite-dimensional normed space, V be a normed space, B be its unit ball, and $f\colon V \to F$ be a linear mapping. Let $G \subset V \times F$ be the graph of f and $F_0 = \{y \in F : (0, y) \in \overline{G}\}$. Then $F_0 = \bigcap_{n\in\mathbb{N}} f(2^{-n}B)$ and f is continuous if and only if $F_0 = \{0\}$.*
Morover, if g is a linear mapping from F to a normed space E and $g_{|F_0} = 0$ then $g\circ f$ is continuous.

Proof. Define $F_0' = \bigcap_{n\in\mathbb{N}} f(2^{-n}B)$. If f is continuous, there exists M such that $\|f(x)\| \le M.\|x\|$. Then $f(2^{-n}B) \subset B_F(0, 2^{-n}.M)$ and

$$F_0' \subset \{y \in F : \|y\| \le \inf_n M.2^{-n}\} = \{0\}\ .$$

If $y \in F_0'$ there exists a sequence (v_n) in V such that $f(v_n) = y$ and $\|v_n\| \le 2^{-n}$. Then $(v_n, y) \in G$ and $(v_n, y) \to (0, y)$. Hence $y \in F_0$.

The linear space $f(V)$ is finite-dimensional hence closed in F. Thus if $y \in F_0$, there is a sequence $(v_n, y_n) \in G$ which converges to $(0, y)$: so $y \in \overline{f(V)} = f(V)$. We can find a linear section $\rho\colon f(V) \to V$ which is continuous since $f(V)$ is finite-dimensional. Put $v_n' = v_n + \rho(y - y_n) \in V$. Then

$$f(v_n') = y_n + f\circ\rho(y - y_n) = y_n + (y - y_n) = y$$

and $v_n' \to 0$ since $\rho(y - y_n) \to \rho(y - y) = 0$. Thus $y \in F_0'$. So $F_0 = F_0'$.

We now prove by induction on the rank $k = \mathrm{rk}(f)$ that f is continuous if $F_0 = \{0\}$. For $k = 0$ nothing is to prove. Assume the statement holds for $\mathrm{rk}(f) < k$ and

$F = f(V)$ has dimension k. Choose any $x_0 \in V \setminus \ker(f)$ and set $y_0 = f(x_0) \neq 0$, then fix n such that $y_0 \notin W = f(2^{-n}B)$. Since W is convex symmetric and generates F, the interior W_0 of W in F contains 0 and by Hahn-Banach's theorem we can find some linear (necessarily continuous) functional ξ on F such that $\langle \xi, y \rangle < 1 = \langle \xi, y_0 \rangle$ for all $y \in W_0$. Since W_0 is dense in W we get $\langle \xi, y \rangle \leq 1$ for $y \in f(2^{-n}B)$, hence $\|\xi \circ f\| \leq 2^n$ and $\xi \circ f(x_0) = 1 \neq 0$.

Thus $\xi \circ f$ is continuous on V, the kernel $V_0 = \ker(\xi \circ f)$ is closed in V, and $\mathrm{rk}(f_{|V_0}) < k$. Since $\bigcap_n f(2^{-n}B \cap V_0) \subset \bigcap_n f(2^{-n}B) = F_0 = \{0\}$ it follows from the induction hypothesis that $f_{|V_0}$ is continuous. Finally we get for $x \in V$:

$$f(x) = f_{|V_0}(x - \xi \circ f(x) \cdot x_0) + \xi \circ f(x) \cdot y_0$$

hence $f = f_{|V_0} \circ (I - \pi) + \xi \circ f \circ \pi \cdot y_0$, where π is the continuous linear mapping $x \mapsto \xi \circ f(x) \cdot x_0$ from V to itself. And this shows the continuity of f.

Assume now that $g \colon F \to E$ is linear and $g = 0$ on F_0. Up to replacing E by $g(F)$ we can also assume that E is finite-dimensional and g is onto. Denote by K the kernel of g and by $G' \subset V \times E$ the graph of $g \circ f$. We have

$$\begin{aligned} G' = \{(v, g \circ f(v)) : v \in V\} = (I \times g)(G) \subset (I \times g)(\overline{G}) \\ \subset (I \times g)\big(G + (\{0\} \times K)\big) \subset (I \times g)(G) \end{aligned}$$

since $\overline{G} = G + (\{0\} \times F_0) \subset G + (\{0\} \times K)$, hence $G' = (I \times g)\big(G + (\{0\} \times K)\big)$. Since F and E are finite-dimensional, it is easy to check that $I \times g$ is an open mapping from $V \times F$ onto $V \times E$. Moreover

$$G' = (I \times g)\big(G + (\{0\} \times K)\big) = (V \times E) \setminus (I \times g)\big((V \times F) \setminus (\overline{G} + (\{0\} \times K))\big)$$

Since $\overline{G}$ is closed and $\{0\} \times K$ finite-dimensional then $\overline{G} + (\{0\} \times K)$ is closed too, and this shows that G' is closed in $V \times E$.

It follows that $\overline{G'} \cap (\{0\} \times E) = G' \cap (\{0\} \times E) = \{0\}$, and the above proof shows that $g \circ f$ is continuous. □

We still define φ, W, Z, Y, P, Q as in Section 2, and B_W as the unit ball of W. Since E is a Banach space, so is its closed linear subspace W, and Lemma 2.2 proves that $\varphi_{|W}$ is monotone, hence continuous by Theorem 1.5. Since Z is finite-dimensional we can equip it with an euclidean norm $\|.\|_2$ and renorm $E \simeq W \times Z$ in an equivalent way by defining $|||w + z||| = \|w\| + \|z\|_2$. Then E^* is identified to $W^* \times Z$, P and $Q \colon E \to E$ are the projections $(w, z) \mapsto (w, 0)$ and $(w, z) \mapsto z$, P^* and $Q^* \colon E^* \to E^*$ are the mappings $(\xi, z) \mapsto (\xi, 0)$ and $(\xi, z) \mapsto (0, z)$. In the sequel of this section we assume silently that (7) holds.

The linear space $F = Q^* \varphi P(E) = Q^* \varphi(W) \subset Z^* \simeq Z$ is a finite-dimensional subspace of E^*. Then $F_1 = \bigcap_n 2^{-n} \cdot Q^* \varphi(B_W)$ is a (necessarily closed) linear subspace of F; we consider the orthogonal projection π from Z onto F_1 and $\pi' = Id_Z - \pi$ is the projection from Z onto $F_0 = F_1^{\perp}$. Recall that $\tilde{K}$ is the closed convex hull of K, which is weakly compact and contained in X.

Theorem 6.7. *Assume that K is convex and weakly compact in E and that $\sup_{y\in K}\langle A(x), x-y\rangle \geq 0$ for all $x \in X \setminus K$.*
Then the set $K_0 = \{x \in K : A(x)_{|W} = 0\}$ is convex and weakly compact, and $Q(K_0) = Q(K)$.

Proof. Since $X' = K + W \subset X$ we have $\sup_{y\in K}\langle A(x), x-y\rangle \geq 0$ for all $x \in X' \setminus K$. So we can and do replace X by X' and get $Q(X) = Q(K)$.

For any $z \in F_1$ consider the sets $E(z) = (\pi Q)^{-1}(z)$, $X(z) = X \cap E(z)$ and $K(z) = K \cap E(z)$ which is a convex weakly compact subset of $X(z)$. It is clear that $x + W \subset X(z)$ as soon as $x \in X(z)$. So $X(z)$ has non-empty weak interior in the affine space it spans. Moreover it follows from Lemma 6.6 that $A_{|E(z)}$ is continuous: indeed $\bigcap_n \Big(A(x + 2^{-n}B_E) - A(x)\Big) \cap \ker(\pi Q) = \{0\}$.

Claim 6.8. *Assume $K(z) \neq \emptyset$. Then for every $x \in X(z) \setminus K(z)$ the inequality $\sup_{y\in K(z)}\langle A(x), x-y\rangle \geq 0$ holds.*

Proof. Assume to achieve a contradiction that

$$\sup_{y\in K(z)} \langle A(x_*), x_* - y\rangle < 0$$

for some $x_* \in X(z)$. Then the weakly compact convex set K is disjoint from the closed affine subspace $H = \{y \in E(z) : \langle A(x_*), x_* - y\rangle = 0\}$ of E since $K \cap H \subset K(z)$. It follows from Hahn-Banach's theorem that there exists a continuous linear functional $\ell \in E^*$ which separates K from H: $\sup_{x\in H}\langle \ell, x\rangle < \inf_{x\in K}\langle \ell, x\rangle$.

In particular the linear functional ℓ is bounded from above on the linear space $H - x^* = \ker(A(x_*)) \cap \ker(\pi Q)$ hence vanishes on $\ker(A(x_*)) \cap \ker(\pi Q)$. It follows that $\ell = \alpha A(x_*) + \ell_1$, where $\alpha \in \mathbb{R}$ and $\ell_1 \in F_1$. Since $Q(x_*) \in Q(X) = Q(K)$, there exists some $x' \in K$ such that $Q(x^*) = Q(x')$.

Thus we have $x_* \in H$ and $x' \in K(z)$, and hence

$$0 > \langle \ell, x^*\rangle - \langle \ell, x'\rangle = \alpha\langle A(x^*), x^* - x'\rangle + \langle \ell_1, x^* - x'\rangle = \alpha\langle A(x^*), x^* - x'\rangle$$

since $\pi Q(x_* - x') = 0$. Then $A(x_*, x_* - x') < 0$ so $\alpha > 0$. Replacing ℓ by $\ell' = \alpha^{-1}\ell$ we get

$$\begin{aligned}\sup_{x\in H}\langle \ell', x\rangle &= \sup_{x\in H}\big(\langle A(x_*), x\rangle + \alpha^{-1}\langle \ell_1, x\rangle\big) = \langle A(x_*), x_*\rangle + \alpha^{-1}\langle \ell_1, x_*\rangle \\ &< \inf_{y\in K}\langle \ell', y\rangle = \inf_{y\in K}\big(\langle A(x_*, y\rangle + \alpha^{-1}\langle \ell_1, y\rangle\big).\end{aligned}$$

Hence $-\delta = \sup_{y\in K}\langle A(x_*) + \alpha^{-1}\ell_1, x_* - y\rangle < 0$.

Since $\alpha^{-1}\ell_1 \in F_1$ we can find a sequence $(w_n) \in W$ such that $\|w_n\| < 2^{-n}$ and $\pi Q(w_n) \to \alpha^{-1}\ell_1$. Then $x_n = x_* + w_n \to x_*$ and $x_n \in X \setminus K$ for n large enough. Moreover

$$\begin{aligned}A(x_n) = A(x_*) + \varphi(w_n) &= A(x_*) + P^*\varphi(w_n) + \pi' Q^*\varphi(w_n) + \pi Q^*\varphi(w_n) \\ &\to A(x_*) + \alpha^{-1}\ell_1 = \ell\end{aligned}$$

since $\varphi' = (P^*\varphi + \pi' Q^*\varphi)_{|W} = (\varphi - \pi Q^*\varphi)_{|W}$ is continuous. For $y \in K$ we have

$$\begin{aligned}\langle A(x_n), x_n - y\rangle &\leq \langle \ell, x_n - y\rangle + \|A(x_n) - \ell\|.\|x_n - y\| \\ &\leq \langle \ell, x_* - y\rangle + \|\ell\|.\|w_n\| + \|A(x_n) - \ell\|.\big(d(x_n, K) + \operatorname{diam}(K)\big) \\ &\leq -\delta + 2^{-n}\|\ell\| + \|A(x_n) - \ell\|.\big(d(x_*, K) + \operatorname{diam}(K) + \|w_n\|\big),\end{aligned}$$

hence $\sup_{y\in K}\langle A(x_n), x_n - y\rangle \leq -\delta/2$ for n large, in contradiction to the hypothesis of the Theorem. This completes the proof of the claim. □

Then it follows from the continuity of $A_{|E(z)}$ and from Theorem 4.1 that for all $y \in Q(K)$ such that $\pi(y) = z$ we have $y \in Q(K_0)$. And this achieves the proof of Theorem 6.7. □

Define now

$$E_0 = \{x \in E : A(x) \in W^o\} = \{x \in E : A(x)_{|W} = 0\} = \{x \in E : P^*A(x) = 0\}$$

which is a closed affine manifold of E since P^*A is affine and continuous. Then we put $W_0 = W \cap E_0$, $X_0 = X \cap E_0$ and $K_0 = K \cap E_0$ as before. Up to a translation we can assume $0 \in K$ hence $0 \in Q(K) = Q(K_0)$ by Theorem 6.7, and even $0 \in K_0 \subset E_0$. Then E_0 appears as a Banach space, and we define the affine mapping $A_0\colon E_0 \to E_0^*$ by $\langle A_0(x), y\rangle = \langle A(x), y\rangle$ for x and y in E_0.

Lemma 6.9. *Assume that K is convex and weakly compact in E and that $\sup_{y\in K}\langle A(x), x - y\rangle \geq 0$ for all $x \in X \setminus K$. Then $Q(X_0) = Y = Q(X)$.*

Proof. Let a be any point in X and replace K by $K' = \operatorname{conv}(K \cup \{a\}) \subset X$: the condition $\sup_{y\in K'}\langle A(x), x - y\rangle \geq 0$ is clearly satisfied for all $x \in X \setminus K'$ and we conclude from Theorem 6.7 applied to K' that $a \in K' \subset K'_0 + W$ hence that there exists some $a' \in X$ satisfying $Q(a') = Q(a)$ and $A(a')_{|W} = 0$. Thus $Y = Q(X_0)$. □

Lemma 6.10. *The mapping A_0 satisfies* (7) *with respect to K_0 and X_0.*

Proof. Let $x_0 \in X_0 \setminus K_0$. Since A satisfies (7) and

$$x_0 \in X_0 \setminus K_0 \subset X \cap \big(E_0 \setminus K_0\big) = X \cap \big(E_0 \setminus E_0 \cap K\big) \subset X \setminus K$$

there exists some $y \in K$ such that $\langle A(x_0), x_0 - y\rangle > 0$ and since $Q(K) = Q(K_0)$ one can find some $y_0 \in K_0$ such that $Q(y) = Q(y_0)$ hence $w = y - y_0 \in W$. Since $x_0 \in E_0$ we have $A(x_0) \in W^o$ hence $\langle A(x_0), w\rangle = 0$ and

$$\begin{aligned}\langle A_0(x_0), x_0 - y_0\rangle &= \langle A(x_0), x_0 - y_0\rangle = \langle A(x_0), x_0 - y\rangle + \langle A(x_0), w\rangle \\ &= \langle A(x_0), x_0 - y\rangle > 0.\end{aligned}$$ □

Proof of Theorem 6.4. It is clear that A_0 takes values in W_0^o: indeed for any $x_0 \in E_0$ and any $w_0 \in W_0 = W \cap E_0$ we have $A(x_0) \in W^o$, hence $\langle A(x_0), w_0\rangle = 0$

and $\langle A_0(x_0), w_0\rangle = 0$. Then we deduce from Theorem 5.2 that for some $x_0^* \in K_0$ we have $\langle A_0(x_0^*), x_0^* - y_0\rangle \leq 0$ for all $y_0 \in X_0$. Since $K_0 = K \cap E_0$, then $x_0^* \in K$ and $A_0(x_0^*) = A(x_0^*)$, hence $\langle A(x_0^*), x_0^* - y\rangle \leq 0$ for all $y \in X_0$.

If $y \in X$ we have $Q(y) \in Y = Q(X_0)$ by Lemma 6.9, and we can find $y_0 \in X_0$ such that $w = y - y_0 \in W$. Again $\langle A(x_0^*), w\rangle = 0$ since $x_0^* \in K_0 \subset E_0$, and

$$\begin{aligned}\langle A(x_0^*), x_0^* - y\rangle &= \langle A(x_0^*), x_0^* - y_0\rangle - \langle A(x_0^*), w\rangle = \langle A(x_0^*), x_0^* - y_0\rangle \\ &= \langle A_0(x_0^*), x_0^* - y_0\rangle \leq 0\end{aligned}$$

what shows that x_0^* satisfies the requested properties for (8) and completes the proof of Theorem 6.4. □

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