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The Minimax Estimation Method for a Class of Inverse Helmholtz Transmission Problems

Oleksandr Nakonechnyi
T. Shevchenko University, Kyiv, Ukraine
a.nakonechniy@gmail.com

Yuri Podlipenko
National Taras Shevchenko University, Kiev, Ukraine
yourip@mail.ru

Yury Shestopalov
University of Gävle, Gävle, Sweden
yuyshv@hig.se

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We present complete mathematical statements and perform detailed investigations of the minimax estimation problems of unknown data for the Helmholtz transmission problems from indirect noisy observations of their solutions. We construct optimal, in certain sense, estimates, which are called minimax mean-square estimates, of the values of linear functionals from unknown data. It is established that when unknown data and correlation functions of errors in observations belong to special sets, the minimax mean square estimates are expressed via solutions to certain transmission problems for systems of Helmholtz equations. We prove that these systems are uniquely solvable. Several possible generalizations of the techniques and results are proposed including applications to the problems with incomplete data and pointwise observations.

Keywords: Minimax estimation, noisy observations, inverse Helmholtz transmission problem, incomplete data, minimax mean-square estimates.

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1. Introduction

Determination of electromagnetic or acoustic fields is reduced to the solution of boundary value problems (BVPs) for the Maxwell or Helmholtz equations where the parameters of sources and frequencies are given. Otherwise, additional data are required for the determination of unknown quantities. The latter constitute inverse problems in the classical meaning, they have ill-posed nature and are investigated by many authors.

We study the wave fields in different domains generated by unknown sources that may be situated both inside the domains and on their boundaries. The

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inverse problems in question differ from known classical settings and are reduced to finding the estimates of the right-hand sides of the Helmholtz equations and the boundary conditions under the assumption that they belong to certain subsets of appropriate function spaces. This forms the essence of the proposed technique for the analysis of inverse problems employing optimal estimations.

From the observations of unknown fields or from their linear transformations at certain points or subdomains of the considered domain perturbed by additive random noises, we find optimal estimates both for unknown right-hand sides of the equations entering the BVPs and for the linear functionals from them. In other words, we look for the estimates in the class of linear estimates with respect to observations, for which the maximal mean square error taken over all the realizations of perturbations from certain given sets takes its minimal value. Such estimates are called the minimax mean square or guaranteed estimates.

In the present paper we show that, under some restrictions on the right-hand sides of the equations entering the statement of Helmholtz transmission problems and on the statistical characteristics of observations errors, the linear mean square estimates and estimation errors are expressed via solutions to certain well-posed problems.

Note that the minimax estimation problem of linear functionals from solutions to transmission problems for the Helmholtz equation with inexact data was investigated in [5] and [7]. Neumann boundary value problems for the Helmholtz equation and for pointwise observations similar estimation problems were studied in [6].

For some other types of partial differential equations such an approach was applied in [8] and [9] for obtaining the minimax mean square estimates of their unknown solutions and right-hand sides.

2. Statement of the estimation problem

Assume that in bounded subdomains $\Omega^1_{j_1}$, $j_1 = 1, \ldots, m_1$, and $\Omega^2_{j_2}$, $j_2 = 1, \ldots, m_2$, of domains D and $\mathbb{R}^n \setminus \bar{D}$, respectively, the following functions are observed

$$y^1_{j_1}(x) = C^1_{j_1}\varphi_1(x) + \xi^1_{j_1}(x), \quad x \in \Omega^1_{j_1}, \quad j_1 = 1, \ldots, m_1, \tag{1}$$

$$y^2_{j_2}(x) = C^2_{j_2}\varphi_2(x) + \xi^2_{j_2}(x), \quad x \in \Omega^2_{j_2}, \quad j_2 = 1, \ldots, m_2. \tag{2}$$

Here $D \in \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with smooth boundary Γ of class C^2, $C^i_{j_i} \in \mathcal{L}(L^2(\Omega^i_{j_i}), L^2(\Omega^i_{j_i}))$, $i = 1, 2$, are linear continuous operators, φ_1 and φ_2 correspond to a transmission problem for the Helmholtz equation

$$-(\Delta + k_1^2)\varphi_1(x) = f_1(x) \text{ in } D, \tag{3}$$

$$-(\Delta + k_2^2)\varphi_2(x) = f_2(x) \text{ in } \mathbb{R}^n \setminus \bar{D}, \tag{4}$$

$$\mu_2\varphi_2 - \mu_1\varphi_1 = g_1 \text{ on } \Gamma, \tag{5}$$

$$\frac{\partial \varphi_2}{\partial \nu} - \frac{\partial \varphi_1}{\partial \nu} = g_2 \text{ on } \Gamma, \tag{6}$$

$$\frac{\partial \varphi_2}{\partial r} - ik_2\varphi_2 = o(1/r^{(n-1)/2}),\ r = |x|,\ r \to \infty, \tag{7}$$

where ν is the unit normal to Γ drawn in the direction from D to $\mathbb{R}^n \setminus \bar{D}$, f_1 and f_2 are unknown source terms distributed, correspondingly, in bounded domains D and $D_0 \in \mathbb{R}^n \setminus \bar{D}$, g_1 and g_2 are unknown functions on Γ, $\xi^i_{j_i}(x)$ are the choice functions of unknown random complex-valued fields $\xi^1_{j_1}(x) = \xi^1_{j_1}(\omega, x)$ and $\xi^2_{j_2}(\omega, x)$. In addition, f_1, f_2, g_1, g_2, and $\xi^i_{k_i}$ are assumed to be such that

$$F := (f_1, f_2, g_1, g_2) \in G_0, \quad \xi := (\xi^1_1, \ldots, \xi^1_{m_1}, \xi^2_1, \ldots, \xi^2_{m_2}) \in G_1, \tag{8}$$

where

$$G_0 := \Big\{ \tilde{F} := (\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2) : \tilde{f}_1 \in L^2(D),\ \tilde{f}_2 \in L^2(D_0),\ \tilde{g}_1 \in H^{3/2}(\Gamma),\ \tilde{g}_2 \in H^{1/2}(\Gamma),$$
$$\int_D Q_1(\tilde{f}_1 - f^0_1)(x)\overline{(\tilde{f}_1 - f^0_1)}(x)\, dx + \int_{D_0} Q_2(\tilde{f}_2 - f^0_2)(x)\overline{(\tilde{f}_2 - f^0_2)}(x)\, dx$$
$$+ \int_\Gamma Q_3(\tilde{g}_1 - g^0_1)\overline{(\tilde{g}_1 - g^0_1)}\, d\Gamma + \int_\Gamma Q_4(\tilde{g}_2 - g^0_2)\overline{(\tilde{g}_2 - g^0_2)}\, d\Gamma \le \epsilon_1 \Big\},$$

and G_1 denotes the set of random variables $\tilde{\xi} = (\tilde{\xi}^1_1, \ldots, \tilde{\xi}^1_{j_1}, \ldots, \tilde{\xi}^1_{m_1}, \tilde{\xi}^2_1, \ldots, \tilde{\xi}^2_{j_2}, \ldots, \tilde{\xi}^2_{m_2})$ with values in the Hilbert space[1]

$$H := L^2(\Omega^1_1) \times \cdots \times L^2(\Omega^1_{m_1}) \times L^2(\Omega^2_1) \times \cdots \times L^2(\Omega^2_{m_2})$$

whose components have zero means, $\mathbb{E}\tilde{\xi}^1_{j_1} = 0$, and $\mathbb{E}\tilde{\xi}^2_{j_2} = 0$, with finite second moments $\mathbb{E}\|\tilde{\xi}^1_{j_1}\|^2_{L^2(\Omega^1_{j_1})}$ and $\mathbb{E}\|\tilde{\xi}^2_{j_2}\|^2_{L^2(\Omega^2_{j_2})}$ and unknown correlation functions $R^1_{j_1}(x, y) = \mathbb{E}\tilde{\xi}^1_{j_1}(x)\bar{\tilde{\xi}}^1_{j_1}(y)$ and $\tilde{R}^2_{j_2}(x, y) = \mathbb{E}\tilde{\xi}^2_{j_2}(x)\bar{\tilde{\xi}}^2_{j_2}(y)$, $j_i = 1, \ldots m_i$, $i = 1, 2$, satisfying the condition

$$\sum_{i=1}^{2} \sum_{j_i=1}^{m_i} \int_{\Omega^i_{j_i}} (r^i_{j_i}(x))^2 \tilde{R}^i_{j_i}(x, x)\, dx \le \epsilon_2. \tag{9}$$

Here $f^0_1 \in L^2(D,)$ $f^0_2 \in L^2(D_0,)$ $g^0_1 \in H^{3/2}(\Gamma)$, and $g^0_2 \in H^{1/2}(\Gamma)$ are prescribed functions, Q_1, Q_2, and Q_3, Q_4 are Hermitian positive definite operators in $L^2(D)$,

[1]If u is an element belonging to the Hilbert space H of the form $u = (u^1_1, \ldots, u^1_{m_1}, u^2_1, \ldots, u^2_{m_2})$ then its norm is defined by

$$\|u\|_H = \left\{ \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} |u^1_{j_1}(x)|^2\, dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} |u^2_{j_2}(x)|^2\, dx \right\}^{1/2}.$$

$L^2(D_0)$, and $L^2(\Gamma)$, respectively, Q_3^{-1} is such that it maps the space $H^{1/2}(\Gamma)$ into itself, and $r^i_{j_i}(x)$ are the functions continuous on the sets $\bar{\Omega}^i_{j_i}$, $j_i = 1, \dots, m_i$, $i = 1, 2$, which do not vanish there, $\epsilon_k > 0$, $k = 1, 2$, are given constants. We also assume that k_1 and k_2, μ_1 and μ_2 are given nonzero complex numbers with $\operatorname{Im} k_1, \operatorname{Im} k_2 \geq 0$, such that

$$\rho := \frac{\mu_1 \bar{k}_1^2}{\mu_2 \bar{k}_2^2} \in \mathbb{R},$$

where $\rho \geq 0$ (< 0) if $\operatorname{Re} k_1 \operatorname{Re} k_2 \geq 0$ (< 0), It can be shown, using the results by R. Kress and G. F. Roach [2], that under our assumptions problem (3)–(7) is uniquely solvable and $\varphi_1 \in H^2(D)$ and $\varphi_2 \in H^2_{\mathrm{loc}}(\mathbb{R}^n \setminus D)$.[2]

Let l_1, l_2, l_3, and l_4 be given functions such that $l_1 \in L^2(D)$, l_2 has a compact support $D_0 \subset \mathbb{R}^n \setminus \bar{D}$, $l_2 \in L^2(D_0)$, and $l_3, l_4 \in L^2(\Gamma)$. The problem of estimation consists in the following. From observations (1) and (2) of the state of the system described by BVP (3)–(7) under conditions (8) it is necessary to estimate the quantity

$$l(F) = \int_D \overline{l_1(x)} f_1(x)\, dx + \int_{D_0} \overline{l_2(x)} f_2(x)\, dx + \int_\Gamma \bar{l}_3 g_1 d\Gamma + \int_\Gamma \bar{l}_4 g_2 d\Gamma \qquad (10)$$

in the class of the estimates linear with respect to observations which have the form

$$\widehat{l(F)} = \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)} y^1_{j_1}(x) + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{u^2_{k_2}(x)} y^2_{k_2}(x)\, dx + c, \qquad (11)$$

where $u^i_{j_i} \in L^2(\Omega^i_{j_i})$, $j_i = 1, \dots, m_i$, $i = 1, 2$, $c \in \mathbb{C}$. An estimate

$$\widehat{\widehat{l(F)}} = \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{\hat{u}^1_{j_1}(x)} y^1_{j_1}(x) + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{\hat{u}^2_{j_2}(x)} y^2_{j_2}(x)\, dx + \hat{c} \qquad (12)$$

is called a *minimax mean square* or *guaranteed estimate* of the $l(F)$ if an element $u := (u^1_1, \dots, u^1_{m_1}, u^2_1, \dots, u^2_{m_2}) \in H$ and a number $\hat{c} \in \mathbb{C}$ are determined from the condition

$$\inf_{u \in H,\, c \in \mathbb{C}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

where
$$\sigma(u, c) = \sup_{\tilde{F} \in G_0,\, \tilde{\xi} \in G_1} \mathbb{E} |l(\tilde{F}) - \widehat{l(\tilde{F})}|^2,$$

$$\widehat{l(\tilde{F})} = \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)} \tilde{y}^1_{j_1}(x) + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)} \tilde{y}^2_{j_2}(x)\, dx + c, \qquad (13)$$

$$\tilde{y}^1_{j_1}(x) = C^1_{j_1} \tilde{\varphi}^1_1(x) + \tilde{\xi}^1_{j_1}(x), \quad x \in \Omega^1_{j_1}, \quad j_1 = 1, \dots, m_1,$$
$$\tilde{y}^2_{j_2}(x) = C^2_{j_2} \tilde{\varphi}^2_2(x) + \tilde{\xi}^2_{j_2}(x), \quad x \in \Omega^2_{j_2}, \quad j_2 = 1, \dots, m_2,$$

[2]Here, we denote by $H^s(\Gamma)$, $s \geq 0$, $H^m(D)$, and $H^m_{\mathrm{loc}}(\mathbb{R}^n \setminus D)$, $m \geq 0$, usual Sobolev spaces in the corresponding sets.

$(\tilde{\varphi}_1, \tilde{\varphi}_2)$ satisfies problem (3)–(7) when $f_1 = \tilde{f}_1$, $f_2 = \tilde{f}_2$, $g_1 = \tilde{g}_1$, $g_2 = \tilde{g}_2$. The quantity $\sigma := [\sigma(\hat{u}, \hat{c})]^{1/2}$ is called *error of the minimax estimation.*

Thus, a minimax estimate is an estimate minimizing the maximal mean-square estimation error calculated for the worst-case realization of the perturbations.

3. Main results

For any fixed $u \in H$ we introduce $z_1(\cdot\,; u) \in H^2(D)$ and $z_2(\cdot\,; u) \in H^2_{\text{loc}}(\mathbb{R}^n \setminus D)$ as solution functions of the following transmission problem:

$$(\Delta + \bar{k}_1^2) z_1(x; u) = \sum_{j_1=1}^{m_1} \chi_{\Omega^1_{j_1}}(x)(C^1_{j_1})^* u^1_{j_1}(x), \quad x \in D, \tag{14}$$

$$(\Delta + \bar{k}_2^2) z_2(x; u) = \sum_{j_2=1}^{m_2} \chi_{\Omega^2_{j_2}}(x)(C^2_{j_2})^* u^2_{j_2}(x), \quad x \in \mathbb{R}^n \setminus \bar{D}, \tag{15}$$

$$z_2(\cdot; u) - z_1(\cdot; u) = 0 \quad \text{on} \quad \Gamma, \tag{16}$$

$$\bar{\mu}_1 \frac{\partial z_2(\cdot\,; u)}{\partial \nu} - \bar{\mu}_2 \frac{\partial z_1(\cdot; u)}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{17}$$

$$\frac{\partial z_2(x; u)}{\partial r} + i\bar{k}_2 z_2(x; u) = o(1/r^{(n-1)/2}), \quad r = |x|, \quad r \to \infty, \tag{18}$$

where $(C^i_{j_i})^*$ is the operator adjoint of $C^i_{j_i}$, $j_i = 1, \ldots, m_i$, $i = 1, 2$,

$$\chi_M(x) = \begin{cases} 1, & x \in M \\ 0, & x \notin M \end{cases}$$

is a characteristic function of the set $M \subset \mathbb{R}^n$.

It is easy to see that under the assumptions made on k_i and μ_i, $i = 1, 2$, the unique solvability of the transmission problem (14)–(18) follows from the unique solvability of the problem (3)–(7) and for any R there exists a positive constant $\alpha > 0$ independent of f and g (but dependent on R) such that

$$\begin{aligned} &\|z_1(\cdot; u)\|_{H^2(D)} + \|z_2(\cdot; u)\|_{H^2(D_R \setminus D)} \\ &\leq \alpha \left(\sum_{i_1=1}^{m_1} \left\| (C^1_{j_1})^* u^1_{j_1} \right\|_{L^2(\Omega^1_{j_1})} + \sum_{j_2=1}^{m_2} \left\| (C^2_{j_2})^* u^2_{j_2} \right\|_{L^2(\Omega^2_{j_2})} \right). \end{aligned} \tag{19}$$

Here $D_R := \{x \in \mathbb{R}^n : |x| < R\}$ and R is chosing so that $\bar{D}_0, \bar{\Omega}^1_{j_1}, \Omega^2_{j_2} \subset D_R \setminus \bar{D}$, $j_i = 1, ..., m_i$, $i = 1, 2$. In fact, taking the complex conjugates of both sides in (3)–(7) and using the substitution

$$\bar{\varphi}_1(x) = \frac{1}{\bar{\mu}_1} z_1(x), \quad \bar{\varphi}_2(x) = \frac{1}{\bar{\mu}_2} z_2(x), \quad \bar{f}_i(x) = \frac{1}{\bar{\mu}_i} \sum_{j_i=1}^{m_i} \chi_{\Omega^i_{j_i}}(x)(C^i_{j_i})^* u^i_{j_i}(x),$$

$i = 1, 2$, $\bar{g}_1 = 0$, and $\bar{g}_2 = 0$ in the resulting equations, we come to the conclusion of the unique solvability of the problem (14)–(18). The estimate (19) follows from the results obtained in [1], [2] and [8].

Lemma 3.1. *The problem of finding the minimax mean squre estimate* $\widehat{\widehat{l(F)}}$ *of* $l(F)$ *is equivalent to the problem of optimal control of a system described by a problem* (14)–(18) *with the quality criterion:*

$$\begin{aligned}I(u) := \epsilon_1 \Bigg(\int_D Q_1^{-1}(l_1 + z_1(\cdot;u))(x)\overline{(l_1(x) + z_1(x;u))}\,dx \\ + \int_{D_0} Q_2^{-1}(l_2 + z_2(\cdot;u))(x)\overline{(l_2(x) + z_2(x;u))}\,dx \\ + \int_\Gamma Q_3^{-1}\Big(l_3 + \frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)\overline{\Big(l_3 + \frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)}\,d\Gamma \\ + \int_\Gamma Q_4^{-1}\Big(l_4 - z_1(\cdot;u)\Big)\overline{\Big(l_4 - z_1(\cdot;u)\Big)}\,d\Gamma \Bigg) \\ + \epsilon_2 \Bigg(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2} |u^1_{j_1}(x)|^2\,dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2} |u^2_{j_2}(x)|^2\,dx \Bigg) \to \inf_{u\in H}.\end{aligned} \tag{20}$$

Proof. Taking into account (10) and (13), we obtain

$$\begin{aligned}l(\tilde F) - \widehat{l(\tilde F)} &= \int_D \overline{l_1(x)}\tilde f_1(x)\,dx + \int_{D_0} \overline{l_2(x)}\tilde f_2(x)\,dx + \int_\Gamma \bar l_3 \tilde g_1 d\Gamma + \int_\Gamma \bar l_4 \tilde g_2 d\Gamma \\ &\quad - \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde y^1_{j_1}(x) - \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde y^2_{j_2}(x)\,dx - c \\ &= \int_D \overline{l_1(x)}\tilde f_1(x)\,dx + \int_{D_0} \overline{l_2(x)}\tilde f_2(x)\,dx + \int_\Gamma \bar l_3 \tilde g_1 d\Gamma + \int_\Gamma \bar l_4 \tilde g_2 d\Gamma \\ &\quad - \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{(C^1_{j_1})^* u^1_{j_1}(x)}\tilde\varphi_1(x)\,dx - \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{(C^2_{j_2})^* u^2_{j_2}(x)}\tilde\varphi_2(x)\,dx \\ &\quad - \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde\xi^1_{j_1}(x) - \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde\xi^2_{j_2}(x)\,dx - c.\end{aligned} \tag{21}$$

Let D_R be such as above and $\Gamma_R := \partial D_R$. Taking into consideration equations (3)–(4), (14)–(15) and applying the second Green formula to $\tilde\varphi_1(x)$ and $z_1(x;u)$ and $\tilde\varphi_2(x)$ and $z_2(x;u)$ in domains D and $D_R\backslash\bar D$, respectively, we obtain

$$\begin{aligned}&- \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \overline{(C^1_{j_1})^* u^1_{j_1}(x)}\tilde\varphi_1(x)\,dx - \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \overline{(C^2_{j_2})^* u^2_{j_2}(x)}\tilde\varphi_2(x)\,dx \\ &= -\int_D \tilde\varphi_1(x)\overline{(\Delta z_1(x;u) + \bar k_1^2 z_1(x;u))}\,dx - \int_{D_R\backslash\bar D} \tilde\varphi_2(x)\overline{(\Delta z_2(x;u) + \bar k_2^2 z_2(x;u))}\,dx \\ &= \int_D (-\Delta\tilde\varphi_1(x) - k_1^2\tilde\varphi_1(x))\overline{z_1(x;u)}\,dx - \int_{D_R\backslash\bar D} (-\Delta\tilde\varphi_2(x) - k_2^2\tilde\varphi_2(x))\overline{z_2(x;u)}\,dx\end{aligned}$$

$$-\int_\Gamma \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\tilde\varphi_1\,d\Gamma+\int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_1\,d\Gamma+\int_\Gamma \frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\tilde\varphi_2\,d\Gamma$$
$$-\int_\Gamma \frac{\partial \tilde\varphi_2(\cdot;u)}{\partial\nu}\bar z_2\,d\Gamma+\Sigma_R(z_2(\cdot;u),\tilde\varphi_2),$$
$$=\int_D \tilde f_1\overline{z_1(x;u)}\,dx-\int_{D_0}\tilde f_2\overline{z_2(x;u)}\,dx-\int_\Gamma \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\tilde\varphi_1\,d\Gamma+\int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_1\,d\Gamma$$
$$+\int_\Gamma \frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\tilde\varphi_2\,d\Gamma-\int_\Gamma \frac{\partial \tilde\varphi_2(\cdot;u)}{\partial\nu}\bar z_2\,d\Gamma+\Sigma_R(z_2(\cdot;u),\tilde\varphi_2), \tag{22}$$

where

$$\Sigma_R(z_2(\cdot;u),\tilde\varphi_2):=-\int_{\Gamma_R}\left(\overline{z_2(x;u)}\frac{\partial\tilde\varphi_2(x)}{\partial\nu}-\tilde\varphi_2(x)\overline{\left(\frac{\partial z_2(x;u)}{\partial\nu}\right)}\right)d\Gamma_R.$$

Since $z_2(\cdot;u)$ and $\tilde\varphi_2$ satisfy the Sommerfeld radiation conditions (18) and (7), respectively, and $d\Gamma_R=R^{n-1}dS_1$, where S_1 denotes the sphere of unit radius centered at the origin, we obtain the following estimate for $\Sigma_R(z_2(\cdot;u),\tilde\varphi_2)$:

$$\Sigma_R(z_2(\cdot;u),\tilde\varphi_2)=\int_{\Gamma_R}\overline{z_2(x;u)}\left(\frac{\partial\tilde\varphi_2(x)}{\partial R}-ik_2\tilde\varphi_2(x)\right)d\Gamma_R$$
$$-\int_{\Gamma_R}\tilde\varphi_2(x)\overline{\left(\frac{\partial z_2(x;u)}{\partial R}+i\bar k_2 z_2(x;u)\right)}\,d\Gamma_R$$
$$=\int_{\Gamma_R}O(1/R^{(n-1)/2})o(1/R^{(n-1)/2})d\Gamma_R$$
$$-\int_{\Gamma_R}O(1/R^{(n-1)/2})o(1/R^{(n-1)/2})d\Gamma_R=o(1)\quad\text{as } R\to\infty.$$

From here, passing to the limit as $R\to\infty$ in (22), we get

$$-\sum_{k_1=1}^{m_1}\int_{\Omega^1_{k_1}}\overline{u^1_{k_1}(x)}\int_{\Omega^1_{k_1}}h^1_{k_1}(x,y)\tilde\varphi_1(y)\,dy\,dx-\sum_{k_2=1}^{m_2}\int_{\Omega^2_{k_2}}\overline{u^2_{k_2}(x)}\int_{\Omega^2_{k_2}}h^2_{k_2}(x,y)\tilde\varphi_2(y)\,dy\,dx$$
$$=\int_D \tilde f_1\overline{z_1(x;u)}\,dx+\int_{D_0}\tilde f_2\overline{z_2(x;u)}\,dx+\int_\Gamma \frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\tilde\varphi_2\,d\Gamma$$
$$-\int_\Gamma \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\tilde\varphi_1\,d\Gamma+\int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_1\,d\Gamma-\int_\Gamma \frac{\partial \tilde\varphi_2(\cdot;u)}{\partial\nu}\bar z_2\,d\Gamma. \tag{23}$$

We transform the sum of the last four terms in the right-hand side of (23) using equalities (5), (6) and (16), (17) and obtain

$$\int_\Gamma \frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\tilde\varphi_2\,d\Gamma-\int_\Gamma \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\tilde\varphi_1\,d\Gamma+\int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_1\,d\Gamma-\int_\Gamma \frac{\partial \tilde\varphi_2(\cdot;u)}{\partial\nu}\bar z_2\,d\Gamma$$
$$=\int_\Gamma \frac{1}{\mu_2}g_1\frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\,d\Gamma+\int_\Gamma \frac{\mu_1}{\mu_2}\tilde\varphi_1\frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\,d\Gamma-\int_\Gamma \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\tilde\varphi_1\,d\Gamma$$

$$
\begin{aligned}
&- \int_\Gamma g_2 \bar z_2(\cdot;u)\,d\Gamma - \int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_2(\cdot;u)\,d\Gamma + \int_\Gamma \frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\bar z_1(\cdot;u)\,d\Gamma\\
&= \int_\Gamma \frac{1}{\mu_2} g_1 \frac{\partial \bar z_2(\cdot;u)}{\partial\nu}\,d\Gamma + \int_\Gamma \Big(\frac{\mu_1}{\mu_2}\frac{\partial \bar z_2(\cdot;u)}{\partial\nu} - \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big)\tilde\varphi_1\,d\Gamma\\
&- \int_\Gamma g_2 \bar z_2(\cdot;u)\,d\Gamma + \int_\Gamma (\bar z_1(\cdot;u) - \bar z_2(\cdot;u))\frac{\partial \tilde\varphi_1(\cdot;u)}{\partial\nu}\,d\Gamma\\
&= \frac{1}{\mu_1}\int_\Gamma g_1 \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\,d\Gamma - \int_\Gamma g_2 \bar z_1(\cdot;u)\,d\Gamma. \qquad (24)
\end{aligned}
$$

Equalities (21), (23), and (24) yield

$$
\begin{aligned}
l(\tilde F) - \widehat{l(\tilde F)} &= \int_D \overline{l_1(x)}\tilde f_1(x)\,dx + \int_{D_0} \overline{l_2(x)}\tilde f_2(x)\,dx + \int_\Gamma \bar l_3 \tilde g_1 d\Gamma + \int_\Gamma \bar l_4 \tilde g_2 d\Gamma\\
&+ \int_D \tilde f_1(x)\overline{z_1(x;u)}\,dx + \int_{D_0} \tilde f_2(x)\overline{z_2(x;u)}\,dx + \frac{1}{\mu_1}\int_\Gamma g_1 \frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\,d\Gamma\\
&- \int_\Gamma g_2 \bar z_1(\cdot;u)\,d\Gamma - \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde\xi^1_{j_1}(x) - \sum_{k_j=1}^{m_2}\int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde\xi^2_{j_2}(x)\,dx - c\\
&= \int_D \overline{(l_1(x) + z_1(x;u))}\tilde f_1(x)\,dx + \int_{D_0} \overline{(l_2(x) + z_2(x;u))}\tilde f_2(x)\,dx\\
&+ \int_\Gamma \Big(\bar l_3 + \frac{1}{\mu_1}\frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big)\tilde g_1\,d\Gamma + \int_\Gamma (\bar l_4 - \bar z_1(\cdot;u))\tilde g_2 d\Gamma\\
&- \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde\xi^1_{j_1}(x) - \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde\xi^2_{j_2}(x)\,dx - c\\
&= \int_D \overline{(l_1(x) + z_1(x;u))}(\tilde f_1(x) - f^0_1(x))\,dx + \int_{D_0} \overline{(l_2(x) + z_2(x;u))}(\tilde f_2(x) - f^0_2(x))\,dx\\
&+ \int_\Gamma \Big(\bar l_3 + \frac{1}{\mu_1}\frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big)(\tilde g_1 - g^0_1)\,d\Gamma + \int_\Gamma (\bar l_4 - \bar z_1(\cdot;u))(\tilde g_2 - g^0_2) d\Gamma\\
&+ \int_D \overline{(l_1(x) + z_1(x;u))} f^0_1(x)\,dx + \int_{D_0} \overline{(l_2(x) + z_2(x;u))} f^0_2(x)\,dx\\
&+ \int_\Gamma \Big(\bar l_3 + \frac{1}{\mu_1}\frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big) g^0_1\,d\Gamma + \int_\Gamma (\bar l_4 - \bar z_1(\cdot;u)) g^0_2 d\Gamma - c\\
&- \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde\xi^1_{j_1}(x) - \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde\xi^2_{j_2}(x)\,dx. \qquad (25)
\end{aligned}
$$

Taking into consideration the relationship $\mathbb{D}\eta = \mathbb{E}|\eta - \mathbb{E}\eta|^2 = \mathbb{E}|\eta|^2 - |\mathbb{E}\eta|^2$ that couples the variance $\mathbb{D}\eta$ of the complex random variable $\eta = \eta_1 + i\eta_2$ and its expectation $\mathbb{E}\eta = \mathbb{E}\eta_1 + i\mathbb{E}\eta_2$, we obtain from the last formulas

$$
\mathbb{E}\left| l(\tilde F) - \widehat{l(\tilde F)}\right|^2 = \Big|\int_D \overline{(l_1(x) + z_1(x;u))}(\tilde f_1(x) - f^0_1(x))\,dx
$$

$$+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}(\tilde{f}_2(x)-f_2^0(x))\,dx$$
$$+\int_\Gamma\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{z}_1(\cdot;u)}{\partial\nu}\Big)(\tilde{g}_1-g_1^0)\,d\Gamma+\int_\Gamma(\bar{l}_4-\bar{z}_1(\cdot;u))(\tilde{g}_2-g_2^0)d\Gamma$$
$$+\int_D\overline{(l_1(x)+z_1(x;u))}f_1^0(x)\,dx+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}f_2^0(x)\,dx$$
$$+\int_\Gamma\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{z}_1(\cdot;u)}{\partial\nu}\Big)g_1^0\,d\Gamma+\int_\Gamma(\bar{l}_4-\bar{z}_1(\cdot;u))g_2^0d\Gamma-c\Big|^2$$
$$+\mathbb{E}\left|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{u^1_{j_1}(x)}\tilde{\xi}^1_{j_1}(x)+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{u^2_{j_2}(x)}\tilde{\xi}^2_{j_2}(x)\,dx\right|^2.$$

Therefore

$$\inf_{c\in\mathbb{C}}\sup_{\tilde{F}\in G_0,\,\tilde{\xi}\in G_1}\mathbb{E}\left|l(\tilde{F})-\widehat{l(\tilde{F})}\right|^2$$
$$=\inf_{c\in\mathbb{C}}\sup_{\tilde{F}\in G_0}\Big|\int_D\overline{(l_1(x)+z_1(x;u))}(\tilde{f}_1(x)-f_1^0(x))\,dx$$
$$+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}(\tilde{f}_2(x)-f_2^0(x))\,dx$$
$$+\int_\Gamma\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{z}_1(\cdot;u)}{\partial\nu}\Big)(\tilde{g}_1-g_1^0)\,d\Gamma+\int_\Gamma(\bar{l}_4-\bar{z}_1(\cdot;u))(\tilde{g}_2-g_2^0)d\Gamma$$
$$+\int_D\overline{(l_1(x)+z_1(x;u))}f_1^0(x)\,dx+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}f_2^0(x)\,dx$$
$$+\int_\Gamma\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{z}_1(\cdot;u)}{\partial\nu}\Big)g_1^0\,d\Gamma+\int_\Gamma(\bar{l}_4-\bar{z}_1(\cdot;u))g_2^0d\Gamma-c\Big|^2$$
$$+\sup_{\tilde{\xi}\in G_1}\mathbb{E}\left|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{u^1_{j_1}(x)}\tilde{\xi}^1_{j_1}(x)+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{u^2_{j_2}(x)}\tilde{\xi}^2_{j_2}(x)\,dx\right|^2. \tag{26}$$

In order to calculate the right-hand side of (26), let us first introduce the notation

$$y=\int_D\overline{(l_1(x)+z_1(x;u))}(\tilde{f}_1(x)-f_1^0(x))\,dx+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}(\tilde{f}_2(x)-f_2^0(x))\,dx$$
$$+\int_\Gamma\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{z}_1(\cdot;u)}{\partial\nu}\Big)(\tilde{g}_1-g_1^0)\,d\Gamma+\int_\Gamma(\bar{l}_4-\bar{z}_1(\cdot;u))(\tilde{g}_2-g_2^0)d\Gamma.$$

Then applying the generalized Cauchy-Bunyakovsky inequality, we have

$$|y|\le\Big\{\int_D Q_1^{-1}(l_1+z_1(\cdot;u))(x)\overline{(l_1(x)+z_1(x;u))}\,dx$$
$$+\int_{D_0}Q_2^{-1}(l_2+z_2(\cdot;u))(x)\overline{(l_2(x)+z_2(x;u))}\,dx$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)\overline{\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)}\,d\Gamma$$
$$+\int_\Gamma Q_4^{-1}\Big(l_4-z_1(\cdot;u)\Big)\overline{\Big(l_4-z_1(\cdot;u)\Big)}\,d\Gamma\Big\}^{1/2}$$
$$\times\Big\{\int_D Q_1(\tilde f_1-f_1^0)(x)\overline{(\tilde f_1-f_1^0)}(x)\,dx+\int_{D_0} Q_2(\tilde f_2-f_2^0)(x)\overline{(\tilde f_2-f_2^0)}(x)\,dx$$
$$+\int_\Gamma Q_3(\tilde g_1-g_1^0)\overline{(\tilde g_1-g_1^0)}\,d\Gamma+\int_\Gamma Q_4(\tilde g_2-g_2^0)\overline{(\tilde g_2-g_2^0)}\,d\Gamma\Big\}^{1/2}\le \epsilon_1^{1/2}a, \qquad (27)$$

where
$$a=\Big\{\int_D Q_1^{-1}(l_1+z_1(\cdot;u))(x)\overline{(l_1(x)+z_1(x;u))}\,dx$$
$$+\int_{D_0} Q_2^{-1}(l_2+z_2(\cdot;u))(x)\overline{(l_2(x)+z_2(x;u))}\,dx$$
$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)\overline{\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)}\,d\Gamma$$
$$+\int_\Gamma Q_4^{-1}\Big(l_4-z_1(\cdot;u)\Big)\overline{\Big(l_4-z_1(\cdot;u)\Big)}\,d\Gamma\Big\}^{1/2}.$$

The direct substitution shows that inequality (27) is transformed to an equality on the element $\hat F:=(\hat f_1,\hat f_2,\hat g_1,\hat g_2)\in G_0$, where

$$\hat f_1=\frac{\epsilon_1^{1/2}}{a}Q_1^{-1}(l_1+z_1(\cdot;u))+f_1^0,\quad \hat f_2=\frac{\epsilon_1^{1/2}}{a}Q_2^{-1}(l_2+z_2(\cdot;u))+f_2^0,$$

$$\hat g_1=\frac{\epsilon_1^{1/2}}{a}Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)+g_1^0,\quad \hat g_2=\frac{\epsilon_1^{1/2}}{a}Q_4^{-1}\Big(l_4-z_2(\cdot;u)\,|_\Gamma\Big)+g_2^0.$$

Hence,

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0}\Big|\int_D \overline{(l_1(x)+z_1(x;u))}(\tilde f_1(x)-f_1^0(x))\,dx$$
$$+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}(\tilde f_2(x)-f_2^0(x))\,dx$$
$$+\int_\Gamma\Big(\bar l_3+\frac{1}{\mu_1}\frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big)(\tilde g_1-g_1^0)\,d\Gamma+\int_\Gamma(\bar l_4-\bar z_1(\cdot;u))(\tilde g_2-g_2^0)d\Gamma$$
$$+\int_D\overline{(l_1(x)+z_1(x;u))}f_1^0(x)\,dx+\int_{D_0}\overline{(l_2(x)+z_2(x;u))}f_2^0(x)\,dx$$
$$+\int_\Gamma\Big(\bar l_3+\frac{1}{\mu_1}\frac{\partial \bar z_1(\cdot;u)}{\partial\nu}\Big)g_1^0\,d\Gamma+\int_\Gamma(\bar l_4-\bar z_1(\cdot;u))g_2^0d\Gamma-c\Big|^2$$

$$= \inf_{c\in\mathbb{C}} \sup_{|y|\le \epsilon_1^{1/2}a} \Big| y + \int_D \overline{(l_1(x)+z_1(x;u))} f_1^0(x)\,dx + \int_{D_0} \overline{(l_2(x)+z_2(x;u))} f_2^0(x)\,dx$$

$$+ \int_\Gamma \Big(\bar{l}_3 + \frac{1}{\mu_1}\frac{\partial \bar{z}_1(\cdot;u)}{\partial\nu}\Big) g_1^0\,d\Gamma + \int_\Gamma (\bar{l}_4 - \bar{z}_1(\cdot;u)) g_2^0 d\Gamma - c\Big|^2 = \epsilon_1 a^2$$

$$= \epsilon_1 \Bigg(\int_D Q_1^{-1}(l_1 + z_1(\cdot;u))(x)\overline{(l_1(x)+z_1(x;u))}\,dx$$

$$+ \int_{D_0} Q_2^{-1}(l_2 + z_2(\cdot;u))(x)\overline{(l_2(x)+z_2(x;u))}\,dx$$

$$+ \int_\Gamma Q_3^{-1}\Big(l_3 + \frac{1}{\bar{\mu}_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)\overline{\Big(l_3 + \frac{1}{\bar{\mu}_1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)}\,d\Gamma$$

$$+ \int_\Gamma Q_4^{-1}\Big(l_4 - z_1(\cdot;u)\Big)\overline{\Big(l_4 - z_1(\cdot;u)\Big)}\,d\Gamma \Bigg), \tag{28}$$

with $$c = \int_D \overline{(l_1(x)+z_1(x;u))} f_1^0(x)\,dx + \int_{D_0} \overline{(l_2(x)+z_2(x;u))} f_2^0(x)\,dx$$

$$+ \int_\Gamma g_1^0\Big(\bar{l}_3 + \frac{1}{\mu_1}\frac{\partial \bar{z}_1(\cdot;u)}{\partial\nu}\Big)\,d\Gamma + \int_\Gamma g_2^0(\bar{l}_4 - \bar{z}_1(\cdot;u))\,d\Gamma. \tag{29}$$

Similarly, due to the Cauchy-Bunyakovsky inequality,

$$\mathbb{E}\Bigg|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde{\xi}^1_{j_1}(x) + \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde{\xi}^2_{j_2}(x)\,dx\Bigg|^2$$

$$\le \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2}|u^1_{j_1}(x)|^2\,dx \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{2}\mathbb{E}|\tilde{\xi}^1_{j_1}(x)|^2\,dx$$

$$+ \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2}|u^2_{j_2}(x)|^2\,dx \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{2}\mathbb{E}|\tilde{\xi}^2_{j_2}(x)|^2\,dx,$$

$$= \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2}|u^1_{j_1}(x)|^2\,dx \sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} (r^1_{j_1}(x))^{2}\tilde{R}^1_{j_1}(x,x)\,dx$$

$$+ \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2}|u^2_{j_2}(x)|^2\,dx \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} (r^2_{j_2}(x))^{2}\tilde{R}^2_{j_2}(x,x)\,dx,$$

the latter implies

$$\sup_{\tilde{\xi}\in G_1}\mathbb{E}\Bigg|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \overline{u^1_{j_1}(x)}\tilde{\xi}^1_{j_1}(x) + \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \overline{u^2_{j_2}(x)}\tilde{\xi}^2_{j_2}(x)\,dx\Bigg|^2$$

$$\le \epsilon_2\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2}|u^1_{j_1}(x)|^2\,dx + \sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2}|u^2_{j_2}(x)|^2\,dx\Bigg).$$

However,

$$\mathbb{E}\left|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{u^1_{j_1}(x)}\hat{\xi}^1_{j_1}(x)+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{u^2_{j_2}(x)}\hat{\xi}^2_{j_2}(x)\,dx\right|^2$$

$$=\epsilon_2\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}|u^1_{j_1}(x)|^2\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}|u^2_{j_2}(x)|^2\,dx\right),$$

where
$$\hat{\xi}^i_{j_i}(x)=\frac{\epsilon_2^{1/2}\nu\left(r^i_{j_i}(x)\right)^{-2}u^i_{j_i}(x)}{\{\sum_{i=1}^2\sum_{j_i=1}^{m_i}\int_{\Omega^i_{j_i}}\left(r^i_{j_i}(x)\right)^{-2}|u^i_{j_i}(x)|^2\,dx\}^{1/2}},$$

$x\in\Omega^i_{j_i}$, $j_i=1,\dots,m_i$, $i=1,2$, $\hat{\xi}=(\hat{\xi}^1_1,\dots,\hat{\xi}^1_{m_1},\hat{\xi}^2_1,\dots,\hat{\xi}^2_{m_2})\in G_1$, and ν is a random variable with $\mathbb{E}\nu=0$ and $\mathbb{E}|\nu|^2=1$. Therefore

$$\sup_{\tilde{\xi}\in G_1}\mathbb{E}\left|\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{u^1_{j_1}(x)}\tilde{\xi}^1_{j_1}(x)+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{u^2_{j_2}(x)}\tilde{\xi}^2_{j_2}(x)\,dx\right|^2$$

$$=\epsilon_2\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}|u^1_{j_1}(x)|^2\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}|u^2_{j_2}(x)|^2\,dx\right).\quad(30)$$

The validity of Lemma 3.1 follows now from the relationships (26), (28), and (30). □

Starting from this lemma, we obtain the following result.

Theorem 3.2. *The minimax mean square estimate of $l(F)$ has the form*

$$\widehat{\widehat{l(F)}}=\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{\hat{u}^1_{j_1}(x)}y^1_{j_1}(x)+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{\hat{u}^2_{j_2}(x)}y^2_{j_2}(x)\,dx+\hat{c}=l(\hat{F}),$$

where

$$\hat{u}^1_{j_1}(x)=\epsilon_2^{-1}(r^1_{j_1}(x))^2C^1_{j_1}p_1(x),\quad j_1=1,\dots,m_1,$$
$$\hat{u}^2_{j_2}(x)=\epsilon_2^{-1}(r^2_{j_2}(x))^2C^2_{j_2}p_2(x),\quad j_2=1,\dots,m_2,$$

$$\hat{c}=\int_D(\overline{l_1(x)+\hat{z}_1(x)})f^0_1(x)\,dx+\int_{D_0}(\overline{l_2(x)+\hat{z}_2(x)})f^0_2(x)\,dx$$
$$+\int_\Gamma g^0_1\Big(\bar{l}_3+\frac{1}{\mu_1}\frac{\partial\bar{\hat{z}}_1}{\partial\nu}\Big)\,d\Gamma+\int_\Gamma g^0_2(\bar{l}_4-\bar{\hat{z}}_1)\,d\Gamma,$$

$$\hat{F}=(\hat{f}_1,\hat{f}_2,\hat{g}_1,\hat{g}_2),\quad \hat{f}_1(x)=\epsilon_1Q_1^{-1}\hat{p}_1(x)+f^0_1(x),\quad \hat{f}_2(x)=\epsilon_1Q_2^{-1}\hat{p}_2(x)+f^0_2(x),$$

$$\hat{g}_1=\frac{1}{\bar{\mu}_1}\epsilon_1Q_3^{-1}\frac{\partial\hat{p}_1}{\partial\nu}+g^0_1,\ \hat{g}_2=-\epsilon_1Q_4^{-1}\hat{p}_1|_\Gamma+g^0_2,\ \hat{z}_1\in H^2(D),\ \hat{z}_2\in H^2_{\text{loc}}(\mathbb{R}^n\setminus D),$$

$p_1 \in H^1(D,\Delta)$, $p_2 \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus D,\Delta)$ [*note that* $H^1(D,\Delta) := \{u\colon u \in H^1(D),\ \Delta u \in L^2(D)\}$, $H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus D,\Delta) := \{u\colon u \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus D,\Delta),\ \Delta u \in L^2_{\mathrm{loc}}(\mathbb{R}^n \setminus D,\Delta)\}$], *and* $\hat p_1 \in H^2(D)$, $\hat p_2 \in H^2_{\mathrm{loc}}(\mathbb{R}^n \setminus D)$ *are determined from the solution to the following problems*:

$$(\Delta + \bar k_1^2)\hat z_1(x) = \epsilon_2^{-1} \sum_{j_1=1}^{m_1} \chi_{\Omega^1_{j_1}}(x)(C^1_{j_1})^*(r^1_{j_1}(\cdot))^2 C^1_{j_1} p_1(x) \quad \text{in } D, \tag{31}$$

$$(\Delta + \bar k_2^2)\hat z_2(x) = \epsilon_2^{-1} \sum_{i_2=1}^{m_2} \chi_{\Omega^2_{i_2}}(x)(C^2_{j_2})^*(r^2_{j_2}(\cdot))^2 C^1_{j_2} p_2(x) \quad \text{in } \mathbb{R}^n \setminus \bar D, \tag{32}$$

$$\hat z_2 - \hat z_1 = 0 \quad \text{on} \quad \Gamma, \tag{33}$$

$$\bar\mu_1 \frac{\partial \hat z_2}{\partial \nu} - \bar\mu_2 \frac{\partial \hat z_1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{34}$$

$$\frac{\partial \hat z_2}{\partial r} + i\bar k_2 \hat z_2 = o(1/r^{(n-1)/2}), \quad r = |x|, \quad r \to \infty, \tag{35}$$

$$-(\Delta + k_1^2)p_1(x) = \epsilon_1 Q_1^{-1}(l_1 + \hat z_1)(x) \quad \text{in} \quad D, \tag{36}$$

$$-(\Delta + k_2^2)p_2(x) = \epsilon_1 \chi_{D_0}(x) Q_2^{-1}(l_2 + \hat z_2)(x) \quad \text{in} \quad \mathbb{R}^n \setminus \bar D, \tag{37}$$

$$\mu_2 p_2 - \mu_1 p_1 = \epsilon_1 Q_3^{-1}\Big(l_3 + \frac{1}{\bar\mu_1}\frac{\partial \hat z_1}{\partial \nu}\Big) \quad \text{on} \quad \Gamma, \tag{38}$$

$$\frac{\partial p_2}{\partial \nu} - \frac{\partial p_1}{\partial \nu} = \epsilon_1 Q_4^{-1}(l_4 - \hat z_1) \quad \text{on} \quad \Gamma, \tag{39}$$

$$\frac{\partial p_2}{\partial r} - i k_2 p_2 = o(1/r^{(n-1)/2}), \quad r = |x|, \quad r \to \infty, \tag{40}$$

and

$$-(\Delta + \bar k_1^2)\hat p_1(x) = \epsilon_2^{-1} \sum_{j_1=1}^{m_1} \chi_{\Omega^1_{j_1}}(x)(C^1_{j_1})^*(r^1_{j_1})^2 \Big[y^1_{j_1} - C^1_{j_1}\hat\varphi_1\Big](x) \quad \text{in } D, \tag{41}$$

$$-(\Delta + \bar k_2^2)\hat p_2(x) = \epsilon_2^{-1} \sum_{j_2=1}^{m_2} \chi_{\Omega^2_{j_2}}(x)(C^2_{j_2})^*(r^2_{j_2})^2 \Big[y^2_{j_2} - C^2_{j_2}\hat\varphi_2\Big](x) \quad \text{in } \mathbb{R}^n \setminus \bar D, \tag{42}$$

$$\hat p_2 - \hat p_1 = 0 \quad \text{on} \ \Gamma, \tag{43}$$

$$\bar\mu_1 \frac{\partial \hat p_2}{\partial \nu} - \bar\mu_2 \frac{\partial \hat p_1}{\partial \nu} = 0 \quad \text{on} \quad \Gamma, \tag{44}$$

$$\frac{\partial \hat p_2}{\partial r} + i\bar k_2 \hat p_2 = o(1/r^{(n-1)/2}), \quad r = |x|, \quad r \to \infty, \tag{45}$$

$$-(\Delta + k_1^2)\hat\varphi_1(x) = \epsilon_1 Q_1^{-1}\hat p_1(x) + f_1^0(x) \quad \text{in} \quad \mathbb{D}, \tag{46}$$

$$-(\Delta + k_2^2)\hat\varphi_2(x) = \chi_{D_0}(x)\Big(\epsilon_1 Q_2^{-1}\hat p_2(x) + f_2^0(x)\Big) \quad \text{in} \quad \mathbb{R}^n \setminus \bar D, \tag{47}$$

$$\mu_2 \hat\varphi_2 - \mu_1 \hat\varphi_1 = \epsilon_1 \frac{1}{\bar\mu_1} Q_3^{-1} \frac{\partial \hat p_1}{\partial \nu} + g_1^0 \quad \text{on} \ \Gamma, \tag{48}$$

$$\frac{\partial \hat\varphi_2}{\partial \nu} - \frac{\partial \hat\varphi_1}{\partial \nu} = -\epsilon_1 Q_4^{-1}\hat p_1 + g_2^0, \ \text{on } \Gamma, \tag{49}$$

$$\frac{\partial \hat{\varphi}_2}{\partial r} - ik_2\hat{\varphi}_2 = o(1/r^{(n-1)/2}),\ r = |x|,\ r \to \infty, \tag{50}$$

respectively, where $\hat{\varphi}_1 \in H^1(D, \Delta)$, *and* $\hat{\varphi}_2 \in H^1_{\text{loc}}(\mathbb{R}^n \setminus D, \Delta)$. *Problems* (31)–(40) *and* (41)–(50) *are uniquely solvable. Equations* (41)–(50) *are fulfilled with probability 1. The estimation error* σ *is determined by the formula*

$$\sigma = l(P)^{1/2}, \tag{51}$$

where

$$P = \left(\epsilon_1 Q_1^{-1}(l_1(x)+\hat{z}_1(x)), \epsilon_1 Q_2^{-1}(l_2(x)+\hat{z}_2(x)), \epsilon_1 Q_3^{-1}\Big(l_3+\frac{1}{\bar{\mu}_1}\frac{\partial \hat{z}_1}{\partial \nu}\Big), \epsilon_1 Q_4^{-1}(l_4-\hat{z}_1)\right).$$

Proof. First notice that that functional $I(u)$ in Lemma 1 can be represented in the form

$$\begin{aligned}
I(u) : = \epsilon_1 \Bigg(& \int_D Q_1^{-1}(l_1 + z_1(\cdot;u))(x)\overline{(l_1(x) + z_1(x;u))}\, dx \\
& + \int_{D_0} Q_2^{-1}(l_2 + z_2(\cdot;u))(x)\overline{(l_2(x) + z_2(x;u))}\, dx \\
& + \int_\Gamma Q_3^{-1}\Big(l_3 + \frac{1}{\bar{\mu}_1}\frac{\partial z_1(\cdot;u)}{\partial \nu}\Big)\overline{\Big(l_3 + \frac{1}{\bar{\mu}_1}\frac{\partial z_1(\cdot;u)}{\partial \nu}\Big)}\, d\Gamma \\
& + \int_\Gamma Q_4^{-1}\Big(l_4 - z_1(\cdot;u)\Big)\overline{\Big(l_4 - z_1(\cdot;u)\Big)}\, d\Gamma \Bigg) \\
& + \epsilon_2 \left(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2} |u^1_{j_1}(x)|^2\, dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2} |u^2_{j_2}(x)|^2\, dx \right) \\
= \tilde{I}(u) + L(u) + C, &
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
\tilde{I}(u) = \epsilon_1 \Bigg(& \int_D Q_1^{-1} z_1(\cdot;u)(x)\overline{z_1(x;u)}\, dx + \int_{D_0} Q_2^{-1} z_2(\cdot;u)(x)\overline{z_2(x;u)}\, dx \\
& + \frac{1}{|\mu_1|^2}\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial \nu}\overline{\frac{\partial z_1(\cdot;u)}{\partial \nu}}\, d\Gamma + \int_\Gamma Q_4^{-1} z_1(\cdot;u)\overline{z_1(\cdot;u)}\, d\Gamma \Bigg) \\
& + \epsilon_2 \left(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^{-2} |u^1_{j_1}(x)|^2\, dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^{-2} |u^2_{j_2}(x)|^2\, dx \right)
\end{aligned}$$

is a quadratic functional in space H corresponding to a sesquilinear continuous Hermitian form

$$\pi(u,v) = \epsilon_1 \Bigg(\int_D Q_1^{-1} z_1(\cdot;u)(x)\overline{z_1(x;v)}\, dx + \int_{D_0} Q_2^{-1} z_2(\cdot;u)(x)\overline{z_2(x;v)}\, dx$$

$$+\frac{1}{|\mu_1|^2}\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma+\int_\Gamma Q_4^{-1}z_1(\cdot;u)\overline{z_1(\cdot;v)}\,d\Gamma\Bigg)$$

$$+\epsilon_2\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}u^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}u^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\right)$$

on $H\times H$ which satisfies the inequality

$$\tilde I(u)\geq\alpha_1\|u\|^2_H\quad\forall u\in H,\quad\alpha_1=\mathrm{const}>0,\tag{53}$$

$$L(u)=\epsilon_1\Bigg(2\mathrm{Re}\int_D Q_1^{-1}z_1(\cdot;u)(x)\overline{l_1(x)}\,dx+2\mathrm{Re}\int_{D_0}Q_2^{-1}z_2(\cdot;u)(x)\overline{l_2(x)}\,dx$$

$$+2\mathrm{Re}\Big(\frac{1}{\bar\mu_1}\int_\Gamma\overline{l_3}Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\Big)\,d\Gamma-2\mathrm{Re}\int_\Gamma Q_4^{-1}z_1(\cdot;u)\overline{l_4}\,d\Gamma\Bigg)$$

is a linear continuous functional in H, and

$$C=\epsilon_1\left(\int_D Q_1^{-1}l_1(x)\overline{l_1(x)}\,dx+\int_{D_0}Q_2^{-1}l_2(x)\overline{l_2(x)}\,dx+\int_\Gamma Q_3^{-1}l_3\overline{l_3}\,d\Gamma+\int_\Gamma Q_4^{-1}l_4\overline{l_4}\,d\Gamma\right),$$

where $(z_1(\cdot;v),z_2(\cdot;v))$ is the unique solution to problem (14)–(18) at

$$u=v=(v^1_1,\dots,v^1_{m_1},v^2_1,\dots,v^2_{m_2}),$$
$$\alpha_1:=\min_{1\leq i\leq 2}\min_{1\leq j_i\leq m_i}\min_{x\in\bar\Omega^i_{j_i}}\left(r^i_{j_i}(x)\right)^{-2}>0.$$

We prove, for example, the continuity of form $\pi(u,v)$; namely, the inequality

$$|\pi(u,v)|\leq c\|u\|_H\|v\|_H\quad\forall u,v\in H,\ c=\mathrm{const}\tag{54}$$

(the continuity of the linear functional $L(u)$ is proved in a similar manner). Using the Cauchy-Bunyakovsky inequality we have

$$|\pi(u,v)|\leq\epsilon_1\left(\int_D Q_1^{-1}z_1(x;u)\overline{z_1(x;u)}\,dx\right)^{1/2}\left(\int_D Q_1^{-1}z_1(x;v)\,\overline{z_1(x;v)}\,dx\right)^{1/2}$$

$$+\epsilon_1\left(\int_{D_0}Q_2^{-1}z_2(x;u)\overline{z_2(x;u)}\,dx\right)^{1/2}\left(\int_{D_0}Q_2^{-1}z_2(x;v)\,\overline{z_2(x;v)}\,dx\right)^{1/2}$$

$$+\epsilon_1\frac{1}{|\mu_1|^2}\left(\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\overline{\frac{\partial z_1(\cdot;u)}{\partial\nu}}\,d\Gamma\right)^{1/2}\left(\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma\right)^{1/2}$$

$$+\epsilon_1\left(\int_\Gamma Q_4^{-1}z_1(\cdot;u)\overline{z_1(\cdot;u)}\,d\Gamma\right)^{1/2}\left(\int_\Gamma Q_4^{-1}z_1(\cdot;v)\overline{z_1(\cdot;v)}\,d\Gamma\right)^{1/2}$$

$$+\epsilon_2\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}u^1_{j_1}(x)\overline{u^1_{j_1}(x)}\,dx\right)^{1/2}\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}v^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx\right)^{1/2}$$

$$+\epsilon_2\left(\sum_{j_2=1}^{m_2}\int_{\Omega^1_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}u^2_{j_2}(x)\overline{u^2_{j_2}(x)}\,dx\right)^{1/2}\left(\sum_{j_2=1}^{m_2}\int_{\Omega^1_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}v^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\right)^{1/2}$$

$$\leq\left\{\epsilon_1^2\int_D Q_1^{-1}z_1(x;u)\overline{z_1(x;u)}\,dx+\epsilon_1^2\int_{D_0}Q_2^{-1}z_2(x;u)\overline{z_2(x;u)}\,dx\right.$$

$$+\epsilon_1^2\frac{1}{|\mu_1|^2}\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\overline{\frac{\partial z_1(\cdot;u)}{\partial\nu}}\,d\Gamma+\epsilon_1^2\int_\Gamma Q_4^{-1}z_1(\cdot;u)\overline{z_1(\cdot;u)}\,d\Gamma$$

$$\left.+\epsilon_2^2\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}u^1_{j_1}(x)\overline{u^1_{j_1}(x)}\,dx+\epsilon_2^2\sum_{j_2=1}^{m_2}\int_{\Omega^1_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}u^2_{j_2}(x)\overline{u^2_{j_2}(x)}\,dx\right\}^{1/2}$$

$$\times\left\{\int_D Q_1^{-1}z_1(x;v)\overline{z_1(x;v)}\,dx+\int_{D_0}Q_2^{-1}z_2(x;v)\overline{z_2(x;v)}\,dx\right.$$

$$+\frac{1}{|\mu_1|^2}\int_\Gamma Q_3^{-1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma+\int_\Gamma Q_4^{-1}z_1(\cdot;v)\overline{z_1(\cdot;v)}\,d\Gamma$$

$$\left.+\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}v^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}v^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\right\}^{1/2}$$

$$\leq\left\{\epsilon_1^2\left(\int_D|Q_1^{-1}z_1(x;u)|^2\,dx\right)^{1/2}\left(\int_D|z_1(x;u)|^2\,dx\right)^{1/2}\right.$$

$$+\epsilon_1^2\left(\int_D|Q_2^{-1}z_2(x;u)|^2\,dx\right)^{1/2}\left(\int_D|z_2(x;u)|^2\,dx\right)^{1/2}$$

$$+\epsilon_1^2\frac{1}{|\mu_1|^2}\left(\int_\Gamma\left|Q_3^{-1}\frac{\partial z_1(\cdot;u)}{\partial\nu}\right|^2d\Gamma\right)^{1/2}\left(\int_\Gamma\left|\frac{\partial z_1(\cdot;u)}{\partial\nu}\right|^2d\Gamma\right)^{1/2}$$

$$+\epsilon_1^2\left(\int_\Gamma|Q_4^{-1}z_1(\cdot;u)|^2\,d\Gamma\right)^{1/2}\left(\int_\Gamma|z_1(\cdot;u)|^2\,d\Gamma\right)^{1/2}$$

$$\left.+\epsilon_2^2\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}|u^1_{j_1}(x)|^2\right)^{1/2}\left(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}|u^1_{j_1}(x)|^2\,dx\right)^{1/2}\right\}^{1/2}$$

$$\left.+\epsilon_2^2\left(\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}|u^2_{j_2}(x)|^2\right)^{1/2}\left(\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}|u^2_{j_2}(x)|^2\,dx\right)^{1/2}\right\}^{1/2}$$

$$\times \left\{ \left(\int_D |Q_1^{-1} z_1(x;v)|^2\, dx \right)^{1/2} \left(\int_D |z_1(x;v)|^2\, dx \right)^{1/2} + \left(\int_D |Q_2^{-1} z_2(x;v)|^2\, dx \right)^{1/2} \left(\int_D |z_2(x;v)|^2\, dx \right)^{1/2} \right.$$

$$+ \frac{1}{|\mu_1|^2} \left(\int_\Gamma \left| Q_3^{-1} \frac{\partial z_1(\cdot;v)}{\partial \nu} \right|^2 d\Gamma \right)^{1/2} \left(\int_\Gamma \left| \frac{\partial z_1(\cdot;v)}{\partial \nu} \right|^2 d\Gamma \right)^{1/2} + \left(\int_\Gamma |Q_4^{-1} z_1(\cdot;v)|^2\, d\Gamma \right)^{1/2} \left(\int_\Gamma |z_1(\cdot;v)|^2\, d\Gamma \right)^{1/2}$$

$$+ \left(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \left(r^1_{j_1}(x) \right)^{-2} |v^1_{j_1}(x)|^2 \right)^{1/2} \left(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} |v^1_{j_1}(x)|^2\, dx \right)^{1/2} \Bigg\}^{1/2} + \left(\sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \left(r^2_{j_2}(x) \right)^{-2} |v^2_{j_2}(x)|^2 \right)^{1/2} \left(\sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} |v^2_{j_2}(x)|^2\, dx \right)^{1/2} \Bigg\}^{1/2}$$

$$\leq C \left\{ \int_D |z_1(x;u)|^2\, dx + \int_{D_0} |z_2(x;u)|^2\, dx + \int_\Gamma \left| \frac{\partial z_1(\cdot;u)}{\partial \nu} \right|^2 d\Gamma + \int_\Gamma |z_1(\cdot;u)|^2\, d\Gamma + \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} |u^1_{j_1}(x)|^2\, dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} |u^2_{j_2}(x)|^2\, dx \right\}^{1/2}$$

$$\times \left\{ \int_D |z_1(x;v)|^2\, dx + \int_{D_0} |z_2(x;v)|^2\, dx + \int_\Gamma \left| \frac{\partial z_1(\cdot;v)}{\partial \nu} \right|^2 d\Gamma + \int_\Gamma |z_1(\cdot;v)|^2\, d\Gamma + \sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} |v^1_{j_1}(x)|^2\, dx + \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} |v^2_{j_2}(x)|^2\, dx \right\}^{1/2}$$

$$\leq C \big\{ \|z_1(\cdot;u)\|^2_{H^2(D)} + \|z_2(\cdot;u)\|^2_{H^2(D_0)} + \|\gamma_0 z_1(\cdot;u)|^2_{L^2(\Gamma)} + \|\gamma_1 z_1(\cdot;u)|^2_{L^2(\Gamma)} + \|u\|^2_H \big\}^{1/2}$$

$$\times \big\{ \|z_1(\cdot;v)\|^2_{H^2(D)} + \|z_2(\cdot;v)\|^2_{H^2(D_0)} + \|\gamma_0 z_1(\cdot;v)|^2_{L^2(\Gamma)} + \|\gamma_1 z_1(\cdot;v)|^2_{L^2(\Gamma)} + \|v\|^2_H \big\}^{1/2}, \quad (55)$$

where $\gamma_0 z_1(\cdot;u) = z_1(\cdot;u)|_\Gamma, \quad \gamma_1 z_1(\cdot;u) = \dfrac{\partial z_1(\cdot;u)}{\partial\nu}|_\Gamma,$

$$C = \max\{\epsilon_1^2\|Q_1^{-1}\|,\ \epsilon_1^2\|Q_2^{-1}\|,\ \epsilon_1^2\frac{1}{|\mu_1|^2}\|Q_3^{-1}\|,\ \epsilon_1^2\|Q_4^{-1}\|,\ \epsilon_2^2\beta_1, \epsilon_2^2\beta_2\},$$

$$\beta_1 := \max_{1\le j_1\le m_1}\max_{x\in\bar\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2} > 0, \quad \beta_2 := \max_{1\le j_2\le m_2}\max_{x\in\bar\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2} > 0.$$

Applying inequality (19) with some fixed $R = R_0$, we find

$$\begin{aligned}\|z_1(\cdot;u)\|_{H^2(D)} + \|z_2(\cdot;u)\|_{H^2(D_0)} &\le |z_1(\cdot;u)\|_{H^2(D)} + \|z_2(\cdot;u)\|_{H^2(D_{R_0}\setminus\bar D)}\\ &\le \alpha_0\left(\sum_{i_1=1}^{m_1}\left\|(C^1_{i_1})^* u^1_{i_1}\right\|_{L^2(\Omega^1_{i_1})} + \sum_{i_2=1}^{m_2}\left\|(C^1_{i_2})^* u^2_{i_2}\right\|_{L^2(\Omega^2_{i_2})}\right) \le c_1\|u\|_H, \quad (56)\end{aligned}$$

where $c_1 > 0$ is a constant. The trace theorem and inequality (56) imply

$$\|\gamma_0 z_1(\cdot;u)\|_{L^2(\Gamma)} + \|\gamma_1 z_1(\cdot;u)\|_{L^2(\Gamma)} \le c_2\|z_1(\cdot;u)\|_{H^2(D)} \le c_3\|u\|_H, \quad (57)$$

where c_2 and c_3 are constants. From (55)–(56) we have

$$\begin{aligned}|\pi(u,v)| &\le C(c_1\|u\|^2_H + c_3\|u\|^2_H + \|u\|^2_H)^{1/2} \times (c_1\|v\|^2_H + c_3\|v\|^2_H + \|v\|^2_H)^{1/2}\\ &\le c\|u\|_H\|v\|_H,\end{aligned}$$

where $c = $ const. Thus inequality (54) and, consequently, the continuity of the form $\pi(u,v)$ is proved.

In line with Remark 1.4 to Theorem 1.1 proved in [5], p. 11, the latter statements imply the existence of the unique element $\hat u \in H$ such that

$$I(\hat u) = \inf_{u\in H} I(u).$$

Therefore, for any $\tau_1, \tau_2 \in \mathbb{R}$ and $v \in H$ the relations

$$\frac{d}{d\tau_1} I(\hat u + \tau_1 v)\Big|_{\tau_1=0} = 0 \quad \text{and} \quad \frac{d}{d\tau_2} I(\hat u + i\tau_2 v)\Big|_{\tau_2=0} = 0 \quad (58)$$

hold. Since we have the relations $z_1(x;\hat u + \tau_1 v) = z_1(x;\hat u) + \tau_1 z_1(x;v)$ and $z_2(x;\hat u + \tau_1 v) = z_2(x;\hat u) + \tau_1 z_2(x;v)$, the first relation in (58) yields

$$0 = \frac{1}{2}\frac{d}{d\tau_1} I(\hat u + \tau_1 v)\Big|_{\tau_1=0}$$

$$= \epsilon_1 \mathrm{Re}\Bigg(\int_D Q_1^{-1}(l_1 + z_1(\cdot;\hat u))(x)\overline{z_1(x;v)}\,dx + \int_{D_0} Q_2^{-1}(l_2 + z_2(\cdot;\hat u))(x)\overline{z_2(x;v)}\,dx$$

$$+ \int_\Gamma Q_3^{-1}\Big(l_3 + \frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;\hat u)}{\partial\nu}\Big)\overline{\Big(\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\Big)}\,d\Gamma - \int_\Gamma Q_4^{-1}\Big(l_4 - z_1(\cdot;\hat u)\Big)\overline{z_1(\cdot;\tilde v)}\,d\Gamma\Bigg)$$

$$+\epsilon_2 \mathrm{Re}\Big\{\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}\hat u^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}\hat u^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\Bigg).$$

Similarly, taking into account that $z_1(x;\hat u+i\tau_2 v)=z_1(x;\hat u)+i\tau_2 z_1(x;v)$ and $z_2(x;\hat u+i\tau_2 v)=z_2(x;\hat u)+i\tau_2 z_2(x;v)$, we find from the second relation in (58)

$$0=\frac{1}{2}\frac{d}{d\tau_2}I(\hat u+i\tau_2 v)\Big|_{\tau_2=0}$$

$$=\epsilon_1 \mathrm{Im}\Bigg(\int_D Q_1^{-1}(l_1+z_1(\cdot;\hat u))(x)\overline{z_1(x;v)}\,dx+\int_{D_0}Q_2^{-1}(l_2+z_2(\cdot;\hat u))(x)\overline{z_2(x;v)}\,dx$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;\hat u)}{\partial\nu}\Big)\overline{\Big(\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\Big)}\,d\Gamma-\int_\Gamma Q_4^{-1}\Big(l_4-z_1(\cdot;\hat u)\Big)\overline{z_1(\cdot;\tilde v)}\,d\Gamma\Bigg)$$

$$+\epsilon_2 \mathrm{Im}\Big\{\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}\hat u^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}\hat u^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\Bigg);$$

consequently,

$$\epsilon_1\Bigg(\int_D Q_1^{-1}(l_1+z_1(\cdot;\hat u))(x)\overline{z_1(x;v)}\,dx+\int_{D_0}Q_2^{-1}(l_2+z_2(\cdot;\hat u))(x)\overline{z_2(x;v)}\,dx \quad (59)$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;\hat u)}{\partial\nu}\Big)\overline{\Big(\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\Big)}\,d\Gamma-\int_\Gamma Q_4^{-1}\Big(l_4-z_1(\cdot;\hat u)\Big)\overline{z_1(\cdot;\tilde v)}\,d\Gamma\Bigg)$$

$$+\epsilon_2\Big\{\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}\hat u^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}\hat u^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\Bigg)=0.$$

We introduce a pair of functions $(p_1,p_2)\in H^1(D,\Delta)\times H^1_{\mathrm{loc}}(\mathbb{R}^n\setminus D,\Delta)$ as the unique solution to the BVP

$$-(\Delta+k_1^2)p_1(x)=\epsilon_1 Q_1^{-1}(l_1+z_1(\cdot;\hat u))(x)\quad\text{in}\quad D, \tag{60}$$

$$-(\Delta+k_2^2)p_2(x)=\epsilon_1\chi_{D_0}(x)Q_2^{-1}(l_2+z_2(\cdot;\hat u))(x)\quad\text{in}\quad \mathbb{R}^n\setminus\bar D, \tag{61}$$

$$\mu_2 p_2-\mu_1 p_1=\epsilon_1 Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;\hat u)}{\partial\nu}\Big)\quad\text{on}\quad\Gamma, \tag{62}$$

$$\frac{\partial p_2}{\partial\nu}-\frac{\partial p_1}{\partial\nu}=\epsilon_1 Q_4^{-1}(l_4-z_1(\cdot;\hat u))\quad\text{on}\quad\Gamma, \tag{63}$$

$$\frac{\partial p_2}{\partial r}-ik_2p_2=o(1/r^{(n-1)/2}),\quad r=|x|,\quad r\to\infty, \tag{64}$$

Then

$$\epsilon_1\Bigg(\int_D Q_1^{-1}(l_1+z_1(\cdot;\hat u))(x)\overline{z_1(x;v)}\,dx+\int_{D_0} Q_2^{-1}(l_2+z_2(\cdot;\hat u))(x)\overline{z_2(x;v)}\,dx$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;\hat u)}{\partial\nu}\Big)\overline{\Big(\frac{1}{\bar\mu_1}\frac{\partial z_1(\cdot;v)}{\partial\nu}\Big)}\,d\Gamma-\int_\Gamma Q_4^{-1}\Big(l_4-z_1(\cdot;\hat u)\Big)\overline{z_1(\cdot;\tilde v)}\,d\Gamma\Bigg)$$

$$=\int_D(-\Delta p_1(x)-k_1^2p_1(x))\overline{z_1(x;v)}\,dx+\int_{D_R\setminus\bar D}(-\Delta p_2-k_2^2p_2(x))\overline{z_2(x;v)}\,dx$$

$$+\int_\Gamma\Big(\frac{\mu_2}{\mu_1}p_2-p_1\Big)\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma-\int_\Gamma\Big(\frac{\partial p_2}{\partial\nu}-\frac{\partial p_1}{\partial\nu}\Big)\overline{z_1(\cdot;\tilde v)}\,d\Gamma$$

$$=\int_D\overline{(-\Delta z_1(x;v)-\bar k_1^2z_1(x;v))}p_1(x)\,dx+\int_{D_R\setminus\bar D}\overline{(-\Delta z_2(x;v)-\bar k_2^2z_2(x;v))}p_2(x)\,dx$$

$$-\int_\Gamma\frac{\partial p_1}{\partial\nu}\overline{z_1(\cdot;v)}\,d\Gamma+\int_\Gamma\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}p_1\,d\Gamma+\int_\Gamma\frac{\partial p_2}{\partial\nu}\overline{z_2(\cdot;v)}\,d\Gamma-\int_\Gamma\overline{\frac{\partial z_2(\cdot;v)}{\partial\nu}}p_2\,d\Gamma$$

$$+\int_\Gamma\Big(\frac{\mu_2}{\mu_1}p_2-p_1\Big)\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma-\int_\Gamma\Big(\frac{\partial p_2}{\partial\nu}-\frac{\partial p_1}{\partial\nu}\Big)\overline{z_1(\cdot;v)}\,d\Gamma$$

$$=-\int_D\overline{\sum_{j_1=1}^{m_1}\chi_{\Omega^1_{j_1}}(x)(C^1_{j_1})^*v^1_{j_1}(x)}p_1(x)\,dx-\int_{D_R\setminus\bar D}\overline{\sum_{j_2=1}^{m_2}\chi_{\Omega^2_{j_2}}(x)(C^2_{j_2})^*v^2_{j_2}(x)}p_2(x)\,dx$$

$$+\int_\Gamma\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}p_1\,d\Gamma-\int_\Gamma\overline{\frac{\partial z_2(\cdot;v)}{\partial\nu}}p_2\,d\Gamma+\int_\Gamma\Big(\frac{\mu_2}{\mu_1}p_2-p_1\Big)\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma$$

$$=-\sum_{i_1=1}^{m_1}\int_{\Omega^1_{i_1}}\overline{(C^1_{i_1})^*v^1_{i_1}(x)}p_1(x)\,dx-\sum_{i_2=1}^{m_2}\int_{\Omega^2_{i_2}}\overline{C^2_{i_2}v^2_{i_2}(x)}p_2(x)\,dx$$

$$+\int_\Gamma\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}p_1\,d\Gamma-\int_\Gamma\frac{\mu_2}{\mu_1}\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}p_2\,d\Gamma+\int_\Gamma\frac{\mu_2}{\mu_1}p_2\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma-\int_\Gamma p_1\overline{\frac{\partial z_1(\cdot;v)}{\partial\nu}}\,d\Gamma$$

$$=-\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{v^1_{j_1}(x)}C^1_{j_1}p_1(x)\,dx-\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{v^2_{j_2}(x)}C^2_{i_2}p_2(x)\,dx. \qquad (65)$$

From (59) and (65) it follows that

$$+\epsilon_2\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\left(r^1_{j_1}(x)\right)^{-2}\hat u^1_{j_1}(x)\overline{v^1_{j_1}(x)}\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\left(r^2_{j_2}(x)\right)^{-2}\hat u^2_{j_2}(x)\overline{v^2_{j_2}(x)}\,dx\Bigg)$$

$$=\sum_{j_1=1}^{m_1}\int_{\Omega^1_{j_1}}\overline{v^1_{j_1}(x)}C^1_{j_1}p_1(x)\,dx+\sum_{j_2=1}^{m_2}\int_{\Omega^2_{j_2}}\overline{v^2_{j_2}(x)}C^2_{j_2}p_2(x)\,dx.$$

Hence,

$$\hat u^1_{j_1}(x)=\epsilon_2^{-1}(r^1_{j_1}(x))^2C^1_{j_1}p_1(x),\quad j_1=1,\dots,m_1,$$

$$\hat u^2_{j_2}(x)=\epsilon_2^{-1}(r^2_{j_2}(x))^2C^2_{j_2}p_2(x),\quad j_2=1,\dots,m_2.$$

Setting $u = \hat{u} = (\hat{u}_1^1, \ldots, \hat{u}_{m_1}^1, \hat{u}_1^2, \ldots, \hat{u}_{m_2}^2)$ in (14)–(18) and (29), taking into account equations (60)–(64), and denoting $\hat{z}_1(x) = z_1(x;\hat{u})$, $\hat{z}_2(x) = z_2(x;\hat{u})$, we see that $\hat{z}_1(x)$, $\hat{z}_2(x)$ and $p_1(x)$, $p_2(x)$ satisfy system (31)–(40); the unique solvability of this system follows from the fact that the functional $I(u)$ has one minimum point $\hat{u}$.

Now we establish the validity of formula (51). We have

$$\sigma^2 = \sigma(\hat{u},\hat{c}) = I(\hat{u}) = \tag{66}$$

$$= \epsilon_1\Bigg(\int_D Q_1^{-1}(l_1+\hat{z}_1)(x)\overline{(l_1(x)+\hat{z}_1(x))}\,dx + \int_{D_0} Q_2^{-1}(l_2+\hat{z}_2)(x)\overline{(l_2(x)+\hat{z}_2(x))}\,dx$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar{\mu}_1}\frac{\partial\hat{z}_1}{\partial\nu}\Big)\overline{\Big(l_3+\frac{1}{\bar{\mu}_1}\frac{\partial\hat{z}_1}{\partial\nu}\Big)}\,d\Gamma + \int_\Gamma Q_4^{-1}\Big(l_4-\hat{z}_1\Big)\overline{\Big(l_4-\hat{z}_1\Big)}\,d\Gamma\Bigg)$$

$$+\epsilon_2^{-1}\Bigg(\sum_{j_1=1}^{m_1}\int_{\Omega_{j_1}^1}\big(r_{j_1}^1(x)\big)^2\Big|C_{j_1}^1 p_1(x)\,dx\Big|^2dx + \sum_{j_2=1}^{m_2}\int_{\Omega_{j_2}^2}\big(r_{j_2}^2(x)\big)^2\Big|\int_{\Omega_{j_2}^2}C_{j_2}^2 p_2(x)\,dx\Big|^2dx\Bigg).$$

We transform the sum of the first four terms on the right-hand side of (66). Making use of equations (31)–(40) we obtain

$$\epsilon_1\Bigg(\int_D Q_1^{-1}(l_1+\hat{z}_1)(x)\overline{(l_1(x)+\hat{z}_1(x))}\,dx + \int_{D_0} Q_2^{-1}(l_2+\hat{z}_2)(x)\overline{(l_2(x)+\hat{z}_2(x))}\,dx$$

$$+\int_\Gamma Q_3^{-1}\Big(l_3+\frac{1}{\bar{\mu}_1}\frac{\partial\hat{z}_1}{\partial\nu}\Big)\overline{\Big(l_3+\frac{1}{\bar{\mu}_1}\frac{\partial\hat{z}_1}{\partial\nu}\Big)}\,d\Gamma + \int_\Gamma Q_4^{-1}\Big(l_4-\hat{z}_1\Big)\overline{\Big(l_4-\hat{z}_1\Big)}\,d\Gamma\Bigg)$$

$$= l(P) - \int_D(\Delta p_1(x)+k_1^2p_1(x))\overline{\hat{z}_1(x)}\,dx - \int_{\mathbb{R}^n\setminus D}(\Delta p_2(x)+k_2^2p_2(x))\overline{\hat{z}_2(x)}\,dx$$

$$+\int_\Gamma\Big(\frac{\mu_2}{\mu_1}p_2-p_1\Big)\overline{\frac{\partial\hat{z}_1}{\partial\nu}}\,d\Gamma - \int_\Gamma\Big(\frac{\partial p_2}{\partial\nu}-\frac{\partial p_1}{\partial\nu}\Big)\overline{\hat{z}_1}\,d\Gamma$$

$$= l(P) - \int_D\overline{(\Delta\hat{z}_1(x)+\bar{k}_1^2\hat{z}_1(x))}p_1(x)\,dx - \int_{\mathbb{R}^n\setminus D}\overline{(\Delta\hat{z}_2(x)+\bar{k}_2^2\hat{z}_2(x))}p_2(x)\,dx$$

$$-\int_\Gamma\frac{\partial p_1}{\partial\nu}\overline{\hat{z}_1}\,d\Gamma + \int_\Gamma\overline{\frac{\partial\hat{z}_1}{\partial\nu}}p_1\,d\Gamma + \int_\Gamma\frac{\partial p_2}{\partial\nu}\overline{\hat{z}_2}\,d\Gamma - \int_\Gamma\overline{\frac{\partial\hat{z}_2}{\partial\nu}}p_2\,d\Gamma$$

$$+\int_\Gamma\Big(\frac{\mu_2}{\mu_1}p_2-p_1\Big)\overline{\frac{\partial\hat{z}_1}{\partial\nu}}\,d\Gamma - \int_\Gamma\Big(\frac{\partial p_2}{\partial\nu}-\frac{\partial p_1}{\partial\nu}\Big)\overline{\hat{z}_1}\,d\Gamma$$

$$= l(P) - \int_D \overline{\sum_{j_1=1}^{m_1} \chi_{\Omega^1_{j_1}}(x)(C^1_{j_1})^* \hat{u}^1_{j_1}(x) p_1(x)\,dx}$$

$$- \int_{\mathbb{R}^n \setminus D} \overline{\sum_{j_2=1}^{m_2} \chi_{\Omega^2_{j_2}}(x)(C^2_{j_2})^* \hat{u}^2_{j_2}(x) p_2(x)\,dx}$$

$$+ \int_\Gamma \overline{\frac{\partial \hat{z}_1}{\partial \nu}} p_1\,d\Gamma - \int_\Gamma \overline{\frac{\partial \hat{z}_2}{\partial \nu}} p_2\,d\Gamma + \int_\Gamma \Big(\frac{\mu_2}{\mu_1} p_2 - p_1\Big) \overline{\frac{\partial \hat{z}_1}{\partial \nu}}\,d\Gamma$$

$$= l(P) - \epsilon_2^{-1} \Bigg(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^2 \Big| C^1_{j_1} p_1(x)\,dx \Big|^2 dx$$

$$+ \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^2 \Big| \int_{\Omega^2_{j_2}} C^2_{j_2} p_2(x)\,dx \Big|^2 dx \Bigg)$$

$$+ \int_\Gamma \overline{\frac{\partial \hat{z}_1}{\partial \nu}} p_1\,d\Gamma - \int_\Gamma \frac{\mu_2}{\mu_1} \overline{\frac{\partial \hat{z}_1}{\partial \nu}} p_2\,d\Gamma + \int_\Gamma \Big(\frac{\mu_2}{\mu_1} p_2 - p_1\Big) \overline{\frac{\partial \hat{z}_1}{\partial \nu}}\,d\Gamma$$

$$= l(P) - \epsilon_2^{-1} \Bigg(\sum_{j_1=1}^{m_1} \int_{\Omega^1_{j_1}} \big(r^1_{j_1}(x)\big)^2 \Big| C^1_{j_1} p_1(x)\,dx \Big|^2 dx$$

$$+ \sum_{j_2=1}^{m_2} \int_{\Omega^2_{j_2}} \big(r^2_{j_2}(x)\big)^2 \Big| \int_{\Omega^2_{j_2}} C^2_{j_2} p_2(x)\,dx \Big|^2 dx \Bigg). \tag{67}$$

Finally, from (66) and (67) we obtain $\sigma = l(P)^{1/2}$. The proof of the rest part of this theorem is similar to the proof of the first part. □

Remark 3.3. Notice that in the representation $\widehat{\widehat{l(F)}} = l(\hat{F})$ of the minimax mean square estimate of $l(F)$ function $\hat{F}$ does not depend on specific form of functional l. Therefore, the functions

$$\hat{f}_1(x) = \epsilon_1 Q_1^{-1} \hat{p}_1(x) + f_1^0(x), \quad \hat{f}_2(x) = \epsilon_1 Q_2^{-1} \hat{p}_2(x) + f_2^0(x),$$

and

$$\hat{g}_1 = \epsilon_1 \frac{1}{\bar{\mu}_1} Q_3^{-1} \frac{\partial \hat{p}_1}{\partial \nu} + g_1^0, \quad \hat{g}_2 = -\epsilon_1 Q_4^{-1} \hat{p}_1|_\Gamma + g_2^0$$

can be taken as good estimates of the right-hand sides f_1, f_2, g_1, and g_2 of equations (3)–(6), respectively.

Remark 3.4. Problems (31)–(40) and (41)–(50) can be solved numerically using a coupled FEM-BEM approach.

Remark 3.5. Analogous results can be obtained for the estimation problem in which instead of observations (1) and (2) one can use pointwise observations of unknown solutions or observations distributed on a certain system of surfaces located in the domains D and $\mathbb{R}^n \setminus \bar{D}$.

All the results of the present paper can also be generalized to this case of more general observations of the form

$$y^1_{j_1} = C^1_{j_1}\varphi_1 + \xi^1_{j_1}, \quad j_1 = 1, \ldots, m_1,$$
$$y^2_{j_2} = C^2_{j_2}\varphi_2 + \xi^2_{j_2}, \quad j_2 = 1, \ldots, m_2,$$

where $C^i_{j_i} \in \mathcal{L}(L^2(\Omega^i_{j_i}), H^i_{j_i})$, $i = 1, 2$, are linear continuous operators, $H^i_{j_i}$, are some Hilbert spaces, $\xi^i_{j_i}$ are realizations of random variables with values in $H^i_{j_i}$.

Remark 3.6. All the results of the paper remain valid if we assume that in the definition of the set G_1 the random fields $\xi^1_{j_1}$, $j_1 = 1, \ldots, m_1$, and $\xi^2_{j_2}$, $j_2 = 1, \ldots, m_2$, are pairwise uncorrelated and, instead of condition (9), the following conditions

$$\int_{\Omega^1_{j_1}} (r^1_{j_1}(x))^2 \tilde{R}^1_{j_1}(x, x)\, dx \leq \epsilon_1, \quad j_1 = 1, \ldots, m_1,$$
$$\int_{\Omega^2_{j_2}} (r^2_{j_2}(x))^2 \tilde{R}^2_{j_2}(x, x)\, dx \leq \epsilon_2, \quad j_2 = 1, \ldots, m_2.$$

are fulfilled.

Remark 3.7. Using the technique elaborated in the present paper in combination with Lagrange multipliers method, we can generalize all the obtained results to the case when the restrictions on unknown deterministic functions $f_1(x)$, $f_2(x)$, g_1, and g_2 in equations (3)–(6) are given partly, for example, when the information about right-hand sides g_1, and g_2 of interface conditions (5) and (6) is absent and right-hand sides $f_1(x)$ and $f_2(x)$ of equations (3) and (4) are subject to quadratic restrictions.

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