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Summation Bounds for Integral Powers of Metallic Ratios

S.U. Malini[1], T.V. Sateesh Kumar[2*], R. Sivaraman[3], J. Suganthi[4]

[1]Assistant Professor, Sri Ramachandra Faculty of Engineering and Technology, Sri Ramachandra Institute of Higher Education and Research, Chennai – 600 116, Email: malini@sriramachandra.edu.in
[2]Assistant Professor, Sri Ramachandra Faculty of Engineering and Technology, Sri Ramachandra Institute of Higher Education and Research, Chennai – 600 116, Email: sateeshkumarassistantprofessor@gmail.com
[3]Associate Professor, Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, India, Email: rsivaraman1729@yahoo.co.in
[4]J. Suganthi, Assistant Professor and Head, Department of Mathematics, S.S.K.V. College of Arts and Science for Women, Kilambi, Kanchipuram, Tamil Nadu ,Email: suganthi271076@gmail.com
*Corresponding Author

Abstract
The problem of determining the sum of powers of natural numbers has long intrigued mathematicians. In this paper, we extend this classical study to irrational numbers by focusing on metallic ratios. We propose a method to derive upper and lower bounds for the sum of the first n terms of first four powers of metallic ratios by using their recurrence relations and known approximations. These bounds are then employed to obtain exact expressions for the summation of the floor and ceiling functions applied to these powers. Our results show that the leading and sub-leading coefficients in the derived expressions closely resemble those in classical power sums of natural numbers. This resemblance highlights a surprising structural harmony between rational and irrational summation behaviors, offering new insights into the additive properties of metallic ratios.

Keywords: Metallic Ratios, Recurrence Relation, Floor Function, Ceiling Function, Upper and Lower Bounds,

1. INTRODUCTION

The paper extends classical results on the summation of natural number powers to the domain of metallic ratios, a family of irrational numbers defined as the positive roots of $x^2 = qx + 1$, for $q \in \mathbb{N}$. While Faulhaber, Euler, and Riemann historically developed techniques for summing powers over integers and even complex numbers, direct summation involving irrational values like metallic ratios remains challenging. Recent studies have shown that metallic ratios share structural properties with Fibonacci-type sequences, and they have inspired growing interest in generalizing classical summation techniques. This paper addresses the challenge by deriving upper and lower bounds for the sum of the first n terms of the first four powers of metallic ratios, and then uses these bounds to construct exact expressions for summations involving floor and ceiling functions of these powers.

A notable outcome is the observation that the resulting formulas' coefficients bear a resemblance to those found in classical power sums, hinting at a deeper structural parallel between rational and irrational number summations.While classical approaches focus on rational and integer sequences, contemporary studies have turned toward exploring similar summation properties in the realm of irrational numbers. One prominent family of such numbers is the class of metallic ratios, which generalizes the golden ratio. These are defined as the positive roots of quadratic equations of the form$x^2 = qx + 1$, for $q \in \mathbb{N}$,and yield well-known constants such as the silver, bronze, and copper ratios (1), (2). The summation of powers of natural numbers is a well-established topic in mathematics, with historical roots dating back to early European scholars. Johann Faulhaber was among the first to derive general formulas for such summations, now known as Faulhaber's formulas, which express the sum of powers as polynomials in n. These were later generalized by Leonhard Euler, who connected them to Bernoulli numbers and extended their validity to negative integers (3). Bernhard Riemann further broadened the scope of these ideas by introducing the concept of analytic continuation via the Riemann zeta function, enabling summation over complex exponents (4).

The metallic ratios are not only structurally intriguing but also exhibit recurrence relations and continued fraction representations similar to those of Fibonacci and Lucas numbers (5). As such, they naturally arise in recursive sequences, geometry, and even aesthetic design. Analytical studies have been conducted on the series expansions and generating functions associated with metallic ratios (6), (7). However, due to their irrational

nature, computing exact sums of their powers remains challenging, especially when involving integer-valued functions such as floor and ceiling.

In this paper, we address these challenges by analyzing the sum of the first n terms of first four powers of metallic ratios. Our approach begins with establishing tight upper and lower bounds for these summations using recurrence properties and analytical approximations. We then leverage these bounds to derive exact closed-form expressions for the summation of the floor and ceiling functions applied to these powers, thus converting irrational components into discrete integer-valued expressions.

One key result is the observation that the leading and sub-leading coefficients in these summation formulas exhibit structural similarity to those found in the classical sum of powers of natural numbers:

$$\sum_{i=1}^{n} i^{p} = \frac{n^{p+1}}{p+1} + \frac{n^{p}}{2} + \cdots,$$

which strongly suggests a deeper algebraic connection between rational and irrational additive sequences (8)

The main contributions of this paper are:

- Computation of upper and lower bounds for the sums of first four powers of metallic ratios.
- Formulation of exact expressions for the sum of floor and ceiling function values applied to these powers.
- Demonstration of structural analogies between classical power sum polynomials and the derived expressions involving metallic ratios.

The study of metallic ratios has evolved significantly in recent years, with researchers exploring their algebraic, geometric, and analytic properties. Gil and Worley introduced the concept of generalized metallic means, extending traditional metallic ratios through invert transforms on Fibonacci-like sequences, which allowed for the definition of higher-degree algebraic metallic constants (9). Rajput examined the geometric interpretations of metallic ratios by associating them with right triangles that satisfy specific proportional conditions, forming what he termed as ``triads" that relate to Pythagorean triples (10).

Sivaraman investigated summations involving powers of metallic ratios and developed bounds using recurrence relations. He further linked these results to Ramanujan summation techniques and noted their similarity to classical power summation formulas (11). Assiry and Chandoul studied the continued fraction expansions of metallic ratios, proving that they take the purely periodic form $[k, k, k, \dots]$. They demonstrated that these expansions are closely related to solutions of Pell-type Diophantine equations (12). In a further geometric elaboration, Rajput provided structural justifications for the construction of metallic-ratio-based triangles and explored their implications in discrete geometry (13).

Gil and Worley later extended their earlier work to analyze the behavior of generalized metallic means under different recurrence systems and transformations, thereby deepening the theoretical foundation of metallic ratios (14). De Spinadel provided important insights into the practical and aesthetic applications of metallic ratios, especially in architectural design, tilings, and visual harmony, highlighting their role in both classical and contemporary structures (15). Finally, Özkan and Yilmaz explored metallic structures on differentiable manifolds, defining them through endomorphisms satisfying the relation $J^2 = aJ + bI$, and demonstrated their relevance in differential geometry and polynomial structure theory (16), (17)

Collectively, these studies illustrate the rich mathematical landscape that metallic ratios inhabit, ranging from continued fractions and Diophantine identities to recurrence relations, geometry, and even manifold theory. This survey provides a comprehensive background for our current study, which builds upon these developments to derive precise bounds and closed-form expressions for summations involving the floor and ceiling functions of metallic ratio powers.

This manuscript is written as follows: Section 2 outlines the basic definitions and notations related to metallic ratios and integer functions. Sections 3 through 6 present and prove the main theorems for bounds and summations for powers from one through four. Section 7 discusses the implications of our results and possible extensions to higher powers or other irrational sequences.

2. PROPOSED METHOD

Overcoming the computing challenge presented by their irrationality, the suggested technique concentrates on assessing the sum of the first n n terms of the first four powers of metallic ratios. Starting with strict upper and

lower constraints for these sums, the method uses existing analytical approximations and recurrence relations fulfilled by metallic ratios. Approximating the behavior of irrational sums in a systematic and predictable way requires certain limits.Following the formulation of the limits, the approach derives accurate closed-form equations for the summation of the floor and ceiling functions applied to the powers of metallic ratios. This stage converts the irrational elements into integer-based formulations, therefore enabling computable and interpretable sums. Every power—first through fourth—is handled independently, with the boundaries allowing exact assessment of floor and ceiling performance throughout every term in the sequence.Especially in the matching of leading and sub-leading polynomial coefficients, the suggested technique not only produces correct summation formulae but also uncovers structural similarities with classical summations of natural number powers. This suggests that, particularly when bound and discretized, the metallic ratio sequences—despite their irrationality—behaviorally resemble integer sequences under summing operations.

2.1 The qth term of sequence of metallic ratios denoted by M_q is defined as the positive root of the equation $x^2 - qx - 1 = 0$ (1). Thus $M_q = \frac{q+\sqrt{q^2+4}}{2}$ (2) and $-\frac{1}{M_q} = \frac{q-\sqrt{q^2+4}}{2}$ (3).

2.2 The real valued functionof a real number x denoted by $\lfloor x \rfloor$ is defined as the greatest integer $\leq x$.

2.3 The real valued functionof a real number xdenoted by $\lceil x \rceil$ is defined as the least integer $\geq x$.

In the following sections, we will prove interesting results regarding obtaining bounds for summation of positive integral powers of metallic ratios.

3. Theorem 1 (Bounds for Summation of first power of metallic ratios)

If M_q is the qth metallic ratio then $\frac{p(p+1)}{2} \leq \sum_{q=1}^{p} M_q \leq \frac{p(p+3)}{2}$ (4)

In particular, $\sum_{q=1}^{p} \lfloor M_q \rfloor = \frac{p(p+1)}{2}$ (5) and $\sum_{q=1}^{p} \lceil M_q \rceil = \frac{p(p+3)}{2}$ (6)

Proof: If q is any natural number then we have $q^2 \leq q^2 + 4$. Thus, $M_q = \frac{q+\sqrt{q^2+4}}{2} \geq \frac{q+q}{2} = q$.

Similarly, $q^2 + 4 \leq (q+2)^2$. Hence, $M_q = \frac{q+\sqrt{q^2+4}}{2} \leq q+1.$

Thus, $q \leq M_q \leq q+1$ (7).

From (7), we get $\sum_{q=1}^{p} q \leq \sum_{q=1}^{p} M_q \leq \sum_{q=1}^{p} (q+1) = \sum_{q=1}^{p} q + p$

In view of**[3]**, **[4]**we observe that

$\frac{p(p+1)}{2} \leq \sum_{q=1}^{p} M_q \leq \frac{p(p+1)}{2} + p = \frac{p(p+3)}{2}$. This proves (4)

Moreover from (7), we have $\lfloor M_q \rfloor = q, \lceil M_q \rceil = q+1$ (8)

Hence, $\sum_{q=1}^{p}\lfloor M_q \rfloor = \sum_{q=1}^{p} q = \frac{p(p+1)}{2}$ and $\sum_{q=1}^{p}\lceil M_q \rceil = \sum_{q=1}^{p}(q+1) = \sum_{k=1}^{n} q + p = \frac{p(p+3)}{2}$

This completes the proof.

4. Theorem 2 (Bounds for Summation of second power of metallic ratios)

If M_q is the qth metallic ratio then $\frac{p(2p^2+3p+7)}{6} \le \sum_{q=1}^{p} M_q^2 \le \frac{p(p^2+3p+5)}{3}$ (9)

In particular, $\sum_{q=1}^{p}\lfloor M_q^2 \rfloor = \frac{p(2p^2+3p+7)}{6}$ (10) and $\sum_{q=1}^{p}\lceil M_q^2 \rceil = \frac{p(2p^2+3p+13)}{6}$ (11)

Proof: From (7) of theorem 1, for any natural number q, we get $q \le M_q \le q+1$. From this, we obtain $q^2+1 \le qM_q+1 \le q^2+q+1$. From (2), we find that $M_q^2 = qM_q+1$ (12). Hence, we see that $q^2+1 \le M_q^2 \le q^2+q+1$ (13).

From (13), we find that $\sum_{q=1}^{p}(q^2+1) \le \sum_{q=1}^{p} M_q^2 \le \sum_{q=1}^{p}(q^2+q+1)$

$$\sum_{q=1}^{p} q^2 + p \le \sum_{q=1}^{p} M_q^2 \le \sum_{q=1}^{p} q^2 + \sum_{q=1}^{n} q + p$$

In view of **[3]**, **[4]** we observe that

$\frac{p(2p^2+3p+7)}{6} \le \sum_{q=1}^{p} M_q^2 \le \frac{p(p^2+3p+5)}{3}$ This proves (9)

Since $q \ge 1$, from (13) we find that $q^2+1 \le M_q^2 \le q^2+2$.

Hence, we find that $\sum_{q=1}^{p}\lfloor M_q^2 \rfloor = \sum_{q=1}^{p}\left(q^2+1\right) = \sum_{q=1}^{p} q^2 + p = \frac{p(2p^2+3p+7)}{6}$ and

$$\sum_{q=1}^{p}\lceil M_q^2 \rceil = \sum_{q=1}^{p}\left(q^2+2\right) = \sum_{q=1}^{p} q^2 + 2p = \frac{p(2p^2+3p+13)}{6}$$

This completes the proof.

5. Theorem 3 (Bounds for Summation of third power of metallic ratios)

If M_q is the qth metallic ratio then

$$\frac{p(p+1)(p^2+p+4)}{4} \le \sum_{q=1}^{p} M_q^3 \le \frac{p(3p^3+10p^2+21p+26)}{12} \quad (14)$$

In particular, we have

$$\sum_{q=1}^{p}\left\lfloor M_q^3 \right\rfloor = \frac{p(p^3+2p^2+7p+6)}{4} \quad (15) \text{ and } \sum_{q=1}^{p}\left\lceil M_q^3 \right\rceil = \frac{p(p^3+2p^2+7p+10)}{4} \quad (16)$$

Proof: Using (12) from theorem 2, we get

$$M_q^3 = M_q^2 \times M_q = \left(qM_q+1\right)\times M_q = qM_q^2 + M_q = q\left(qM_q+1\right)+M_q = \left(q^2+1\right)M_q + q$$

Now from (7) of theorem 1, we get $q\left(q^2+1\right)+q \le \left(q^2+1\right)M_q + q \le (q+1)\left(q^2+1\right)+q$ Hence, we obtain $\left(q^3+2q\right) \le M_q^3 \le \left(q^3+q^2+2q+1\right)$ (17)

From (17) we see that $\sum_{q=1}^{p}\left(q^3+2q\right) \le \sum_{q=1}^{p} M_q^3 \le \sum_{q=1}^{p}\left(q^3+q^2+2q+1\right)$

In view of**[3]**, **[4]**we observe that

$$\frac{p^2(p+1)^2}{4} + p(p+1) \le \sum_{q=1}^{p} M_q^3 \le \frac{p^2(p+1)^2}{4} + \frac{p(p+1)(2p+1)}{6} + p(p+1) + p$$

Hence, $\frac{p(p+1)(p^2+p+4)}{4} \le \sum_{q=1}^{p} M_q^3 \le \frac{p(3p^3+10p^2+21p+26)}{12}$

This proves (14). Since $q \ge 1$ from (17) we find that

$$q^3+3q \le M_q^3 \le q^3+3q+1$$

Hence, we get

$$\sum_{q=1}^{p}\left\lfloor M_q^3 \right\rfloor = \sum_{q=1}^{p}\left(q^3+3q\right) = \frac{p^2(p+1)^2}{4} + \frac{3p(p+1)}{2} = \frac{p(p^3+2p^2+7p+6)}{4} \text{ and}$$

$$\sum_{q=1}^{p}\left\lceil M_q^3 \right\rceil = \sum_{q=1}^{p}\left(q^3+3q+1\right) = \frac{p^2(p+1)^2}{4} + \frac{3p(p+1)}{2} + p = \frac{p(p^3+2p^2+7p+10)}{4}$$

This proves (15) and (16) and hence completes the proof.

6. Theorem 4 (Bounds for Summation of fourth power of metallic ratios)

If M_q is the qth metallic ratio then

$$\frac{p\left(6p^4+15p^3+40p^2+45p+44\right)}{30} \le \sum_{q=1}^{p} M_q^4 \le \frac{p\left(12p^4+45p^3+110p^2+165p+148\right)}{60} \quad (18)$$

In particular, we have

$$\sum_{q=1}^{p}\left\lfloor M_q^4 \right\rfloor = \frac{p\left(6p^4+15p^3+50p^2+60p+49\right)}{30} \quad (19) \quad \text{and}$$

$$\sum_{q=1}^{p}\left\lceil M_q^4 \right\rceil = \frac{p\left(6p^4+15p^3+50p^2+60p+79\right)}{30} \quad (20)$$

Proof: Using (12) of theorem 2 and the value of M_q^3 from theorem 3, we get

$$M_q^4 = M_q^3 \times M_q = \left[\left(q^2+1\right)M_q + q\right] \times M_q = \left(q^2+1\right)\left(qM_q+1\right) + qM_q = \left(q^3+2q\right)M_q + \left(q^2+1\right)$$

Using (7) of theorem 1, we have

$$q\left(q^3+2q\right)+\left(q^2+1\right) \le \left(q^3+2q\right)M_q + \left(q^2+1\right) \le (q+1)\left(q^3+2q\right)+\left(q^2+1\right).$$

Hence we obtain $q^4+3q^2+1 \le M_q^4 \le q^4+q^3+3q^2+2q+1$ (21)

From (6.4), $\sum_{q=1}^{p}\left(q^4+3q^2+1\right) \le \sum_{q=1}^{p} M_q^4 \le \sum_{q=1}^{p}\left(q^4+q^3+3q^2+2q+1\right)$

In view of**[3]**, **[4]**we observe that

$$\frac{p(p+1)(2p+1)(3p^2+3p-1)}{30} + \frac{p(p+1)(2p+1)}{2} + p \le \sum_{q=1}^{p} M_q^4$$

$$\le \frac{p(p+1)(2p+1)(3p^2+3p-1)}{30} + \frac{p^2(p+1)^2}{4} + \frac{p(p+1)(2p+1)}{2} + p(p+1) + p$$

Hence, $\frac{p\left(6p^4+15p^3+40p^2+45p+44\right)}{30} \le \sum_{q=1}^{p} M_q^4 \le \frac{p\left(12p^4+45p^3+110p^2+165p+148\right)}{60}$

This proves (18). Since $q \ge 1$ from (21) we find that

$$q^4+4q^2+1 \le M_q^4 \le q^4+4q^2+2$$

Hence, we get

$$\sum_{q=1}^{p}\left\lfloor M_q^4 \right\rfloor = \sum_{q=1}^{p}\left(q^4+4q^2+1\right) = \frac{p\left(6p^4+15p^3+50p^2+60p+49\right)}{30} \text{ and}$$

$$\sum_{q=1}^{p}\left\lceil M_q^4 \right\rceil = \sum_{q=1}^{p}\left(q^4+4q^2+2\right) = \frac{p\left(6p^4+15p^3+50p^2+60p+79\right)}{30}$$

This proves (19) and (20) and hence completes the proof.

From (2), it is easy to see that the metallic ratios are irrational numbers. Hence it would be difficult to determine sum of first n terms of such numbers. But it is possible to obtain the lower and upper bounds for such sums. This is exactly done in this paper. In fact, equations (4), (9), (14) and (18) provide such bounds for the sum of first four powers of metallic ratios respectively. These bounds will provide a good estimate of the growth of metallic ratios when they are added up. Further, making use of the floor and ceiling functions, we had obtained exact formulas for sum of first n terms of first four powers of metallic ratios through equations (5), (6), (10), (11), (15), (16), (19), (20) through the theorems 1, 2, 3 and 4 of this paper. Moreover, we notice that just like as in sum of powers of first n natural numbers, the coefficients of highest degree terms in the lower and upper bounds

of sum of first four powers of metallic ratios of $M_q^r, \lfloor M_q^r \rfloor, \lceil M_q^r \rceil$ are $\frac{1}{r+1}$. Similarly, the coefficients of second highest degree terms of $\lfloor M_q^r \rfloor$ are $\frac{1}{2}$ just like in sum of first n natural numbers. These results and observations will provide new insights in understanding the properties of metallic ratios further.

CONCLUSION

In this paper, we investigated the summation behavior of powers of metallic ratios—an important family of irrational numbers defined as the positive roots of quadratic equations of the form $x^2 = qx + 1$. Due to their irrational nature, direct calculation of such summations presents significant challenges. To address this, we derived tight upper and lower bounds for the sum of the first nnn terms of the first four powers of metallic ratios, using recurrence relations and analytical approximations.Building upon these bounds, we established exact closed-form expressions for the summation of the floor and ceiling functions of these powers. These results not only provide accurate integer-valued approximations for inherently irrational sums but also reveal structural similarities to classical power summation formulas for natural numbers. Notably, the leading and second-leading coefficients in our formulas mirror those found in Faulhaber's expressions, indicating deeper algebraic relationships.Finding such expressions for general positive integral powers would be an interesting exercise and scope for further improvements to this paper. The findings of this study contribute to a broader understanding of summation techniques involving irrational sequences and may hold value in fields such as discrete mathematics, number theory, and mathematical modeling. Future research could extend these methods to higher-order powers, investigate links with continued fractions and Diophantine equations, or explore applications in geometry and algebraic structures such as manifold theory and symbolic computation.

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